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Arithmetic structure of CMSZ fake projective planes

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Abstract

We show that the fake projective planes that are constructed from dyadic discrete subgroups discovered by Cartwright, Mantero, Steger, and Zappa are realized as connected components of certain unitary Shimura surfaces. As a corollary we show that these fake projective planes have models defined over the number field $\mathbb{Q}(\sqrt{-3}, \sqrt{5})$.

Keywords: Non-Archimedean uniformization; Discrete group

1. Introduction

In [2], Cartwright, Mantero, Steger, and Zappa discovered a unitary group in three variables with respect to the quadratic extension $\mathbb{Q}(\sqrt{-15})/\mathbb{Q}$ whose integral model over the integer ring with the prime 2 inverted gives rise to a dyadic discrete subgroup of $\mathbf{PGL}_3(\mathbb{Q}_2)$ that acts transitively on vertices of Bruhat–Tits building over $\mathbb{Q}_2$. It has, moreover, three subgroups of finite index that act on the set of vertices of Bruhat–Tits building simply transitively, and exactly two of them act freely also on simplicies of the other dimensions. The last-mentioned two subgroups are at our interest in this paper.

A discrete subgroup of $\mathbf{PGL}_3(\mathbb{Q}_2)$ having the same properties as the above-mentioned two subgroups, but coming from a different unitary group, was obtained by Mumford [8], and this group was used to construct, by dyadic non-archimedean uniformization, an example of algebraic

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surfaces with $p_g = q = 0$, $c_1^2 = 3c_2 = 9$, and with the ample canonical class, so-called, *fake projective planes*, which are among the most interesting classes of algebraic surfaces. It can be shown that the above-mentioned two discrete subgroups also yield fake projective planes, which we call *CMSZ fake projective planes*. As shown in [4], these fake projective planes, the Mumford's one and the two CMSZ ones, are not isomorphic to each other.

On the other hand, Klingler [7] showed that all possible fake projective planes are *arithmetic* quotients of the complex unit-ball. In fact, in Mumford's case, it is quite clear from the construction that Mumford's fake projective plane should be realized as a connected component of a unitary Shimura surface, for Mumford's discrete subgroup is not only arithmetic but a congruence subgroup in the integral model of the unitary group. In [5] the first-named author described the Shimura surface, and showed that Mumford's fake projective plane has a model defined over the 7th cyclotomic field $\mathbb{Q}(\zeta_7)$. However, it is not clear whether the groups in [2] are characterized by congruence conditions or not, because the definition of the groups therein only gives generators in matrices, although they are certainly arithmetic. This point should be pursued in order to settle whether CMSZ fake projective planes also allow such a nice arithmetic description in terms of Shimura surfaces. This is exactly what we do in this paper.

In the next section we will construct congruence subgroups in the unitary group characterized by conditions in modulo 3-reduction, and will prove that they coincide with the above-mentioned discrete subgroups. In other words, we give another way of construction of these groups, and hence the main stream of our argument is independent from that of [2], although we have been much motivated by it. From this we proceed to construct the Shimura varieties in the following section, which mimics the argument in [5]. Our main theorem (Theorem 3.3.3) states that, for each of the two groups, there exists a Shimura variety with the reflex field $\mathbb{Q}(\sqrt{-15})$ consisting of two connected components isomorphic to the corresponding fake projective plane; moreover, we will also see that these connected components have $\mathbb{Q}(\sqrt{-3}, \sqrt{5})$ as a field of definition.

Appendix A is an appendix to the next section, in which we will concentrate on proving a technical result.

Notation. (1) We use the notation $\mathbf{GL}_d$, $\mathbf{SL}_d$, $\mathbf{PGL}_d$, etc., as usual. For a ring R, we denote by $\mathbf{Mat}_{n\times m}(R)$ the set of all $(n \times m)$-matrices with entries in R. We especially write $\mathbf{Mat}_d(R) = \mathbf{Mat}_{d\times d}(R)$.

(2) For an *étale quadratic extension* L/F of a field F (i.e., either L is a separable quadratic field extension of F, or $L \cong F \times F$ together with the diagonal mapping $F \hookrightarrow L$), a central simple algebra A over L, and a *unitary F-involution* $\star$ on A (i.e., an anti-automorphism $a \mapsto a^\star$ of A of period 2 whose restriction on L coincides with the non-trivial element of $\mathrm{Gal}(L/F)$ if L is a field, or with the exchange map $(a,b) \mapsto (b,a)$ otherwise), we denote by $\mathbf{GU}(A,\star)$ the *F-algebraic group of unitary similitudes* associated to the pair $(A,\star)$; that is, for any commutative F-algebra R, we set $\mathbf{GU}(A,\star)(R) = \{\gamma \in (A \otimes_F R)^\times \mid \gamma\gamma^\star \in R^\times\}$. The *unitary group* associated to $(A,\star)$ is denoted by $\mathbf{U}(A,\star)$; that is, $\mathbf{U}(A,\star)(R) = \{\gamma \in (A \otimes_F R)^\times \mid \gamma\gamma^\star = 1\}$ for any commutative F-algebra R.

(3) Let L/F be an étale quadratic extension. For any matrix $X = (x_{ij}) \in \mathbf{Mat}_{n\times m}(L)$, we denote by X^* the matrix $(\bar{x}_{ji})$, where $\bar{\cdot}$ denotes the non-trivial element of $\mathrm{Gal}(L/F)$.

(4) For a field extension F/E, we denote by $\mathrm{Res}_{F/E}$ the Weil restriction.

(5) We denote by $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ the adele ring, and by $\mathbb{A}_f$ its finite part. For a subset S of finite places of $\mathbb{Q}$, we denote by $\mathbb{A}_f^S$ the prime-to-S part of $\mathbb{A}_f$. Similarly, we denote by $\widehat{\mathbb{Z}}^S$ the prime-to-S part of the profinite group $\widehat{\mathbb{Z}}$.

2. The CMSZ group

2.1. Discrete group from unitary group

Let $K \subset \mathbb{C}$ be an imaginary quadratic extension of $\mathbb{Q}$, and p a rational prime having the prime decomposition of the form $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ in K. We consider the unitary group $\mathbf{U} = \mathbf{U}(H)$ associated to a positive definite Hermitian matrix H with entries in K. The group $\mathbf{U}$ is the $\mathbb{Q}$-algebraic group such that

$$\mathbf{U}(\mathbb{Q}) = \{g \in \mathbf{GL}_d(K) \mid g^* H g = H\}.$$

Let $\mathscr{U} = \mathscr{U}_{\mathscr{O}_K[1/p]}$ be the subgroup of $\mathbf{U}(\mathbb{Q})$ consisting of H-unitary matrices lying in $\mathbf{GL}_d(\mathscr{O}_K[1/p])$, that is,

$$\mathscr{U} = \mathbf{U}(\mathbb{Q}) \cap \mathbf{GL}_d\left(\mathscr{O}_K\left[\frac{1}{p}\right]\right).$$

Since p decomposes in $\mathscr{O}_K$, the group $\mathbf{U}(\mathbb{Q}_p)/(\mathbb{Q}_p \otimes_{\mathbb{Q}} K)^\times$ is, once we choose $K \hookrightarrow \mathbb{Q}_p$ continuous with respect to the $\mathfrak{p}$-adic topology on K, identified with $\mathbf{PGL}_d(\mathbb{Q}_p)$, and thus we have a homomorphism $\mathbf{U}(\mathbb{Q}) \to \mathbf{PGL}_d(\mathbb{Q}_p)$. Let Γ be the image of $\mathscr{U}$ in $\mathbf{PGL}_d(\mathbb{Q}_p)$. Then, it is well known that the group Γ is a discrete and co-compact subgroup.

2.2. The CMSZ situation

The so-called *CMSZ group* is the group Γ as above in $d = 3$, $p = 2$, and the following setting. First we set $K = \mathbb{Q}(\sqrt{-15})$. The integer ring $\mathscr{O}_K$ is $\mathbb{Z}[\lambda]$ with $\lambda = (1 - \sqrt{-15})/2$. The prime 2 is decomposed as $(2) = \mathfrak{p}\bar{\mathfrak{p}}$, where $\mathfrak{p} = 2\mathbb{Z} + \lambda\mathbb{Z}$. The Hermitian matrix H is the one given by

$$H = \begin{bmatrix} 10 & -2(\lambda+2) & \lambda+2 \\ -2(\bar{\lambda}+2) & 10 & -2(\lambda+2) \\ \bar{\lambda}+2 & -2(\bar{\lambda}+2) & 10 \end{bmatrix}.$$

Note that $\det H = 2^2 \cdot 3 \cdot 5^2$. The embedding $K \hookrightarrow \mathbb{Q}_2$ is chosen to be the $\mathfrak{p}$-adic completion; hence, $\lambda/2$ gives a uniformizer, and $\bar{\lambda} = 1 - \lambda$ is a dyadic unit. The significance of the group Γ thus obtained is the following fact.

Theorem 2.2.1. [2, Section 5] *The group Γ acts transitively on the vertices of the Bruhat–Tits building Δ attached to* $\mathbf{PGL}_3(\mathbb{Q}_2)$.

Here, we insert the proof for the reader's convenience.

