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Monoids of IG-type and maximal orders

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Abstract

Let G be a finite group that acts on an abelian monoid A. If $\phi : A \to G$ is a map so that $\phi(a\phi(a)(b)) = \phi(a)\phi(b)$, for all $a, b \in A$, then the submonoid $S = \{(a, \phi(a)) \mid a \in A\}$ of the associated semidirect product $A \rtimes G$ is said to be a monoid of IG-type. If A is a finitely generated free abelian monoid of rank n and G is a subgroup of the symmetric group Sym_n of degree n, then these monoids first appeared in the work of Gateva-Ivanova and Van den Bergh (they are called monoids of I-type) and later in the work of Jespers and Okniński. It turns out that their associated semigroup algebras share many properties with polynomial algebras in finitely many commuting variables.

In this paper we first note that finitely generated monoids S of IG-type are epimorphic images of monoids of I-type and their algebras $K[S]$ are Noetherian and satisfy a polynomial identity. In case the group of fractions SS^{-1} of S is torsion-free abelian then it is characterized when $K[S]$ also is a maximal order. It turns out that they often are, and hence these algebras again share arithmetical properties with natural classes of commutative algebras. The characterization is in terms of prime ideals of S, in particular G-orbits of minimal prime ideals in A play a crucial role. Hence, we first describe the prime ideals of S. It also is described when the group SS^{-1} is torsion-free.
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1. Introduction

In [9] Gateva-Ivanova and Van den Bergh introduced a new class of monoids T, called monoids of I-type, with the aim of constructing non-commutative algebras that share many properties with polynomial algebras in finitely many commuting variables. In particular, the semigroup algebras $K[T]$ are Noetherian maximal orders that satisfy a polynomial identity. Moreover, these monoids are intimately connected with set theoretic solutions of the quantum Yang–Baxter equation and Bieberbach groups. In this paper, we consider a much wider class of semigroups S and show that often their algebras $K[S]$ still are Noetherian maximal orders that satisfies a polynomial identity. Earlier recent results on the construction of such algebras can be found in [13,14,16], as well as an extensive literature on the topic.

To put things into context, we first recall the definition of a monoid of I-type. By FaM_n we denote the free abelian monoid of rank n with basis $\{u_1, \ldots, u_n\}$. A monoid S, generated by a set $X = \{x_1, \ldots, x_n\}$, is said to be of left I-type if there exists a bijection (called a left I-structure) $v : \mathrm{FaM}_n \to S$ such that $v(1) = 1$ and $\{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\}$, for all $a \in \mathrm{FaM}_n$. Similarly one defines monoids of right I-type. In [9] it was shown that a monoid S of left I-type has a presentation $S = \langle x_1, \ldots, x_n \mid R\rangle$, where R is a set of $\binom{n}{2}$ defining relations of the type $x_i x_j = x_k x_l$, so that every word $x_i x_j$ with $1 \leqslant i, j \leqslant n$ appears at most once in one of the relations. Hence, one obtains an associated bijective map $r : X \times X \to X \times X$, defined by $r(x_i, x_j) = (x_k, x_l)$ if $x_i x_j = x_k x_l$ is a defining relation for S, otherwise one defines $r(x_i, x_j) = (x_i, x_j)$. For every $x \in X$, denote by $f_x : X \to X$ and by $g_x : X \to X$ the mappings defined by $f_x(x_i) = p_1(r(x, x_i))$ and $g_x(x_i) = p_2(r(x_i, x))$, where p_1 and p_2 denote the projections onto the first and second component, respectively. So, $r(x_i, x_j) = (f_{x_i}(x_j), g_{x_j}(x_i))$. One says that r (or simply S) is left non-degenerate if each g_x is bijective. In case each f_x is bijective then r (or S) is said to be right non-degenerate. Also, one says that r is a set theoretic solution of the Yang–Baxter equation if $r_1 r_2 r_1 = r_2 r_1 r_2$, where $r_i : X^m \to X^m$ is defined as $id_{X^{i-1}} \times r \times id_{X^{m-i-1}}$ and id_{X^j} denotes the identity map on the Cartesian product X^j.

In [9] the equivalence of the first two statements of the following theorem has been proven. The equivalence with the third statement has been proven in [13].

Theorem 1.1. *The following conditions are equivalent for a monoid S.*

1. *S is a monoid of left I-type.*
2. *S is finitely generated, say by $x_1, \ldots, x_n$, and is defined by $\binom{n}{2}$ homogeneous relations of the form $x_i x_j = x_k x_l$ so that every word $x_i x_j$ with $1 \leqslant i, j \leqslant n$ appears at most once in one of the relations and the associated bijective map r is a solution of the Yang–Baxter equation and is left non-degenerate.*
3. *S is a submonoid of a semidirect product of a free abelian monoid FaM_n of rank n and a symmetric group of degree n, so that the projection onto the first component is bijective. That is, $S = \{(a, \phi(a)) \mid a \in \mathrm{FaM}_n\}$ where ϕ is a mapping from FaM_n to Sym_n so that*

$$\phi(a)\phi(b) = \phi\big(a\phi(a)(b)\big), \tag{1}$$

or equivalently

$$\phi(ac) = \phi(a)\phi\big(\phi(a)^{-1}(c)\big), \tag{2}$$

for all $a, b, c \in \mathrm{FaM}_n$.

It follows that a monoid is of left I-type if and only if it is of right I-type. Such monoids are simply called monoids of I-type (as in [13]).

Note that the above mentioned semidirect product $\mathrm{FaM}_n \rtimes \mathrm{Sym}_n$ is defined via the natural action of Sym_n on a chosen basis $\{u_1, \ldots, u_n\}$ of the free abelian monoid FaM_n, that is, $\phi(a)(u_i) = u_{\phi(a)(i)}$. Let Fa_n denote the free abelian group with the same basis. Then, the monoid S has a group of quotients SS^{-1} contained in $\mathrm{Fa}_n \rtimes \mathrm{Sym}_n$ and $SS^{-1} = \{(a, \bar{\phi}(a)) \mid a \in \mathrm{Fa}_n\}$, where $\bar{\phi} : \mathrm{Fa}_n \to \mathrm{Sym}_n$ is a mapping that extends the map $\mathrm{FaM}_n \to \mathrm{Sym}_n$ and it also satisfies (1). In [13] such groups are called groups of I-type. In [9] and [13] it is shown that SS^{-1} is a solvable Bieberbach group, that is, SS^{-1} is a finitely generated solvable torsion-free group. These groups also have been investigated by Etingof, Guralnick, Schedler and Soloviev in [6,7], where they are called structural groups.

Gateva-Ivanova and Van den Bergh [9], proved that the semigroup algebra of such a monoid shares a lot of properties with commutative polynomial algebras in finitely many variables. In particular, it is a Noetherian domain that satisfies a polynomial identity and it is a maximal order.

Jespers and Okniński in [14] investigated when an arbitrary semigroup algebra satisfies these latter properties. The assumptions on S say that $K[S]$ is a Noetherian domain that satisfies a polynomial identity. For details we refer to [14] and [17] (see also the introduction of Section 3).

Theorem 1.2. *Let K be a field and S a submonoid of a torsion-free finitely generated abelian-by-finite group. The monoid algebra $K[S]$ is a Noetherian maximal order if and only if the following conditions are satisfied*:

1. *S satisfies the ascending chain condition on one sided ideals,*
2. *S is a maximal order in its group of quotients $H = SS^{-1}$,*
3. *for every minimal prime P in S,*

$$S_P = \{g \in H \mid Cg \subseteq S \text{ for some } H\text{-conjugacy class } C \text{ of } H \\ \text{contained in } S \text{ and with } C \nsubseteq P\}$$

has only one minimal prime ideal.

It is worth mentioning that Brown in [3] proved that, for a field K, a group algebra $K[G]$ of a torsion-free polycyclic-by-finite group G is a maximal order in its classical ring of quotients (which is a domain). A characterization of commutative semigroup algebras $K[A]$ that are Noetherian domains and maximal orders can be found in [10]. It turns out that $K[A]$ is such an algebra if and only if A is finitely generated and a maximal order in its torsion-free group of quotients AA^{-1} (see also the comments given in Section 3). Extensions of this result to Krull orders have been proved by Chouinard [4].

2. Monoids of IG-type

We begin with introducing the larger class of monoids of interest. Let G be a group and A a monoid. Recall that G is said to act on A if there exists a monoid morphism $\varphi : G \to \mathrm{Aut}(A)$. The associated semidirect product $A \rtimes_\varphi G$ we often simply denote by $A \rtimes G$.

Definition 2.1. Suppose G is a finite group acting on a cancellative abelian monoid A. A submonoid S of $A \rtimes G$ so that the natural projection on the first component is bijective is said to be a monoid of IG-type. Thus,

$$S = \{(a, \phi(a)) \mid a \in A\},$$

with $\phi: A \to G$ a mapping satisfying (1). (We denote the action of $g \in G$ on $a \in A$ as $g(a)$.)

