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The weak Lefschetz property of equigenerated monomial ideals

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ABSTRACT

We determine a sharp lower bound for the Hilbert function in degree d of a monomial algebra failing the weak Lefschetz property over a polynomial ring with n variables and generated in degree d, for any $d \geq 2$ and $n \geq 3$. We consider artinian ideals in the polynomial ring with n variables generated by homogeneous polynomials of degree d invariant under an action of the cyclic group $\mathbb{Z}/d\mathbb{Z}$, for any $n \geq 3$ and any $d \geq 2$. We give a complete classification of such ideals in terms of the weak Lefschetz property depending on the action.

1. Introduction

The weak Lefschetz property (WLP) for an artinian graded algebra A over a field $\mathbb{K}$, says there exists a linear form ℓ that induces, for each degree i, a multiplication map $\times\ell : (A)_i \longrightarrow (A)_{i+1}$ that has maximal rank, i.e. that is either injective or surjective. Though many algebras are expected to have the WLP, establishing this property for a specific class of algebras is often rather difficult. In this paper we study the WLP of the

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specific class of algebras which are the quotients of a polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ over field $\mathbb{K}$ of characteristic zero by artinian monomial ideals generated in the same degree d. For this class of artinian algebras, E. Mezzetti and R. M. Miró-Roig [9], showed that $2n-1$ is the sharp lower bound for the number of generators of I when the injectivity fails for S/I in degree $d-1$. In fact they give the lower bound for the number of generators for the minimal monomial Togliatti systems $I \subset \mathbb{K}[x_1, \dots, x_n]$ of the forms of degree d. For more details see the original articles of Togliatti [14,15]. In the first part of this article we establish the lower bound for the number of monomials in the cobasis of the ideal I in the ring S or equivalently, lower bound for the Hilbert function of S/I in degree d, which is $H_{S/I}(d) := \dim_{\mathbb{K}}(S/I)_d$, where surjectivity fails in degree $d-1$. Observe that once multiplication by a general linear form on a quotient of S is surjective, then it remains surjective in the next degrees. This implies that all these algebras with the Hilbert function $H_{S/I}(d)$ below our bound satisfy the WLP.

In the main theorems of the first part of this paper, we provide a sharp lower bound for $H_{S/I}(d)$ for artinian monomial algebra S/I, where the surjectivity fails for S/I in degree $d-1$. For the cases when the number of variables is less than three the bound is known. The first main theorem provides the bound when the polynomial ring has three variables.

Theorem 1.1. *Let $I \subset S = \mathbb{K}[x_1, x_2, x_3]$ be an artinian monomial ideal generated in degree d, for $d \geq 2$ such that S/I fails to have the WLP. Then we have that*

$$H_{S/I}(d) \geq \begin{cases} 3d-3 & \text{if } d \text{ is odd} \\ 3d-2 & \text{if } d \text{ is even.} \end{cases}$$

Furthermore, the bounds are sharp.

In the second theorem we provide a sharp bound when the number of variables is more than three.

Theorem 1.2. *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be an artinian monomial ideal generated in degree d, for $d \geq 2$ and $n \geq 4$ such that S/I fails to have the WLP. Then we have that*

$$H_{S/I}(d) \geq 2d.$$

Furthermore, the bound is sharp.

In [11], Mezzetti, Miró-Roig and Ottaviani describe a connection between projective varieties satisfying at least one Laplace equation and homogeneous artinian ideals generated by polynomials of the same degree d failing the WLP by failing injectivity of the multiplication map by a linear form in degree $d-1$. In [10], Mezzetti and Miró-Roig construct a class of examples of Togliatti systems in three variables of any degree. More precisely, they consider the action on $S = \mathbb{K}[x, y, z]$ of cyclic group $\mathbb{Z}/d\mathbb{Z}$ defined by

$[x, y, z] \mapsto [\xi^a x, \xi^b y, \xi^c z]$, where ξ is a primitive d-th root of unity and $\gcd(a, b, c, d) = 1$. They prove that the ideals generated by forms of degree d invariant by such actions are all defined by monomial Togliatti systems of degree d. In [1], Colarte, Mezzetti, Miró-Roig and Salat show that in $S = \mathbb{K}[x_1, \dots, x_n]$ the ideal fixed by the action of cyclic group $\mathbb{Z}/d\mathbb{Z}$, defined by $[x_1, \dots, x_n] \mapsto [\xi^{a_1} x_1, \dots, \xi^{a_n} x_n]$, where $\gcd(a_1, \dots, a_n, d) = 1$ and there are at most $\binom{n+d-2}{n-2}$ fixed monomials is a monomial Togliatti system.

In this article, we generalize the result in [1] and in Theorem 7.8 we prove that these ideals satisfy the WLP if and only if at least $n-1$ of the integers a_i are equal. In addition, in the polynomial ring with three variables we give a formula for the number of fixed monomials and we provide bounds for such numbers.

2. Preliminaries

We consider standard graded algebras S/I, where $S = \mathbb{K}[x_1, \dots, x_n]$, I is a homogeneous ideal of S, $\mathbb{K}$ is a field of characteristic zero and the x_i's all have degree 1. Our ideal I will be an artinian monomial ideal generated in a single degree d. Given a polynomial f we denote the set of monomials with non-zero coefficients in f by $\mathrm{Supp}(f)$.

Now let us define the weak and strong Lefschetz properties for artinian algebras.

Definition 2.1. Let $I \subset S$ be a homogeneous artinian ideal. We say that S/I has the *Weak Lefschetz Property* (WLP) if there is a linear form $\ell \in (S/I)_1$ such that, for all integers j, the multiplication map

$$\times \ell : (S/I)_j \longrightarrow (S/I)_{j+1}$$

has maximal rank, i.e. it is injective or surjective. In this case the linear form ℓ is called a *Lefschetz element* of S/I. If for general linear form $\ell \in (S/I)_1$ and for an integer j the map $\times \ell$ does not have the maximal rank we will say that S/I fails the WLP in degree j.

We say that S/I has the *Strong Lefschetz Property* (SLP) if there is a linear form $\ell \in (S/I)_1$ such that, for all integers j and k the multiplication map

$$\times \ell^k : (S/I)_j \longrightarrow (S/I)_{j+k}$$

has maximal rank, i.e. it is injective or surjective. We often abuse the notation and say that I fails or satisfies the WLP or SLP, when we mean that S/I does so.

In the case of one variable, the WLP and SLP hold trivially since all ideals are principal. Harima, Migliore, Nagel and Watanabe in [5, Proposition 4.4], proved the following result in two variables.

Proposition 2.2. *Every artinian ideal in* $\mathbb{K}[x, y]$ *(*$\operatorname{char} \mathbb{K} = 0$*) has the Strong Lefschetz property (and consequently also the Weak Lefschetz property).*

In a polynomial ring with more than two variables, it is not true in general that every artinian monomial algebra has the SLP or WLP. Also it is often rather difficult to determine whether a given algebra satisfies the SLP or even WLP. One of the main general results in a ring with more than two variables is proved by Stanley in [13].

Theorem 2.3. *Let $S = \mathbb{K}[x_1, \dots, x_n]$, where $\operatorname{char}(\mathbb{K}) = 0$. Let I be an artinian monomial complete intersection, i.e. $I = (x_1^{a_1}, \dots, x_n^{a_n})$. Then S/I has the SLP.*

Because of the action of the torus $(\mathbb{K}^*)^n$ on monomial algebras, there is a canonical linear form that we have to consider. In fact we have the following result in [12, Proposition 2.2], proved by Migliore, Miró-Roig and Nagel.

Proposition 2.4. *Let $I \subset S$ be an artinian monomial ideal. Then S/I has the weak Lefschetz property if and only if $x_1 + x_2 + \cdots + x_n$ is a weak Lefschetz element for S/I.*

Let us now recall some facts of the theory of the *inverse system*, or *Macaulay duality*, which will be a fundamental tool in this paper. For a complete introduction, we refer the reader to [3] and [6].

Let $R = \mathbb{K}[y_1, \dots, y_n]$, and consider R as a graded S-module where the action of x_i on R is partial differentiation with respect to y_i.

Since we assumed that $\operatorname{char}(\mathbb{K}) = 0$ there is a one-to-one correspondence between graded artinian algebras S/I and finitely generated graded S-submodules M of R, where $I = \operatorname{Ann}_S(M)$ and is the annihilator of M in S and, conversely, $M = I^{-1}$ is the S-submodule of R which is annihilated by I (cf. [3, Remark 1]), p.17). Since the map $\circ\ell : R_{i+1} \longrightarrow R_i$ is dual of the map $\times\ell : (S/I)_i \longrightarrow (S/I)_{i+1}$ we conclude that the injectivity (resp. surjectivity) of the first map is equivalent to the surjectivity (resp. injectivity) of the second one. Here by "$\circ\ell$" we mean that the linear form ℓ acts on R.

For a monomial ideal I the inverse system module $(I^{-1})_d$ is generated by the dual elements in R_d to the monomials in $S_d \setminus I_d$.

Recall that for an n-dimensional variety $X \subseteq \mathbb{P}^N$ and $m \geq 1$ the *m-th osculating space* of X at a point $p \in X$, $T_p^m(X)$, is the subspace of $\mathbb{P}^N$ spanned by p and all the derivative points of order $\leq m$ of a local parametrization of X evaluated at p. The expected dimension of the m-th osculating space at a general point $p \in X$ is equal to $\min\{N, \binom{n+m}{n} - 1\}$. If for some positive integer δ and a general point $p \in X$ we have $\dim T_p^m(X) = \exp\dim T_p^m(X) - \delta$, then X is said to satisfy δ *Laplace equation* of order m.

Mezzetti, Miró-Roig and Ottaviani in [11] describe a relation between existence of artinian ideals $I \subset S$ generated by homogeneous forms of degree d failing the WLP and the existence of projections of the Veronese variety $V(n-1, d) \subset \mathbb{P}^{\binom{n+d-1}{d}-1}$ satisfying at least one Laplace equation of order $d-1$.

For an artinian ideal $I \subset S$, they make the following construction. Assume that I is minimally generated by the homogeneous polynomials $f_1, \dots, f_r$ of degree d and denote by I^{-1}, the inverse system module of I. Since I is artinian, the polynomials $f_1, \ldots, f_r$ define a regular morphism

$$\varphi_{I_d} : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{r-1}.$$

Denote by X_{n-1,I_d}, the closure of the image of φ_{I_d}. There is a rational map

$$\varphi_{(I^{-1})_d} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{\binom{n+d-1}{d}-r-1}$$

associated to $(I^{-1})_d$. Denote by $X_{n-1,(I^{-1})_d}$, the closure of the image of $\varphi_{(I^{-1})_d}$.

With notations as above, in [11], Theorem 3.2, Mezzetti, Miró-Roig and Ottaviani prove the following theorem.

Theorem 2.5. *Let $I \subset S$ be an artinian ideal generated by r forms $f_1, \ldots, f_r$ of degree d. If $r \leq \binom{n+d-2}{n-2}$, then the following conditions are equivalent:*

(1) *The ideal I fails the WLP in degree $d-1$,*
(2) *The forms $f_1, \ldots, f_r$ become $\mathbb{K}$-linearly dependent on a general hyperplane H of $\mathbb{P}^{n-1}$,*
(3) *The $n-1$-dimensional variety $X_{n-1,(I^{-1})_d}$ satisfies at least one Laplace equation of order $d-1$.*

A monomial ideal I satisfying the equivalent conditions in the above theorem is called a *monomial Togliatti system* and moreover if no proper subset satisfies the equivalent conditions it is called a *minimal monomial Togliatti system*.

3. On the support of form f annihilated by ℓ and its higher powers

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring where $n \geq 3$ and $\mathbb{K}$ is a field of characteristic zero. In this section we give some definitions and notations and prove some results about the number of monomials in the support of polynomials $f \in (I^{-1})_d$ with $(x_1 + \cdots + x_n)^a \circ f = 0$ for some $1 \leq a \leq d$. Now let us define a specific type of well known integer matrices which we use throughout this section.

Definition 3.1. For a non-negative integer k and positive integer m, where $k \leq m$, we define the Toeplitz matrix $T_{k,m}$, to be the following $(k+1) \times (m+1)$ matrix

$$T_{k,m} = \begin{bmatrix} \binom{m-k}{0} & \binom{m-k}{1} & \binom{m-k}{2} & \cdots & \binom{m-k}{m-k} & 0 & \cdots & 0 \\ 0 & \binom{m-k}{0} & \binom{m-k}{1} & \cdots & \binom{m-k}{m-k-1} & \binom{m-k}{m-k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{m-k}{m-k-3} & \binom{m-k}{m-k-2} & \binom{m-k}{m-k-1} & \binom{m-k}{m-k} \end{bmatrix}$$

where the $(i, j)th$ entry of this matrix is $\binom{m-k}{j-i}$ and we use the convention that $\binom{m}{i} = 0$ if $i < 0$ or $m > i$.

We have the following useful lemma which proves the maximal minors of $T_{k,m}$ are non-zero.