Proof. We introduce two elements that belong to $\mathscr{U}$:

$$\rho = \begin{bmatrix} 0 & 0 & \frac{\lambda}{2} \\ 0 & -1 & 1+\frac{\lambda}{2} \\ 1 & -1 & 1 \end{bmatrix}, \qquad \tau = \begin{bmatrix} 0 & -1 & \frac{\lambda}{2} \\ 1 & -1 & 1+\frac{\lambda}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $\rho^3 = \frac{\lambda}{2}$ and $\tau^3 = 1$. Let Λ_0 be the vertex of Δ that is the similarity class of the standard lattice $(\mathbb{Z}_2)^3$. It suffices to show that, for each vertex Λ adjacent to Λ_0, there exists an element $g \in \mathscr{U}$ such that $\Lambda = g\Lambda_0$. Let us define elements $g_i \in \mathscr{U}$ for $i \in \mathbb{Z}/7\mathbb{Z}$ as follows: we set $g_3 = \rho$, and

$$\tau^{-1} g_{2i} \tau = g_i \tag{1}$$

for any i. These elements satisfy the following relations:

$$\begin{aligned} g_3 g_3 g_3 &= g_6 g_6 g_6 = g_5 g_5 g_5 = \frac{\lambda}{2}, \\ g_1 g_1 g_0 &= g_2 g_2 g_0 = g_4 g_4 g_0 = 1, \\ g_1 g_3 g_6 &= g_2 g_6 g_5 = g_4 g_5 g_3 = 1. \end{aligned} \tag{2}$$

Then one can easily show that the set $\{g_i^{\pm 1}\Lambda_0 \mid i \in \mathbb{Z}/7\mathbb{Z}\}$ coincides with the set of vertices adjacent to Λ_0. $\square$

Remark 2.2.2. [1,2] Let $\mathscr{F}$ be the subset of $(\mathbb{Z}/7\mathbb{Z})^3$ given by

$$\mathscr{F} = \left\{(i, j, k) \in (\mathbb{Z}/7\mathbb{Z})^3 \mid \text{either } g_i g_j g_k,\ g_j g_k g_i, \text{ or } g_k g_i g_j \text{ appears in (2)}\right\}.$$

Then $\mathscr{F}$ consists of $21 = 3 + 6 \cdot 3$ elements. Note that $g_i g_j g_k$ is either 1 or $\frac{\lambda}{2}$ for each $(i, j, k) \in \mathscr{F}$. This set has the following meaning: for each $(i, j, k) \in \mathscr{F}$, the set $\{g_i\Lambda_0, \Lambda_0, g_k^{-1}\Lambda_0\}$ forms a chamber in Δ, which we denote by $C(i, j, k)$; this gives rise to a bijection between $\mathscr{F}$ and the set of all (non-oriented) chambers containing Λ_0. The following facts are easy to see.

(a) We have $g_i^{-1} C(i, j, k) = C(j, k, i)$. In particular, g_3 (respectively g_6, g_5) stabilizes the chamber $C(3, 3, 3)$ (respectively $C(6, 6, 6)$, $C(5, 5, 5)$).
(b) We have $\tau C(i, j, k) = C(2i, 2j, 2k)$ (this follows from (1) and the fact that τ fixes Λ_0).

Proposition 2.2.3. *Let $\mathscr{U}$ act on Δ through $\mathscr{U} \to \mathbf{PGL}_3(\mathbb{Q}_2)$. Then the stabilizer* $\mathrm{Stab}(\Lambda_0)$ *in $\mathscr{U}$ of the vertex Λ_0 is the subgroup generated by τ and scalars of norm* 1. *In particular, the group $\mathscr{U}$ is generated by ρ, τ, and scalars of norm* 1.

Proof. As the proof of Theorem 2.2.1 indicates, the subgroup of $\mathscr{U}$ generated by ρ and τ acts transitively on the vertices of Δ. Hence the second assertion follows from the first. Let $g \in \mathscr{U}$ be an element that fixes $\Lambda_0 = [(\mathbb{Z}_2)^3]$. Multiplying by a scalar of the form $(\lambda/2)^k$ if necessary, we may assume $g(\mathbb{Z}_2)^3 = (\mathbb{Z}_2)^3$. Therefore, the proposition follows from the following lemma. $\square$

Lemma 2.2.4. *A matrix $g \in \mathscr{U}$ with $g(\mathbb{Z}_2)^3 = (\mathbb{Z}_2)^3$ must be of the form $\pm\tau^i$ $(i = 0, 1, 2)$.*

The proof of the lemma is elementary, but is technical. We postpone it to Appendix A.

Recall that a *labelling* of Δ is a map l from the set of vertices to $\mathbb{Z}/3\mathbb{Z}$ that maps $\Lambda = [g(\mathbb{Z}_2)^3]$ with $g \in \mathbf{PGL}_3(\mathbb{Q}_2)$ to $\nu(\det g) \bmod 3$, where ν is the normalized valuation $\nu : \mathbb{Q}_2^\times \to \mathbb{Z}$. The map l restricted to each chamber is bijective, and hence l gives rise to an orientation in Δ. Since

$l(g\Lambda) = l(\Lambda) + (\nu(\det g) \bmod 3)$ for any $g \in \mathbf{PGL}_3(\mathbb{Q}_2)$ and any Λ, the action of $\mathbf{PGL}_3(\mathbb{Q}_2)$ on Δ preserves the orientation.

Proposition 2.2.5. *Let us consider, for $(i, j, k) \in \mathscr{F}$, the stabilizer* $\mathrm{Stab}(C(i, j, k))$ *in $\mathscr{U}$ of the chamber $C(i, j, k)$. If $i = j = k = 3$, 6, or 5, then* $\mathrm{Stab}(C(i, j, k))$ *is the subgroup generated by g_i and scalars of norm* 1; *otherwise, it consists only of scalars of norm* 1.

Proof. In view of (a) and (b) in Remark 2.2.2, it suffices to show the proposition in the cases $(i, j, k) = (3, 3, 3)$, $(1, 1, 0)$, and $(1, 3, 6)$. Let $g \in \mathscr{U}$ stabilizes $C(i, j, k)$, and assume that g is not a scalar. By (b) in Remark 2.2.2 and Proposition 2.2.3, g does not fix the vertex Λ_0, but g^3 does, since g preserves the orientation by the labelling. Then, taking inverse if necessary, one may assume $g\Lambda_0 = g_i\Lambda_0$. By Proposition 2.2.3, we deduce $g = g_i\tau^j c$ for some $j \in \{0, 1, 2\}$ and $c \in (\mathscr{O}_K[1/2])^\times$ with $c\bar{c} = 1$. Since $g^3 \in \mathrm{Stab}(\Lambda_0)$, it follows from Proposition 2.2.3 that $(g_i\tau^j)^9$ is a scalar. In case $i = 1$, this can be shown to be impossible by direct calculation. If $i = 3$, this is only the case when $j = 0$ (hence g^3 is a scalar), as one can check directly. □

Theorem 2.2.6. *Any non-trivial finite subgroup of Γ is conjugate to either the image of $\langle\rho\rangle$ or of $\langle\tau\rangle$.*

Proof. Let G be a finite subgroup in Γ. Then it is well known that G stabilizes a simplex in Δ. Since Γ preserves the orientation, G stabilizes either a vertex or a chamber. Since Γ acts on the vertices transitively, the result follows from Propositions 2.2.3 and 2.2.5. □

2.3. Reduction of the unitary group

For an ideal $\mathfrak{c}$ of $\mathscr{O}_K[1/2]$ that is stable under the complex conjugation, and for a non-negative integer n, the Artinian ring $R_n = \mathscr{O}_K[1/2]/\mathfrak{c}^{n+1}$ has the induced involution, which we again denote by $\alpha \mapsto \bar{\alpha}$. There is the obvious modulo $\mathfrak{c}^{n+1}$-reduction morphism $\pi_n : \mathscr{U} \to \mathbf{GL}_3(R_n)$, whose image lies in the subgroup of the unitary group

$$\mathscr{U}_n = \{g \in \mathbf{GL}_3(R_n) \mid g^* H_n g = H_n\},$$

where H_n is the modulo $\mathfrak{c}^{n+1}$-reduction of H (note that H has coefficients in $\mathscr{O}_K[1/2]$). Our particular interest is in the case $\mathfrak{c} = 3\mathbb{Z} + (\lambda + 1)\mathbb{Z}$ and $n = 0$ or 1. Note that $\mathfrak{c}^2 = (3)$ gives the prime decomposition of the ideal (3) in $\mathscr{O}_K$. Our goal in the rest of this section is to construct subgroups in $\mathscr{U}$ that acts freely on Δ and transitively on the vertices in terms of congruence conditions in these reductions. To this end, we will first determine the subgroups $\pi_n(\mathscr{U})$ $(n = 0, 1)$ and then, ask for nice subgroups of them of which the inverse images by π_n keep the transitivity on vertices and rule out torsion elements.

Before doing this, we change our presentation of matrices into more convenient form: let us define a matrix

$$U = \begin{bmatrix} 1 & 2\lambda - 1 & -2 \\ -\frac{4\lambda-3}{2} & \frac{2\lambda+1}{2} & -1 \\ -\frac{2\lambda-1}{2} & \frac{1}{2} & -2 \end{bmatrix},$$

and the new Hermitian matrix $H' = U^* H U$, which appears to be

$$H' = \begin{bmatrix} 90 & 2\bar{\lambda} - 1 & -15 \\ 2\lambda - 1 & 90 & 15(2\lambda - 1) \\ -15 & 15(2\bar{\lambda} - 1) & 70 \end{bmatrix}.$$

We set $\mathscr{U}' = U^{-1} \mathscr{U} U$. (Note that, since $\det U = 2^2 \cdot 7$, this twist is not admissible over $\mathscr{O}_K[1/2]$; however, it is harmless because we work only over the prime above 3.) Set

$$\mathscr{U}'_n = U^{-1} \mathscr{U}_n U = \{ g \in \mathbf{GL}_3(R_n) \mid g^* H'_n g = H'_n \},$$

and let $\pi'_n : \mathscr{U} \to \mathscr{U}'_n$ be the composite of the reduction map π_n followed by the isomorphism $\mathscr{U}_n \xrightarrow{\sim} \mathscr{U}'_n$.