Note that for every $a \in A$,

$$(a, \phi(a))^{|G|} = (a\phi(a) \cdots \phi(a)^{|G|-1}(a), \phi(a)^{|G|}) = (a\phi(a) \cdots \phi(a)^{|G|-1}(a), 1).$$

It follows that $b = a\phi(a) \cdots \phi(a)^{|G|-1}(a) \in A$ is such that $\phi(b) = 1$ and, for every $a^{-1}b_1 \in AA^{-1}$ (the group of quotients of A), we have $a^{-1}b_1 = b^{-1}(\phi(a) \cdots \phi(a)^{|G|-1}(a)b_1)$. So, any element of AA^{-1} can be written as $a^{-1}b$ with $a, b \in A$ and $\phi(a) = 1$. Furthermore, if $a_1^{-1}b_1 = a_2^{-1}b_2$, with $a_i, b_i \in A$ and $\phi(a_i) = 1$, then by Eq. (1), it is easily verified that $\phi(b_1) = \phi(b_2)$. We hence can extend the action of G onto A to an action of G onto AA^{-1} and thus obtain a mapping $\overline{\phi}: AA^{-1} \to G$ so that

$$\overline{\phi}(a^{-1}b) = \phi(b),$$

for every $a, b \in A$ with $\phi(a) = 1$. This mapping again satisfies (1).

Hence S is a submonoid of the group $AA^{-1} \rtimes G$. Since $AA^{-1} \rtimes G$ is abelian-by-finite and S is cancellative, Lemma 7.1 in [17] yields that S has a group of fractions $SS^{-1} \subseteq AA^{-1} \rtimes G$. Furthermore, because of Theorem 15 in [17], the algebra $K[S]$ satisfies a polynomial identity, and if $K[S]$ is prime then $SS^{-1} = SZ(S)^{-1}$, with $Z(S)$ the center of S [16].

We claim that the natural projection of $SS^{-1} \to AA^{-1}$ is a one-to-one mapping. Indeed, if $a_1^{-1}b_1 = a_2^{-1}b_2$ (with $a_i, b_i \in A$, $\phi(a_i) = 1$) then, by (1), $\phi(b_1) = \phi(b_2)$. So,

$$(a_1^{-1}b_1, \overline{\phi}(a_1^{-1}b_1)) = (a_1^{-1}b_1, \phi(b_1)) = (a_2^{-1}b_2, \phi(b_2)) = (a_2^{-1}b_2, \overline{\phi}(a_2^{-1}b_2)).$$

This proves the injectiveness. The surjectiveness follows from the fact that

$$\begin{aligned}(a_1^{-1}b_1, \overline{\phi}(a_1^{-1}b_1)) &= (a_1^{-1}b_1, \phi(b_1)) \\ &= (a_1, 1)^{-1}(b_1, \phi(b_1)) \\ &= (a_1, \phi(a_1))^{-1}(b_1, \phi(b_1)) \in SS^{-1},\end{aligned}$$

for $a_i, b_i \in A$ with $\phi(a_i) = 1$. Groups of the type SS^{-1} we call groups of IG-type. So we have shown the following.

Corollary 2.2. *A group H is of IG-type if and only if H is a subgroup of a semidirect product $A \rtimes G$ of a finite group G with an abelian group A so that*

$$H = \{(a, \overline{\phi}(a)) \mid a \in A\}$$

and

$$\overline{\phi}\big(a\overline{\phi}(a)(b)\big) = \overline{\phi}(a)\overline{\phi}(b),$$

for all $a, b \in A$. Of course, such a group is abelian-by-finite.

A subset B of A is said to be ϕ-invariant if $\phi(a)(B) = B$ for all $a \in A$. In case B is a subgroup of A then this condition is equivalent with B being a normal subgroup of SS^{-1}.

Note also that if $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ is a monoid of IG-type, with A an abelian monoid and $G = \{\phi(a) \mid a \in A\}$ a finite group then $\prod_{\phi(a)\in G} \phi(a)(b)$ is an invariant element of A, for every $b \in A$. (In this case we will also use G-invariant as ϕ-invariant.) It follows that every element of SS^{-1} can be written as $(z, 1)^{-1}(a, \phi(a))$ with $z, a \in A$ and z and invariant element in A. So $(z, 1)$ is a central element of S.

We now describe when the semigroup algebra $K[S]$ of a monoid of IG-type is Noetherian. This easily can be deduced from the following lemma and the recent result of Jespers and Okniński proved in [12] which says that, for a submonoid T of a polycyclic-by-finite group, the semigroup algebra $K[T]$ is left Noetherian if and only if $K[T]$ is right Noetherian, or equivalently, T satisfies the ascending chain condition on left (or right) ideals. An algebra which is left and right Noetherian we simply call Noetherian. However, for completeness' sake we include a simple proof for the monoids under consideration. The subgroup generated by a set X of elements in a group G is denoted $\mathrm{gr}(X)$. By $\langle X\rangle$ we denote the monoid generated by X.

Lemma 2.3. *Let $A = \langle u_1, \dots, u_n\rangle$ be a finitely generated abelian monoid, G a finite group acting on A. Let $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ be a monoid of IG-type. Put $B = \{\phi(a)(u_i) \mid a \in A,\ 1 \leqslant i \leqslant n\}$. Then the following conditions hold*:

1. *G acts on the set B, that is, $\phi(a)(B) = B$, for all $a \in A$.*
2. *$S = \langle (b, \phi(b)) \mid b \in B\rangle$.*
3. *For some divisor k of $|G|$, the subgroup $\mathrm{gr}\{(b^k, 1) \mid b \in B\}$ is normal and of finite index in SS^{-1}.*
4. *$S = \bigcup_{f\in F} \langle (b^k, 1) \mid b \in B\rangle (f, \phi(f))$ and $(f, \phi(f))\langle (b^k, 1) \mid b \in B\rangle = \langle (b^k, 1) \mid b \in B\rangle (f, \phi(f))$, for some finite subset F of A.*

Proof. The first and second part follow at once from the equalities (1) and (2). Put $N = \{a \in AA^{-1} \mid \overline{\phi}(a) = 1\}$, the kernel of the natural homomorphism $SS^{-1} \to G$. So, N is an abelian subgroup of finite index k in AA^{-1}, with k a divisor of $|G|$. It follows that $\phi(a^k) = 1$, for any $a \in A$ and that the abelian monoid $C = \langle (b^k, 1) \mid b \in B\rangle$ is contained in S and its group of quotients $CC^{-1} = \mathrm{gr}\{(b^k, 1) \mid b \in B\}$ is normal and of finite index in SS^{-1}. This proves the third part. Part four is now also clear. $\square$

If, in the previous lemma, $U(A) = \{1\}$ then one can take $\{u_1, \dots, u_n\}$ to be the set of indecomposable elements, that is, the set consisting of those elements $f \in A$ so that Af is a maximal principal ideal. Indeed, since A is finitely generated, we know that A satisfies the ascending chain condition on ideals. Hence, A has finitely many indecomposable elements, say $u_1, \dots, u_n$, and $A = \langle u_1, \dots, u_n\rangle$ [15]. Clearly any automorphism of A permutes the indecomposable elements. It follows that $S = \langle (u_1, \phi(u_1)), \dots, (u_n, \phi(u_n))\rangle$.

Let S be a submonoid of a group G. Assume N is a normal subgroup of G. If $N \subseteq S$ then, as in group theory we denote by S/N the monoid consisting of the cosets $sN = Ns$, with $s \in S$.

Proposition 2.4. *Let A be an abelian monoid and G a finite group acting on A. Let $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ be a monoid of IG-type. Then, the semigroup algebra $K[S]$ is Noetherian if and only if the abelian monoid A is finitely generated, or, equivalently, S is finitely generated.*

Proof. Suppose that $K[S]$ is right Noetherian. Because $K[L]$ is a right ideal of $K[S]$ for every right ideal L of S, it follows easily that S satisfies the ascending chain condition on right ideals. Consequently, also A satisfies the ascending chain on right ideals. Indeed, if L is a right ideal of A, then $\lambda_L = \{(a, \phi(a)) \in S \mid a \in L\}$ is a right ideal of S and $\lambda_L \subset \lambda_{L'}$ if and only if $L \subset L'$. So A is an abelian and cancellative monoid that satisfies the ascending chain condition on ideals. Hence, so is the monoid $A/U(A)$. Because $U(A/U(A)) = \{1\}$, it follows (see the remark above) that $A/U(A)$ is finitely generated by its indecomposable elements. Clearly $U(S) = \{(a, \phi(a)) \mid a \in U(A)\}$ and because $IK[S] \cap K[U(S)] = I$ for any right ideal I of $K[U(S)]$, it follows that the group algebra $K[U(S)]$ is Noetherian. Hence it is well known that $U(S)$ is a finitely generated monoid. Consequently, $U(A)$ and thus also A is finitely generated.

For the converse, suppose that $A = \langle u_1, \dots, u_n\rangle$ is finitely generated. From Lemma 2.3 it follows that the algebra $K[S]$ is a finite module over the commutative Noetherian algebra $K[\langle (b^k, 1) \mid b \in B\rangle]$. Hence $K[S]$ is Noetherian. $\square$

We now give a link with monoids of I-type by proving another characterization of finitely generated monoids S of IG-type.