Lemma 3.2. *For each non-negative integer k and positive integer m where $k \leq m$, all maximal minors of the Toeplitz matrix $T_{k,m}$ are non-zero.*

Proof. Let $R = \mathbb{K}[x, y]$ be the polynomial ring in variables x and y and choose monomial bases $\mathcal{A} := \{x^j y^{k-j}\}_{j=0}^{k}$ and $\mathcal{B} := \{x^i y^{m-i}\}_{i=1}^{m}$ for the $\mathbb{K}$-vector spaces R_k and R_m, respectively. Observe that $T_{k,m}$ is the matrix representing the multiplication map $\times(x+y)^{m-k} : R_k \to R_m$ with respect to the bases $\mathcal{A}$ and $\mathcal{B}$. Given any square submatrix M of size $k+1$, define the ideal $J \subset R$ generated by the subset of monomials in $\mathcal{B}$, called $\mathcal{B}'$, corresponding to the columns of $T_{k,m}$ not in M. Therefore, $\mathcal{A}$ and $\mathcal{B} \setminus \mathcal{B}'$ form monomial bases for $(R/J)_k$ and $(R/J)_m$, respectively and M is the matrix representing the multiplication map $\times(x+y)^{m-k} : (R/J)_k \to (R/J)_m$ with respect to $\mathcal{A}$ and $\mathcal{B} \setminus \mathcal{B}'$. Note that by the definition of J we have $(R/J)_k = R_k$. Since by Proposition 2.2, any monomial R-algebra has the SLP, and by Proposition 2.4 $x + y$ is a Lefschetz element for R/J, the multiplication map by $x + y$ is a bijection and therefore the matrix M has non-zero determinant. This implies that all the maximal minors of $T_{k,m}$ are non-zero. $\square$

Consider a non-zero homogeneous polynomial f of degree d in the dual ring $R = \mathbb{K}[y_1, \ldots, y_n]$ for which we have $(x_1 + \cdots + x_n) \circ f = 0$. We use the following notations and definitions to prove some properties of such polynomial f.

Definition 3.3. For an ideal I of S, we denote the Hilbert function of S/I in degree d by $H_{S/I}(d) := \dim_{\mathbb{K}}(S/I)_d$, and the set of all artinian monomial ideals of S generated in a single degree d by $\mathcal{I}_d$. In addition, for an artinian ideal I we define $\phi(I, d) : \times(x_1 + \cdots + x_n) : (S/I)_{d-1} \to (S/I)_d$ and

$$\nu(n, d) := min\{H_{(S/I)}(d) \mid \phi(I, d) \quad \text{is not surjective, for} \quad I \in \mathcal{I}_d\}.$$

Definition 3.4. In a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$, for any monomial m and variable x_i, we define

$$\deg_i(m) := \max\{e \mid x_i^e | m\}$$

Define the set $\mathcal{M}_d$ to be the set of monomials of degree d in R and denote the set of monomials of degree k with respect to the variable y_i by,

$$\mathcal{L}_{i,d}^{k} := \{m \in \mathcal{M}_d \mid \deg_i(m) = k\} \subset \mathcal{M}_d.$$

Lemma 3.5. *Consider f be a form of degree $d \geq 2$ in the dual ring $R = \mathbb{K}[y_1, \ldots, y_n]$ of $S = \mathbb{K}[x_1, \ldots, x_n]$ and let the linear forms $\ell := x_1 + \cdots + x_n$ and $\ell' := \ell - x_j$ for $1 \leq j \leq n$. Write $f = \sum_{i=0}^{d} y_j^i g_i$, where g_i is a polynomial of degree $d - i$ in the variables different from y_j, then for every $0 \leq c \leq d$, we have*

$$\ell^c \circ f = \sum_{k=0}^{d-c} \sum_{i=0}^{c} \frac{(k+c-i)!}{k!} \binom{c}{i} y_j^k \left({\ell'}^i \circ g_{k+c-i} \right). \tag{3.1}$$

In particular, $\ell^c \circ f = 0$ if and only if,

$$\sum_{i=0}^{c} \frac{(k+c-i)!}{k!} \binom{c}{i} \left({\ell'}^i \circ g_{k+c-i} \right) = 0, \quad 0 \leq k \leq d-c. \tag{3.2}$$

Proof. We prove the lemma using induction on c. For $c = 0$ the equality (3.1) is trivial. For $c = 1$, we have

$$\ell \circ f = \sum_{k=0}^{d} \left(k y_j^{k-1} g_k + y_j^k \left(\ell' \circ g_k \right) \right) = \sum_{k=0}^{d-1} \left((k+1) y_j^k g_{k+1} + y_j^k \left(\ell' \circ g_k \right) \right). \tag{3.3}$$

Assume the equality holds for $c - 1$ then we have $\ell^c \circ f = \ell \circ (\ell^{c-1} \circ f)$ and

$$\begin{aligned}
&\ell \circ (\ell^{c-1} \circ f) \\
&= \ell \circ \left(\sum_{k=0}^{d-c+1} \sum_{i=0}^{c-1} \frac{(k+c-1-i)!}{k!} \binom{c-1}{i} y_j^k \left({\ell'}^i \circ g_{k+c-1-i} \right) \right) \\
&= \sum_{k=0}^{d-c+1} \sum_{i=0}^{c-1} \frac{(k+c-1-i)!}{k!} \binom{c-1}{i} \left(k y_j^{k-1} \left({\ell'}^i \circ g_{k+c-1-i} \right) + y_j^k \left({\ell'}^{i+1} \circ g_{k+c-1-i} \right) \right) \\
&= \sum_{k=0}^{d-c} \sum_{i=0}^{c-1} \left[\frac{(k+c-i)!}{(k+1)!} \binom{c-1}{i} (k+1) y_j^k \left({\ell'}^i \circ g_{k+c-i} \right) \right. \\
&\quad \left. + \frac{(k+c-1-i)!}{k!} \binom{c-1}{i} y_j^k \left({\ell'}^{i+1} \circ g_{k+c-1-i} \right) \right] \\
&= \sum_{k=0}^{d-c} \sum_{i=0}^{c} \left(\frac{(k+1)(k+c-i)!}{(k+1)!} \binom{c-1}{i} + \frac{(k+c-i)!}{k!} \binom{c-1}{i-1} \right) y_j^k \left({\ell'}^i \circ g_{k+c-i} \right) \\
&= \sum_{k=0}^{d-c} \sum_{i=0}^{c} \frac{(k+c-i)!}{k!} \binom{c}{i} y_j^k \left({\ell'}^i \circ g_{k+c-i} \right). \quad \square
\end{aligned}$$

Using the above lemma the following proposition gives properties about the form f.

Proposition 3.6. *Let f be a non-zero form of degree d in the dual ring $R = \mathbb{K}[y_1, \dots, y_n]$ of $S = \mathbb{K}[x_1, \dots, x_n]$ such that $(x_1 + \cdots + x_n) \circ f = 0$. Then the following conditions hold:*

(i) *If $y_i^d \notin \mathrm{Supp}(f)$, then the sum of the coefficients of f corresponding to the monomials in $\mathcal{L}_{i,d}^k \cap \mathrm{Supp}(f)$ is zero; for each $0 \le k \le d-1$.*
(ii) *If $a = \max\{\deg_i(m) \mid m \in \mathrm{Supp}(f)\}$, then $\mathcal{L}_{i,d}^k \cap \mathrm{Supp}(f) \neq \emptyset$; for all $0 \le k \le a$.*

Proof. Write the form f as, $f = \sum_{k=0}^{d} y_i^k g_k$, where g_k is a degree $d-k$ polynomial in variables different from y_i. Denote $\ell = x_1 + \cdots + x_n$ and $\ell' = \ell - x_i$. Since $\ell \circ f = 0$, Lemma 3.5 implies that

$$(k+1)y_i^k g_{k+1} + y_i^k (\ell' \circ g_k) = 0, \quad \forall\, 0 \le k \le d-1. \tag{3.4}$$

To show (i) we act on each equation by $(\ell')^{d-k-1}$ and we get that

$$(k+1)(d-k-1)! g_{k+1}(1, \dots, 1) + (d-k)! g_k(1, \dots, 1) = 0 \quad \forall\, 0 \le k \le d-1. \tag{3.5}$$

Since we assumed $g_d = 0$ we get that $g_k(1, \dots, 1) = 0$ for all $0 \le k \le d-1$, which implies that for all $0 \le k \le d-1$ sum of the coefficients of f corresponding to the monomials in $\mathcal{L}_{i,d}^d \cap \mathrm{Supp}(f)$ is zero and proves part (i).

To show part (ii), note that $a = \max\{\deg_i(m) \mid m \in \mathrm{Supp}(f)\}$ implies that $g_a \neq 0$. Using Equation (3.4) recursively we get that $g_j \neq 0$ for all $0 \le j \le a$, which means that $\mathcal{L}_{i,d}^j \cap \mathrm{Supp}(f) \neq \emptyset$ for all $0 \le j \le a$. $\square$

In the following theorem we provide a bound for the number of monomials with non-zero coefficients in the non-zero form in the kernel of the map $\circ(x_1 + \cdots + x_n)^{d-a} \colon (I^{-1})_d \longrightarrow (I^{-1})_a$. In particular it provides a bound on the number of generators for an equigenerated monomial ideal in S failing the WLP.

Theorem 3.7. *Let $f \neq 0$ be a form of degree d in the dual ring $R = \mathbb{K}[y_1, \dots, y_n]$ of the ring $S = \mathbb{K}[x_1, \dots, x_n]$. If for the linear form $\ell := x_1 + \cdots + x_n$ we have $\ell^{d-a} \circ f = 0$ for some $0 \le a \le d-1$, then $|\mathrm{Supp}(f)| \ge a+2$.*

Proof. For a variable y_j write $f = \sum_{i=0}^{d} y_j^i g_i$ such that g_i is a polynomial of degree $d-i$ in the variables different from y_j. Since for some $1 \le a \le d-1$ we have $\ell^{d-a} \circ f = 0$ from Lemma 3.5 we have that

$$\sum_{i=0}^{d-a} \frac{(k+d-a-i)!}{k!} \binom{d-a}{i} \left({\ell'}^{i} \circ g_{k+d-a-i} \right) = 0, \quad 0 \le k \le a. \tag{3.6}$$

For every j with $1 \le j \le a+1$ we act on each equation in the above system by $(\ell')^{j-k-1}$, so we have

$$\sum_{i=0}^{d-a} \frac{(k+d-a-i)!}{k!} \binom{d-a}{i} \left(\ell'^{i+j-k-1} \circ g_{k+d-a-i}\right) = 0, \quad 0 \le k \le a, \tag{3.7}$$

equivalently for each j with $1 \le j \le a+1$ we have that

$$\sum_{i=a-k}^{d-k} \frac{(d-i)!}{k!} \binom{d-a}{i-(a-k)} \left(\ell'^{i+j-(a+1)} \circ g_{d-i}\right) = 0, \quad 0 \le k \le j-1. \tag{3.8}$$

Note that for $k \ge j$ the equations in (3.7) are zero.

For each $0 \le j \le a+1$ the coefficient matrix of the system in (3.8) in the forms $(d-i)!\ell'^{i+j-(a+1)} \circ g_{d-i}$ is the Toeplitz matrix $T_{(j-1)\times(d-a+j-1)}$ up to multiplication of k-th row by $\frac{1}{k!}$. Using Lemma 3.2 we get that all the maximal minors of this coefficient matrix are non-zero. This implies that in each system of equations either all the terms are zero or there are at least $j+1$ non-zero terms.

Now we want to prove the statement by induction on the number of variables n. Suppose $n = 2$ then each g_i is a monomial of degree $d-i$ in one variable. In (3.8) consider the corresponding system of equations for $j = a+1$. If for every $0 \le i \le d$ we have that $\ell'^i \circ g_{d-i} = 0$, it implies that for every $0 \le i \le d$ we have $g_{d-i} = 0$ which contradicts the assumption that $f \ne 0$. Therefore for at least $a+2$ indices $0 \le i \le d$ we have $\ell'^i \circ g_{d-i} \ne 0$ which means $|\operatorname{Supp}(f)| \ge a+2$.

Now we assume that the statement is true for the forms f in polynomial rings with $n-1$ $(n \ge 3)$ variables and we prove it for the form with n variables.

We divide it into two cases. Suppose in the system of equations for every $1 \le j \le a+1$ all terms are zero. In this case for each $1 \le j \le a+1$, letting $i = a-j+1$ implies that $(\ell')^{a-j+1+j-(a+1)} \circ g_{d-(a-j+1)} = g_{d-a+j-1} = 0$ for all $1 \le j \le a+1$. Since we assume that $f \ne 0$ there exists $a+1 \le i \le d$ such that $g_{d-i} \ne 0$, but considering $j = 1$ in (3.8) with the assumption that all terms in this equation is zero we get that $(\ell')^{i-a} \circ g_{d-i} = 0$. Using the induction hypothesis on the polynomial g_{d-i} in $n-1$ variables we get that $|\operatorname{Supp}(f)| \ge |\operatorname{Supp}(g_{d-i})| \ge d-(d-i)-(i-a)+2 = a+2$ as we wanted to prove.

Now we assume that there exists $1 \le j \le a+1$ such that there are at least $j+1$ indices $0 \le i \le d$ such that $\ell'^{i+j-(a+1)} \circ g_{d-i} \ne 0$ in the corresponding system of equations in (3.8). We take the largest index j with this property and we get that for these $j+1$ indices we have that $\ell'^{i+j-(a+1)+1} \circ g_{d-i} = 0$. Now using the induction hypothesis in these polynomials we get that $|\operatorname{Supp}(g_{d-i})| \ge d-(d-i)-(i+j-(a+1)+1)+2 = a+2-j$, therefore

$$|\operatorname{Supp}(f)| \ge \sum_{i=0}^{d} |\operatorname{Supp}(g_i)| \ge (j+1)(a+2-j) \ge a+2. \quad \square$$

4. Bounds on the number of generators of ideals with three variables failing WLP

In this section we consider artinian monomial ideals $I \subset S = \mathbb{K}[x_1, x_2, x_3]$ generated in a single degree d. In [9], Mezzetti and Miró-Roig provided a sharp lower bound for the number of generators of such ideals failing the WLP by failing *injectivity* of the multiplication map on the algebra in degree $d-1$. Here we prove a sharp upper bound for the number of generators of such ideals failing the WLP by failing *surjectivity* in degree $d-1$ equivalently we provide a sharp lower bound for the number of generators of $(I^{-1})_d$ where the map $\circ\ell\colon (I^{-1})_d \longrightarrow (I^{-1})_{d-1}$ is not injective, where $\ell = x_1 + x_2 + x_3$.

First we prove an easy but interesting result. Recall that every polynomial in at most two variables factors as a product of linear forms over an algebraically closed field. Here we note that the same statement holds in three variables if the polynomial vanishes by the action of a linear form on the dual ring. This in some cases corresponds to the failure of WLP. Note that for the WLP, the assumption on the field to be algebraically closed is not necessary, but in order to factor the form as a product of linear forms we need to have this assumption on the field. In addition the statement does not necessarily hold in polynomial rings with more than three variables.