The unitary groups $\mathscr{U}'_0$ and $\mathscr{U}'_1$. First we note that $R_0 \cong \mathbb{F}_3$ and that the induced involution acts on it trivially. Since

$$H'_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the group $\mathscr{U}'_0$ modulo center is isomorphic to the Euclidean motion group on the affine plane over $\mathbb{F}_3$; in fact, we have

$$\mathscr{U}'_0 = \left\{ \begin{bmatrix} A & B \\ 0 & \pm 1 \end{bmatrix} \,\middle|\, A \in \mathbf{GL}_2(\mathbb{F}_3),\ B \in \mathbf{Mat}_{2\times 1}(\mathbb{F}_3) \right\}. \tag{3}$$

The order of $\mathscr{U}'_0$ is therefore $864 = 2^5 \cdot 3^3$.

The Artinian ring R_1, in turn, is isomorphic to $\mathbb{F}_3[t]/(t^2)$, where t is the image of $1 - 2\lambda$ $(= \sqrt{-15}\,)$, which is acted on by the involution $t \mapsto -t$. The group $\mathscr{U}'_1$ is the unitary group with respect to the Hermitian form

$$H'_1 = \begin{bmatrix} & t & \\ -t & & \\ & & 1 \end{bmatrix},$$

whence

$$\mathscr{U}'_1 = \left\{ \begin{bmatrix} A & B \\ tC & d \end{bmatrix} \,\middle|\, \begin{array}{l} A \in \mathbf{GL}_2(R_1),\ B \in \mathbf{Mat}_{2\times 1}(R_1) \text{ and } d \in R_1^\times \text{ with } (A \bmod t) \in \mathbf{SL}_2(\mathbb{F}_3) \\ \text{and } C \equiv {}^t B J A d^{-1} \bmod t \end{array} \right\}, \tag{4}$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ denotes the standard symplectic matrix. Note that for any element α in $R_1^\times$, we have $\alpha\bar{\alpha} = 1$. The order of this group is calculated to be $944\,784 = 2^4 \cdot 3^{10}$.

Special elements. Here we introduce some matrices in $\mathscr{U}_1'$ which will play important roles in analyzing the structure of $\mathscr{U}_1'$:

$$z=\begin{bmatrix}1+t&&\\&1+t&\\&&1+t\end{bmatrix},\quad u=\begin{bmatrix}1&1&\\&1&\\&&1-t\end{bmatrix},\quad w=\begin{bmatrix}&-1&\\1&&\\&&1\end{bmatrix},$$

$$b_1=\begin{bmatrix}1&&t\\&1&\\&&1\end{bmatrix},\quad b_2=\begin{bmatrix}1&&\\&1&t\\&&1\end{bmatrix},\quad c_1=\begin{bmatrix}1&t&1\\&1&\\&-t&1\end{bmatrix},\quad c_2=\begin{bmatrix}1&&\\-t&1&1\\t&&1\end{bmatrix},$$

$$d_1=\begin{bmatrix}1+t&&\\&1&\\&&1\end{bmatrix},\quad d_2=\begin{bmatrix}1&&\\&1+t&\\&&1\end{bmatrix},\quad d_3=\begin{bmatrix}1&t&\\&1&\\&&1\end{bmatrix},\quad d_4=\begin{bmatrix}1&&\\t&1&\\&&1\end{bmatrix}.$$

We set

$$T=\langle b_1,b_2\rangle,\qquad N=\langle z,c_1,c_2\rangle,\qquad M=\langle d_1,d_2,d_3,d_4\rangle,\qquad S=\langle u,w\rangle.$$

Notation 2.3.1. In the sequel, we make use of the following notation:

$$[\alpha,\beta]=\alpha^{-1}\beta^{-1}\alpha\beta,\qquad \alpha^{\delta}=\delta^{-1}\alpha\delta.$$

Relations among the special elements. Here we list up the relations among those elements defined above:

(R1) all elements except w are of order 3, while the order of w is 4;
(R2) z lies in the center of $\mathscr{U}_1'$, and $[c_1,c_2]=z$; therefore, the subgroup N is the Heisenberg group with the center $\langle z\rangle$ and $N/\langle z\rangle\cong\mathbb{F}_3^2$;
(R3) T and M are vector groups; $T\cong\mathbb{F}_3^2$ and $M\cong\mathbb{F}_3^4$; we, moreover, have $[T,N]=[T,M]=1$;
(R4) in S we have the relations $[u,w^2]=1$, $wuw=u^{-1}wu^{-1}$, $wu^{-1}w=uw^{-1}u$; S is isomorphic to $\mathbf{SL}_2(\mathbb{F}_3)$;
(R5) T is normalized by S; actually, we have $b_1^u=b_1$, $b_2^u=b_1^{-1}b_2$, $b_1^w=b_2^{-1}$, $b_2^w=b_1$;
(R6) the action of S on (conjugates of) N is given by $c_1^u=b_1^{-1}c_1$, $c_2^u=b_1^{-1}c_1^{-1}c_2$, $c_1^w=c_2^{-1}$, $c_2^w=c_1$; hence, in particular, the $\langle T,N\rangle$ is normalized by S;
(R7) the action of N on M is by

$$\begin{aligned}&d_1^{c_1}=b_1d_1,\qquad d_2^{c_1}=d_2,\qquad d_3^{c_1}=d_3,\qquad d_4^{c_1}=b_2d_4,\\&d_1^{c_2}=d_1,\qquad d_2^{c_2}=b_2d_2,\qquad d_3^{c_2}=b_1d_3,\qquad d_4^{c_2}=d_4;\end{aligned}$$

(R8) M is normalized by S, since

$$\begin{aligned}&d_1^u=d_1d_3,\qquad d_2^u=d_2d_3^{-1},\qquad d_3^u=d_3,\qquad d_4^u=d_1^{-1}d_2d_3^{-1}d_4,\\&d_1^w=d_2,\qquad d_2^w=d_1,\qquad d_3^w=d_4^{-1},\qquad d_4^w=d_3^{-1}.\end{aligned}$$

Proposition 2.3.2. *The group $\mathscr{U}_1'$ is generated by the subgroups T, N, M, S, and -1. Moreover, there exist isomorphisms*

$$\mathscr{U}_1' \xrightarrow{\sim} \big(\big((T \times M) \rtimes N\big) \rtimes S\big) \times \mathbb{F}_3^\times \xleftarrow{\sim} \big(\big(\mathbb{F}_3^6 \rtimes N\big) \rtimes \mathbf{SL}_2(\mathbb{F}_3)\big) \times \mathbb{F}_3^\times.$$

Proof. Let $\mathscr{U}_1'^{+}$ be the subgroup generated by T, N, M, and S. By the definition of the matrices and the relations as above, we easily see $\mathscr{U}_1'^{+} \cong ((T \times M) \rtimes N) \rtimes S$. But the order of the left-hand side already attains the half of that of $\mathscr{U}_1'$, that is, $2^3 \cdot 3^{10}$. Hence $\mathscr{U}_1'^{+}$ is precisely the kernel of the surjective homomorphism

$$\det \bmod t : \mathscr{U}_1' \longrightarrow \mathbb{F}_3^\times. \tag{5}$$

Clearly, this morphism has a cross-section with the image in the center of $\mathscr{U}_1'$, thereby the assertion. □

Proposition 2.3.3. *The image $\pi_1'(\mathscr{U})$ by the modulo 3-reduction of $\mathscr{U}$ is the subgroup of $\mathscr{U}_1'$ generated by T, N, S, d_1d_2, and scalars.* (*Notice again that any scalar in $R_1^\times$ has norm* 1.) *Hence the order of $\pi_1'(\mathscr{U})$ is $2^4 \cdot 3^7 = 34\,992$.*

Proof. By Proposition 2.2.3, we know that $\pi_1'(\mathscr{U})$ is generated by (ρ^U mod 3), (τ^U mod 3) (which we denote, in this proof, by ρ and τ, respectively, for simplicity), and scalars. As one checks easily,

$$\begin{aligned}
\rho &= \begin{bmatrix} 0 & -1+t & -t \\ 1-t & -1-t & 1+t \\ 0 & -t & 1+t \end{bmatrix} = b_1^{-1} c_2 w u^{-1} (d_1 d_2)^{-1}, \\
\tau &= \begin{bmatrix} 1+t & 1-t & 1+t \\ 0 & 1+t & 0 \\ 0 & -t & 1+t \end{bmatrix} = z^{-1} c_1 u (d_1 d_2)^{-1},
\end{aligned} \tag{6}$$

and hence it follows that $\pi_1'(\mathscr{U}) \subseteq \langle T, N, S, d_1 d_2 \rangle$. For the converse, one calculates

$$\begin{gathered}
z = \rho^{-3}, \qquad w = \rho^4 \big(\tau\rho^{-1}\tau^{-1}\rho^{-1}\big)^2, \\
b_1 = \big[\tau\rho\tau\rho^2\tau\rho\tau^{-1}, \rho\tau\rho\tau^{-1}\rho\tau\rho\tau\big], \qquad b_2 = w b_1 w^{-1}, \\
c_1 = \rho^{-3} \big(\tau^{-1}\rho\tau\rho\tau^{-1}\big)^{(\rho\tau\rho)^{-1}} (\tau\rho)^{-2} b_1 b_2^{-1}, \qquad c_2 = w c_1 w^{-1}, \\
u = \tau\rho^{-1}\tau^{-1}\rho^{-1}\tau^{-1} b_1^{-1} b_2 c_2^{-1} w^2,
\end{gathered}$$

whence $\pi_1'(\mathscr{U}) \supseteq \langle T, N, S, d_1 d_2 \rangle$. □

Now, to find subgroups of $\mathscr{U}$ as in the beginning of this subsection, one has to first look for subgroups that does not contain $\mathscr{U}$-conjugates of τ and ρ; moreover, the indices of such subgroups in $\mathscr{U}$ are required to be 3, since they have to act on the vertices of Δ simply transitively. This leads us to the following kind of statement.