Theorem 2.5. *A finitely generated monoid S is of IG-type if and only if there exists a monoid of I-type $T = \{(x, \psi(x)) \mid x \in \mathrm{FaM}_m\} \subseteq \mathrm{FaM}_m \rtimes \mathrm{Sym}_m$ and a subgroup B of Fa_m that is ψ-invariant so that $S \cong TB/B$.*

Proof. Assume $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ is a finitely generated monoid of IG-type, where G is a finite group acting on the finitely generated abelian monoid $A = \langle u_1, \dots, u_n\rangle$. Of course we may assume that $G = \{\phi(a) \mid a \in A\}$.

Let $m = n|G|$ and let FaM_m be the free abelian monoid of rank m with basis the set $M = \{v_{g,i} \mid g \in G,\ 1 \leqslant i \leqslant n\}$. Clearly the mapping $f : \mathrm{FaM}_m \to A$ defined by $f(v_{g,i}) = gu_i$ is a monoid epimorphism. For $x \in \mathrm{FaM}_m$ define a mapping $\psi(x) : M \to M$ by $\psi(x)(v_{g,i}) = v_{\phi(f(x))g,i}$. Then $\psi(x) \in \mathrm{Sym}_m$, $f(\psi(x)(y)) = \phi(f(x))(f(y))$ and $\psi(x\psi(x)(y)) = \psi(x)\psi(y)$, for any $x, y \in \mathrm{FaM}_m$. So $T = \{(x, \psi(x)) \mid x \in \mathrm{FaM}_m\}$ is a monoid of I-type contained in $\mathrm{FaM}_m \rtimes \mathrm{Sym}_m$. Furthermore, $f^e : T \to S$ defined by $f^e((x, \psi(x))) = (f(x), \phi(f(x)))$ is a monoid epimorphism. Its extension to an epimorphism $TT^{-1} \to SS^{-1}$ we also denote by f^e. Let $B = \ker(f^e)$. Clearly, if $(x, \psi(x)) \in B$ then $\phi(f(x)) = 1$ and thus $\psi(x) = 1$. Thus, $B \subseteq \mathrm{Fa}_m$ and B is ψ-invariant. So $TB = BT$ is a submonoid of TT^{-1} and $TB/B \cong S$. This proves the necessity of the conditions.

Conversely, assume $T = \{(x, \psi(x)) \mid x \in \mathrm{FaM}_m\} \subseteq \mathrm{FaM}_m \rtimes \mathrm{Sym}_m$ is a monoid of I-type and B is a subgroup of Fa_m that is ψ-invariant. Let $A = \mathrm{FaM}_m B/B$ and let $f : \mathrm{FaM}_m \to A$ be the natural monoid epimorphism. Because of (2) we get that $\psi(x) = \psi(y)$ if $f(x) = f(y)$. Hence, for each $a = f(x)$ the mapping $\phi(a) : A \to A$ given by $\phi(a)(f(y)) = f(\psi(x)(y))$ is a well defined bijection of finite order. Furthermore, $\phi(a\phi(a)(b)) = \phi(a)\phi(b)$ for all $a, b \in A$. Hence

$S = \{(a, \phi(a)) \mid a \in A\}$ is a monoid of IG-type contained in $A \rtimes G$, where $G = \{\phi(a) \mid a \in A\}$. The mapping $TB \to S$ defined by mapping $(x, \psi(x))$ onto $(a, \phi(a))$ is a monoid epimorphism and it easily follows that this map induces an isomorphism between TB/B and S. $\square$

We note that the proposition can be formulated using congruence relations as follows. A finitely generated monoid S is of IG-type if and only if there exists a monoid of I-type $T = \{(a, \psi(a)) \mid a \in \mathrm{FaM}_m\} \subseteq \mathrm{FaM}_m \rtimes \mathrm{Sym}_m$ and there exists a congruence relation ρ on FaM_m with

$$a\rho b \quad \text{implies} \quad \psi(a) = \psi(b) \quad \text{and} \quad \psi(x)(a)\rho\psi(x)(b), \tag{3}$$

for every $a, b, x \in \mathrm{FaM}_m$, and so that $S \cong T/\overline{\rho}$ where $\overline{\rho}$ is the congruence relation on T defined by $(a, \psi(a))\overline{\rho}(b, \psi(b))$ if and only if $a\rho b$, for $a, b \in \mathrm{FaM}_m$.

We remark that many monoids of IG-type are not of I-type. Indeed, suppose $S = \{(a, \phi(a)) \mid a \in A\}$ is a monoid of IG-type with A a finitely generated monoid so that $U(A) = \{1\}$. Let $\{u_1, \ldots, u_n\}$ be the set of indecomposable elements of A. So $A = \langle u_1, \ldots, u_n \rangle$. It follows that the elements $(u_i, \phi(u_i))$ are the unique indecomposable elements of S, that is, they cannot be decomposed as a product of two non-invertible elements. So S also has n indecomposables. In particular, the number of indecomposables in a monoid T of I-type equals the torsion-free rank of any abelian subgroup of finite index in TT^{-1}. So, if the torsion-free rank of AA^{-1} is strictly smaller than n then S is not of I-type.

3. Torsion-freeness of groups of IG-type

We recall some notation and terminology on maximal orders (see for example [14]). A cancellative monoid S which has a left and right group of quotients G is called an order. Such a monoid S is called a maximal order if there does not exist a submonoid S' of G properly containing S and such that $aS'b \subseteq S$ for some $a, b \in S$. For subsets A, B of G put $(A :_l B) = \{g \in G \mid gB \subseteq A\}$ and $(A :_r B) = \{g \in G \mid Bg \subseteq A\}$. It turns out that S is a maximal order if and only if $(I :_l I) = (I :_r I) = S$ for every fractional ideal I of S. The latter means that $SIS \subseteq I$ and $cI, Id \subseteq S$ for some $c, d \in S$. If S is a maximal order, then $(S :_l I) = (S :_r I)$ for any fractional ideal I. One simply denotes this fractional ideal by $(S : I)$ or by I^{-1}. Recall that I is said to be divisorial if $I = I^*$, where $I^* = (S : (S : I))$. The divisorial product $I * J$ of two divisorial ideals I and J is defined as $(IJ)^*$.

Also recall that a cancellative monoid S is said to be a Krull order if and only if S is a maximal order satisfying the ascending chain condition on integral divisorial ideals, that is, fractional ideals contained in S. In this case the set $D(S)$ of divisorial ideals is a free abelian group for the $*$ operation. If G is abelian-by-finite, then every ideal of S contains a central element. In this case, it follows that the minimal primes of S form a free basis for $D(S)$. The positive cone of this group (with respect to this basis) is denoted by $D(S)^+$.

In this section, we investigate periodic elements of a monoid $S = \{(a, \phi(a)) \mid a \in A\}$ of IG-type, and we will restrict our attention to the case that A is a finitely generated maximal order (and hence a Krull order) with trivial unit group, AA^{-1} is torsion-free and the action of $G = \{\phi(a) \mid a \in A\}$ on A is faithful. Because of the latter condition we may consider G as a subgroup of the automorphism group of A. Since A is finitely generated, we know that A has only finitely many minimal prime ideals and every prime ideal is a union of minimal prime ideals. Recall from

Theorem 37.5 in [19] that, as SS^{-1} is polycyclic-by-finite, $K[S]$ (or equivalently $K[SS^{-1}]$) is a domain if and only if SS^{-1} is torsion-free.

Proposition 3.1. *Let A be an abelian cancellative monoid. Assume that A is a finitely generated maximal order and $U(A)=\{1\}$. If $S=\{(a,\phi(a)) \mid a\in A\}\subseteq A\rtimes G$ is a monoid of IG-type and the action of $G=\{\phi(a)\mid a\in A\}$ is faithful, then*

$$S\cong \bigl(D(A)^{+}\rtimes G\bigr)\cap SS^{-1},$$

where $D(A)$ is the divisor class group of A and G is a subgroup of the permutation group of the minimal primes of A.

Proof. As A is a Krull order, we know that $D(A)$ is a free abelian group with the set $\mathrm{Spec}^0(A)$ consisting of the minimal primes of A as a free generating set. Of course, for each $a\in A$, $\phi(a)$ induces an automorphism on $\mathrm{Spec}^0(A)$, and thus also on $D(A)$. We denote this again by $\phi(a)$. It follows that, if I is an ideal of S, then $\phi(a)(I^*)=(\phi(a)(I))^*$. We thus obtain a morphism $G\to \mathrm{Sym}(\mathrm{Spec}^0(A))$. This mapping is injective. Indeed, suppose $\phi(a)$ is the identity map on $\mathrm{Spec}^0(A)$, with $a\in A$. For $c\in A$ the ideal Ac is divisorial. hence $A\phi(a)(c)=\phi(a)(Ac)=Ac$. Since, by assumption $U(A)=\{1\}$ it follows that $\phi(a)(c)=c$. Hence it follows that $\phi(a)=1$. Because, also by assumption, the action of G on A is faithful, it follows that $\phi(a)$ is the identity map on G, as desired.