Lemma 4.1. *Let $S = \mathbb{K}[x_1, x_2, x_3]$ and S/I be an artinian algebra over an algebraically closed field $\mathbb{K}$. Let f be a form in the kernel of the map $\circ\ell : (I^{-1})_i \longrightarrow (I^{-1})_{i-1}$ for a linear form ℓ and integer i, then f factors as a product of linear forms each of which is annihilated by ℓ.*

Proof. By a linear change of variables we consider $S = \mathbb{K}[x'_1, x'_2, x'_3]$ and $R = \mathbb{K}[y'_1, y'_2, y'_3]$ simultaneously in such a way that $x'_1 = \ell$. Then we have that $\ell \circ f(y_1, y_2, y_3) = x'_1 \circ f(y'_1, y'_2, y'_3) = 0$ where this implies that f is a polynomial in two variables y'_2 and y'_3. Using the fact that any polynomial in two variables over an algebraically closed field factors as a product of linear forms we conclude that f factors as a product of linear forms in y'_2 and y'_3. Hence all of them are annihilated by $\ell = x'_1$. □

The next proposition provides a bound for the number of non-zero terms in each homogeneous component with respect to one of the variables for a non-zero form f, where $\ell \circ f = 0$.

Proposition 4.2. *Let f be a non-zero form of degree $d \geq 2$ in the dual ring $R = \mathbb{K}[y_1, y_2, y_3]$ of $S = \mathbb{K}[x_1, x_2, x_3]$ such that $(x_1 + x_2 + x_3) \circ f = 0$. Then we have*

$$|\mathcal{L}^k_{i,d} \cap \mathrm{Supp}(f)| \geq d - a_i + 1, \quad \forall\ 0 \leq k \leq a_i,\ 1 \leq i \leq 3,$$

where $a_i = \max\{\deg_i(m) \mid m \in \mathrm{Supp}(f)\}$.

Proof. Write $f = \sum_{k=0}^{a_i} y_i^k g_k$, where g_k is a degree $d-k$ polynomial in two variables different from y_i. Let $\ell' = \ell - x_i$, then we have

$$0 = \ell \circ f = \ell \circ (\sum_{k=0}^{a_i} y_i^k g_k) = \sum_{k=0}^{a_i} \left(k y_i^{k-1} g_k + y_i^k \left(\ell' \circ g_k\right)\right)$$

therefore

$$(k+1)g_{k+1} + \ell' \circ g_k = 0, \quad \forall\, 0 \le k \le a_i \tag{4.1}$$

after linear change of variables to $u := y_\alpha + y_\beta$ and $v := y_\alpha - y_\beta$, Equation (4.1) implies that, $(\partial/\partial u)^{a_i-k+1} \circ g_k = 0$ for any $0 \le k \le a_i$. Therefore, for each $0 \le k \le a_i$ we have

$$g_k = \sum_{j=0}^{a_i-k} \lambda_j u^j v^{a_i-k+j} = v^{d-a_i} \sum_{j=0}^{a_i-k} \lambda_j u^j v^{a_i-k+j} \qquad \lambda_j \in \mathbb{K}.$$

Rewriting g_k in the variables y_α and y_β we get that

$$\begin{aligned} g_k =&(y_\alpha - y_\beta)^{d-a_i} \sum_{j=0}^{a_i-k} \lambda_j (y_\alpha - y_\beta)^j (y_\alpha + y_\beta)^{a_i-k-j} \\ =&\Big(\sum_{s=0}^{d-a_i} (-1)^s \binom{d-a_i}{s} y_\alpha^s y_\beta^{d-a_i-s}\Big)\Big(\sum_{t=0}^{a_i-k} \lambda_t (y_\alpha - y_\beta)^t (y_\alpha + y_\beta)^{a_i-k-t}\Big) \end{aligned}$$

where the second sum is a polynomial of degree $a_i - k$ in the variables y_α and y_β, and since any such polynomial is of the form $\sum_{j=0}^{a_i-k} \mu_j y_\alpha^j y_\beta^{a_i-k-j}$ for some $\mu_j \in \mathbb{K}$. So we have

$$\begin{aligned} g_k =&\Big(\sum_{s=0}^{d-a_i} (-1)^s \binom{d-a_i}{s} y_\alpha^s y_\beta^{d-a_i-s}\Big)\Big(\sum_{j=0}^{a_i-k} \mu_j y_\alpha^j y_\beta^{a_i-k-j}\Big) \\ =&\sum_{s=0}^{d-a_i} \sum_{j=0}^{a_i-k} (-1)^s \mu_j \binom{d-a_i}{s} y_\alpha^{s+j} y_\beta^{d-k-s-j} \\ =&\sum_{j=0}^{a_i-k} \sum_{l=j}^{j+d-a_i} (-1)^{l-j} \mu_j \binom{d-a_i}{l-j} y_\alpha^l y_\beta^{d-k-l}. \end{aligned}$$

We claim that g_k has at most $a_i - k$ coefficients that are zero. Suppose $a_i - k + 1$ coefficients in the above expression of g_k are zero and consider the system of equations in the parameters μ_j corresponding to these coefficients being zero. Observe that the coefficient matrix of this system of equations is the transpose of a square submatrix of maximal rank of the Toeplitz matrix $T_{(a_i-k+1)\times(d-k+1)}$, up to multiplication of every second row and every second column by negative one. Using Lemma 3.2 we get that the determinant of this coefficient matrix is non-zero and this implies that all the parameters μ_j are zero hence g_k is zero. Therefore for all $0 \le k \le a_i$ the polynomial g_k has at most $(a_i - k + 1) - 1 = a_i - k$ zero terms. So we have $|\mathcal{L}_{i,d}^k \cap \mathrm{Supp}(f)| = |\mathrm{Supp}(g_k)| \ge (d-k+1) - (a_i - k) = d - a_i + 1$ for all $0 \le k \le a_i$ and all $1 \le i \le 3$. $\square$

Now we are able to state and prove the main theorem of this section. Recall from Definition 3.3 that $\phi(I,d) = \times(x_1+x_1+x_3) : (S/I)_{d-1} \to (S/I)_d$.

Theorem 4.3. *For $d \geq 2$ we have that*

$$\nu(3,d) = \begin{cases} 3d-3 & \textit{if } d \textit{ is odd} \\ 3d-2 & \textit{if } d \textit{ is even.} \end{cases}$$

Where, $\nu(3,d) = min\{H_{(S/I)}(d) \mid \phi(I,d)$ is not surjective, for $I \in \mathcal{I}_d\}$, and $\mathcal{I}_d$ is the set of all artinian monomial ideals of S generated in degree d.

Proof. First of all we observe that for $f = (y_1-y_2)(y_1-y_3)(y_2-y_3)^{d-2}$ we have $\ell \circ f = 0$, where $\ell = x_1+x_2+x_3$ and since $|\operatorname{Supp}(f)| = 3d-3$ for odd d and $|\operatorname{Supp}(f)| = 3d-2$ for even d, we have $\nu(3,d) \leq 3d-3$ for odd d, and $\nu(3,d) \leq 3d-2$ for even d.

To prove the equality, we check that for any $f \in (I^{-1})_d$ where, $\ell \circ f = 0$, $|\operatorname{Supp}(f)| \geq 3d-3$ for odd d and $|\operatorname{Supp}(f)| \geq 3d-2$ for even d.

We start by showing that $|\operatorname{Supp}(f)| \geq 3d-3$ for all $d \geq 3$. Set $a_i = \max\{\deg_i(m) \mid m \in \operatorname{Supp}(f)\}$ for $1 \leq i \leq 3$. Without loss of generality, we may assume that $a_1 \leq a_2 \leq a_3$. We can see $a_1 \geq 2$. In fact by using Proposition 4.2 we get that $|\mathcal{L}^0_{1,d} \cap \operatorname{Supp}(f)| \geq d-a_1+1$. On the other hand since I is an artinian ideal generated in degree d we have $y_i^d \notin \operatorname{Supp}(f)$ for each $1 \leq i \leq 3$ and this implies that $|\mathcal{L}^0_{1,d} \cap \operatorname{Supp}(f)| \leq d-1$ and therefore, $a_1 \geq 2$.

Write $f = \sum_{j=0}^{a_1} y_1^j g_j$, where g_j is a polynomial of degree $d-j$ in the variables y_2 and y_3. Using Proposition 4.2 we get, $|\mathcal{L}^j_{1,d} \cap \operatorname{Supp}(f)| \geq d-a_1+1$ for all $0 \leq j \leq a_1$. Therefore,

$$|\operatorname{Supp}(f)| \geq \sum_{j=0}^{a_1} |\mathcal{L}^j_{1,d} \cap \operatorname{Supp}(f)| \geq (a_1+1)(d-a_1+1).$$

So $|\operatorname{Supp}(f)| \geq (a_1+1)(d-a_1+1) = 3(d-1)+(a_1-2)(d-2-a_1) \geq 3d-3$, for $2 \leq a_1 \leq d-2$. Furthermore, strict inequality holds for $2 < a_1 < d-2$, which means $|\operatorname{Supp}(f)| \geq 3d-2$, for all $2 < a_1 < d-2$. It remains to consider the cases where $a_1 = 2$ and $a_1 \geq d-2$.

If $a_1 = 2$, the ideal $J = (x_1^3, x_2^d, x_3^d) \subset S$ is an artinian monomial complete intersection and by Theorem 2.3, J has the strong Lefschetz property. The Hilbert series of S/J shows that there is a unique generator for the kernel of the differentiation map $\circ(x_1+x_2+x_3) : (J^{-1})_d \longrightarrow (J^{-1})_{d-1}$. Since the polynomial $(y_1-y_2)(y_1-y_3)(y_2-y_3)^{d-2}$ is in the kernel of this map and has $3d-2$ non-zero terms for even degree d we have $|\operatorname{Supp}(f)| \geq 3d-2$ for any homogeneous degree d form f where $\ell \circ f = 0$.

Now suppose that $a_1 \geq d-2$, then since $a_1 \leq a_2 \leq a_3 \leq d-1$, all possible choices for the triple (a_1,a_2,a_3) are $(d-1,d-1,d-1)$, $(d-2,d-1,d-1)$, $(d-2,d-2,d-1)$ and $(d-2,d-2,d-2)$.

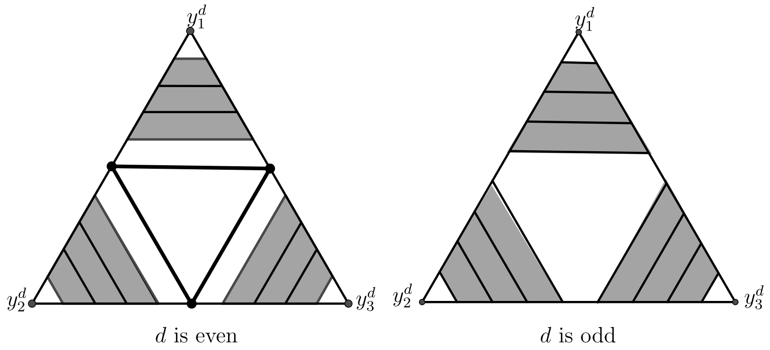

Fig. 1. Monomials of degree d in $\mathbb{K}[y_1, y_2, y_3]$ are considered as the points in the equilateral triangular arrangement with $d+1$ dots on a side.

First, we consider the case $(a_1, a_2, a_3) = (d-1, d-1, d-1)$. Proposition 4.2 implies that

$$|\mathcal{L}^k_{i,d} \cap \operatorname{Supp}(f)| \geq d - (d-1) + 1 = 2, \qquad \text{for all } 0 \leq k \leq d-1 \text{ and } 1 \leq i \leq 3. \quad (4.2)$$

Assume d is odd. Consider the set $\cup_{(d+1)/2}^{d-1} \left(\mathcal{L}^k_{1,d} \cap \operatorname{Supp}(f)\right)$ which consists of all the monomials in $\operatorname{Supp}(f)$ where the exponent of y_1 is between $(d+1)/2$ and $d-1$ (monomials on the horizontal lines shown in the right triangle in Fig. 1). Similarly, for y_2 and y_3. Note that since d is odd the three sets $\cup_{(d+1)/2}^{d-1} \left(\mathcal{L}^k_{1,d} \cap \operatorname{Supp}(f)\right)$, $\cup_{(d+1)/2}^{d-1} \left(\mathcal{L}^k_{2,d} \cap \operatorname{Supp}(f)\right)$ and $\cup_{(d+1)/2}^{d-1} \left(\mathcal{L}^k_{3,d} \cap \operatorname{Supp}(f)\right)$ are disjoint. Thus, using equation (4.2) we get that

$$\begin{aligned} |\operatorname{Supp}(f)| \geq & |\cup_{(d+1)/2}^{d-1} \mathcal{L}^k_{1,d} \cap \operatorname{Supp}(f)| + |\cup_{(d+1)/2}^{d-1} \mathcal{L}^k_{2,d} \cap \operatorname{Supp}(f)| \\ & + |\cup_{(d+1)/2}^{d-1} \mathcal{L}^k_{3,d} \cap \operatorname{Supp}(f)| \\ \geq & 3\big(2(d-1-\frac{d+1}{2}+1)\big) = 3d-3, \end{aligned}$$

which shows the desired inequality for odd d, see Fig. 1.