Lemma 2.3.4. *Every subgroup of $\pi_1'(\mathscr{U})$ of index* 3 *is conjugate to either one of the following subgroups*:

$$V_1 = \langle T, N, P, u, \text{scalars}\rangle,$$
$$V_2 = \langle T, N, P, d_1d_2, \text{scalars}\rangle,$$
$$V_3 = \langle T, N, P, ud_1d_2, \text{scalars}\rangle,$$
$$V_4 = \big\langle T, N, P, u(d_1d_2)^{-1}, \text{scalars}\big\rangle,$$

where P is the unique 2*-Sylow subgroup of S, generated by w and w^u.*

Proof. Let V be a subgroup of $\pi_1'(\mathscr{U})$ of index 3. Since $\langle P, \text{scalars}\rangle \cong P \times \mathbb{F}_3^\times$ is a 2-Sylow subgroup of $\pi_1'(\mathscr{U})$, replacing it by a suitable conjugate, we may assume that V contains $\langle P, \text{scalars}\rangle$. Let us consider the modulo t-reduction map $\psi : \pi_1'(\mathscr{U}) \to \pi_0'(\mathscr{U})$, of which the kernel is $K = \langle T, z, d_1d_2\rangle$. Since $V \cap K$ is a normal subgroup in V, it is in particular a $(P \times \mathbb{F}_3^\times)$-module by conjugation. Since the $(P \times \mathbb{F}_3^\times)$-module K is decomposed into irreducible factors as $T \times \langle z\rangle \times \langle d_1d_2\rangle \cong \mathbb{F}_3^2 \times \mathbb{F}_3 \times \mathbb{F}_3$, we deduce that $V \cap K$ contains T:

$$T \subset V \cap K \subset V. \tag{7}$$

Next we claim that

$$\psi(N) \subset \psi(V). \tag{8}$$

Consider the projection $\pi_0'(\mathscr{U}) \to \mathbf{GL}_2(\mathbb{F}_3) \times \mathbb{F}_3^\times$ that takes the top-left 2×2 component and the scalars. Since its kernel is $\psi(N)$ consisting of 9 elements, and since $\psi(V)$ is of index at most 3 in $\pi_0'(\mathscr{U})$, we deduce $\psi(V) \cap \psi(N) \neq \{1\}$. This intersection is, moreover, stable under conjugation by elements of P. But, as one sees easily, the group $\psi(N)$ is an irreducible P-module, thereby the claim.

By (8), we know that, for each $i = 1, 2$, there exists $k_i \in K$ such that $k_ic_i \in V$. Since K is an abelian group, we deduce that $z = [k_1, k_2][c_1, c_2] = [k_1c_1, k_2c_2] \in V$. Combining with (7), we get $\langle z, T\rangle \subset V \cap K \subset V$. By this and the equality $K = \langle T, z, d_1d_2\rangle$, we may take k_i from $\langle d_1d_2\rangle$. Since $[d_1d_2, w^2] = 1$, we get $c_1 = [c_1, w^2] = [k_1c_1, w^2] \in V$. Then we, moreover, have $c_2 = wc_1w^{-1} \in V$. This implies that V contains also N, and hence

$$\langle T, N, P, \text{scalars}\rangle \subset V \cap K \subset V. \tag{9}$$

The left-hand side of (9) is already a subgroup in $\pi_1'(\mathscr{U})$ of index 9. Then it is now obvious that any subgroup of $\pi_1'(\mathscr{U})$ of index 3 that contains it is one of V_i $(i = 1, 2, 3, 4)$, as stated. □

2.4. The CMSZ fake projective planes

Construction 2.4.1. Now, let us define the subgroups $\widetilde{\Gamma}_i$ $(i = 1, 2, 3, 4)$ of $\mathscr{U}$ by

$$\widetilde{\Gamma}_i = (\pi_1')^{-1}(V_i),$$

and let Γ_i be its image in $\mathbf{PGL}_3(\mathbb{Q}_2)$. By (6) we know that the subgroup Γ_3 contains ρ and that Γ_4 contains τ. Hence these are not torsion-free. The groups Γ_1 and Γ_2, on the other hand, have no element of finite order except 1, since, as one sees from the relations (R1)–(R8), they do not have any conjugates of τ and ρ. Hence the discrete groups Γ_1 and Γ_2 act freely on Δ and simply transitively on the vertices.

Let Ω be the Drinfel'd symmetric space over $\mathbb{Q}_2$ of dimension 2. By [8, Section 1], we deduce that both Ω/Γ_1 and Ω/Γ_2 are, respectively, algebraized to fake projective planes ${}^{I}X_{\mathrm{CMSZ}}$ and ${}^{II}X_{\mathrm{CMSZ}}$ over $\mathbb{Q}_2$, not isomorphic to each other due to [4], since our definition shows that Γ_1 and Γ_2 are not conjugate in $\mathbf{PGL}_3(\mathbb{Q}_2)$.

We call the algebraic surfaces ${}^{I}X_{\mathrm{CMSZ}}$ and ${}^{II}X_{\mathrm{CMSZ}}$ the *CMSZ fake projective planes*.

Remark 2.4.2. One can check by direct calculation that our groups Γ_i for $i = 1, 2, 3$ are exactly the U-conjugate of the groups $\Gamma_{B.i}$ discovered by Cartwright et al. in [2, p. 182]. Indeed, the elements ρ and τ in 2.2.1 are exactly $g_3 s$ and s therein, respectively. The set $\mathscr{F}$ defined in Remark 2.2.2, which was used to analyze the action of $\mathscr{U}$ on the chambers of Δ, is nothing but the triangle presentation $\mathscr{F}_{B.3}$. Notice that the group Γ_4 does not appear in the list, for it does not act on the vertices of the Bruhat–Tits building simply transitively.

3. The Shimura variety

3.1. Situation

Let $\lambda = (1 - \sqrt{-15})/2$. In this section we regard the field $K = \mathbb{Q}(\lambda)$ as a subfield of the 45th-cyclotomic field $\mathbb{Q}(\zeta_{45})$ by the mapping $\lambda \mapsto \zeta_{45}^{21} + \zeta_{45}^{33} + \zeta_{45}^{39} + \zeta_{45}^{42}$. Consider the intermediate field $\mathbb{Q}(\zeta_9)$ with $\zeta_9 = \zeta_{45}^5$, and let L be the composite field of K and the maximal real subfield F of $\mathbb{Q}(\zeta_9)$. The field L is a Galois extension of K of degree 3 generated by $\zeta_9 + \zeta_9^{-1}$. The Galois group $\mathrm{Gal}(L/K)$ is generated by $\sigma : \zeta_9 + \zeta_9^{-1} \mapsto \zeta_9^2 + \zeta_9^{-2}$.

The following construction is parallel to [5, Section 5.3]. Set $B = L$, and $V = L^3$, which we regard as right B-module. Define $A = \mathrm{End}_B(V) = \mathbf{Mat}_3(L)$. We consider the Hermitian form J on V given by

$$J = \begin{bmatrix} -4 & \lambda^3 & 4\bar{\lambda} \\ \bar{\lambda}^3 & -4 & \lambda^3 \\ 4\lambda & \bar{\lambda}^3 & -4 \end{bmatrix}.$$

The characteristic polynomial is $t^3 + 12t^2 - 144t + 300$. In particular, we have $\det J = -2^2 \cdot 3 \cdot 5^2$ and $\operatorname{sign} J = (2, 1)$. Note also that the Hermitian form J is defined over K. Define the $\mathbb{Q}$-algebraic group G by $G = \mathrm{Res}_{F/\mathbb{Q}} \mathbf{GU}(A, \star)$, where $\star$ is the adjoint involution of J on A, that is, an involution on A such that $J(\gamma^\star u, v) = J(u, \gamma v)$ for any $u, v \in V$ and $\gamma \in A$. By definition, for a commutative $\mathbb{Q}$-algebra R, the group of R-points $G(R)$ is given as follows:

$$\begin{aligned} G(R) &= \bigl\{\gamma \in (A \otimes_{\mathbb{Q}} R)^\times \mid J(\gamma u, \gamma v) = c_\gamma J(u, v) \text{ for } u, v \in V \\ &\qquad \text{and } c_\gamma \in R^\times \text{ depending only on } \gamma\bigr\} \\ &= \bigl\{\gamma \in (A \otimes_{\mathbb{Q}} R)^\times \mid \gamma\gamma^\star \in R^\times\bigr\}. \end{aligned}$$

Next, consider the cocycle

$$\alpha : \mathrm{Gal}(F/\mathbb{Q}) \to G_{\mathrm{ad}}(\mathbb{Q}), \quad \sigma \mapsto Q,$$

where

$$Q = \begin{bmatrix} & & \mu \\ 1 & & \\ & 1 & \end{bmatrix},$$

and $\mu = \lambda/\bar{\lambda}$. Here, G_{ad} denotes the adjoint group of G, that is, $G_{\mathrm{ad}} = G/Z(G)$.

One can show that $Q^* J Q = J$, and that $Q^3 = \mu$. We set

$$A^\alpha = \{\gamma \in A = \mathbf{Mat}_3(L) \mid \gamma = \alpha_\sigma^{-1} \gamma^\sigma \alpha_\sigma \text{ for } \sigma \in \mathrm{Gal}(F/\mathbb{Q})\},$$
$$B^\alpha = \mathrm{End}_{A^\alpha}(V).$$

The following statements are easy to see.

Proposition 3.1.1.

(1) *The algebra A^α is isomorphic to a central division algebra D over K given by*

$$D = L \oplus L\Pi \oplus L\Pi^2, \qquad \Pi^3 = \mu, \qquad \Pi z = z^\sigma \Pi \quad \textit{for } z \in L.$$

(2) *The algebra B^α is isomorphic to D (but acting on V from the right) with the involution $*$ characterized by $\Pi^* = \Pi^{-1} = \bar{\mu}\Pi^2$.*

Let $\mathfrak{p} = 2\mathbb{Z} + \lambda\mathbb{Z}$, as in Section 2.2. We put β to be the trivial cocycle, that is, $\beta(\sigma) = 1$.