Again, because $U(A)=\{1\}$, we get a monoid morphism

$$S\to D(A)^{+}\rtimes G:\bigl(a,\phi(a)\bigr)\mapsto\bigl(aA,\phi(a)\bigr).$$

So, identifying S with its image in $D(A)^{+}\rtimes G$ (and also SS^{-1} with its image in $\{(a^{-1}bA, \bar{\phi}(a^{-1}b))\mid a,b\in A\}\subseteq D(A)\rtimes G$), we get that

$$S\subseteq \bigl(D(A)^{+}\rtimes G\bigr)\cap SS^{-1}.$$

Conversely, if $b\in AA^{-1}$ and $(bA,\bar{\phi}(b))\in D(A)^{+}\rtimes G$, then $bA\subseteq A$ and thus $b\in A$. Hence $(bA,\bar{\phi}(b))\in S$. □

Note that this characterization is a non-commutative version of result of Chouinard that describes commutative cancellative semigroups that are Noetherian maximal orders or more generally Krull orders [4].

To investigate the torsion-freeness, we need the following theorem. The authors would like to thank Karel Dekimpe for the proof of this result [5].

Theorem 3.2. *Let H be a group of affine transformations such that $H\cap\mathbb{R}^n$ (the subgroup of pure translations) is of finite index in H. Then the following properties are equivalent.*

1. *H is torsion-free.*
2. *The action of H on $\mathbb{R}^n$ is fixed-point free, that is, if $g\cdot a=a$ for some $a\in\mathbb{R}^n$ and $g\in H$, then $g=1$.*

Proof. Suppose that the action of H on $\mathbb{R}^n$ has a fixed-point. Let therefore $h \neq 1$ and $x \in R^n$ be such that $h \cdot x = x$. Then, also $h^k \cdot x = x$ for every $k \in \mathbb{Z}$. Because $H \cap \mathbb{R}^n$ is of finite index in H there exists a $k\rangle 0$ such that h^k is a pure translation. But as this translation has a fixed-point it follows that h^k should be trivial. Therefore, H has torsion.

Conversely, suppose that H is a finite subgroup of the affine transformations in dimension n. So every element h of H is of the form (t_h, M_h), with t_h the translation part and M_h the linear part. Take $h_1, h_2 \in H$ then:

$$(t_{h_1h_2}, M_{h_1h_2}) = (t_{h_1} + M_{h_1}t_{h_2}, M_{h_1}M_{h_2}).$$

Therefore the map

$$\phi : H \to GL(n, \mathbb{R}) : h \mapsto M_h,$$

is a group morphism. Hence $\mathbb{R}^n$ is an H-module. With this module structure the map $t : H \to \mathbb{R}^n$ becomes a 1-cocycle. Because H is finite, we have that $H^1(H, \mathbb{R}^n) = 0$ and therefore the map t is a 1-coboundary [2]. Consequently there exists a $x \in \mathbb{R}^n$ with $t_h = x - M_h x$ for every $h \in H$ and therefore we have a fixed-point. □

Note that, if SS^{-1} is torsion-free, then so is necessarily AA^{-1}. Indeed, if $a^m = 1$ in AA^{-1}, then, by Lemma 2.3, $(a\phi(a)(a)\phi(a)^2(a) \cdots \phi(a)^{k-1}(a), 1)^m = 1 \in SS^{-1}$, for some divisor k of $|G|$.

Assume now that $S = \{(a, \phi(a)) \mid a \in A\}$ is a monoid of IG-type with faithful action of $G = \{\phi(a) \mid a \in A\}$ on A. Suppose that AA^{-1} is a torsion-free finitely generated abelian group. So, $SS^{-1} \subseteq \mathbb{Z}^k \rtimes G$ and $G \subseteq \mathrm{Aut}(\mathbb{Z}^k) \cong GL_k(\mathbb{Z})$. Hence, every element of G can be seen as a $k \times k$-matrix with values in $\mathbb{Z}$ and the action of SS^{-1} on $\mathbb{Z}^k$ can be extended to $\mathbb{R}^k$ and can be written as:

$$\big(a, \phi(a)\big) \cdot b = \phi(a)b + a,$$

where $a \in \mathbb{Z}^k$, $\phi(a) \in GL_k(\mathbb{Z})$, $b \in \mathbb{Z}^k$ (or $\mathbb{R}^k$) and $\phi(a)b$ is given by the classical matrix multiplication. For convenience sake we use the additive notation on $\mathbb{Z}^k$ and $\mathbb{R}^k$ (instead of the multiplicative on AA^{-1}). So, $SS^{-1} \subseteq \mathbb{R}^k \rtimes GL_k(\mathbb{R})$. Thus, SS^{-1} is a group of affine transformations and every element of SS^{-1} is of the form $(a, \phi(a))$, where a is the translation part and $\phi(a)$ the linear part. Clearly, $(a, A)(b, B) = (a + Ab, AB)$. As G is a finite group, we also have that the subgroup of pure translations is of finite index in SS^{-1}.

As an immediate consequence of Theorem 3.2, we get that the quotient group $SS^{-1} \subseteq AA^{-1} \rtimes G$ is torsion-free if and only if the action of SS^{-1} on $\mathbb{R}^n$ is fixed-point free.

If also $U(A) = \{1\}$ and A is a maximal order in its free abelian group of quotients then, by Proposition 3.1, we can extend the action of SS^{-1} to an action of the semidirect product $D(A) \rtimes G$ and G acts as the symmetric group on the set $\mathrm{Spec}^0(A) = \{P_1, \ldots, P_l\}$. Hence the action can also naturally be extended to an action of the semidirect product $\mathbb{R}^l \rtimes G$.

Theorem 3.3. *Let A be an abelian cancellative monoid. Assume A is a finitely generated maximal order with $U(A) = \{1\}$ and AA^{-1} is torsion-free. If $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ is a monoid of IG-type with faithful action of $G = \{\phi(a) \mid a \in A\}$ on A, then an element $(a, \overline{\phi}(a))$ of SS^{-1} is periodic if and only if there exists a divisorial ideal I of A such that $a\overline{\phi}(a)(I) = I$.*

Proof. Suppose $(a, \bar{\phi}(a))$ is a periodic element. So, because of Proposition 3.1 and the proof of Theorem 3.2, $(aA, \bar{\phi}(a)) \in D(A) \rtimes \mathrm{Sym}_l$ has a fixed point b in $\mathbb{R}^l$. Write $aA = P_1^{\alpha_1} * \cdots * P_l^{\alpha_l}$, with $\alpha_i \in \mathbb{Z}^l$ and $\mathrm{Spec}^0(A) = \{P_1, \ldots, P_l\}$. So $\bar{\phi}(a) \cdot b + \alpha = b$, where $\alpha = (\alpha_1, \ldots, \alpha_l)$. Since $\bar{\phi}(a)$ acts as a permutation on the components of $b \in \mathbb{R}^l$, it is not so difficult to see that $\bar{\phi}(a) \cdot \lfloor b \rfloor + \alpha = \lfloor b \rfloor$, where $\lfloor b \rfloor$ is the integral part of b. Hence we have a fixed point in $\mathbb{Z}^l$. This means that $a\bar{\phi}(a)(I) = I$, where $I = P_1^{\beta_1} * \cdots * P_l^{\beta_l}$, with $(\beta_1, \ldots, \beta_l) = \lfloor b \rfloor$.

Conversely, suppose there exists a divisorial ideal I of A such that $a\bar{\phi}(a)(I) = I$. It follows that

$$a\bar{\phi}(a)(a)\bar{\phi}(a)^2(a)\cdots\bar{\phi}(a)^n(a)\bar{\phi}(a)^{n+1}(I) = I.$$

So if $\bar{\phi}(a)^{n+1} = 1$, then we obtain that $a\bar{\phi}(a)(a)\cdots\bar{\phi}(a)^n(a)A = A$. Again because $U(A) = 1$, it follows that $a\bar{\phi}(a)(a)\cdots\bar{\phi}(a)^n(a) = 1$. Consequently, $(a, \bar{\phi}(a))^{n+1} = 1$, as desired. □

We now give concrete examples of monoids of IG-type that are not of I-type. The first one is based on an example of an abelian finitely generated monoid that is considered by Anderson in [1].

Example 3.4. Let $A = \langle u_1, u_2, u_3, u_4 \mid u_i u_j = u_j u_i,\ u_1u_2 = u_3u_4 \rangle$ and let $||: A \to \mathbb{Z}$ denote the degree function on A defined by $|u_i| = 1$. Put

$$\sigma = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \in Gl_3(\mathbb{Z}).$$

The natural action of $\mathbb{Z}_2 = \langle \sigma \rangle$ on $\mathbb{Z}^3 = AA^{-1} = \mathrm{gr}(u_1, u_2, u_3)$ defines a semidirect product $A \rtimes \mathbb{Z}_2$. Then

$$S = \{(a, \phi(a)) \mid a \in A,\ \phi(a) = 1 \text{ if } |a| \in 2\mathbb{Z},\ \phi(a) = \sigma \text{ if } |a| \in 2\mathbb{Z}+1\} \subseteq A \rtimes \mathbb{Z}_2$$

is a monoid of IG-type (which is not of I-type) and its group of quotients

$$SS^{-1} = \{(a, \phi(a)) \mid a \in \mathbb{Z}^3,\ \phi(a) = 1 \text{ if } |a| \in 2\mathbb{Z},\ \phi(a) = \sigma \text{ if } = |a| \in 2\mathbb{Z}+1\}$$

is torsion-free.