Now we assume d is even. Similar to the odd case we consider the set

$$\cup_{(d+2)/2}^{d-1} \left(\mathcal{L}^k_{1,d} \cap \operatorname{Supp}(f)\right)$$

consisting of all monomials where the exponent of y_1 is between $(d+1)/2$ and $d-1$ (monomials on the horizontal lines in the shaded part of the left triangle in Fig. 1). Consider the corresponding sets for y_2 and y_3. Moreover, denote by $\mathcal{C}$ the set of all monomials in $\operatorname{Supp}(f)$ where the exponent of at least one of the variables is exactly $\frac{d}{2}$, (monomials on the edges of the bold downward triangle in Fig. 1), in other words

$$\mathcal{C} := \left(\mathcal{L}^{d/2}_{1,d} \cup \mathcal{L}^{d/2}_{2,d} \cup \mathcal{L}^{d/2}_{3,d}\right) \cap \operatorname{Supp}(f).$$

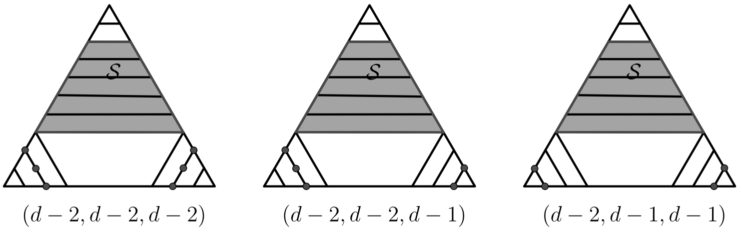

Fig. 2. Three cases for the triple (a_1,a_2,a_3) when d is even.

The four sets

$$\cup_{(d+2)/2}^{d-1}\left(\mathcal{L}_{1,d}^{k}\cap \mathrm{Supp}(f)\right),\cup_{(d+2)/2}^{d-1}\left(\mathcal{L}_{2,d}^{k}\cap \mathrm{Supp}(f)\right),\cup_{(d+2)/2}^{d-1}\left(\mathcal{L}_{3,d}^{k}\cap \mathrm{Supp}(f)\right)$$

and $\mathcal{C}$ are disjoint. Using equation (4.2) and noticing that for each $1\le i\neq j\le 3$ the sets $\mathcal{L}_{i,d}^{d/2}\cap\mathcal{L}_{j,d}^{d/2}$ contains exactly two monomials we get that $|\mathcal{C}|\ge 3\times 2-3=3$. If $|\mathcal{C}|=3$ then these three monomials are exactly the pairwise intersection of $\mathcal{L}_{i,d}^{d/2}$ for $1\le i\le 3$, which are $y_1^{d/2},y_2^{d/2}$ and $y_3^{d/2}$. Proposition 3.6, part (i) implies that the sum of the coefficients of f corresponding to the monomials in $\mathcal{L}_{i,d}^{d/2}\cap \mathrm{Supp}(f)$ is zero for each $1\le i\le 3$, thus the sum of the coefficients of each pair of the monomials in $\mathcal{C}$ is zero and this means the coefficients of all the monomials in $\mathcal{C}$ have to be zero which is a contradiction. So $|\mathcal{C}|\ge 4$. Using this and equation (4.2) we conclude that

$$\begin{aligned}|\,\mathrm{Supp}(f)|\ge&|\cup_{(d+2)/2}^{d-1}\mathcal{L}_{1,d}^{k}\cap \mathrm{Supp}(f)|+|\cup_{(d+2)/2}^{d-1}\mathcal{L}_{2,d}^{k}\cap \mathrm{Supp}(f)|\\&+|\cup_{(d+2)/2}^{d-1}\mathcal{L}_{3,d}^{k}\cap \mathrm{Supp}(f)|+|\mathcal{C}|\\\ge&3\big(2(d-1-\frac{d+2}{2}+1)\big)+4=3d-2.\end{aligned}$$

For the three remaining cases where $a_1=d-2$ we will show for even degree d we get $|\,\mathrm{Supp}(f)|\ge 3d-2$, see Fig. 2. Note that when $d=4$ and $a_1=d-2=2$ we have seen already that $|\,\mathrm{Supp}(f)|\ge 3d-2$. So we can assume $d\ge 6$. Denote by $\mathcal{S}$ the set of all monomials in $\mathrm{Supp}(f)$ where the exponent of y_1 is between 3 and $d-2$, (monomials on the horizontal lines in the shaded part of the triangles in Fig. 2). In other words

$$\mathcal{S}:=\cup_{k=3}^{d-2}\left(\mathcal{L}_{1,d}^{k}\cap \mathrm{Supp}(f)\right).$$

Using Proposition 4.2 we get $|\mathcal{L}_{1,d}^{j}\cap \mathrm{Supp}(f)|\ge d-(d-2)+1=3$ for each $0\le j\le d-2$. This implies that $|\mathcal{S}|\ge 3(d-4)$.

We have three cases as follows.

If $(a_1,a_2,a_3)=(d-2,d-2,d-2)$, consider the set $\cup_{j=d-3}^{d-2}(\mathcal{L}_{2,d}^{j}\cap \mathrm{Supp}(f))$ (the monomials in $\mathrm{Supp}(f)$ where the exponent of y_2 is either $d-3$ or $d-2$) and similarly $\cup_{j=d-3}^{d-2}(\mathcal{L}_{3,d}^{j}\cap \mathrm{Supp}(f))$. Notice that $\cup_{j=d-3}^{d-2}(\mathcal{L}_{2,d}^{j}\cap \mathrm{Supp}(f))$ and $\cup_{j=d-3}^{d-2}(\mathcal{L}_{3,d}^{j}\cap \mathrm{Supp}(f))$ are disjoint. Also the intersection of $\cup_{j=d-3}^{d-2}(\mathcal{L}_{2,d}^{j}\cap \mathrm{Supp}(f))$ and $\mathcal{S}$ is either empty or $y_1^3y_2^{d-3}$. Similarly, $|\left(\cup_{j=d-3}^{d-2}\mathcal{L}_{3,d}^j \cap \operatorname{Supp}(f)\right) \cap \mathcal{S}| \le 1$. Therefore, applying Proposition 4.2 implies that

$$\begin{aligned}|\operatorname{Supp}(f)| &\ge |\mathcal{S} \cup \left(\cup_{j=d-3}^{d-2}(\mathcal{L}_{2,d}^j \cap \operatorname{Supp}(f))\right) \cup \left(\cup_{j=d-3}^{d-2}(\mathcal{L}_{3,d}^j \cap \operatorname{Supp}(f))\right)| \\ &\ge 3(d-4) + (2\times 3 - 1) + (2 \times 3 - 1) = 3d-2.\end{aligned}$$

If $(a_1, a_2, a_3) = (d-2, d-2, d-1)$, we consider two sets $\cup_{j=d-3}^{d-2}(\mathcal{L}_{2,d}^j \cap \operatorname{Supp}(f))$ and $\cup_{j=d-3}^{d-1}(\mathcal{L}_{3,d}^j \cap \operatorname{Supp}(f))$. The similar argument as in the previous case implies that

$$\begin{aligned}|\operatorname{Supp}(f)| &\ge |\mathcal{S} \cup \left(\cup_{j=d-3}^{d-2}(\mathcal{L}_{2,d}^j \cap \operatorname{Supp}(f))\right) \cup \left(\cup_{j=d-3}^{d-1}(\mathcal{L}_{3,d}^j \cap \operatorname{Supp}(f))\right)| \\ &\ge 3(d-4) + (2\times 3 - 1) + (3 \times 2 - 1) = 3d-2.\end{aligned}$$

If $(a_1, a_2, a_3) = (d-2, d-1, d-1)$, we consider two sets $\cup_{j=d-3}^{d-1}(\mathcal{L}_{2,d}^j \cap \operatorname{Supp}(f))$ and $\cup_{j=d-3}^{d-1}(\mathcal{L}_{3,d}^j \cap \operatorname{Supp}(f))$ and we conclude that

$$\begin{aligned}|\operatorname{Supp}(f)| &\ge |\mathcal{S} \cup \left(\cup_{j=d-3}^{d-1}(\mathcal{L}_{2,d}^j \cap \operatorname{Supp}(f))\right) \cup \left(\cup_{j=d-3}^{d-1}(\mathcal{L}_{3,d}^j \cap \operatorname{Supp}(f))\right)| \\ &\ge 3(d-4) + (3\times 2 - 1) + (3 \times 2 - 1) = 3d-2. \quad \square\end{aligned}$$

5. Bound on the number of generators of ideals with more than three variables failing WLP

In this section we consider artinian monomial ideals $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ generated in degree d, for $n \ge 4$. We provide a sharp lower bound for the number of monomials with non-zero coefficients in a non-zero form $f \in (I^{-1})_d$ such that $(x_1 + \cdots + x_n) \circ f = 0$. The next theorem provides such lower bound for the form f in terms of the maximum degree of the variables in f.

Theorem 5.1. *For $n \ge 4$ and $d \ge 2$, let f be a non-zero form of degree d in the dual ring $R = \mathbb{K}[y_1, \dots, y_n]$ of $S = \mathbb{K}[x_1, \dots, x_n]$ such that $(x_1 + \cdots + x_n) \circ f = 0$. Then we have $|\operatorname{Supp}(f)| \ge \max\{(a_i+1)(d-a_i+1) \mid a_i \ne 0\}$, where $a_i = \max\{\deg_i(m) \mid m \in \operatorname{Supp}(f)\}$.*

Proof. We show that for each $1 \le i \le n$ we have $|\operatorname{Supp}(f)| \ge (a_i+1)(d-a_i+1)$. Denote $\ell = x_1 + \cdots + x_n$ and $\ell' = \ell - x_i$ and write $f = \sum_{k=0}^{a_i} y_i^k g_k$, where g_k is a polynomial of degree $d-k$ in the variables different from y_i. Since we have $\ell \circ f = 0$, Lemma 7.8 implies that

$$(k+1)g_{k+1} + \ell' \circ g_k = 0, \quad \forall 0 \le k \le a_i$$

and acting on each equation by $(\ell')^{a_i-k}$ we get that $(\ell')^{a_i-k+1} \circ g_k = 0$ for all $0 \le k \le a_i$. By the definition of a_i we have $g_{a_i} \ne 0$, Proposition 3.6 part (ii) implies that for every

$0 \le k \le a_i$ we have $g_k \ne 0$. Now applying Theorem 3.7 we get that for each $0 \le k \le a_i$, $|\operatorname{Supp}(g_k)| \ge d-k-(a_i-k+1)+2 = d-a_i+1$. Therefore,

$$|\operatorname{Supp}(f)| = |\cup_{k=0}^{a_i} \mathcal{L}_{i,d}^k \cap \operatorname{Supp}(f)| = \sum_{k=0}^{a_i} |\operatorname{Supp}(g_k)| \ge (a_i+1)(d-a_i+1)$$

and we conclude that $|\operatorname{Supp}(f)| \ge \max\{(a_i+1)(d-a_i+1) \mid 1 \le i \le n\}$. □

In general we can prove that the sharp lower bound is always $2d$. Recall from Definition 3.3 that $\phi(I,d) = \times(x_1+\cdots+x_n) : (S/I)_{d-1} \to (S/I)_d$.

Theorem 5.2. *For $n \ge 4$ and $d \ge 2$, we have*

$$\nu(n,d) = 2d.$$

Where, $\nu(n,d) = min\{H_{(S/I)}(d) \mid \phi(I,d)$ is not surjective, for $I \in \mathcal{I}_d\}$, and $\mathcal{I}_d$ is the set of all artinian monomial ideals of S generated in degree d.

Proof. First of all, we observe that for $f = (y_1-y_2)(y_3-y_4)^{d-1}$ we have $\ell \circ f = 0$ and since $|\operatorname{Supp}(f)| = 2d$ we get $\nu(n,d) \le 2d$. To show the equality, let $I \subset S$ be an artinian monomial ideal. We check that for any $f \in (I^{-1})_d$ where, $\ell \circ f = 0$, $|\operatorname{Supp}(f)| \ge 2d$.

Using Theorem 5.1 above, we get that for some $1 \le i \le n$ where $1 \le a_i \le d-1$ we have $|\operatorname{Supp}(f)| \ge (a_i+1)(d-a_i+1)$. Observe that since we have $a_i \le d-1$ we get that $(a_i+1)(d-a_i+1) = d(a_i+1)-(a_i-1)(a_i+1) \ge 2d$, which completes the proof. □

6. Formulations in terms of simplicial complexes and matroids

In [2] Gennaro, Ilardi and Vallès describe a relation between the failure of the SLP of artinian ideals and the existence of special singular hypersurfaces. In particular, for the ideals we consider in this section they proved that in the following cases the ideal I fails the SLP at the range k in degree $d+i-k$ if and only if there exists at any point M a hypersurface of degree $d+i$ with multiplicity $d+i-k+1$ at M given by a form in $(I^{-1})_{d+i}$, see [2] for more details. In [2, Theorem 6.2], they provide a list of monomial ideals $I \subset S = \mathbb{K}[x_1,x_2,x_3]$ generated in degree 5 failing the WLP. Here we give the exhaustive list of such ideals.

Definition 6.1. Let $I \subset S$ be an artinian monomial ideal and $G = \{m_1,\dots,m_r\} \subset R_d$ be a monomial generating set of $(I^{-1})_d$. Assume that I fails the WLP by failing surjectivity in degree $d-1$ thus there is a non-zero polynomial $f \in (I^{-1})_d$ with $\operatorname{Supp}(f) \subset G$ such that $(x_1+\cdots+x_n) \circ f = 0$. We say I fails the WLP *minimally* if the set G is minimal with respect to inclusion.

Remark 6.2. Note that for every artinian monomial ideal $I \subset S$ where the WLP fails minimally, there is a unique form in the kernel of the map $\circ(x_1 + \cdots + x_n)\colon (I^{-1})_d \longrightarrow (I^{-1})_{d-1}$. In fact, if there are two different forms with the same support we can eliminate at least one monomial in one of the forms and get a form where its support is strictly contained in the support of the previous ones, contradicting the minimality.