Lemma 3.1.2. *The following conditions are satisfied:*

(1) *$A^\alpha \otimes_K K_\mathfrak{p}$ has Brauer invariant $1/3$;*
(2) *$A^\beta \otimes_K K_\mathfrak{p}$ is isomorphic to $\mathbf{Mat}_3(K_\mathfrak{p})$;*
(3) *for any finite place v of $\mathbb{Q}$ different from 2, the cohomology class of α localized at v is trivial.*

Proof. The conditions (1) and (2) are clearly satisfied. To see that (3) is satisfied, we first look at primes of K lying over ℓ, where $\ell \neq 2, 3, 5$. Then at these primes the extension L/K is unramified. Since μ is integral at these primes, we can apply [9, Theorem 6.8] to conclude that the cohomology class $[\alpha]$ is locally trivial at these primes. It remains to show that $[\alpha]$ is trivial at the primes over 3 and 5. Let $\mathfrak{q}$ be a prime over one of these rational primes. It suffices to show that the equation $x^3 = \mu$ $(= \frac{1-\sqrt{-15}}{1+\sqrt{-15}})$ is soluble in the local field $K_\mathfrak{q}$. This is done by Hensel's lemma. □

Let † be the adjoint involution on A of the Hermitian form H as in Section 2.2, that is, for $\gamma \in A = \mathbf{Mat}_3(L)$, $\gamma^\dagger = H^{-1}\gamma^* H$. Since H is defined over K, it defines a unitary involution † on $A^\beta = \mathbf{Mat}_3(K)$.

Proposition 3.1.3. *There exists an isomorphism*

$$\left(A^\alpha \otimes_{\mathbb{Q}} \mathbb{A}_f^2, \star\right) \xrightarrow{\sim} \left(A^\beta \otimes_{\mathbb{Q}} \mathbb{A}_f^2, \dagger\right)$$

of $\mathbb{A}_f^2$ algebras with involution, where the involution $\star$ in the left-hand side is the involution induced from $\star$ on A.

To show the proposition, we need the following lemma.

Lemma 3.1.4. *Let $d > 1$ be an integer.*

(1) *Let ℓ be a prime, and k/k_0 a quadratic extension of ℓ-adic field. Let $h \in \mathbf{Mat}_d(k)$ be a non-degenerate Hermitian matrix. Then there exists $u \in \mathbf{GL}_d(k)$ such that $u^*hu = \mathrm{diag}(1, \ldots, 1, a)$ with $a \in k_0$ satisfying $a \equiv \det h \bmod \mathrm{Nm}_{k/k_0}(k^\times)$.*
(2) *Let L/F be a totally imaginary quadratic extension of a totally real number field F, and $h \in \mathbf{Mat}_d(L)$ a non-degenerate Hermitian matrix such that each component of $h_{\mathbb{R}}$ is either positive definite or of signature $(d-1, 1)$. Then there exists $u \in \mathbf{GL}_d(L)$ such that $u^*hu = \mathrm{diag}(1, \ldots, 1, a)$ with $a \in F$ satisfying $a \equiv \det h \bmod \mathrm{Nm}_{L/F}(L^\times)$.*

Proof. (1) Consider the quadratic equation $h(e_1, e_1) = 1$. Solving this amounts to finding zero of a homogeneous quadratic form over $\mathbb{Q}$ of $2d + 1$ variables. Since $d > 1$, this is soluble. Decompose k^d into the direct sum of $\langle e_1 \rangle$ and its orthogonal complement $\langle e_1 \rangle^\perp$. Next we consider $h(e_2, e_2) = 1$ for $e_2 \in \langle e_1 \rangle^\perp$, and repeat the above argument. Inductively, we get an orthogonal basis $e_1, \ldots, e_d$ of k^d such that $h(e_i, e_i) = 1$ for $i < d$.

(2) Consider the quadratic equation $h(e_1, e_1) = 1$ for $e_1 \in L^d$. By Hasse–Minkowski principle, we can find such e_1. By a similar inductive argument, we get the result. $\square$

Proof of Proposition 3.1.3. First we show that, under the condition (3) of Lemma 3.1.2, we have an isomorphism

$$\left(A^\alpha \otimes_{\mathbb{Q}} \mathbb{A}_f^2, \star\right) \xrightarrow{\sim} \left(A^\beta \otimes_{\mathbb{Q}} \mathbb{A}_f^2, \star\right),$$

where the $\star$ in the right-hand side is the one induced from $\star$ on A. Let $v \neq 2$ be a finite place. Then, there exists $\delta_v \in G_{\mathrm{ad}}(\mathbb{Q}_v)$ such that $\alpha_\sigma = (\delta_v^\sigma)^{-1} \beta_\sigma \delta_v$. Consider the isomorphism

$$A^\alpha \otimes_{\mathbb{Q}} \mathbb{Q}_v \xrightarrow{\sim} A^\beta \otimes_{\mathbb{Q}} \mathbb{Q}_v, \quad \gamma \mapsto \delta_v \gamma \delta_v^{-1}.$$

Since $\delta_v \delta_v^\star \in \mathbb{Q}_v^\times$, we have $\delta_v \gamma^\star \delta_v^{-1} = (\delta_v^\star)^{-1} \gamma^\star \delta_v^\star$. Hence the isomorphism commutes with $\star$. Thus we get the isomorphism

$$\left(A^\alpha \otimes_{\mathbb{Q}} \mathbb{Q}_v, \star\right) \xrightarrow{\sim} \left(A^\beta \otimes_{\mathbb{Q}} \mathbb{Q}_v, \star\right).$$

On the other hand, the cohomology classes $[\alpha]$ and $[\beta]$ are integral at almost all finite places. If they are integral at v and if $F/\mathbb{Q}$ is unramified over v, then by [9, Theorem 6.8], we deduce that δ_v as above can be taken from $G_{\mathrm{ad}}(\mathbb{Q}_v) \cap \mathbf{Mat}_3(\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_v)$. Hence the above isomorphisms at any $v \neq 2$ induce the isomorphism at adelic level, as desired.

Next we show

$$\left(A^{\beta} \otimes_{\mathbb{Q}} \mathbb{A}_f^2, \star\right) \xrightarrow{\sim} \left(A^{\beta} \otimes_{\mathbb{Q}} \mathbb{A}_f^2, \dagger\right).$$

First we notice that, for any prime number $\ell \neq 2, 3, 5$, we have $(A^{\beta} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, \star) \cong (A^{\beta} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, \dagger)$. This follows immediately from Lemma 3.1.4(1) and the fact that $\det H = -\det J = 2^2 \cdot 3 \cdot 5^2$. Hence it suffices to show that, for almost all odd prime number $\ell \neq 2, 3, 5$, there exists $P \in \mathbf{GL}_3(\mathscr{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$ such that $P^* H P = -J$. By Lemma 3.1.4(2), there exist $P_1, P_2 \in \mathbf{GL}_3(K)$ such that $P_1^* H P_1 = \mathrm{diag}(1, 1, a)$ and $P_2^* J P_2 = \mathrm{diag}(1, 1, -a)$. Hence, since d is odd, we only need to show the following statement.

Claim 3.1. *For any odd prime ℓ, there exists $W \in \mathbf{GL}_2(\mathscr{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$ such that $W^* W = -1$.*

To show the claim, first we deal with the case where ℓ splits in K. In this case, $\mathscr{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$. Set $W = (-1, 1) \in \mathbf{GL}_2(\mathbb{Z}_\ell) \times \mathbf{GL}_2(\mathbb{Z}_\ell)^{\mathrm{opp}}$. Then $W^* = (1, -1)$, and hence we have $W^* W = -1$. Suppose ℓ does not split. Then $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is a quadratic extension of $\mathbb{Q}_\ell$. If $\ell \equiv 1 \bmod 4$, then the desired W can be found easily for $\sqrt{-1} \in \mathbb{Z}_\ell$. Suppose $\ell \equiv 3 \bmod 4$. Find a natural number s with $1 \leqslant s \leqslant \ell - 1$ such that s is a quadratic residue but $s + 1$ is not. Then $-1 - s$ is a quadratic residue, and $W = \left(\begin{smallmatrix} \sqrt{s} & \sqrt{-1-s} \\ \sqrt{-1-s} & -\sqrt{s} \end{smallmatrix}\right)$ satisfies the desired condition. □

3.2. p-Adic uniformization

We set

$$G^{\flat} = \mathbf{GU}\left(A^{\alpha}, \star\right),$$
$$G^{\sharp} = \mathbf{GU}\left(A^{\beta}, \dagger\right).$$

Note that the subgroup $\mathbf{U}(A^{\beta}, \dagger)$ of the last group coincides with $\mathbf{U}$ introduced in 2.2. By Proposition 3.1.3, we have an isomorphism

$$\Phi : G^{\flat}\left(\mathbb{A}_f^2\right) \xrightarrow{\sim} G^{\sharp}\left(\mathbb{A}_f^2\right).$$

Since the prime 2 splits in K, we have the following isomorphisms:

$$G^{\flat}(\mathbb{A}_f) \cong \left(D_{\mathfrak{p}}^{\times} \times \mathbb{Q}_2^{\times}\right) \times G^{\sharp}\left(\mathbb{A}_f^2\right),$$
$$G^{\sharp}(\mathbb{A}_f) \cong \left(\mathbf{GL}_3(K_{\mathfrak{p}}) \times \mathbb{Q}_2^{\times}\right) \times G^{\sharp}\left(\mathbb{A}_f^2\right),$$

where $D_{\mathfrak{p}}$ is the central simple algebra of degree 3 over $K_{\mathfrak{p}}$ of Brauer invariant $1/3$. Here, in the first isomorphism, we have already identified $G^{\flat}(\mathbb{A}_f^3)$ with $G^{\sharp}(\mathbb{A}_f^3)$ by the isomorphism Φ.