Proof. Because $A = AA^{-1} \cap F^+$, the intersection of the group of quotients AA^{-1} and the positive cone of a free abelian group, we know that A is a maximal order (see [1,4]). Clearly $U(A) = \{1\}$ and A has four minimal primes: $Q_1 = (u_1, u_3)$, $Q_2 = (u_1, u_4)$, $Q_3 = (u_2, u_3)$ and $Q_4 = (u_2, u_4)$. So, these minimal primes generate the free abelian group $D(A) \cong \mathbb{Z}^4$. Because of Theorem 3.3, to prove that SS^{-1} is torsion-free we need to show that if $(a, \phi(a)) \in SS^{-1}$ such that $a\phi(a)(I) = I$ for some divisorial ideal I of A then $a = 1$. Clearly, if $|a| \in 2\mathbb{Z}$ then $\phi(a) = \{1\}$, and thus $aI = I$ implies $a = 1$. So, suppose a has odd degree. Then $a = u_1^{a_1}u_2^{a_2}u_3^{a_3}$ or $a = u_1^{a_1}u_2^{a_2}u_4^{a_4}$. We deal with the former case (the other case is dealt with similarly). It is readily verified that $Au_1 = Q_1 * Q_2$, $Au_2 = Q_3 * Q_4$ and $Au_3 = Q_1 * Q_3$. Write

$I = Q_1^{\gamma_1} * Q_2^{\gamma_2} * Q_3^{\gamma_3} * Q_4^{\gamma_4}$, with each $\gamma_i \in \mathbb{Z}$. Because σ interchanges Q_1 with Q_4 and Q_2 with Q_3, the equality $Aa * \phi(a)(I) = (Aa\phi(a)(I))^* = I$ becomes

$$Q_1^{a_1+a_3} * Q_2^{a_1} * Q_3^{a_2+a_3} * Q_4^{a_2} * Q_4^{\gamma_1} * Q_3^{\gamma_2} * Q_2^{\gamma_3} * Q_1^{\gamma_4} = Q_1^{\gamma_1} * Q_2^{\gamma_2} * Q_3^{\gamma_3} * Q_4^{\gamma_4}.$$

It follows that $a_1 + a_2 + a_3 = 0$, in contradiction with the fact that a is of odd degree.

Note that, as AA^{-1} has torsion-free rank 3, while A, and therefore also S, has 4 indecomposable elements, it follows from the remark at the end of Section 2 that S is not of I-type. $\square$

A second type of examples of monoids of IG-type that are not of I-type can be constructed as a natural class of submonoids of a monoid of I-type. In [13, Section 4] an example in this class is given to show that there exists monoids T of I-type with a group of fractions TT^{-1} that is not poly-infinite cyclic.

Example 3.5. Let $T = \{(a, \phi(a)) \mid a \in \mathrm{FaM}_n\} \subseteq \mathrm{FaM}_n \rtimes G$ be a monoid of I-type with $H = \{\phi(a) \mid a \in A\}$. Suppose B is a G-invariant submonoid of FaM_n. Then, $S = \{(b, \phi(b)) \mid b \in B\}$ is a monoid of IG-type. Note again that if $B = BB^{-1} \cap \mathrm{FaM}_n$ then we know from [4] that B is a maximal order. Clearly, $U(B) = \{1\}$.

We give a concrete example. Let $T = \langle x_1, x_2, x_3, x_4\rangle$ be the monoid defined by the relations $x_1x_2 = x_3x_3$, $x_2x_1 = x_4x_4$, $x_1x_3 = x_2x_4$, $x_1x_4 = x_4x_2$, $x_2x_3 = x_3x_1$, $x_3x_2 = x_4x_1$. We know that T is a monoid of I-type (see [13, Section 4]) and thus, by Theorem 1.1, $T = \{(a, \phi(a)) \mid a \in \mathrm{FaM}_n\}$ for some map $\phi : \mathrm{FaM}_n \to \mathrm{Sym}_n$. Put $\mathrm{FaM}_4 = \langle u_1, u_2, u_3, u_4\rangle$, $\phi(u_i) = \sigma_i$ and $x_i = (u_i, \sigma_i)$. The defining relations allow us to discover the action on FaM_4. For example, $x_1x_2 = (u_1, \sigma_1)(u_2, \sigma_2) = (u_1\sigma_1(u_2), \sigma_1\sigma_2)$ and $x_3x_3 = (u_3, \sigma_3)(u_3, \sigma_3) = (u_3\sigma_3(u_3), \sigma_3\sigma_3)$. Since $x_1x_2 = x_3x_3$ we get that $\sigma_1(u_2) = u_3$ and $\sigma_3(u_3) = u_1$. Going through all the defining relations we obtain that

$$\sigma_1 = (23), \qquad \sigma_2 = (14), \qquad \sigma_3 = (1243), \qquad \sigma_4 = (1342).$$

Clearly, $G = \{\phi(a) \mid a \in \mathrm{FaM}_4\} \cong D_8$, the dihedral group of order 8. Let $B = \langle u_i^3, u_i^2u_j, u_iu_ju_k \mid 1 \leqslant i \neq j \neq k \leqslant 4\rangle$. Then B has a group of quotients $BB^{-1} = \mathrm{gr}(u_1^3, u_1u_4^{-1}, u_2u_4^{-1}, u_3u_4^{-1}) = \{a \in \mathrm{Fa}_4 \mid |a| \in 3\mathbb{Z}\}$, where $|a|$ denotes the natural (total) degree of a. Clearly, B is G-invariant. Hence, $S = \{(b, \phi(b)) \mid b \in B\} \subseteq B \rtimes G$ is a monoid of IG-type. Note that G is now considered as a subgroup of $\mathrm{Aut}(B)$. Because BB^{-1} has torsion-free rank 4 and since B (and thus S) has 20 indecomposable elements, it follows that S is not of I-type.

We now give an example of a monoid S of IG-type so that SS^{-1} has non-trivial periodic elements. On the other hand, SS^{-1} does not contain non-trivial finite normal subgroups and thus $K[SS^{-1}]$ (and $K[S]$) are prime algebras (see for example [17,18]).

Example 3.6. Let $A = \langle u_1, u_2, u_3, u_4 \mid u_1u_2 = u_3u_4\rangle$ be the maximal order as in Example 3.4. Let $D_8 = \langle a, b \mid a^4 = 1,\ b^2 = 1,\ a^3b = ba\rangle$, with $a = (1324)$ and $b = (12)$, the dihedral group of order 8. So D_8 acts naturally on A. In the semidirect product $A \rtimes D_8$ consider the elements $x_i = (u_i, \sigma_i)$, where $\sigma_1 = (1324)$, $\sigma_2 = (12)$, $\sigma_3 = (1423)$ and $\sigma_4 = (34)$ and let $S = \langle x_1, x_2, x_3, x_4\rangle$. Then S is a monoid of IG-type. Furthermore, SS^{-1} has non-trivial periodic elements but SS^{-1} does not have non-trivial finite normal subgroups. So $K[S]$ is a prime ring.

Proof. It is easily verified that $S = \{(a, \phi(a)) \mid a \in A\}$ and thus S is a monoid of IG-type, with $G = \{\phi(a) \mid a \in A\} = D_8$. So, $SS^{-1} \subseteq AA^{-1} \rtimes D_8$. Clearly,

$$(u_3, \sigma_3)\big(\sigma_1^{-1}\big(u_1^{-1}\big), \sigma_1^{-1}\big) = \big(u_3u_2^{-1}, (12)(34)\big),$$

and, as $u_4 = u_1u_2u_3^{-1}$ in AA^{-1}, we have that $(u_3u_2^{-1}, (12)(34))^2 = 1$. So SS^{-1} has non-trivial periodic elements.

We claim now that SS^{-1} does not contain a finite normal subgroup, or equivalently, $K[SS^{-1}]$ is prime. Indeed, since AA^{-1} is torsion-free, it is readily verified that finite normal subgroups N of SS^{-1} must be such that their natural projection onto $G = D_8$ are contained in $\mathrm{gr}(a)$. Furthermore, it then follows that N contains a finite normal subgroup of G that is of order 2. So N contains a central element of order 2. But central elements in SS^{-1} are of the form $(u_1^i u_2^i, 1)$, so they are not periodic. This proves the claim. $\square$

We finish this section by showing that the infinite dihedral group D_∞ is a group of IG-type. It also gives an example with non-trivial torsion.

Example 3.7. Let σ be the non-trivial isomorphism of the infinite cyclic group $\mathbb{Z}$. So $\sigma(1) = -1$. Then $H = \{(a, \phi(a)) \mid a \in \mathbb{Z},\ \phi(a) = \sigma \text{ if } a \in 2\mathbb{Z}+1,\ \phi(a) = 1 \text{ if } a \in 2\mathbb{Z}\}$ is a group of IG-type and $H \cong D_\infty$.

Proof. Put $a = (2, 1)$ and $b = (1, \sigma)$. Then, $H = \mathrm{gr}(a, b)$ and as $bab^{-1} = a^{-1}, b^2 = 1$ we have that $H \cong D_\infty$. $\square$

In [3] it is shown that the group algebra $K[D_\infty]$ is not a maximal order and that this algebra is the key in characterizing when a group algebra of a polycyclic-by-finite group is a prime maximal order.

