



The complex Lorentzian Leech lattice and the bimonster

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Abstract

We find 26 reflections in the automorphism group of the Lorentzian Leech lattice L over $\mathbb{Z}[e^{2\pi i/3}]$ that form the Coxeter diagram seen in the presentation of the bimonster. We prove that these 26 reflections generate the automorphism group of L . We find evidence that these reflections behave like the simple roots and the vector fixed by the diagram automorphisms behaves like the Weyl vector for the reflection group.

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1. Introduction

Let $\omega = e^{2\pi i/3}$ and $\mathcal{E} = \mathbb{Z}[\omega]$ be the ring of Eisenstein integers. Let Λ be the Leech lattice considered as a twelve-dimensional negative definite complex Hermitian lattice over \mathcal{E} . In this paper we study the automorphism group of the Lorentzian complex Leech lattice $L = \Lambda \oplus H$, where H is the lattice $\mathcal{E} \oplus \mathcal{E}$ with gram matrix $\begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix}$ and $\theta = \omega - \bar{\omega}$. We call H the hyperbolic cell. L is a Hermitian Lorentzian lattice over \mathcal{E} ; the underlying integer lattice is $II_{2,26}$. A complex reflection in a lattice vector $r \in L$ of negative norm, fixes r^\perp and multiplies r by a cube root of unity. The vector r is called a *root* of the reflection. The subgroup $R(L)$ of $\text{Aut}(L)$ generated by these complex reflections is the reflection group of L . Modulo multiplication by scalars $R(L)$ is a complex hyperbolic reflection group acting faithfully on 13-dimensional complex hyperbolic space $\mathbb{C}H^{13}$. This and many other complex hyperbolic reflection groups were studied

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by Allcock in [1] and [2]. The group $R(L)$ was shown to be of finite index in the automorphism group of L in [1]. As a consequence the reflection group is arithmetic, and hence is finitely presented.

A detailed study of a specific object like the reflection group of L may call for some explanation. There are two themes explored in this article that suggests that the reflection group of L is interesting.

(1) We find numerical evidence that suggests a link between the reflection group of L and the group known as the bimonster. Let M be the monster, the largest sporadic simple group and $M \wr 2$ denote the wreath product of M with $\mathbb{Z}/2\mathbb{Z}$. Conway, Norton and Soicher [8] conjectured two surprisingly simple presentations of the bimonster $M \wr 2$ which were proved by Ivanov [13], and Conway and Simons [9]. The bimonster is presented on the set of 16 generators satisfying the Coxeter relations of the diagram M_{666} (see Fig. 2) and on 26 generators satisfying the Coxeter relations of the incidence graph of the projective plane over \mathbb{F}_3 (Fig. 3), respectively, with some extra relations. (The M_{666} diagram is also called Y_{555} in literature because the graph looks like the letter Y with 5 nodes on each hand. The name M_{666} was suggested to me by Prof. Conway and Prof. McKay, because it is consistent with the names M_{333} , M_{244} and M_{236} for the affine Dynkin diagrams of E_6 , E_7 and E_8 . The triples $\{3, 3, 3\}$, $\{2, 4, 4\}$ and $\{2, 3, 6\}$ are the only integer solutions to $1/p + 1/q + 1/r = 1$.)

The basic observation, that led us to an investigation of the generators of the automorphism group of L is the following:

One can find 16 reflections in the complex reflection group of L and then extend them to a set of 26 reflections, such that they form exactly the same Coxeter diagrams appearing in the presentation of the bimonster mentioned above.

(However $\text{Aut}(L)$ is an infinite group and the vertices of the diagram for $\text{Aut}(L)$ have order 3 instead of 2 as in the usual Coxeter groups.) Let D be the “Coxeter diagram” of the 26 reflections: it is a graph on 26 vertices (also called nodes) which correspond to reflections of order 3. Two of these reflections a and b either braid ($aba = bab$) or commute ($ab = ba$). The vertices corresponding to a and b are joined if and only if the reflections braid. A lot of the calculations done here are aimed towards proving the following (see 5.7)

1.1. Theorem. *The 16 reflections of the M_{666} diagram generate the automorphism group of L . So the four groups, namely, the group R_1 generated by the 16 reflections of the diagram M_{666} , the group R_2 generated by the 26 reflections of the diagram D , the reflection group $R(L)$ and the automorphism group $\text{Aut}(L)$ are all equal.*

Though the 16 reflections of the M_{666} diagram are enough to generate the automorphism group, we shall see that the 26 node diagram D play the key role in the proofs. The group $PGL_3(\mathbb{F}_3)$ of diagram automorphism act on both $\text{Aut}(L)$ and the bimonster. Analogs of the extra relations needed for the presentation of the bimonster are also available in the group $\text{Aut}(L)$.