Proposition 6.3. *For an artinian monomial ideal $I \subset S$ generated in degree 5 with at least 6 generators, S/I fails the WLP by failing surjectivity in degree 4 if and only if the set of generators for the inverse system module I^{-1} contains the monomials in the support of one of the following forms, up to permutation of variables:*

- $(y_2 - y_3)(y_1 - y_3)^2(y_1 - y_2)(2y_1 - y_2 - y_3)$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)^2(2y_1 + y_2 - 3y_3)$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)(y_1^2 + y_1y_2 + y_2^2 - 3y_1y_3 - 3y_2y_3 + 3y_3^2)$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)(y_1^2 - y_1y_2 - y_2^2 - y_1y_3 + 3y_2y_3 - y_3^2)$
- $(y_2 - y_3)^2(y_1 - y_3)^2(y_1 - y_2)$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)^3$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)(y_1^2 - y_1y_2 + y_2^2 - y_1y_3 - y_2y_3 + y_3^2)$.

Moreover, the support of all the above forms define monomial ideals failing surjectivity minimally.

Proof. We prove the statement using Macaulay2 and considering all artinian monomial ideals generated in degree 5 with at least 6 generators. There are 816 of such ideals but considering the ones failing the WLP by failing surjectivity in degree 4 and considering the forms in the inverse system module $(I^{-1})_5$ there are only 25 distinct non-zero forms $f \in (I^{-1})_5$ such that $(x_1 + x_2 + x_3) \circ f = 0$. Therefore, every ideal where I^{-1} contains the support of each polynomial fails WLP by failing surjectivity in degree 4. Permuting the variables we get only 7 equivalence classes which correspond to the forms given in the statement. □

Remark 6.4. The support of the last three forms in Proposition 6.3 consists of 12 monomials which is the same as $\nu(3, 5) = 12$ given in 4.3. Therefore, the support of each form in the last three cases, up to permutations of the variables generates I^{-1} with lease possible number of generators in degree 5 where I fails the WLP.

Using Proposition 4.1, each of the forms above factors in linear form over an algebraically closed field; e.g. $\mathbb{K} = \mathbb{C}$.

The next result completely classifies monomial ideals $I \subset S = \mathbb{K}[x_1, x_2, x_3, x_4]$, generated in degree 3, failing the WLP which extends Proposition 6.3 in [2].

Proposition 6.5. *For an artinian monomial ideal $I \subset S$ generated in degree 3 with at least 10 generators, surjectivity of the multiplication map by a linear form in degree 2 of*

S/I fails if and only if the set of generators for inverse system module I^{-1} contains the monomials in the support of one of the following forms, up to permutation of variables:

- $(y_2-y_4)^2(y_1-y_3)$
- $(y_2-y_4)(y_1-y_4)(y_1-y_2)$
- $(y_2-y_3)(y_1-y_4)(y_1-2y_3+y_4)$
- $(y_2-y_3)(y_1-y_4)(y_1-y_2-y_3+y_4)$
- $(y_3-y_4)(y_2-y_4)(y_1-y_3)$
- $(y_1-y_4)(y_1y_2+y_1y_3-2y_2y_3-2y_1y_4+y_2y_4+y_3y_4)$
- $(y_1-y_2)(y_1y_2-y_1y_3-y_2y_3+2y_3y_4-y_4^2)$
- $(y_3-y_4)(y_2^2-y_1y_3+y_1y_4-2y_2y_4+y_3y_4)$
- $(y_3-y_4)(y_1^2+y_2^2-2y_1y_3-2y_2y_4+2y_3y_4)$
- $2y_1^2y_2-3y_1y_2^2+2y_2^2y_3-y_1y_3^2-2y_1^2y_4+2y_1y_2y_4+y_2^2y_4+2y_1y_3y_4-4y_2y_3y_4+y_3^2y_4$
- $y_1^2y_2-y_1y_2^2+y_2^2y_3-y_1y_3^2-y_1^2y_4+2y_1y_3y_4-2y_2y_3y_4+y_3^2y_4+y_2y_4^2-y_3y_4^2$
- $y_1^2y_2-y_1^2y_3-2y_1y_2y_3+2y_2y_3^2+4y_1y_3y_4-2y_2y_3y_4-2y_3^2y_4-2y_1y_4^2+y_2y_4^2+y_3y_4^2$
- $y_1^2y_2-y_1^2y_3-y_1y_2y_3+y_2y_3^2-y_1y_2y_4+3y_1y_3y_4-y_2y_3y_4-y_3^2y_4-y_1y_4^2+y_2y_4^2$.

Moreover, the support of all the above forms define monomial ideals failing surjectivity minimally.

Proof. We prove it using the same method as the proof of Proposition 6.3 using Macaulay2. There are 8008 artinian monomial ideals generated in degree 3 with at least 10 generators. Considering the forms in the inverse system module $(I^{-1})_3$ where $(x_1+x_2+x_3+x_4)\circ f=0$ correspond to the ideals failing WLP with failing surjectivity in degree 2, there are 237 distinct non-zero forms. Thus any ideal I where its inverse system module I^{-1} contains the support of each of the forms fails WLP in degree 2. Also considering the permutation of the variables there are 13 distinct forms given in the statement. □

Remark 6.6. The first two forms have 6 monomials which is the same as $\nu(4,3)=6$ given in 5.2. Therefore, each form in the last two cases, up to permutation of variables give the minimal number of generators for the inverse system module I^{-1} where I fails the WLP.

One can check that the factors in the forms given in Proposition 6.5 are irreducible even over the complex numbers (or any algebraically closed field of characteristic zero).

The above results lead us to correspond simplicial complexes to the class of ideals failing the WLP by failing surjectivity. Recall that Theorem 4.3 and Corollary 5.2 imply that in the polynomial ring $S=\mathbb{K}[x_1,\dots,x_n]$ when the Hilbert function of an artinian monomial algebra generated in a single degree d, $H_{S/I}(d)$, is less than $\nu(n,d)$, the monomial algebra S/I satisfies the WLP. First we recall the following definitions:

Definition 6.7. A *matroid* is a finite set of elements M together with the family of subsets of M, called independent sets, satisfying,

- The empty set is independent,
- Every subset of an independent set is independent,
- For every subset A of M, all maximal independent sets contained in A have the same number of elements.

A *simplicial complex* Δ is a set of simplices such that any face of a simplex from Δ is also in Δ and the intersection of any two simplices is a face of both. Note that every matroid is also a simplicial complex with independent sets as its simplices.

Definition 6.8. Recall $\mathcal{M}_d$ from Definition 3.4 which is the set of monomials in degree d in the ring R and define $\mathcal{M}'_d = \mathcal{M}_d \setminus \{y_1^d, \dots, y_n^d\}$. We define *independent* set $s \subset \mathcal{M}'_d$ to be the set of monomials such that the set $\{(x_1 + \dots + x_n) \circ m \mid m \in s\}$ is a linearly independent set. A subset $s \subset \mathcal{M}'_d$ is called *dependent* if it is not an independent set. Then define $\Delta_{d,sur}$ to be the simplicial complex with the monomials in $\mathcal{M}'_d$ as the ground set and all independent sets as its faces. Note that $\Delta_{d,sur}$ forms a matroid.

Define $\Delta^*_{d,sur}$ to be the simplicial complex with the monomials in $S_d \setminus \{x_1^d, \dots, x_n^d\}$ as its ground set and faces of $\Delta^*_{d,sur}$ are the corresponding monomials of $\mathcal{M}'_d \setminus s$ in S where s is a dependent set.

Proposition 6.9. *With the above notations we have*

(*i*) $\dim(\Delta_{d,sur}) \le h_{d-1}(R) - 1$,
(*ii*) $\dim(\Delta^*_{d,sur}) = |S_d| - n - \nu(n,d) - 1$.

Proof. Too show (i) notice that any proper subset of the support of each of the forms in 6.3 and 6.5 forms an independent set. For every independent set s, monomial ideal $I \subset S$ generated by the d-th power of the variables in S and corresponding monomials of $\mathcal{M}'_d \setminus s$ in S form an artinian ideal I, where S/I satisfies the WLP. Since the ground set of $\Delta_{d,sur}$ is the subset of monomials R_d the size of an independent set is bounded from above by the number of monomials in R_{d-1}. Therefore we have $\dim(\Delta_{d,sur}) \le h_{d-1}(R) - 1$.

In order to prove (ii) we observe that an artinian algebra S/I where I is generated by the d-th power of the variables in S together with the monomials in a face of $\Delta^*_{d,sur}$, fails the WLP. In fact the multiplication map on S/I form degree $d-1$ to d is not surjective. Theorem 4.3 and 5.2 imply that every subset $s \subset \mathcal{M}'_d$ with $|s| \le \nu(n,d)$ is independent. Therefore we have $\dim(\Delta^*_{d,sur}) = |S_d| - n - \nu(n,d) - 1$, the equality is because the bound is sharp. □

Example 6.10. The support of each polynomial given in Proposition 6.3 is a minimal non-face of the simplicial complex $\Delta_{5,sur}$ with the ground set $\mathcal{M}'_5 = \mathcal{M}_5 \setminus \{y_1^5, y_2^5, y_3^5\}$. This simplicial complex has 25 minimal non-faces (considering the permutations of variables). $\Delta_{5,sur}$ has 7 minimal non-faces of dimension 11, 6 minimal non-faces of dimension 13 and 12 minimal non-faces of dimension 14.

Remark 6.11. Recall that the *Alexander dual* of a simplicial complex Δ on the ground set V is a simplicial complex with the same ground set and faces are all the subsets of V where their complements are non-faces of Δ. Observe that $\Delta^*_{d,sur}$ is a simplicial complex in S_d and $\Delta_{d,sur}$ is a simplicial complex in the Macaulay dual ring R_d. Note that for any independent set $s \subset \mathcal{M}'_d$ the corresponding monomials of the complement $\mathcal{M}'_d \setminus s$ in the ring S is not a face of $\Delta^*_{d,sur}$ which implies that $\Delta_{d,sur}$ is Alexander dual to $\Delta^*_{d,sur}$.

7. WLP of ideals fixed by actions of a cyclic group

Mezzetti and Miró-Roig in [10] studied artinian ideals of the polynomial ring $\mathbb{K}[x_1, x_2, x_3]$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero generated by homogeneous polynomials of degree d invariant under an action of cyclic group $\mathbb{Z}/d\mathbb{Z}$, for $d \geq 3$ and they proved that if $\gcd(a_1, a_2, a_3, d) = 1$ they define monomial Togliatti systems. In [1], Colarte, Mezzetti, Miró-Roig and Salat consider such ideals in a polynomial ring with at least three variables. Throughout this section $\mathbb{K} = \mathbb{C}$ and $S = \mathbb{K}[x_1, \dots, x_n]$, where $n \geq 3$. Let $d \geq 2$ and $\xi = e^{2\pi i/d}$ to be the primitive d-th root of unity. Consider the diagonal matrix

$$M_{a_1,\dots,a_n} = \begin{pmatrix} \xi^{a_1} & 0 & \cdots & 0 \\ 0 & \xi^{a_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \xi^{a_n} \end{pmatrix}$$

representing the cyclic group $\mathbb{Z}/d\mathbb{Z}$, where $a_1, a_2, \dots, a_n$ are integers and the action is defined by $[x_1, \dots, x_n] \mapsto [\xi^{a_1}x_1, \dots, \xi^{a_n}x_n]$. Since $\xi^d = 1$, we may assume that $0 \leq a_i \leq d-1$, for every $1 \leq i \leq n$. Let $I \subset S$ be the ideal generated by all the forms of degree d fixed by the action of $M_{a_1,\dots,a_n}$. In [10, Theorem 3.1], Mezzetti and Miró-Roig showed that these ideals are monomial ideals when $n = 3$. Here we state it in general for all $n \geq 3$ with a slightly different proof.

Lemma 7.1. *For integer $d \geq 2$, the ideal $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ generated by all the forms of degree d fixed by the action of $M_{a_1,\dots,a_n}$ is artinian and generated by monomials.*

Proof. Since $M_{a_1,\dots,a_n}$ is a monomial action in the sense that for every monomial m of degree d we have $M^r_{a_1,\dots,a_n} m = cm$ for each $0 \leq r \leq d-1$ and for some $c \in \mathbb{K}$, then if we have a form of degree d fixed by $M_{a_1,\dots,a_n}$, all its monomials are fixed by $M_{a_1,\dots,a_n}$. This implies that I is a monomial ideal. Note also that since $\xi^d = 1$, all the monomials $x_1^d, x_2^d, \dots, x_n^d$ are fixed by the action of $M_{a_1,\dots,a_n}$ which means I is an artinian ideal. $\square$

Using the above result, from now on we take the monomial set of generators for I. Observe that for two distinct primitive d-th roots of unity we get different actions, but the set of monomials fixed by both actions are the same. Also the action $M_{a_n+r,\dots,a_n+r}$ which is obtained by multiplying the matrix $M_{a_1,\dots,a_n}$ with a d-th root of unity defines the same action on degree d monomials in S. In [10], Colarte, Mezzetti, Miró-Roig and Salat show that in the case that $n=3$ where a_i's are distinct and $\gcd(a_1,a_2,a_3,d)=1$, these ideals are all monomial Togliatti systems. In fact they show that the WLP of these ideals fails in degree $d-1$ by failing injectivity of the multiplication map by a linear form in that degree. In this section, we study the cases where WLP of such ideals fails by failing surjectivity in degree $d-1$. Then we classify all such ideals in polynomial rings with more than 2 variables, in terms of their WLP.

We start this section by stating some results about the number of monomials of degree d fixed by the action $M_{a_1,\dots,a_n}$ of $\mathbb{Z}/d\mathbb{Z}$ in S. In fact we prove that this number depends on the integers a_i's. In the next result we give an explicit formula computing the number of such monomials where $n=3$.