4. Prime ideals and maximal orders

Throughout this section $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ is a monoid of IG-type, with A a finitely generated abelian cancellative monoid, $G = \{\phi(a) \mid a \in A\}$ a finite group and SS^{-1} is torsion-free. For an ideal I of A, we put $(I, \phi(I)) = \{(a, \phi(a)) \mid a \in I\}$. Note that this is a right ideal of S. By $\mathrm{Spec}(S)$ we denote the set of all prime ideals of S. Recall that the height of $Q \in \mathrm{Spec}(S)$ is, by definition, the largest non-negative integer n, so that S has a chain of primes $Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q$. We denote this height by $\mathrm{ht}(Q)$.

We first describe the prime ideals of S. For this we will make use of the next theorem (Theorem 1.4 in [14]). It is worth mentioning (as is already done in [14]) that, since SS^{-1} is a localization of S with respect to an Ore set of regular elements of Noetherian ring $K[S]$, the prime ideals of the group algebra $K[SS^{-1}]$ are in a one-to-one correspondence with the prime ideals P of $K[S]$ that do not intersect S (see for example [11, Theorems 9.22, 9.20 and Lemma 9.21]). Since prime ideals of group algebras of polycyclic-by-finite groups have been well studied through the work of Rosablade (see [18,19]) we thus get a lot of information on all prime ideals of $K[S]$.

Proposition 4.1. *Let S be a submonoid of a torsion-free finitely generated abelian-by-finite group G and let K be a field.*

1. *If P is a prime ideal in S, then $K[P]$ is a prime ideal in $K[S]$.*
2. *If Q is a prime ideal in $K[S]$ with $Q \cap S \neq \emptyset$, then $K[Q \cap S]$ is a prime ideal in $K[S]$.*
3. *The height one prime ideals of $K[S]$ intersecting S are of the form $K[P]$, where P is a minimal prime ideal of S.*

Theorem 4.2. *Let $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ be a monoid of IG-type and suppose SS^{-1} is torsion-free. The prime ideals P of S of height m are the sets $(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$ so that*

1. *each Q_i is a prime ideal of A of height m,*
2. *$a\phi(a)(Q_1 \cap \cdots \cap Q_n) \subseteq Q_1 \cap \cdots \cap Q_n$, for every $a \in A$ (that is, $(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$ is an ideal of S),*
3. *condition* (2) *is not satisfied for an intersection over a proper subset of $\{Q_1, \ldots, Q_n\}$ (that is, $(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$ is a maximal set satisfying conditions* (1) *and* (2)).

Proof. Let P be a prime ideal of S and let K be a field. Because of Proposition 4.1, $K[P]$ is a prime ideal of $K[S]$. Let $A^k = \{a^k \mid a \in A\}$, where k is a divisor of the order of the group G such that $\phi(a^k) = 1$, for every $a \in A$ (see Lemma 2.3). We identify the group $A^k A^{-k}$ with its natural image in SS^{-1}. The algebra $K[S]$ has a natural gradation by the finite group $SS^{-1}/A^k A^{-k}$. The homogeneous component of degree e (the identity of $A^k A^{-k}$) is the semigroup algebra $K[A^k]$. So, by Theorem 17.9 in [19],

$$K[P] \cap K\left[A^k\right] = P_1 \cap \cdots \cap P_n,$$

an intersection of primes P_i of $K[A^k]$, each of the same height as $K[P]$ (these are all the primes of $K[A^k]$ minimal over $K[P \cap A^k]$). Clearly,

$$P \cap A^k = \bigcap_{i=1}^{n} \left(P_i \cap A^k\right),$$

each $P_i \cap A^k$ is a prime ideal of A^k and thus $K[P_i \cap A^k]$ is a prime ideal of $K[A^k]$. From Proposition 4.1 and Theorem 17.9 in [19] we verified that $\mathrm{ht}(P) = \mathrm{ht}(P_i \cap A^k)$. As every P_i is minimal over $K[P \cap A^k]$ and because $K[P_i \cap A^k]$ also is a prime over $K[P \cap A^k]$ and it is contained in P_i it follows that $K[P_i \cap A^k] = P_i$. So

$$P \cap A^k = Q_1^{(k)} \cap \cdots \cap Q_n^{(k)},$$

with $Q_i^{(k)} = P_i \cap A^k$.

We also make another remark. Let Q be a prime ideal of A, then $Q^{(k)} = \{q^k \mid q \in Q\} \subseteq Q \cap A^k$ and $Q^{(k)}$ is a prime ideal of A^k. Furthermore $Q \cap A^k$ is a nil ideal modulo $Q^{(k)}$. Since A^k is commutative it follows that $Q \cap A^k \subseteq Q^{(k)}$. Hence $Q \cap A^k = Q^{(k)}$. So we have a bijection between the primes of Q and $Q^{(k)}$ (and corresponding primes have the same height).

If $(a, \phi(a)) \in P$ then

$$\left(a, \phi(a)\right)\left(\phi(a)^{-1}(a), \phi\left(\phi(a)^{-1}(a)\right)\right) \cdots = \left(a^k, 1\right) \in P \cap A^k.$$

Hence $a^k \in Q_1^{(k)} \cap \cdots \cap Q_n^{(k)}$, and thus $a \in Q_1 \cap \cdots \cap Q_n$. Therefore $P \subseteq (Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$. Conversely, if $(b, \phi(b)) \in (Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$, then

$$\left(\left(b, \phi(b)\right)^k\right)^k \in \left(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n)\right) \cap A^k \subseteq \left(\bigcap_{i=1}^{n}\left(Q_i \cap A^k\right), 1\right) \subseteq P.$$

So $(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$ is a right ideal of S that is nil modulo P. Since S/P satisfies the ascending chain condition on one sided ideals, it follows that $(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n)) \subseteq P$ (see for example 17.22 in [8]). Hence $P = (Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$.

Since P is a left ideal we also have that $a\phi(a)(Q_1 \cap \cdots \cap Q_n) \subseteq Q_1 \cap \cdots \cap Q_n$.

Next we show that if $P_1 = (Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$ and $P_2 = (Q'_1 \cap \cdots \cap Q'_m, \phi(Q'_1 \cap \cdots \cap Q'_m))$ are different prime ideals (of the same height) of S then $\{Q_1, \ldots, Q_n\} \cap \{Q'_1, \ldots, Q'_m\} = \emptyset$. Indeed, suppose the contrary, then, without loss of generality, we may assume that $Q_1 = Q'_1$. As $P_1 \neq P_2$, and because they are of the same height, we thus get that say $n > 1$ and $m > 1$.

Clearly $(Q_2 \cap \cdots \cap Q_n, \phi(Q_2 \cap \cdots \cap Q_n))$ is a right ideal of S and

$$\begin{aligned}
&\left(Q_2 \cap \cdots \cap Q_n, \phi(Q_2 \cap \cdots \cap Q_n)\right)P_2 \\
&\quad \subseteq \left\{\left(a\phi(a)\left(Q'_1 \cap \cdots \cap Q'_m\right), \phi\left(a\phi(a)\left(Q'_1 \cap \cdots \cap Q'_m\right)\right)\right) \bigm| a \in Q_2 \cap \cdots \cap Q_n\right\} \\
&\quad \subseteq \left(\left(Q'_1 \cap \cdots \cap Q'_m\right) \cap (Q_2 \cap \cdots \cap Q_n), \phi\left(\left(Q'_1 \cap \cdots \cap Q'_m\right) \cap (Q_2 \cap \cdots \cap Q_n)\right)\right) \\
&\quad \subseteq P_1.
\end{aligned}$$

But as $\mathrm{ht}(Q_1) = \mathrm{ht}(Q_2) = \cdots = \mathrm{ht}(Q_n)$ and the primes $Q_1, \ldots, Q_n$ are distinct it follows that

$$Q_2 \cap \cdots \cap Q_n \nsubseteq Q_1 \cap \cdots \cap Q_n.$$

As P_1 is prime we thus get that $P_2 \subseteq P_1$. But since they are of the same height, it follows that $P_1 = P_2$, a contradiction. The above claim of course implies the minimality as stated in the theorem.

To end the proof, we need to show that ideals $(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$ with the listed properties are prime ideals of S. We know that $Q_1^{(k)} = Q_1 \cap A^k$ is a prime ideal of A^k of the same height as Q_1. Also $K[Q_1^{(k)}]$ is a prime ideal of $K[A^k]$. Again using graded techniques and [19, 17.9] we know that there exists a prime ideal P of $K[S]$ that lies over $K[Q_1^k]$. So $K[Q_1^k]$ is a minimal prime over $P \cap K[A^k]$ and $P \cap K[A^k] = K[Q_1^k] \cap X_2 \cap \cdots \cap X_m$, where $X_2, \ldots, X_m$ are minimal primes over $P \cap K[A^k]$ and they are of the same height as $K[Q_1^k]$. Clearly $P \cap S$ is a prime ideal of S and $(P \cap S) \cap A^k = Q_1^{(k)} \cap (X_2 \cap A^k) \cap \cdots \cap (X_m \cap A^k)$. Hence, by the first part of the proof, $P_1 = P \cap S = (Q_1 \cap Q'_2 \cap \cdots \cap Q'_m, \phi(Q_1 \cap Q'_2 \cap \cdots \cap Q'_m))$ with $(Q_i^{(k)})' = (X_2 \cap A^k)$.