Set $T_n = 1 + \pi^n \mathscr{O}_{D_{\mathfrak{p}}}$ for $n > 0$ and $T_0 = \mathscr{O}_{D_{\mathfrak{p}}}^{\times}$, where π denotes the uniformizer of $K_{\mathfrak{p}}$. Let $S \subset \mathbb{Q}_2^{\times} \times G^{\sharp}(\mathbb{A}_f^2)$ be an open compact subgroup. Then one can consider the Shimura variety $M^{T_n \times S}(G^{\flat}, \mathscr{B})$ (where $\mathscr{B}$ is the 2-dimensional complex unit-ball) over the reflex field E, identified with K by the embedding $K \hookrightarrow \mathbb{C}$ (cf. [5, Section 2]). Since $K_{\mathfrak{p}} \cong \mathbb{Q}_p$, its base change $M^{T_n \times S}(G^{\flat}, \mathscr{B}) \otimes_K K_{\mathfrak{p}}^{\mathrm{ur}}$ has the associated rigid analytic space $(M^{T_n \times S}(G^{\flat}, \mathscr{B}) \otimes_K K_{\mathfrak{p}}^{\mathrm{ur}})^{\mathrm{an}}$, where $K_{\mathfrak{p}}^{\mathrm{ur}}$ denotes the maximal unramified extension of $K_{\mathfrak{p}}$.

Theorem 3.2.1. (Varshavsky [10].) *There exists a $K_{\mathfrak{p}}$-rational isomorphism*

$$\left(M^{T_n \times S}\left(G^\flat, \mathscr{B}\right) \otimes_K K_{\mathfrak{p}}^{\mathrm{ur}}\right)^{\mathrm{an}} \xrightarrow{\sim} \left[\Sigma^{3,n}_{K_{\mathfrak{p}}^{\mathrm{ur}}} \times \left(G^\sharp(\mathbb{Q}) \setminus G^\sharp(\mathbb{A}_f)/S\right)\right]/\mathbf{GL}_3(K_{\mathfrak{p}})$$

of $K_{\mathfrak{p}}^{\mathrm{ur}}$-analytic spaces.

Here, $\Sigma^{3,n}_{K_{\mathfrak{p}}^{\mathrm{ur}}}$ is the Drinfel'd's cover of the Drinfel'd symmetric space $\Sigma^{3,0}_{K_{\mathfrak{p}}^{\mathrm{ur}}} = \Omega^3_{K_{\mathfrak{p}}^{\mathrm{ur}}}$ of dimension 2. The right-hand side is the disjoint union of rigid analytic spaces of the form $\Sigma^{3,n}_{K_{\mathfrak{p}}^{\mathrm{ur}}}/\Gamma$, where Γ is a uniform lattice of $\mathbf{PGL}_3(K_{\mathfrak{p}})$. Indeed, one can find finitely many $\gamma_1, \ldots, \gamma_m \in \mathbb{Q}_2^\times \times G^\sharp(\mathbb{A}_f^2)$ (cf. [10, Lemma 1.1.9(a)]) such that, if we set Γ_j for $j = 1, \ldots, m$ to be the image of

$$G^\sharp(\mathbb{Q}) \cap \gamma_j S \gamma_j^{-1} \hookrightarrow G^\sharp(\mathbb{A}_f) \xrightarrow{\mathrm{pr}_1} \mathbf{GL}_3(K_{\mathfrak{p}}) \to \mathbf{PGL}_3(K_{\mathfrak{p}}),$$

then it is the disjoint union of $\Sigma^{3,n}_{K_{\mathfrak{p}}^{\mathrm{ur}}}/\Gamma_j$ for $j = 1, \ldots, m$.

3.3. CMSZ level structures

We are going to construct open compact subgroups ${}^{I}S, {}^{II}S \subset \mathbb{Q}_2^\times \times G^\sharp(\mathbb{A}_f^2)$. These subgroups are of the following form:

$${}^{I}S = S_2 \cdot {}^{I}S_3 \cdot S^{2,3}, \qquad {}^{II}S = S_2 \cdot {}^{II}S_3 \cdot S^{2,3}.$$

Here, we set S_2 and $S^{2,3}$ to be the maximal ones; that is,

$$S_2 = \mathbb{Z}_2^\times, \qquad S^{2,3} = G^\sharp\left(\mathbb{A}_f^{2,3}\right) \cap \mathbf{GL}_3\left(\widehat{\mathbb{Z}}^{2,3} \otimes_{\mathbb{Z}} \mathscr{O}_K\right).$$

The components ${}^{I}S_3$ and ${}^{II}S_3$ are subgroups of $\widetilde{S}_3 = G^\sharp(\mathbb{Q}_3) \cap \mathbf{GL}_3(\mathbb{Z}_3 \otimes_{\mathbb{Z}} \mathscr{O}_K)$ that we are going to define.

Let us use the notation as in Section 2.3. Consider the modulo $\mathfrak{c}^{n+1}$-reduction mapping $\pi_n : \widetilde{S}_3 \to \mathbf{GL}_3(R_n)$. The image lies in the following subgroup:

$$\widetilde{\mathscr{U}}_n = \left\{g \in \mathbf{GL}_3(R_n) \mid g^* H_n g = c(g) H_n,\ c(g) \in F_n^\times\right\},$$

which contains $\mathscr{U}_n$, where $F_n = \{c \in R_n \mid \bar{c} = c\}$. Set $\widetilde{\mathscr{U}}_n' = U^{-1} \widetilde{\mathscr{U}}_n U$, and define

$$\widetilde{\pi}_1' : \widetilde{S}_3 \to \widetilde{\mathscr{U}}_1'$$

similarly as in Section 2.3. We set

$${}^{I}S_3 = (\widetilde{\pi}_1')^{-1}(V_1), \qquad {}^{II}S_3 = (\widetilde{\pi}_1')^{-1}(V_2).$$

Proposition 3.3.1. *Let $\widetilde{\Gamma}_i$ $(i=1,2)$ be the subgroup of $\mathscr{U}$ defined in* 2.4.1. *Then we have*

$$\widetilde{\Gamma}_1 = G^\sharp(\mathbb{Q}) \cap {}^I S, \qquad \widetilde{\Gamma}_2 = G^\sharp(\mathbb{Q}) \cap {}^{II} S.$$

Proof. We prove the proposition for the group $\widetilde{\Gamma}_1$. The proof for the other one is similar. The inclusion $\widetilde{\Gamma}_1 \subseteq G^\sharp(\mathbb{Q}) \cap {}^I S$ is clear. To show the other inclusion, we only need to show that the right-hand side is contained in $\mathbf{U}(H)(\mathbb{Q})$, for the group $\widetilde{\Gamma}_1$ is defined by the same congruence relation as a subgroup of $\mathscr{U}$. Take an element $\gamma \in G^\sharp(\mathbb{Q}) \cap {}^I S$. By [6, III.(14.1)], we have $G^\sharp(\mathbb{Q}) = \mathbf{GU}(H)(\mathbb{Q}) = K^\times \mathbf{U}(H)(\mathbb{Q})$. Hence we can write $\gamma = a\delta$, where $a \in K^\times$ and $\delta \in \mathbf{U}(H)(\mathbb{Q})$. Since $(a\bar{a})^3 = (\gamma\gamma^\dagger)^3 = \det\gamma\overline{\det\gamma}$ belongs to $(\mathscr{O}_K[1/2])^\times$ and $\mathbb{Z}_2^\times$, we have $a\bar{a} \in \mathscr{O}_K^\times \cap \mathbb{Q}^\times = \{\pm 1\}$; hence we have $a\bar{a} = 1$. But this means that $\gamma\gamma^\dagger = 1$ and hence $\gamma \in \mathbf{U}(H)(\mathbb{Q})$, whence the result. $\square$

Corollary 3.3.2. *The analytic space $[\Omega^3_{K_\mathfrak{p}^{\mathrm{ur}}} \times (G^\sharp(\mathbb{Q}) \setminus G^\sharp(\mathbb{A}_f)/{}^I S)]/\mathbf{GL}_3(K_\mathfrak{p})$ (respectively $[\Omega^3_{K_\mathfrak{p}^{\mathrm{ur}}} \times (G^\sharp(\mathbb{Q}) \setminus G^\sharp(\mathbb{A}_f)/{}^{II} S)]/\mathbf{GL}_3(K_\mathfrak{p})$) has a connected component that is $K_\mathfrak{p}$-rationally isomorphic to $({}^I X_{\mathrm{CMSZ}} \otimes_{K_\mathfrak{p}} K_\mathfrak{p}^{\mathrm{ur}})^{\mathrm{an}}$ (respectively $({}^{II} X_{\mathrm{CMSZ}} \otimes_{K_\mathfrak{p}} K_\mathfrak{p}^{\mathrm{ur}})^{\mathrm{an}}$).*

Proof. This follows from Proposition 3.3.1 and the fact mentioned after Theorem 3.2.1. $\square$

Theorem 3.3.3. *Set $K' = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$, and let us fix an embedding $K' \hookrightarrow \mathbb{C}$ extending $K \hookrightarrow \mathbb{Q}$ and an embedding $K' \hookrightarrow K_\mathfrak{p}^{\mathrm{ur}}$. There exists a K'-morphism $M^{T_0 \times {}^I S}(G^\flat, \mathscr{B}) \otimes_K K' \to \operatorname{Spec} K' \otimes_K K'$ (respectively $M^{T_0 \times {}^{II} S}(G^\flat, \mathscr{B}) \otimes_K K' \to \operatorname{Spec} K' \otimes_K K'$) such that the following conditions are satisfied*:

(1) *it is smooth and projective*;
(2) *all fibers of the induced morphism $M^{T_0 \times {}^I S}(G^\flat, \mathscr{B}) \otimes_K \mathbb{C} \to \operatorname{Spec} K' \otimes_K \mathbb{C}$ (respectively $M^{T_0 \times {}^{II} S}(G^\flat, \mathscr{B}) \otimes_K \mathbb{C} \to \operatorname{Spec} K' \otimes_K \mathbb{C}$) are connected and isomorphic to each other*;
(3) *one of the fibers of the induced morphism $M^{T_0 \times {}^I S}(G^\flat, \mathscr{B}) \otimes_K K_\mathfrak{p}^{\mathrm{ur}} \to \operatorname{Spec} K' \otimes_K K_\mathfrak{p}^{\mathrm{ur}}$ (respectively $M^{T_0 \times {}^{II} S}(G^\flat, \mathscr{B}) \otimes_K K_\mathfrak{p}^{\mathrm{ur}} \to \operatorname{Spec} K' \otimes_K K_\mathfrak{p}^{\mathrm{ur}}$) is isomorphic to ${}^I X_{\mathrm{CMSZ}} \otimes_{K_\mathfrak{p}} K_\mathfrak{p}^{\mathrm{ur}}$ (respectively ${}^{II} X_{\mathrm{CMSZ}} \otimes_{K_\mathfrak{p}} K_\mathfrak{p}^{\mathrm{ur}}$).*

Proof. In this proof, we set ${}^I C = T_0 \times {}^I S$ and ${}^{II} C = T_0 \times {}^{II} S$. In the following, we discuss only on $M^{T_0 \times {}^I S}(G^\flat, \mathscr{B})$, and the other case is proven similarly.

First we set

$$T = \{(k, f) \in \operatorname{Res}_{K/\mathbb{Q}} \mathbf{G}_{m,K} \times \mathbf{G}_{m,\mathbb{Q}} \mid k\bar{k} = f^3\} \cong \operatorname{Res}_{K/\mathbb{Q}} \mathbf{G}_{m,K},$$

where the last isomorphism is given by $(k, f) \mapsto kf^{-1}$. This is a $\mathbb{Q}$-algebraic torus, which admits a morphism $\vartheta : G^\flat \to T$ defined by $\gamma \mapsto (\mathrm{Nm}^\circ_{D^{\mathrm{opp}}|K}(\gamma), c(\gamma))$. The kernel of ϑ is the derived group of $G^\flat$. We shall claim that the image $\vartheta({}^I C)$ is a maximal open compact subgroup of $\operatorname{Res}_{K/\mathbb{Q}} \mathbf{G}_{m,K}(\mathbb{A}_f)$. To this end, we only have to look at the component at 3. What to prove is that the homomorphism ϑ localized at 3

$$\vartheta : {}^I C_3 \ni k\theta \mapsto (k\bar{k})^{-1} k^3 \det\theta \in \mathscr{O}_{K_\mathfrak{c}}^\times,$$

where $\mathfrak{c} = 3\mathbb{Z} + (\lambda + 1)\mathbb{Z}$ is the prime lying over 3, is surjective (here θ has been taken such that $\theta^* J\theta = J$; cf. [6, III.(14.1)]). First, note that $\mathcal{O}_{K_\mathfrak{c}}$ is a ramified quadratic extension of $\mathbb{Z}_3$. Since $\mathbb{Z}_3^\times$ is contained in ${}^I C_3$, we have $\mathbb{Z}_3^\times \subset \vartheta({}^I C_3)$. Also, since $\mathcal{O}_{K_\mathfrak{c}}^\times$ is contained in ${}^I C_3$, we know the elements of the form $(k\bar{k})^{-1}k^3$ for $k \in \mathcal{O}_{K_\mathfrak{c}}^\times$ belongs to $\vartheta({}^I C_3)$; but since $k\bar{k} \in \mathbb{Z}_3^\times$ is in it, we deduce $\{k^3 \mid k \in \mathcal{O}_{K_\mathfrak{c}}^\times\} = \pm 1 + 3\mathcal{O}_{K_\mathfrak{c}} \subset \vartheta({}^I C_3)$. Hence it suffices to prove that $(\vartheta({}^I C_3) \bmod 3)$ is the whole $(\mathbb{F}_3[t]/(t^2))^\times \cong (\mathcal{O}_{K_\mathfrak{c}}/(3))^\times$. But this can be checked easily; for instance, factoring through $\mathcal{U}_1'$, we get the value $1 - t$ by the element u (cf. Section 2.3), and hence get $1 + t$ and 1. Multiplying with -1, we get all.

By [3, Section 2] we know that the morphism ϑ induces the canonical morphism of Shimura varieties

$$M^{{}^I C}\big(G^\flat, \mathcal{B}\big)(\mathbb{C}) = G^\flat(\mathbb{Q}) \setminus \big(\mathcal{B} \times G^\flat(\mathbb{A}_f)/{}^I C\big) \longrightarrow T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/\vartheta\big({}^I C\big)$$

that induces the bijection between the sets of all connected components. The right-hand Shimura variety has the canonical model $M^{\vartheta({}^I C)}(T, \cdot)$, whose reflex field is given by K' (the Hilbert class field of K). By what we have seen above and class field theory, the set $T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/\vartheta({}^I C) = M^{\vartheta(C)}(T, \cdot)(\mathbb{C})$ consists of two points, and the scheme $M^{\vartheta({}^I C)}(T, \cdot)$ is isomorphic to $\operatorname{Spec} K' \otimes_K K'$. Hence, by [3, 5.4], we have the desired K-morphism $M^{T_0 \times {}^I S}(G^\flat, \mathcal{B}) \otimes_K K' \to \operatorname{Spec} K' \otimes_K K'$.

Let us show (2). By the well-known fact on connected components of Shimura varieties (cf. [3, Section 2]), the induced morphism $M^{{}^I C}(G^\flat, \mathcal{B}) \otimes_K \mathbb{C} \to \operatorname{Spec} K' \otimes_K \mathbb{C}$ has connected fibers. To show that the fibers are all isomorphic, we refer to the fact that $M^{{}^I C}(G^\flat, \mathcal{B})(\mathbb{C})$ as a complex manifold is the disjoint union of complex manifolds of the form $\Gamma_i \setminus \mathcal{B}$ $(i = 0, 1)$, where $\Gamma_i = \gamma_i {}^I C \gamma_i^{-1} \cap G^\flat(\mathbb{Q})$ and $\gamma_0, \gamma_1 \in G^\flat(\mathbb{A}_f)$ are such that $G^\flat(\mathbb{A}) = \coprod_{i=0,1} G^\flat(\mathbb{Q})\gamma_i G(\mathbb{R}) {}^I C$. Since the prime ideal $\mathfrak{c}$ is not a principal ideal, we may choose these $\gamma_i's$ as follows: for any finite place v of $\mathbb{Q}$ different from 3, we set $\gamma_{i,v} = 1$, and $\gamma_{0,3} = 1$ and $\gamma_{1,3}$ is a uniformizer of $\mathcal{O}_{K_\mathfrak{c}}$ regarded as a scalar matrix. Indeed, since ϑ maps each non-zero scalar a to $a^3/a\bar{a} = a^2/\bar{a}$ and hence a uniformizer to a uniformizer at the ramified prime 3, one sees that $\vartheta(\gamma_0)$ and $\vartheta(\gamma_1)$ belong to different cosets in the idele group $T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/\vartheta({}^I C)$. Since γ_i's are taken to be component-wise scalar, we see that $\gamma_0 {}^I C \gamma_0^{-1} = \gamma_1 {}^I C \gamma_1^{-1}$, and hence $\Gamma_0 = \Gamma_1$. This implies that the connected components of $M^{{}^I C}(G^\flat, \mathcal{B}) \otimes_K \mathbb{C}$ are isomorphic, as desired.

The assertion (3) follows from Theorem 3.2.1 and Corollary 3.3.2. Then, since we know that the fake projective plane ${}^I X_{\mathrm{CMSZ}}$ is smooth and projective, (1) follows from (2). □

Corollary 3.3.4. *The CMSZ fake projective planes have models defined over the number field* $K' = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$.

Proof. By Theorem 3.3.3(3), there exists a connected component of $M^{T_0 \times {}^I S}(G^\flat, \mathcal{B}) \otimes_K K'$ (respectively $M^{T_0 \times {}^{II} S}(G^\flat, \mathcal{B}) \otimes_K K'$) that gives such a model of ${}^I X_{\mathrm{CMSZ}} \otimes_{K_\mathfrak{p}} K_\mathfrak{p}^{\mathrm{ur}}$ (respectively ${}^{II} X_{\mathrm{CMSZ}} \otimes_{K_\mathfrak{p}} K_\mathfrak{p}^{\mathrm{ur}}$). □

Appendix A. Proof of Lemma 2.2.4

Step 1

Let $G_1 = \{g \in \mathscr{U} \mid g(\mathbb{Z}_2)^3 = (\mathbb{Z}_2)^3\}$ and $G_2 = \{\pm\tau^i \mid i = 0, 1, 2\}$. What to prove is the equality $G_1 = G_2$. The inclusion $G_2 \subset G_1$ is clear. We will prove $G_1 \subset G_2$. Let g be an element in G_1. Then g has entries in

$$\mathscr{O}_K\left[\frac{1}{2}\right] \cap \mathbb{Z}_2 = \mathbb{Z}\left[\lambda, \frac{1}{2}\right] \cap \mathbb{Z}_2 = \mathbb{Z}[\bar{\lambda}, \bar{\lambda}^{-1}].$$

Since g is H-unitary (i.e., $g^* = Hg^{-1}H^{-1}$), and since $30H^{-1}$ takes its entries in $\mathbb{Z}[\bar{\lambda}, \bar{\lambda}^{-1}]$, the entries of g actually belong to $\mathbb{Z}[\bar{\lambda}, \bar{\lambda}^{-1}] \cap \frac{1}{30}\mathbb{Z}[\lambda, \lambda^{-1}] = \mathbb{Z} \oplus \frac{\lambda}{2}\mathbb{Z}$, which we denote by $\mathbf{L}$. Set

$$V = \left\{ v \in \mathbf{L}^3 \;\middle|\; v^* H v = 10,\ \tau v \text{ and } \tau^{-1} v \text{ lie in } \mathbf{L}^3,\ \text{and } v \notin \frac{\lambda}{2}\mathbb{Z}^3 \right\}.$$

Then each column vector of g belongs to V; indeed, $v^*Hv = 10$ is from $g \in \mathscr{U}$, and τv, $\tau^{-1}v \in \mathbf{L}^3$ are from $\tau g, \tau^{-1}g \in G_1$, and, if g were in $(\lambda/2)\mathbb{Z}^3$, then $\det g$ would be a multiple of $\lambda/2$, which is not invertible in $\mathbb{Z}_2$.