Proposition 7.2. *For integers a_1,a_2,a_3 and $d\geq 2$, the number of monomials in $S=\mathbb{K}[x_1,x_2,x_3]$ of degree d fixed by the action of M_{a_1,a_2,a_3} is*

$$1+\frac{\gcd(a_2-a_1,a_3-a_1,d)\cdot d+\gcd(a_2-a_1,d)+\gcd(a_3-a_1,d)+\gcd(a_3-a_2,d)}{2}. \tag{7.1}$$

Proof. From the discussion above, the number of monomials of degree d fixed by M_{0,a_2-a_1,a_3-a_1} and M_{a_1,a_2,a_3} are the same. Thus, we count the number of monomials of degree d fixed by M_{0,a_2-a_1,a_3-a_1}. Any monomial of degree d in S can be written as $x^{d-m-n}y^mz^n$ with $0\leq m,n\leq d$ and $m+n\leq d$ and it is invariant under the action of M_{0,a_2-a_1,a_3-a_1} if and only if $(a_2-a_1)m+(a_3-a_1)n\equiv 0 \pmod d$. In [8, Chapter 3], we find that the number of congruent solutions of $(a_2-a_1)m+(a_3-a_1)n\equiv 0 \pmod d$ is $\gcd(a_2-a_1,a_3-a_1,d)\cdot d$ but since the solutions $(0,0)$, $(0,d)$ and $(d,0)$ (corresponding to the powers of variables) are all congruent to d and fixed by M_{0,a_2-a_1,a_3-a_1} we get two more solutions than $\gcd(a_2-a_1,a_3-a_1,d)\cdot d$. In order to count the monomials of degree d invariant under the action of M_{0,a_2-a_1,a_3-a_1} we need to count the number of solutions of $(a_2-a_1)m+(a_3-a_1)n\equiv 0 \pmod d$ satisfying the extra condition $m+n\leq d$.

First we count the number of such solutions when $m=0$ and $n\neq 0$. So every $1\leq n<\gcd(a_3-a_1,d)$ is a solution of $(a_3-a_1)n\equiv 0 \pmod d$. Therefore there are $\gcd(a_3-a_1,d)-1$ solutions in this case. Similarly, there are $\gcd(a_2-a_1,d)-1$ solutions when $n=0$ and $m\neq 0$. Counting the solutions when $m+n=d$ is equivalent to counting the solutions of $(a_3-a_2)m\equiv 0 \pmod d$ which is similar to the previous case and is equal to $\gcd(a_3-a_2,d)-1$. There is also one solution when $m=n=0$.

Now rest of the solutions (where $m\neq 0$ and $n\neq 0$ and $m+n\neq d$) by [8, Chapter 3] is equal to $\gcd(a_2-a_1,a_3-a_1,d)\cdot d-\gcd(a_3-a_2,d)-\gcd(a_2-a_1,d)-\gcd(a_3-a_1,d)+2$ but we need to count the number of those satisfying $0<m+n<d$. Note that if

$0 < m_0 < d$ and $0 < n_0 < d$ is a solution of $(a_2 - a_1)m + (a_3 - a_1)n \equiv 0 \pmod d$ then $0 < d - m_0 < d$ and $0 < d - n_0 < d$ is also a solution but one and only one of the two conditions $0 < m_0 + n_0 < d$ and $0 < d - m_0 + d - n_0 < d$ is satisfied. Therefore, there are

$$\frac{\gcd(a_2 - a_1, a_3 - a_1, d) \cdot d - \gcd(a_3 - a_2, d) - \gcd(a_2 - a_1, d) - \gcd(a_3 - a_1, d) + 2}{2}$$

solutions satisfying $0 < m + n < d$. Adding this with the solutions where $m = 0$ or $n = 0$ or $m + n = d$ which we have counted them above together with two more pairs $(0, d)$ and $(d, 0)$ (explained in the beginning of the proof) we get what we wanted to prove. □

For a fixed integer $d \geq 2$ Proposition 7.2 shows how the number of fixed monomials of degree d depends on the integers a_1, a_2, a_3. In the following example we see how they are distributed.

Example 7.3. Using Formula (7.1) we count the number of monomials of degree $d = 15$ in $\mathbb{K}[x_1, x_2, x_3]$ fixed by the action $M_{0,a,b}$ for every $0 \leq a, b \leq 14$. We see the distribution of them in terms of $\mu(I)$ in the following table:

m	10	11	12	13	17	28	34	46	51	136
d_m	24	72	24	48	24	12	12	2	6	1

where $d_m = |\{(a, b) \mid \mu(I) = m\}|$. Note that the last column of the table corresponds to the action $M_{0,0,0}$ where we get $\mu(I) = (\mathbb{K}[x_1, x_2, x_3])_{15} = 136$. There are exactly 24 pairs (a, b) where either at least one of them is zero or $a = b$, which in these cases we get $\mu(I) = 17$. We have $\gcd(a, b, d) \neq 1$ for all the cases with $\mu(I) > 17$ and $\gcd(a, b, d) = 1$ for all the cases with $\mu(I) < 17$.

As we saw in the above example the distribution of the number of monomials of degree d fixed by M_{a_1,a_2,a_3} is quite difficult to understand but we prove that such numbers are bounded from above depending on the prime factors of d in the case that a_i's are distinct and $\gcd(a_1, a_2, a_3, d) = 1$.

Proposition 7.4. *For $d \geq 3$ and distinct integers $0 \leq a_1, a_2, a_3 \leq d - 1$ with $\gcd(a_1, a_2, a_3, d) = 1$, let $\mu(I)$ be the number of monomials of degree d fixed by M_{a_1,a_2,a_3}. Then*

$$\mu(I) \leq \begin{cases} \frac{(p+1)d+p^2+3p}{2p} & \text{if } p^2 \nmid d \\ \frac{(p+1)d+4p}{2p} & \text{if } p^2 \mid d, \end{cases}$$

where p is the smallest prime dividing d. Moreover, the bounds are sharp.

Proof. Using Proposition 7.2 we provide an upper bound for $\gcd(a_2-a_1,d)+\gcd(a_3-a_1,d)+\gcd(a_3-a_2,d)$. For some integer t we have $d=\gcd(a_2-a_1,d)\cdot\gcd(a_3-a_1,d)\cdot\gcd(a_3-a_2,d)\cdot t$. Since $\gcd(a_3-a_1,d)\cdot\gcd(a_3-a_2,d)=\frac{d}{\gcd(a_2-a_1,d)\cdot t}$, we have

$$\gcd(a_3-a_1,d)+\gcd(a_3-a_2,d)\leq 1+\frac{d}{\gcd(a_3-a_2,d)\cdot t}.$$

Therefore,

$$\begin{aligned}
&\gcd(a_2-a_1,d)+\gcd(a_3-a_1,d)+\gcd(a_3-a_2,d)\\
&\quad\leq \gcd(a_2-a_1,d)+\frac{d}{\gcd(a_3-a_1,d)\cdot t}+1\\
&\quad\leq \gcd(a_2-a_1,d)+\frac{d}{\gcd(a_2-a_1,d)}+1\\
&\quad\leq p+\frac{d}{p}+1.
\end{aligned}$$

Note that, $\gcd(a_2-a_1,d)+\gcd(a_3-a_1,d)+\gcd(a_3-a_2,d)=d+2>p+\frac{d}{p}+1$ if and only if at least two integers a_i are the same which contradicts the assumption. Since for every $q\geq p$ we have $p+\frac{d}{p}+1\geq q+\frac{d}{q}+1$ we get $\gcd(a_2-a_1,d)+\gcd(a_3-a_1,d)+\gcd(a_3-a_2,d)\leq p+\frac{d}{p}+1$.

Now assume that $p^2\nmid d$, to reach the bound we let $a_2-a_1=p$ and $a_3-a_1=\frac{d}{p}$. In this case since we have that $\gcd(a_3-a_2,d)=1$, Proposition 7.2 implies that $\mu(I)\leq\frac{(p+1)d+p^2+3p}{2p}$.

If $p^2\mid d$, choosing $a_2-a_1=p$ and $a_3-a_1=\dfrac{d}{p}$ implies that $\gcd(a_3-a_2,d)=p$. So the given bound can not be sharp. Observe that for $q>p$ and $q\mid d$ we have $q+\frac{d}{q}+1\leq 1+\frac{d}{p}+1$. Therefore in this case we have $\gcd(a_2-a_1,d)+\gcd(a_3-a_1,d)+\gcd(a_3-a_2,d)\leq 1+\frac{d}{p}+1$, and equality holds for $a_2-a_1=1$ and $a_3-a_1=\frac{d}{p}$ so by Proposition 7.2 we have that $\mu(I)\leq\frac{(p+1)d+4p}{2p}$. $\square$

In the proof of Proposition 7.2, we used the fact that the number of solutions (m,n) for $(a_2-a_1)m+(a_3-a_1)n\equiv 0 \pmod d$ (corresponding to the action by M_{a_1,a_2,a_3}) where $m,n\neq 0$ and $m+n\neq d$ is exactly twice the number of solutions of $(b-a)m+(c-a)n\equiv 0 \pmod d$ satisfying $0<m+n<d$. But in the polynomial ring with more than three variables this is no longer the case that the solutions of the corresponding equation of $M_{a_1,\ldots,a_n}$ are distributed in a nice way so we do not have the explicit formula as in Proposition 7.2 in higher number of variables. In Proposition 7.5 below we provide an upper bound for this number in the polynomial ring with four variables where $\gcd(a_1,a_2,a_3,a_4,d)=1$. The bound implies $H_{S/I}(d-1)\leq H_{S/I}(d)$ and therefore the WLP in degree $d-1$ is an assertion of injectivity. In [1, Theorem 4.8], Colarte, Mezzetti,

Miró-Roig and Salat show that the number of monomials in $(\mathbb{K}[x_1, \dots, x_n])_{n+1}$ fixed by the action $M_{0,1,2,\dots,n}$ of $\mathbb{Z}/(n+1)\mathbb{Z}$ is bounded above by $\binom{2n-1}{n-1}$, for any $n \geq 3$.

Proposition 7.5. *For $d \geq 2$ and integers $0 \leq a_1, a_2, a_3, a_4 \leq d-1$, where at most two of the integers among a_i's are equal and $\gcd(a_1, a_2, a_3, a_4, d) = 1$. Let $\mu(I)$ be the number of monomials of degree d in $S = \mathbb{K}[x_1, x_2, x_3, x_4]$ fixed by M_{a_1,a_2,a_3,a_4}. Then*

$$\mu(I) \leq 1 + \frac{(d+2)(d+1)}{2}.$$

Proof. Any monomial of degree d in S can be written as $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{d-m_1-m_2-m_3}$ with $0 \leq m_1, m_2, m_3 \leq d$ and $m_1 + m_2 + m_3 \leq d$. Monomial $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{d-m_1-m_2-m_3}$ is invariant under the action of M_{a_1,a_2,a_3,a_4} or equivalently $M_{a_1-a_4,a_2-a_4,a_3-a_4,0}$ if and only if

$$(a_1 - a_4)m_1 + (a_2 - a_4)m_2 + (a_3 - a_4)m_3 \equiv 0 \pmod{d}, \quad m_1 + m_2 + m_3 \leq d. \tag{7.2}$$

In [8, Chapter 3] we find that the number of congruent solutions of $(a_1 - a_4)m_1 + (a_2 - a_4)m_2 + (a_3 - a_4)m_3 \equiv 0 \pmod{d}$ is d^2. We first count the number of congruent solutions of (7.2) where at least one of m_1, m_2 or m_3 is zero. Suppose $m_1 = 0$ then by [8, Chapter 3], the number of congruent solutions of $(a_2 - a_4)m_2 + (a_3 - a_4)m_3 \equiv 0 \pmod{d}$ is $gcd(a_2 - a_4, a_3 - a_4, d) \cdot d$. Similarly, by [8, Chapter 3], the number of congruent solutions of (7.2) having two coordinates zero, for example $m_1 = m_2 = 0$, is $gcd(a_3 - a_4, d)$. All together the number of congruent solutions of (7.2) where at least one of the coordinates m_1, m_2, m_3 is zero is as follows

$$\begin{aligned} &d\,(gcd(a_1 - a_4, a_2 - a_4, d) + gcd(a_2 - a_4, a_3 - a_4, d) + gcd(a_1 - a_4, a_3 - a_4, d)) \\ &- gcd(a_1 - a_4, d) - gcd(a_2 - a_4, d) - gcd(a_3 - a_4, d) + 1. \end{aligned}$$

Note that if (m_{10}, m_{20}, m_{30}) is a solution of (7.2) such that $m_{i0} \neq 0$ for $i = 1, 2, 3$, then $(d - m_{10}, d - m_{20}, d - m_{30})$ is a solution of $(a_1 - a_4)m_1 + (a_2 - a_4)m_2 + (a_3 - a_4)m_3 \equiv 0 \pmod{d}$ where $3d - m_{10} - m_{20} - m_{30} \geq d$. Therefore, the number of congruent solutions of (7.2) where no m_i is zero is bounded from above by

$$\begin{aligned} &[d^2 - (d(gcd(a_1 - a_4, a_2 - a_4, d) + gcd(a_2 - a_4, a_3 - a_4, d) + gcd(a_1 - a_4, a_3 - a_4, d)) \\ &\qquad - gcd(a_1 - a_4, d) - gcd(a_2 - a_4, d) - gcd(a_3 - a_4, d) + 1)]/2. \end{aligned}$$

Using Proposition 7.2, we count the number of solutions (7.2) where at least one of the coordinates m_i is zero. If $m_1 = 0$ then by Proposition 7.2 the number of solutions of $(a_2 - a_4)m_2 + (a_3 - a_4)m_3 = 0 \pmod{d}$ where $0 \leq m_2, m_3 \leq d$ and $m_2 + m_3 \leq d$ is

$$\frac{\gcd(a_2 - a_4, a_3 - a_4, d) \cdot d + \gcd(a_2 - a_4, d) + \gcd(a_3 - a_4, d) + \gcd(a_2 - a_3, d) + 2}{2}.$$

Similarly we can count the number of such solutions when $m_2 = 0$ or $m_3 = 0$. Now suppose that $m_1 = m_2 = 0$ then we get $\gcd(a_3 - a_4, d) + 1$ where $0 \leq m_3 \leq d$. All together the number of solutions of (7.2) where at least one m_i is zero is

$$\begin{aligned}&[d(\gcd(a_1 - a_4, a_2 - a_4, d) + \gcd(a_2 - a_4, a_3 - a_4, d) + \gcd(a_1 - a_4, a_3 - a_4, d)) \\ &+ \gcd(a_1 - a_2, d) + \gcd(a_1 - a_3, d) + \gcd(a_2 - a_3, d) + 2]/2.\end{aligned}$$