(2) The other theme that we pursue in this article is an analogy with the theory of Dynkin diagrams of Weyl groups. The incidence graph of the projective plane over \mathbb{F}_3 emerges as sort of a “Coxeter–Dynkin diagram” for the reflection group of L .

The proof of the theorem above follows the analogy with Weyl groups. The proof proceeds as follows. Using the work done in [1] one can write down a set of 50 reflections that generate

the reflection group of L . The job is to show that these reflections are in the group generated by the reflections in the 26 roots of D . For this purpose we find an analog of Weyl vector $\bar{\rho}$ fixed by the group of diagram automorphisms, and use it to define the height of a root r by the formula $\text{ht}(r) = |\langle r, \bar{\rho} \rangle|/|\bar{\rho}|^2$. The 26 roots of D then become the roots of minimal height, that is, they are the analogs of “simple roots.” Then one runs a “height reduction algorithm” on the set of 50 generators found before to show that they are in the group generated by the 26 nodes of D . The algorithm tries to reduce the height of a vector by reflecting it in the simple roots and repeats the process until it reaches one of the simple roots. (The algorithm gets stuck once in a while and then one has to perturb the vector somehow.) The codes for the computer calculations done are available at <http://www.math.berkeley.edu/~tathagat>.

The analogy with Dynkin diagram is further substantiated by the following (see Proposition 6.1)

1.2. Proposition. *All the roots of L have height greater than or equal to one and the 26 nodes of D are the only ones with minimal height. This amounts to the statement that the mirrors of the 26 reflections of D are the only ones that are closest to the Weyl vector $\bar{\rho}$.*

We found a second example of the phenomenon described above for the quaternionic Lorentzian Leech lattice that is discussed in [4]. Here the ring \mathcal{E} is replaced by the noncommutative ring of quaternions of the form $(a + bi + cj + dk)/2$ where a, b, c, d are integers congruent modulo 2; the diagram turns out to be the incidence graph of projective plane over \mathbb{F}_2 with 14 vertices (where each vertex represent a quaternionic reflection of order 4) which is obtained by extending an M_{444} diagram and we have exactly parallel results.

I would like to remark upon the element of surprise that is contained in the above results. Suppose we are given the roots of L forming the M_{666} diagram and want to find the other 10 roots to get the 26 node diagram. Finding these roots amount to solving a system of highly overdetermined system of linear and quadratic equations over integers. This system of equations happens to be consistent. This is one of the many coincidences which suggest that the reflection group of L and the diagram D are worth exploring in detail. One should also emphasize that the proof of the theorem and the proposition mentioned above go through because the numbers work out nicely in our favor. For example, by the above definition, the squared height is a positive element of $\mathbb{Z}[\sqrt{3}]$, so it is not even a priori obvious that all the roots have height greater than or equal to one. What is more intriguing is that all the above remarks hold true for the example of the quaternionic lattice too (see [4]).

A few words of caution: though we use the terminology from the theory of Weyl groups like “simple roots” and “Weyl vector,” how far the analogy with Dynkin diagram goes is not clearly understood. Some of the facts that are true for Weyl groups are certainly not true here. For example, one cannot reduce the height of the root always by a single reflection in one of the 26 roots of D . But experimentally it seems that one almost always can.

The connection between the bimonster and the automorphism group of L observed here is still based on numerical observations. It is possible though unlikely that this is just a coincidence. We do not yet know if the Coxeter relations coming from the diagram D along with the extra relators we found so far, following the analogy with the bimonster suffice to give a presentation of the automorphism group of L .