Note that $\mathbf{L}^3 \cap \tau^{-1}(\mathbf{L}^3) = \mathbf{L}^3 \cap \tau^{-1}(\mathbf{L}^3) \cap \tau(\mathbf{L}^3) = \mathbf{L}^2 \oplus \mathscr{O}_K$, where $\mathscr{O}_K = \mathbb{Z} \oplus \mathbb{Z}\lambda$.

Claim A.1. *The set V consists of the following* 24 *vectors*:

$$G_2\begin{bmatrix} \frac{\lambda}{2} \\ 1 \\ 0 \end{bmatrix}, \qquad G_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad G_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \textit{and} \quad G_2\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

The proof of the claim is done in the following sections.

Step 2 (Change of coordinates)

We denote a vector v in $\mathbf{L}^2 \oplus \mathscr{O}_K$ as $v = (a_1 + (\lambda/2)b_1, a_2 + (\lambda/2)b_2, a_3 + \lambda b_3)$ with $(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{Z}^6$.

We define the map $\psi : \mathbb{Z}^6 \ni (a_1, a_2, a_3, b_1, b_2, b_3) \mapsto z = (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{Z}^6$ by

$$\begin{aligned}
z_1 &= a_1 \\
z_2 &= -a_2 - 2b_3 \\
z_3 &= -a_1 + a_2 - a_3 \\
z_4 &= a_3 - b_2 + b_3 \\
z_5 &= -b_1 + b_2 - 2b_3 \\
z_6 &= b_1
\end{aligned}$$

then ψ gives a bijection between $\mathbb{Z}^6$ and $\{z \in \mathbb{Z}^6 \mid \sum_{i=1}^{6} z_i \in 3\mathbb{Z}\}$. The action of τ on $\mathbf{L}^2 \oplus \mathscr{O}_K \cong \mathbb{Z}^6$ induces the action $(z_1, z_2, z_3, z_4, z_5, z_6) \mapsto (z_2, z_3, z_1, z_5, z_6, z_4)$, and the action of -1 does the action $z \mapsto -z$. The norm v^*Hv is equal to the quadratic polynomial $5f(z)$, which can be calculated as $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \sum_{i=1}^{6} z_i^2 - \sum_{i=1}^{3} z_i z_{i+3},$$

$$f_2(z) = \left(\sum_{i=1}^{6} z_i\right)^2 \Big/ 3.$$

Note that f_1 and f_2 are positive semi-definite.

Suppose $v \in V$. This implies $f(z) = 2$. Then, $\sum_{i=1}^{6} z_i = 0$, since otherwise $f(z) \geqslant f_2(z) \geqslant 3$. This shows $\psi(V) = \{z \in \mathbb{Z}^6 \mid \sum_{i=1}^{6} z_i = 0, f_1(z) = 2, (z_1, z_2, z_3) \neq (0,0,0)\}$ since $v \in \frac{\lambda}{2}\mathbb{Z}^3$ if and only if $a_1 = a_2 = a_3 = 0$ if and only if $z_1 = z_2 = z_3 = 0$ under the condition $\sum_{i=1}^{6} z_i = -3b_3 = 0$. Then Claim A.1 follows from the following.

Claim A.2.

(i) *We have* $\{z \in \mathbb{Z}^6 \mid \sum_{i=1}^{6} z_i = 0, 0 < f_1(z) < 2\} = \emptyset$.
(ii) *The set* $\{z \in \mathbb{Z}^6 \mid \sum_{i=1}^{6} z_i = 0, f_1(z) = 2\}$ *consists of the following* 30 *vectors*:

$$G_2\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad G_2\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad G_2\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad G_2\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \textit{and} \quad G_2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

(iii) *The set* $\{z \in \mathbb{Z}^6 \mid \sum_{i=1}^{6} z_i = 0,\ f_1(z) = 2,\ (z_1, z_2, z_3) = (0,0,0)\}$ *consists of the following* 6 *vectors*:

$$G_2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

We will prove this claim in the following section.

Step 3 (Proof of Claim A.2)

Let $z_+ = (z_1, z_2, z_3)$ and $z_- = (z_4, z_5, z_6)$ such that $z = (z_+, z_-) \in \{z \in \mathbb{Z}^6 \mid \sum_{i=1}^{6} z_i = 0, f_1(z) \leqslant 2\}$. Let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{Z}^3$ and $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ be the corresponding norm. Then $f_1(z_+, z_-) = \|z_+\|^2 + \|z_-\|^2 - \langle z_+, z_-\rangle$. We see that $|\langle z_+, z_-\rangle| \leqslant \|z_+\|\,\|z_-\|$. Then,

$$2 \geqslant f_1(z_+, z_-) \geqslant \|z_+\|^2 + \|z_-\|^2 - \|z_+\|\,\|z_-\| \geqslant \big(\|z_+\| - \|z_-\|/2\big)^2 + (3/4)\|z_-\|^2.$$

This shows $\|z_-\|^2 \leqslant 8/3$, that is, $\|z_+\|^2$ and $\|z_-\|^2$ take values 0, 1 or 2. In particular z_i^2 is 0 or 1. This shows

$$\|z_+\|^2 \equiv \sum_{i=1}^{3} z_i \equiv -\sum_{i=4}^{6} z_i \equiv \|z_-\|^2 \mod 2.$$

Then the possibility of the pairs $(\|z_+\|^2, \|z_-\|^2)$ is

$$(0,0), \qquad (0,2), \qquad (2,0), \qquad (2,2), \qquad (1,1).$$

Let us discuss each case individually:

(B1) if $(\|z_+\|^2, \|z_-\|^2) = (0,2)$, then $z_+ = 0$ and z_- belongs to the set

$$\mathbf{S} = \{(1,-1,0), (0,1,-1), (-1,0,1), (-1,1,0), (0,-1,1), (1,0,-1)\}.$$

The case $(\|z_+\|^2, \|z_-\|^2) = (2,0)$ is similar;

(B2) if $(\|z_+\|^2, \|z_-\|^2) = (2,2)$, then the condition $f_1 \leqslant 2$ implies $\|z_+\| = \|z_-\| = \langle z_+, z_-\rangle = 2$; we have $z_+ = z_-$ and then $\sum_{i=1}^{3} z_i = 0$, which imply that $z_+ = z_- \in \mathbf{S}$;

(B3) if $(\|z_+\|^2, \|z_-\|^2) = (1,1)$, then the condition $f_1 \leqslant 2$ implies $\langle z_+, z_-\rangle \geqslant 0$; if $\langle z_+, z_-\rangle$ were 1, then $z_+ = z_-$, which contradicts to $\sum_{i=1}^{3} z_i = 0$; hence we have $\langle z_+, z_-\rangle = 0$, and thus there are 6 choices of z_+ with $\|z_+\| = 1$, and for each z_+ we have two choices of z_-.

This proves Claims A.2 and A.1.

Step 4

Now, let us turn back to the proof of $G_1 \subset G_2$. We ask for, for any $v_2 \in V$, two vectors $v_1, v_3 \in V$ such that $v_1^* H v_2 = v_2^* H v_3 = -2(\lambda + 2)$ to get the matrix $g = (v_1, v_2, v_3)$ which is presumed to be H-unitary. Due to Claim A.1 it suffices to check the following four cases:

(C1) if $v_2 = {}^t(\lambda/2, 1, 0)$, there does not exist such v_1;

(C2) if $v_2 = {}^t(0,0,1)$, then $v_3 = {}^t(0,-1,-1)$ and v_1 is either ${}^t(0,1,0)$ or ${}^t(0,-\lambda/2,-1)$; but in this case, (v_1, v_2, v_3) is not an invertible matrix;

(C3) if $v_2 = {}^t(0,-1,-1)$, then $v_1 = {}^t(0,0,1)$ and $v_3 = {}^t(0,1,0)$ or ${}^t(\lambda/2, 1+\lambda/2, 1)$; but, also in this case, $\det(v_1, v_2, v_3)$ is not invertible in $\mathbb{Z}_2$;

(C4) if $v_2 = {}^t(0,1,0)$, then v_1 is either ${}^t(1,0,0)$, ${}^t(0,-1,-1)$, or ${}^t(-\bar{\lambda}, -1, 0)$, and v_3 is either ${}^t(0,0,1)$ or ${}^t(-1,-1,0)$; among them, $v_1^* H v_3 = \lambda + 2$ is satisfied only when $g = (v_1, v_2, v_3) = I_3$.

All of these cases are verified without so much pain. Since any member in G_2 is H-unitary, we immediately see that the possible (v_1, v_2, v_3) are only among elements in G_2. Therefore, we conclude $G_1 = G_2$, as desired.

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