Therefore, the number of solutions of (7.2) is bounded from above by

$$\frac{d^2 + \sum_{i=1}^{3} \gcd(a_i - a_4, d) + \sum_{1\leq i<j\leq 3} \gcd(a_i - a_j, d) + 1}{2}. \tag{7.3}$$

To show the assertion of the theorem we need to show (7.3) is bounded from above by $\frac{(d+2)(d+1)+2}{2}$ where at most 2 of integers among a_i's are equal and $\gcd(a_1, a_2, a_3, a_4, d) = 1$. So we need to show that

$$\sum_{i=1}^{3} \gcd(a_i - a_4, d) + \sum_{1\leq i<j\leq 3} \gcd(a_i - a_j, d) = \sum_{1\leq i\leq j\leq 4} \gcd(a_i - a_j, d) \leq 3d + 3. \tag{7.4}$$

To show this we consider the following cases:

(1) Suppose at least two terms in the left hand side of (7.4) are equal to d then at least three integers among a_i's are equal which contradicts the assumption.
(2) Suppose that one of the terms in the left hand side is equal to d. By relabeling the indices we may assume that $\gcd(a_1 - a_2, d) = d$, this implies that $a_1 = a_2$ then we need to show that

$$d + 2\gcd(a_1 - a_3, d) + 2\gcd(a_1 - a_4, d) + \gcd(a_3 - a_4, d) \leq 3d + 3$$

since we assume that $\gcd(a_1, a_2, a_3, a_4, d) = 1$ we have that $\gcd(a_1 - a_4, d)$, $\gcd(a_3 - a_4, d)$ and $\gcd(a_1 - a_3, d)$ are all distinct and strictly less than d. Thus we have

$$d + 2\gcd(a_1 - a_3, d) + 2\gcd(a_1 - a_4, d) + \gcd(a_3 - a_4, d) \leq d + 2\frac{d}{2} + 2\frac{d}{3} + \frac{d}{4} < 3d + 3.$$

(3) Suppose all the terms in the left hand side of (7.4) are strictly less than d. Then the assumption $\gcd(a_1, a_2, a_3, a_4, d) = 1$ implies that at most two terms can be $d/2$ and assuming the other terms are $d/3$ we get

$$\sum_{1\leq i\leq j\leq 4} \gcd(a_i - a_j, d) \leq 3d + 3 \leq 2(d/2) + 4(d/3) = d + d/2 < 3d + 3. \quad \square$$

In the rest of this section we study the WLP of ideals in $S = \mathbb{K}[x_1, \dots, x_n]$ for $n \geq 3$ generated by all forms of degree $d \geq 3$ invariant by the action $M_{a_1,\dots,a_n}$ of $\mathbb{Z}/d\mathbb{Z}$. First we prove the following key lemma.

Lemma 7.6. *For integer $d \geq 2$ and distinct integers $0 \leq a_1, a_2, a_3 \leq d-1$, let M_{a_1,a_2,a_3} be a representation of $\mathbb{Z}/d\mathbb{Z}$. Define the linear form*

$$L = \sum_{j=0}^{l} \xi^j x_1 + \sum_{j=l+1}^{l+k+1} \xi^j x_2 + \sum_{j=l+k+2}^{2d-1} \xi^j x_3,$$

where l and k are the residues of $a_2 - a_3 - 1$ and $a_3 - a_1 - 1$ modulo d. Then the support of the form $F = L^d - \overline{L}^d$ are exactly the monomials of degree d in $\mathbb{K}[x_1, x_2, x_3]$ which are not invariant under the action of M_{a_1,a_2,a_3}, where $\overline{L}$ is the conjugate of L and ξ is a primitive d-th root of unity.

Proof. First, note that for a rational number j we let $\xi^j = e^{j\frac{2\pi i}{2}}$. We observe that for integers $0 \leq p \leq q$ we have $\sum_p^q \xi^i = \xi^{\frac{p+q}{2}} \sum_p^q \xi^{i-\frac{p+q}{2}}$, where $\sum_p^q \xi^{i-\frac{p+q}{2}} = \xi^{\frac{p-q}{2}} + \xi^{\frac{p-q}{2}+1} + \cdots + \xi^{\frac{q-p}{2}-1} + \xi^{\frac{q-p}{2}}$ which is invariant under conjugation, so it is a real number. Therefore, we have

$$L = \sum_{j=0}^{l} \xi^j x_1 + \sum_{j=l+1}^{l+k+1} \xi^j x_2 + \sum_{j=l+k+2}^{2d-1} \xi^j x_3 = r_1 \xi^{\frac{l}{2}} x_1 + r_2 \xi^{\frac{2l+k+2}{2}} x_2 + r_3 \xi^{\frac{l+k+1}{2}} x_3$$

where r_1, r_2 and r_3 are non-zero real numbers. In fact, using the assumption that a_1, a_2 and a_3 are distinct we get that $0 \leq l, k \leq d-2$ which implies that the r_i's are all non-zero. The form F can be written as

$$\begin{aligned} F &= L^d - \overline{L}^d \\ &= \left(r_1 \xi^{\frac{l}{2}} x_1 + r_2 \xi^{\frac{2l+k+2}{2}} x_2 + r_3 \xi^{\frac{l+k+1}{2}} x_3\right)^d \\ &\quad - \left(r_1 \xi^{-\frac{l}{2}} x_1 + r_2 \xi^{-\frac{2l+k+2}{2}} x_2 + r_3 \xi^{-\frac{l+k+1}{2}} x_3\right)^d. \end{aligned}$$

Consider monomial $m = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ of degree d in $\mathbb{K}[x_1, x_2, x_3]$. The coefficient of m in F is zero if and only if the coefficients of m in L^d is real. The coefficient of m in L^d is real if and only if

$$\alpha_1 \frac{l}{2} + \alpha_2 \frac{2l+k+2}{2} + \alpha_3 \frac{l+k+1}{2} \equiv \alpha_1 \frac{-l}{2} + \alpha_2 \frac{-(2l+k+2)}{2} + \alpha_3 \frac{-(l+k+1)}{2} \pmod{d}$$

which is equivalent to have

$$\alpha_1 l + \alpha_2 (2l+k+2) + \alpha_3 (l+k+1) \equiv 0, \quad (\bmod\ d).$$

Therefore, the monomials with non-zero coefficients in F are exactly the monomials of degree d in $\mathbb{K}[x_1, x_2, x_3]$, which are not fixed by the action of $M_{l,2l+k+2,l+k+1}$. Substituting l, k we get that $M_{l,2l+k+2,l+k+1}$ is equivalent to the action $M_{a_2-a_3-1,2a_2-a_3-a_1-1,a_2-a_1-1}$

and by adding $a_1 - a_2 + a_3 + 1$ to the indices the last one is also equivalent to M_{a_1,a_2,a_3} which proves what we wanted. $\square$

Remark 7.7. The assumption in Lemma 7.6 that a_i's are distinct is necessary to have the form F non-zero. If at least two of the integers a_i are equal then in the linear form L at least the coefficient of one of the variables x_1, x_2 and x_3 is zero. Then we conclude that in L^d all the monomials have real coefficients which implies $F = 0$.

Lemma 7.6, can be extended to any polynomial ring with odd number of variables. In fact in this case we can find $n-1$ integers l_i in terms of the integers a_i defining the action $M_{a_1,\dots,a_n}$ in such a way that a similar linear form as L in the lemma in n variables does the same.

In [1, Proposition 4.6], Colarte, Mezzetti, Miró-Roig and Salat show that the WLP of I fails by failing injectivity in degree $d-1$ in the polynomial ring $\mathbb{K}[x_1,\dots,x_n]$. In fact they provide the non-zero form $f = \prod_{i=1}^{d-1}(\xi^{ia_1}x_1 + \cdots + \xi^{ia_n}x_n)$ in the kernel of the multiplication map by a linear form on artinian algebra $\mathbb{K}[x_1,\dots,x_n]/I$ from degree $d-1$ to degree d. So all the monomials with non-zero coefficient in $(x_1 + \cdots + x_n)f$ are fixed by the action $M_{a_1,\dots,a_n}$.

We can now state and prove our main theorem which generalizes [10, Proposition 3.2] and [1, Proposition 4.6] and gives the complete classification of ideals in $S = \mathbb{K}[x_1,\dots,x_n]$ generated by all forms of degree d fixed by the action of $M_{a_1,\dots,a_n}$, for every $n \geq 3$ and $d \geq 2$, in terms of their WLP.

Theorem 7.8. *For integers $d \geq 2$, $n \geq 3$ and $0 \leq a_1,\dots,a_n \leq d-1$, let $M_{a_1,\dots,a_n}$ be a representation of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ and $I \subset S = \mathbb{K}[x_1,\dots,x_n]$ be the ideal generated by all forms of degree d fixed by the action of $M_{a_1,\dots,a_n}$. Then, I satisfies the WLP if and only if at least $n-1$ of the integers a_i are equal.*

Proof. Suppose at least $n-1$ of the integers a_i's are equal and by relabeling the variables we may assume that $a_1 = a_2 = \cdots = a_{n-1}$. For $n = 3$, Lemma 5.2 [10] shows that I satisfies the WLP. Similarly for $n \geq 3$ the ideal I contains $(x_1, x_2, \dots, x_{n-1})^d$, and then all the monomials in $(S/I)_d$ are divisible by x_n which implies that the map $\times x_n : (S/I)_{d-1} \longrightarrow (S/I)_d$ is surjective. Since $[(S/I)/x_n(S/I)]_d = 0$ we have that $[(S/I)/x_n(S/I)]_j = 0$ for all $j \geq d$ and then $\times x_n : (S/I)_{j-1} \longrightarrow (S/I)_j$ is surjective for all $j \geq d$. On the other hand, since I is generated in degree d, the map $\times x_n : (S/I)_{j-1} \longrightarrow (S/I)_j$ is injective, for every $j < d$. Therefore, I has the WLP.

To show the other implication, we assume that at most $n-2$ integers a_i are equal and we prove that I fails WLP by showing that map $\times(x_1 + \cdots + x_n) : (S/I)_{d-1} \longrightarrow (S/I)_d$ is neither injective nor surjective.

By [1, Proposition 4.6], for the non-zero form $f = \prod_{i=1}^{d-1}(\xi^{ia_1}x_1 + \cdots + \xi^{ia_n}x_n)$ of degree $d-1$ we have that $(x_1 + \cdots + x_n)f$ is a form of degree d in I. Therefore the map $\times(x_1 + \cdots + x_n) : (S/I)_{d-1} \longrightarrow (S/I)_d$ is not injective.

Now it remains to show the failure of surjectivity. To do so by Macaulay duality equivalently we show that the map $\circ(x_1+\cdots+x_n):(I^{-1})_d \longrightarrow (I^{-1})_{d-1}$ is not injective. Note that the inverse module $(I^{-1})_d$ is generated by all the monomials of degree d in the dual ring $R=\mathbb{K}[y_1,\dots,y_n]$ which are not fixed by the action $M_{a_1,\dots,a_n}$.

We consider two cases depending on a_i's. First, assume that there are at least three distinct integers among a_i's and by relabeling the variables we may assume that $a_1 < a_2 < a_3$.

By applying Lemma 7.6 on the ring R, we get the linear form

$$L=\sum_{j=0}^{l}\xi^j y_1+\sum_{j=l+1}^{l+k+1}\xi^j y_2+\sum_{j=l+k+2}^{2d-1}\xi^j y_3,$$

where l and k are the residues of a_2-a_3-1 and a_3-a_1-1 modulo d and ξ is a primitive d-th root of unity. Since a_1,a_2 and a_3 are distinct $F=L^d-\overline{L}^d$ is non-zero form of degree d. The monomials with non-zero coefficients in F are exactly the monomials of degree d in $\mathbb{K}[y_1,y_2,y_3]$ which are not fixed by the action M_{a_1,a_2,a_3}. Therefore, all the monomials of degree d in R fixed by the action $M_{a_1,\dots,a_n}$ have coefficient zero in F and thus we get that $F\in(I^{-1})_d$. Moreover, sum of the coefficients in L and $\overline{L}$ is exactly $2(1+\xi^1+\xi^2+\cdots+\xi^{d-1})=0$. Therefore,

$$\begin{aligned}(x_1+\cdots+x_n)\circ F&=(x_1+\cdots+x_n)\circ(L^d)-(x_1+\cdots+x_n)\circ(\overline{L}^d)\\&=d\cdot L^{d-1}\cdot\big((x_1+\cdots+x_n)\circ L\big)-d\cdot\overline{L}^{d-1}\cdot\big((x_1+\cdots+x_n)\circ\overline{L}\big)\\&=d\cdot(L^{d-1}-\overline{L}^{d-1})\cdot\big(2(1+\xi^1+\xi^2+\cdots+\xi^{d-1})\big)\\&=0,\end{aligned}$$

and this implies that $\times(x_1+\cdots+x_n):(S/I)_{d-1}\longrightarrow(S/I)_d$ is not surjective in this case. Now assume that there are only two distinct integers among a_i's. Without loss of generality we may assume that $a_1=a_2=\cdots=a_m<a_{m+1}=a_{m+2}=\cdots=a_n$. Since we assume that at most $n-2$ of the integers a_i's are equal, we have $m,n-m\geq 2$ and so $a_1=a_2\neq a_{n-1}=a_n$. Consider the element $H=(y_1-y_2)(y_n-y_{n-1})^{d-1}\in R$. Acting by $M^r_{a_1,\dots,a_n}$ on H we get that $M^r_{a_1,\dots,a_n}(y_1-y_2)(y_n-y_{n-1})^{d-1}=\xi^{r(a_1-a_n)}(y_1-y_2)(y_n-y_{n-1})^{d-1}$ for every $0\leq r\leq d-1$. So H is fixed by the action $M_{a_1,\dots,a_n}$ if and only if $a_1=a_n$ which we assumed $a_1\neq a_n$. This implies that H is not fixed by $M_{a_1,\dots,a_n}$. Moreover, notice that every monomial m with non-zero coefficient in H is mapped to $\xi^{a_1+(d-1)a_n}m$ by the action $M_{a_1,\dots,a_n}$. So since $0\leq a_i\leq d-1$, for every $1\leq i\leq n$, then monomial m is fixed by the action $M_{a_1,\dots,a_n}$ if and only if $a_1=a_n$ but we assumed $a_1<a_n$. We conclude that none of the monomials in H is fixed by $M_{a_1,\dots,a_n}$, therefore $H\in(I^{-1})_d$. Moreover, we have that $(x_1+\cdots+x_n)\circ H=0$ and then the map $\times(x_1+\cdots+x_n):(S/I)_{d-1}\longrightarrow(S/I)_d$ is not surjective. □

We illustrate Theorem 7.8 in the next example for the ideal in the polynomial ring with three variables failing the WLP.