We can now do the same for $Q_2, \ldots, Q_n$. Hence, we get primes $P_2, \ldots, P_n$ of S so that $P_2 = (Q_2 \cap J_2, \phi(Q_2 \cap J_2)), \ldots, P_n = (Q_n \cap J_n, \phi(Q_n \cap J_n))$, with each J_i an intersection of primes of A that are of the same height as Q_i. Furthermore, because of the assumptions we get that

$$\left(J_2, \phi(J_2)\right)\left(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n)\right) \subseteq (J_2 \cap Q_1 \cap \cdots \cap Q_n) \subseteq P_2.$$

Since $(J_2, \phi(J_2)) \nsubseteq P_2$ this yields that $(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n)) \subseteq P_2 = (Q_2 \cap J_2, \phi(Q_2 \cap J_2))$. Since $\{Q_1, \ldots, Q_n\}$ satisfies the minimality condition as stated in the theorem, we obtain that $Q_2 \cap J_2 = Q_1 \cap \cdots \cap Q_n$. Thus $P_1 = P_2 = \cdots = P_n = (Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$ and thus this is a prime ideal of S. □

The following is an immediate consequence from the previous result (and its proof).

Corollary 4.3. *If $L = \{Q_1, \ldots, Q_n\}$ is a full G-orbit of primes of the same height in A, then there exists a partition $\{X_1, \ldots, X_n\}$ of L, so that*

$$\left(\bigcap_{Q \in X_i} Q, \phi\left(\bigcap_{Q \in X_i} Q\right)\right) = P_i$$

are prime ideals of S.

We now can prove the main result. It provides a characterization of semigroup algebras $K[S]$ of monoids of IG-type that are a maximal order.

Theorem 4.4. *Let $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ be a monoid of IG-type. Suppose that SS^{-1} is torsion-free and suppose that the abelian monoid A is finitely generated and a maximal order. Then, the Noetherian PI-domain $K[S]$ is a maximal order if and only if the minimal primes of S are of the form*

$$P = \big(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n)\big),$$

where $\{Q_1 \cdots Q_n\} = \{\phi(a)(Q_1) \mid a \in A\} \subseteq \operatorname{Spec}^0(A)$.

Proof. Because of the assumption, the results in the first section show that $K[S]$ is a Noetherian domain that satisfies a polynomial identity. In particular, S satisfies the ascending chain condition on one sided ideals.

We first prove the sufficiency of the mentioned condition. So, suppose that the minimal primes of S are of the from $(Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$, where $\{Q_1 \cdots Q_n\} = \{\phi(a)(Q_1) \mid a \in A\} \subseteq \operatorname{Spec}^0(A)$. To prove that $K[S]$ is a maximal order, it is sufficient to verify conditions (2) and (3) of Theorem 1.2. The former says that S is a maximal order. Because of Lemma 4.4 in [14], in order to prove this property, it is sufficient to show that $(P :_l P) = (P :_r P) = S$ for every prime ideal P of S. From Theorem 4.2 we know that $P = (Q, \phi(Q))$ with Q an intersection of prime ideals in A of the same height, say n. Assume $(x, \phi(x)) \in (P :_l P)$. Then $x\phi(x)(Q) \subseteq Q$. If $n \neq 0$ (so Q is an intersection of primes that are not minimal) then the divisorial closure of both Q and $\phi(x)(Q)$ equals A. As $x(\phi(x)(Q))^* \subseteq Q^*$ we thus get that $x \in A$ and thus $(x, \phi(x)) \in S$. If $n = 0$, then, by assumption, Q is G-invariant and thus we get that $xQ \subseteq Q$. Since A is a maximal order, this yields that $x \in A$ and again $(x, \phi(x)) \in S$. So $(P :_l P) = S$. On the other hand, suppose $(Q, \phi(Q))(x, \phi(x)) \subseteq (Q, \phi(Q))$. Then $A(Q \cap A^k)x \subseteq Q$. If $n \neq 0$ then $Q \cap A^k$ is not contained in a minimal prime ideal of A and thus the divisorial closure of both $A(Q \cap A^k)$ and Q is A. Since $(A(Q \cap A^k))^* x \subseteq Q^*$ we get that $(x, \phi(x)) \in S$. So it remains to show that $(P :_r P) = S$ for a minimal prime ideal P. Hence, by assumption $P = (Q, \phi(Q))$ with Q a G-invariant ideal. More generally, we prove that $(I :_r I) = S$ for any ideal $I = (M, \phi(M))$ of S with M a G-invariant ideal of A.

We now prove this by contradiction. So suppose that I is such an ideal of S with $Ig \subseteq I$ for some $g \in SS^{-1} \setminus S$.

Now, as $g \in SS^{-1}$, we know that $g = (a, \phi(a))(z, 1)^{-1}$ with z an invariant element of A, and thus $(z, 1)$ central in S. As A is a maximal order, we have that the minimal primes of A freely generate the abelian group $D(A)$. So, in the divisor group $D(A)$, we can write Az as a product of minimal primes. Because Az is invariant, the minimal primes in a G-orbit have the same exponent. Hence,

$$Az = \left(J_1^{n_1}\right)^* * \cdots * \left(J_l^{n_l}\right)^*,$$

where each J_i is an intersection of all minimal primes of A in a G-orbit. So, because of the assumption and Theorem 4.2, each $(J_i, \phi(J_i))$ is a minimal prime of S. Of course also Aa is a divisorial product of minimal primes of A. If necessary, canceling some common factors of Aa and Az, we may assume that $Aaz^{-1} = K * L^{-1} \nsubseteq A$, and thus $KL^{-1} \nsubseteq A$ with $L = (J_1^{n_1})^* * \cdots * (J_l^{n_l})^*$, $L^{-1} = (A : L)$ and K is not contained in J_i, for every i with $1 \leqslant i \leqslant l$. Note that, also, L^{-1} is G-invariant and thus $(L^{-1}, \phi(L^{-1}))$ is a fractional ideal of S. Of course, $I(K, \phi(K))(L^{-1}, \phi(L^{-1})) \subseteq I$. Because S satisfies the ascending chain condition on ideals, we can choose I maximal with respect to the property that such K and L exist with $KL^{-1} \nsubseteq A$.

Clearly we obtain that

$$\begin{aligned} I\big(K, \phi(K)\big)\big(L^{-1}, \phi\big(L^{-1}\big)\big)\big(L, \phi(L)\big) &\subseteq I\big(L, \phi(L)\big) \subseteq S\big(L, \phi(L)\big) \\ &\subseteq \big(J_i^{n_i}, \phi\big(J_i^{n_i}\big)\big) \\ &\subseteq \big(J_i, \phi(J_i)\big). \end{aligned}$$

Since, $(J_i, \phi(J_i))$ is a prime ideal of S, we get that either $(KL^{-1}L, \phi(KL^{-1}L)) \subseteq (J_i, \phi(J_i))$ or $I \subseteq (J_i, \phi(J_i))$. Because of the above, the former is excluded. Hence $I \subseteq (J_i, \phi(J_i))$. As J_i is G-invariant, we get again that $(J_i^{-1}, \phi(J_i^{-1}))$ is a fractional ideal of S that contains S. Therefore, we get that $I \subseteq (J_i^{-1}, \phi(J_i^{-1}))I$ is an ideal of S. Since

$$\big(J_i^{-1}, \phi\big(J_i^{-1}\big)\big)I\big(K, \phi(K)\big)\big(L^{-1}, \phi\big(L^{-1}\big)\big) \subseteq \big(J_i^{-1}, \phi\big(J_i^{-1}\big)\big)I,$$

the maximality condition on I thus implies that

$$\big(J_i^{-1}, \phi\big(J_i^{-1}\big)\big)I = I.$$

Since M is G-invariant this yields that $J_i^{-1}M = M$ and thus $J_i^{-1} * M^* = M^*$. So $J_i^{-1} = A$, a contradiction.