Example 7.9. Let $I \subset S = \mathbb{K}[x_1, x_2, x_3]$ be the ideal generated by forms of degree $d = 10$ fixed by the action of $M_{0,2,4}$. Theorem 7.1 implies that I is generated by all monomials of degree d fixed by the action of $M_{0,2,4}$. By Theorem 7.8 above we get that I fails WLP form degree 9 to degree 10. Since by Theorem 7.2 we have $H_{S/I}(10) = 52 < 55 = H_{S/I}(9)$, failing WLP is an assertion of failing surjectivity of the multiplication map $\times(x_1+x_2+x_3) : (S/I)_9 \longrightarrow (S/I)_{10}$. We equivalently show that the map $\circ(x_1+x_2+x_3) : (I^{-1})_{10} \longrightarrow (I^{-1})_9$ is not injective.

Using Lemma 7.6, we let L be the linear form $L = \sum_{j=0}^{7} \xi^j y_1 + \sum_{j=8}^{11} \xi^j y_2 + \sum_{j=12}^{19} \xi^j y_3$ for $l = 7$ and $k = 3$ in the dual ring $R = \mathbb{K}[y_1, y_2, y_3]$. Then we get the non-zero form $F = L^{10} - \overline{L}^{10}$ in the kernel of the map $\circ(x_1+x_2+x_3) : (I^{-1})_{10} \longrightarrow (I^{-1})_9$. Computations by Macaulay2 software, show that the kernel of this map has dimension 2. We can actually get the other form in the kernel by changing ξ with $\xi' = \xi^3 = e^{6\pi i/d}$. Therefore we have $L' = \sum_{j=0}^{7} \xi^{3j} y_1 + \sum_{j=8}^{11} \xi^{3j} y_2 + \sum_{j=12}^{19} \xi^{3j} y_3$ and then $G = L'^d - \overline{L'}^d$ is another form of degree 10 in the kernel where $(x_1 + x_2 + x_3) \circ G = 0$.

8. Dihedral group acting on $\mathbb{K}[x, y, z]$

In the previous section we have studied the WLP of ideals generated by invariant forms of degree d under an action of cyclic group of order d. In this section we study an action of dihedral group D_{2d} on the polynomial ring with three variables $S = \mathbb{K}[x, y, z]$ where $\mathbb{K} = \mathbb{C}$ and $d \geq 2$. Let $\xi^{2\pi i/d}$ be a primitive d-th root of unity and

$$A_d = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_d = \begin{pmatrix} 0 & \xi^{-1} & 0 \\ \xi & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

be a representation of dihedral group D_{2d}. Let $F = \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)(\xi^j x + \xi^{-j} y - z)$ which is a polynomial of degree $2d$ invariant by the action A_d and B_d of dihedral group D_{2d}. We study the WLP of the artinian monomial ideal in S generated by all the monomials in F with non-zero coefficients. First we count the number of generators of such ideals.

Proposition 8.1. *For integer $d \geq 2$, let A_d and B_d be a representation of D_{2d} and let $I \subset S$ be the artinian monomial ideal generated by all monomial with non-zero coefficients in $F = \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)(\xi^j x + \xi^{-j} y - z)$. Then $\mu(I) = d + 3$, if $d = 2k + 1$; and $\mu(I) = 2d + 5$, if $d = 2k$.*

Proof. Fist, assume $d = 2k + 1$ and consider the action by

$$M_{2,2d-2,d} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^{2d-2} & 0 \\ 0 & 0 & \omega^d \end{pmatrix}$$

of a cyclic group $\mathbb{Z}/2d\mathbb{Z}$ where $\omega = e^{2\pi i/2d}$ is a primitive $2d$-root of unity. Then consider the form $H = \prod_{j=0}^{2d-1}(\omega^{2j}x + \omega^{(2d-2)j}y + \omega^{dj}z)$. We have that

$$\begin{aligned} H &= \prod_{j=0}^{2d-1} (\xi^j x + \xi^{-j} y + (-1)^j z) \\ &= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + (-1)^j z)(\xi^{j+d} x + \xi^{-j+d} y + (-1)^{j+d} z) \\ &= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + (-1)^j z)(\xi^j x + \xi^{-j} y + (-1)^{j+d} z) \\ &= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y - z)(\xi^j x + \xi^{-j} y + z) = F. \end{aligned}$$

Note that the monomials fixed by the action of $M_{2,2d-2,d}$, $M_{0,2d-4,d-2}$ and $M_{0,1,a}$ are the same, where $(2d-4)a = (d-2)$, since d is odd such an integer a exists. By Theorem 7.2 we get that the number of monomials fixed by any of those actions is $d+3$. On the other hand Theorem 2, in [7] implies that the number of terms with non-zero coefficient in H and then in F is exactly $d+3$ which implies that $\mu(I) = d+3$.

Now assume that $d = 2k$ and consider the action by $M_{2,2d-2,0} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^{2d-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ of a cyclic group $\mathbb{Z}/2d\mathbb{Z}$. Consider the form $G = \prod_{j=0}^{2d-1}(\omega^{2j}x + \omega^{(2d-2)j}y + z)$ then we have

$$\begin{aligned} G &= \prod_{j=0}^{2d-1} (\xi^j x + \xi^{-j} y + z) \\ &= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)(\xi^{j+d} x + \xi^{-j+d} y + z) \\ &= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)(-\xi^j x - \xi^{-j} y + z) \\ &= (-1)^d \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)(\xi^j x + \xi^{-j} y - z) = F \end{aligned}$$

also we have that $F = G = (\prod_{j=0}^{d-1}(\xi^j x+\xi^{-j}y+z))^2$ and denote $f := \prod_{j=0}^{d-1}(\xi^j x+\xi^{-j}y+z)$. Theorem 2 in [7], implies that the monomials in f with non-zero coefficients are exactly the monomials of degree d fixed by the action $M_{1,d-1,0} = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ of a cyclic group $\mathbb{Z}/d\mathbb{Z}$. Therefore, using Theorem 7.2 we get that there are $3+d/2$ monomials with non-zero coefficients in f, and they are exactly the monomials of the form $(xy)^\alpha z^{d-2\alpha}$ and x^d and y^d.

We now count the monomials in $F = f^2$. First we claim that the form f has alternating sign in the variable z. To show this we evaluate the form in $x = y = 1$ then we get

$$\begin{aligned}\prod_{j=0}^{d-1}(\xi^j + \xi^{-j} + z) &= \prod_{j=0}^{d/2-1} (\xi^j + \xi^{-j} + z)(\xi^{j+d/2} + \xi^{-j+d/2} + z) \\ &= (-1)^{d/2} \prod_{j=0}^{d/2-1} (w^2 + a_j^2)\end{aligned}$$

where $a_j = \xi^j + \xi^{-j}$ and $z^2 = -w$. Then this expression proves the claim.

Multiplying x^d and y^d in f with $d/2+1$ monomials $(xy)^\alpha z^{d-2\alpha}$ gives $2(d/2+1) = d+2$ monomials in F. Using the claim above we get that all $d + 1$ monomials of degree $2d$ of the form $(xy)^\beta z^{2d-2\beta}$ have non-zero coefficients in F. Adding 2 corresponding to the monomials x^{2d} and y^{2d} we get that there are exactly $2d+5$ monomials in F with non-zero coefficients or equivalently $\mu(I) = 2d + 5$. $\square$

Proposition 8.2. *For integer $d \geq 2$ let $I \subset \mathbb{K}[x, y, x]$ be the ideal generated by all the monomials with non-zero coefficients in $F = \prod_{j=0}^{d-1}(\xi^j x + \xi^{-j}y + z)(\xi^j x + \xi^{-j}y - z)$, introduced in Proposition 8.1. Then I fails WLP from degree $2d-1$ to degree $2d$.*

Proof. Suppose that $d = 2k + 1$, using Proposition 8.1 we have that

$$\begin{aligned}H_{S/I}(2d) &= H_S(2d) - \mu(I) = (2d^2 + 3d + 1) - (d + 3) \\ &= 2(d^2 + d - 1) > d(2d + 1) = H_{S/I}(2d - 1).\end{aligned}$$

Consider the form $K = (x+y-z)\prod_{i=1}^{d-1}(\xi^i x+\xi^{-i}y+z)(\xi^{-i}x+\xi^i y-z)$ of degree $2d-1$. Since we have $(x + y + z)K = F$, the map $\times(x + y + z) : (S/I)_{2d-1} \longrightarrow (S/I)_{2d}$ is not injective.

Now assume $d = 2k$, then Proposition 8.1 implies that

$$\begin{aligned}H_{S/I}(2d) &= H_S(2d) - \mu(I) = (2d^2 + 3d + 1) - (2d + 5) \\ &= 2d^2 + d - 4 < d(2d + 1) = H_{S/I}(2d - 1).\end{aligned}$$

Therefore, in order to prove S/I fails the WLP we need to prove that the multiplication map by $x+y+z$ on the algebra from degree $2d-1$ to degree $2d$ is not surjective. To do so we use the representation theory of the symmetric group S_2 where 1 acts trivially and -1 interchanges x and y. Note that this action fixes the form $x+y+z$. We look at the multiplicity of the alternating representation of S/I in degree $2d-1$ and $2d$. In degree $2d-1$ there are $d(2d+1)$ monomials in S/I that are fixed by the identity permutation and there are d monomials of the form $(xy)^{\alpha}z^{2d-1-2\alpha}$ that are fixed by interchanging x and y. Therefore the multiplicity of the alternating representation in degree $2d-1$ is $(d(2d+1)-d)/2 = d^2$. In degree $2d$ there are $(2d+1)(d+1)-(2d+5) = 2d^2+d-4$ monomials in S/I that are all fixed by the identity permutation and there is no monomial of the form $(xy)^{\alpha}z^{2d-2\alpha}$ in the algebra which is fixed by interchanging x and y, since they all belong to I. So the multiplicity of the alternating representation of S/I in degree $2d$ is $(2d^2+d-4)/2$. Since $(2d^2+d-4)/2 > d^2$ for $d \geq 5$ the multiplication by $x+y+z$ cannot be surjective by Schur's lemma.

For $d = 4$ computations in Macaulay2 show the multiplication map by $x+y+z$ is not surjective on S/I from degree 7 to degree 8. □

Remark 8.3. In Proposition 8.2, we have proved that for odd integer d the monomial ideals generated by the monomials of degree $2d$ with non-zero coefficients in F fail WLP by failing injectivity in degree $2d-1$, therefore such ideals define minimal monomial Togliatti systems.

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References

[1] L. Colarte, E. Mezzetti, R.M. Miró-Roig, M. Salat, On the coefficients of the permanent and the determinant of a circulant matric. Applications, arXiv e-prints, June 2018.
[2] Roberta Di Gennaro, Giovanna Ilardi, Jean Vallès, Singular hypersurfaces characterizing the Lefschetz properties, J. Lond. Math. Soc. 89 (1) (Feb 2014) 194–212.
[3] Anthony V. Geramita, Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals, The Curves Seminar at Queen's 10 (1996) 2–114.
[4] Daniel R. Grayson, Michael E. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.
[5] Tadahito Harima, Juan C. Migliore, Uwe Nagel, Junzo Watanabe, The weak and strong Lefschetz properties for Artinian K-algebras, J. Algebra 262 (1) (Apr 2003) 99–126.
[6] Anthony Iarrobino, Vassil Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, Lecture Notes in Mathematics, vol. 1721, Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.
[7] N.A. Loehr, G.S. Warrington, H.S. Wilf, The combinatorics of a three-line circulant determinant, Isr. J. Math. 143 (1) (2004) 141–156.
[8] P.J. McCarthy, Introduction to Arithmetic Functions, Springer-Verlag, New York, 1986.
[9] Emilia Mezzetti, Rosa M. Miró-Roig, The minimal number of generators of a Togliatti system, Ann. Mat. Pura Appl. 195 (6) (Dec 2016) 2077–2098.
[10] Emilia Mezzetti, Rosa M. Miró-Roig, Togliatti systems and Galois coverings, J. Algebra 509 (2018) 263–291.
[11] Emilia Mezzetti, Rosa M. Miró-Roig, Giorgio Ottaviani, Laplace equations and the weak Lefschetz property, Can. J. Math. 65 (3) (Jun 2013) 634–654.

[12] Juan C. Migliore, Rosa M. Miró-Roig, Uwe Nagel, Monomial ideals, almost complete intersections and the weak Lefschetz property, Trans. Am. Math. Soc. 363 (01) (Jan 2011) 229–257.
[13] Richard P. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, SIAM J. Algebraic Discrete Methods 1 (2) (Jun 1980) 168–184.
[14] Eugenio Togliatti, Alcune osservazioni sulle superficie razionali che rappresentano equazioni di Laplace, Ann. Mat. Pura Appl. (4) 25 (1) (Dec 1946) 325–339.
[15] Eugenio G. Togliatti, Alcuni esempi di superficie algebriche degli iperspazi che rappresentano un' equazione di Laplace, Comment. Math. Helv. 1 (1929) 255–272.