We now show that, if P is a minimal prime ideal of S then the monoid S_P has only one minimal prime. As A^k is G-invariant, Lemma 2.3 in [14] gives that $S_P = S_{(P)}$ with

$$\begin{aligned} S_{(P)} = \big\{g \in SS^{-1} \bigm| Cg \subseteq S,\ C \nsubseteq P,\ C \subseteq A^k, \\ \text{for some conjugacy class } C \text{ of } SS^{-1}\big\}. \end{aligned}$$

Again, by assumption, $P = (Q_1 \cap \cdots \cap Q_n, \phi(Q_1 \cap \cdots \cap Q_n))$, where $\{Q_1, \ldots, Q_n\}$ is a full G-orbit of minimal primes of A. First we prove that

$$S_{(P)} = \big(A_{Q_1} \cap \cdots \cap A_{Q_n}, \phi(A_{Q_1} \cap \cdots \cap A_{Q_n})\big). \tag{4}$$

So suppose that $(b, \phi(b)) \in S_{(P)}$. Then there exists a conjugacy class C of SS^{-1} in A^k with

$$C\big(b, \phi(b)\big) \subseteq S$$

and C not contained in P. But as C is contained in A^k it is easily verified that $C = \{(\phi(a)(c^k), 1) \mid \phi(a) \in G\}$, for some $c \in A$. Since $C \nsubseteq P$ and because $\{Q_1, \ldots, Q_n\}$ is a G-orbit, this yields that $C \nsubseteq Q_i$, for every $i \in \{1, \ldots, n\}$. Hence, it follows that $b \in A_{Q_1} \cap \cdots \cap A_{Q_n}$. Conversely, suppose that $b \in A_{Q_1} \cap \cdots \cap A_{Q_n}$. Then there exist $c_i \in A \setminus Q_i$ with $c_i^k b \in A$ for every $i \in \{1, \ldots, n\}$. As $M = A^k c_1^k \cup \cdots \cup A^k c_n^k \nsubseteq Q_1^{(k)} \cap \cdots \cap Q_n^{(k)}$, it follows that, in $D(A^k)$, M^* is a product of minimal primes that do not belong to $\{Q_1^{(k)}, \ldots, Q_n^{(k)}\}$. So, $N = \prod_{g \in G} g(M^*)$ is an invariant ideal of A^k and $N \nsubseteq Q_1 \cap \cdots \cap Q_n$. Clearly, $NbA \subseteq A$. Choose $d \in N \setminus P$. Then $C' = \{\phi(a)(d) \mid a \in A\}$ is a SS^{-1}-conjugacy class contained in A^k, but not in P. Since, $C'(b, \phi(b)) \subseteq S$, we get that $(b, \phi(b)) \in S_{(P)}$, as desired. This finishes the proof of (4).

The monoid $B = A_{Q_1} \cap \cdots \cap A_{Q_n}$ is a maximal order with minimal prime ideals $P_i = A_{Q_1} \cap \cdots \cap Q_i A_{Q_i} \cap \cdots \cap A_{Q_n}$, $1 \leqslant i \leqslant n$. From Lemma 2.2 in [14] we know that $I(P) = \{(x, \phi(x)) \in S_P \mid (x, \phi(x))C \subseteq P,\ \text{for some } G\text{-conjugacy class } C \subseteq S \text{ with } C \nsubseteq P\}$ is a prime ideal of S_P. It is easily seen that $I(P) = \{(x, \phi(x)) \in S_P \mid (x, \phi(x))C \subseteq P,\ \text{for some } G\text{-conjugacy class } C \subseteq A^k \text{ with } C \nsubseteq P\}$. From (4) it then follows that $I(P) = (BQ_1 \cap \cdots \cap BQ_n, \phi(BQ_1 \cap \cdots \cap BQ_n))$. Therefore, Theorem 4.2 implies that this is the only minimal prime ideal of S_P. This finishes the proof of the sufficiency of the conditions.

To prove the necessity, assume $K[S]$ is a maximal order. Let $P = (M, \phi(M))$ be a minimal prime ideal of S. Theorem 1.2 yields that S_P has a unique minimal prime. Furthermore, since A^k is G-invariant, Lemma 2.5 in [14] yields that $(M, \phi(M)) \cap A^k = M \cap A^k$ is G-invariant. Consequently, M is G-invariant and Theorem 4.2 yields that M is the intersection of a full G-orbit of minimal primes. This finishes the proof. □

Remark 4.5. Let $S = \{(a, \phi(a)) \mid a \in A\} \subseteq A \rtimes G$ be a monoid of IG-type. Suppose that the abelian monoid A is finitely generated and a maximal order. If the minimal primes of the monoid S are as stated in the sufficient condition of Theorem 4.4 then S is a maximal order. (The assumption SS^{-1} is torsion-free, is not needed in the proof of this part of the result.)

As an application of Theorem 4.4 we give two examples.

Example 4.6. Let S be the monoid of IG-type considered in Example 3.4. The semigroup S is a maximal order and $P_1 = (Q_1 \cap Q_4, \phi(Q_1 \cap Q_4))$ and $P_2 = (Q_2 \cap Q_3, \phi(Q_2 \cap Q_3))$ are its minimal prime ideals. Furthermore, $K[S]$ is a maximal order for any field K.

Proof. From Example 3.4 (and its proof) we know that A is a finitely generated maximal order with four minimal primes: $Q_1 = (u_1, u_3)$, $Q_2 = (u_1, u_4)$, $Q_3 = (u_2, u_3)$ and $Q_4 = (u_2, u_4)$. Because SS^{-1} is torsion-free, Theorem 4.2 yields a description of the prime ideals of S. Clearly, $a\phi(a)(P_i) \subseteq P_i$, for $i \in \{1, 2\}$. Hence because of Theorem 4.4, to prove that P_1 and P_2 are the only minimal primes of S, it is now sufficient to note that for every Q_i, there exists an $a \in A$ such that $a\phi(a)(Q_i) \nsubseteq Q_i$. Indeed, $u_2(12)(34)Q_1 \nsubseteq Q_1$, $u_2(12)(34)Q_2 \nsubseteq Q_2$,, $u_4(12)(34)Q_3 \nsubseteq Q_3$, and $u_3(12)(34)Q_4 \nsubseteq Q_4$. □

Example 4.7. Let S be the monoid of IG-type defined in Example 3.5. Then $K[S]$ is a maximal order for any field K.

Proof. It is readily verified that the minimal primes of S are of the form as required in Theorem 4.4. $\square$

We finish this paper with an example of a monoid of IG-type that is not a maximal order.

Example 4.8. Let $A = \langle u_1, u_2, u_3, u_4 \mid u_1u_2u_3 = u_4^2\rangle$. Then A is a maximal order (in its torsion-free group of quotients) with minimal prime ideals $Q_1 = (u_1, u_4)$, $Q_2 = (u_2, u_4)$ and $Q_3 = (u_3, u_4)$. Let $S = \{(a, \phi(a)) \mid a \in A\}$, with $\phi(a) = 1$ if $a \in A$ has even degree in u_4, otherwise, $\phi(a) = (12)$ (the transposition interchanging u_1 with u_2). Then, $S \subseteq A \rtimes \mathbb{Z}_2$ is a monoid of IG-type which is not a maximal order and the group of quotients SS^{-1} is torsion-free. Thus, $K[S]$ is not a maximal order for any field K. The minimal prime ideals of S are $P_i = (Q_i, \phi(Q_i))$ with $1 \leqslant i \leqslant 3$.

Proof. Clearly, $AA^{-1} = \mathrm{gr}(u_1, u_2, u_4)$ is a free abelian group of rank 4. Furthermore, every element of AA^{-1} has a unique presentation of the form $u_1^i u_2^j u_3^k u_4^m$, with $m \in \{0, 1\}$ and $i, j, k \in \mathbb{Z}$; elements of A are those with i, j, k non-negative. It is easily seen that Q_1, Q_2 and Q_3 are the minimal primes of A and the localizations of A with respect to these prime ideals are $A_{Q_1} = A\langle u_2^{-1}, u_3^{-1}\rangle$, $A_{Q_2} = A\langle u_1^{-1}, u_3^{-1}\rangle$ and $A_{Q_3} = A\langle u_1^{-1}, u_2^{-1}\rangle$. Furthermore, $A_{Q_1} \cap A_{Q_2} \cap A_{Q_3} = A$ and each A_{Q_i} is a maximal order with unique minimal prime ideal $u_4A_{Q_i}$. It follows that A is a maximal order. As $\mathbb{Z}_2 = \mathrm{gr}((12))$ induces a faithful action on the finitely generated monoid $A = \langle u_1, u_2, u_3, u_4 \mid u_1u_2u_3 = u_4^2\rangle$, we thus get that $S \subseteq A \rtimes \mathbb{Z}_2$ is a monoid of IG-type.

Suppose that there exists a non-trivial periodic element in the group of quotients SS^{-1}. Such an element must be of the form $(u_1^{\alpha_1} u_2^{\alpha_2} u_4^{\alpha_4}, (12))$, with $\alpha_i \in \mathbb{Z}$ and α_4 odd. Then, by Theorem 3.2 (see also the remarks stated after its proof), $(\alpha_1, \alpha_2, \alpha_4) + (12)(\alpha_1', \alpha_2', \alpha_4') = (\alpha_1', \alpha_2', \alpha_4')$, for some $\alpha_1', \alpha_2', \alpha_4' \in \mathbb{R}$. Hence $\alpha_4 + \alpha_4' = \alpha_4'$ and thus $\alpha_4 = 0$, a contradiction. So, SS^{-1} indeed is torsion-free.

Let I be the ideal generated by $((u_1, 1), (u_4, (12)))$. Then $(u_1u_3u_4^{-1}, (12))I \subseteq I$ and thus S is not a maximal order.

As an immediate consequence of Theorem 4.4 one sees that $P_1 = (Q_1, \phi(Q_1))$, $P_2 = (Q_2, \phi(Q_2))$ and $P_3 = (Q_3, \phi(Q_3))$ are the minimal prime ideals of S. Clearly Q_1 and Q_2 are not $\mathbb{Z}_2$-invariant, as corresponds with Remark 4.5. $\square$

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