The sections are organized as follows. In Section 2 we set up the notation and sketch the steps of a computer calculation to find an explicit isomorphism from $3E_8 \oplus H$ to $\Lambda \oplus H$ over \mathcal{E} , that is used later. Section 3 describes the diagram D in detail and a new construction of L starting

with D . Section 4 contains some preliminary results about the reflection group of L ; in particular we give an elementary proof to show that the reflection group of L acts transitively on the roots. Sections 5 and 6 contain the main results of the paper mentioned above. The calculations done here bear resemblance with the theory of Weyl groups. In Section 7 we mention some relations satisfied by the generators of $\text{Aut}(L)$ that are analogous to those in the bimonster and state a conjecture by Daniel Allcock about a possible relation between the reflection group of L and the bimonster. Some numerical details are given in Appendix A.

2. Notations and basic definitions

2.1. Notation

$\text{Aut}(L)$	automorphism group of a lattice L ;
$\mathbb{C}H^n$	the complex hyperbolic space of dimension n ;
D	the 26 node diagram isomorphic to the incidence graph of $P^2(\mathbb{F}_3)$;
\mathcal{E}	the ring $\mathbb{Z}[e^{2\pi i/3}]$ of Eisenstein integers;
\mathbb{F}_q	finite field of cardinality q ;
$\text{ht}(x)$	height of a vector x given by $\text{ht}(x) = \langle \bar{\rho}, x \rangle / \bar{\rho} ^2$;
G	$G \cong PGL_3(\mathbb{F}_3)$ is the automorphism group of $P^2(\mathbb{F}_3)$;
H	the hyperbolic cell over Eisenstein integers with Gram matrix $\begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix}$;
$II_{m,n}$	the even unimodular integer lattice of signature (m, n) ;
l_i	the 13 lines of $P^2(\mathbb{F}_3)$ or the roots corresponding to them;
L	the Lorentzian Leech lattice over Eisenstein integers: $L = \Lambda \oplus H$;
L'	the dual lattice of L ;
Λ	the Leech lattice as a negative definite Hermitian lattice over Eisenstein integers;
ϕ_x	ω -reflection in the vector x ;
Φ	a root system (Φ_L is the set of roots of L);
Q	the finite group of automorphisms generated by G and σ acting on L having a unique fixed point in the complex hyperbolic space $\mathbb{C}H^{13}$ given by $\bar{\rho}$;
r	a root;
R_1	the group generated by reflections in the 26 roots of D ;
$R(L)$	reflection group of the lattice L ;
$\text{rad}(D)$	the set of roots obtained by repeatedly reflecting the roots of D in each other;
ρ_i	$(\rho_1, \dots, \rho_{26}) = (p_1, \dots, p_{13}, \xi l_1, \dots, \xi l_{13})$;
$\bar{\rho}$	the Weyl vector (also the average of the ρ_i 's);
σ	an automorphism of L of order 12 that interchanges the lines and points;
θ	$\sqrt{-3}$;
$T_{\lambda,z}$	a translation, an element of the Heisenberg group;
\mathbb{T}	Heisenberg group: automorphisms fixing $\rho = (0^{12}; 0, 1)$ and acting trivially on ρ^\perp/ρ ;
ω	$e^{2\pi i/3}$;
\wr	the wreath product;
x_i	the 13 points of $P^2(\mathbb{F}_3)$ or the roots corresponding to them;
ξ	$e^{-\pi i/6}$.

2.2. Basic definitions about complex lattices

The following definitions are mostly taken from [1, Section 2]. They are included here for completeness. Let $\omega = e^{2\pi i/3}$. Let $\mathcal{E} = \mathbb{Z}[\omega]$ be the ring of Eisenstein integers. A lattice K over

\mathcal{E} is a free module over \mathcal{E} with an \mathcal{E} valued Hermitian form $\langle \cdot, \cdot \rangle$, that is conjugate linear in the first variable and linear in the second. Let $V = K \otimes_{\mathcal{E}} \mathbb{C}$ be the complex inner product space underlying K . All lattices considered below are lattices over \mathcal{E} with nonsingular Hermitian form unless otherwise stated.

For a complex number z , we let $\text{Re}(z) = (z + \bar{z})/2$ be the real part and $\text{Im}(z) = z - \text{Re}(z)$. Let $[\cdot, \cdot]$ be the real valued alternating form given by $[a, b] = (1/\theta) \text{Im}\langle a, b \rangle$ where $\theta = \sqrt{-3}$. The *real form* or \mathbb{Z} form of a complex lattice is the underlying \mathbb{Z} -module with the Euclidean inner product $\text{Tr} \circ \langle \cdot, \cdot \rangle$, that is, twice the real part of the Hermitian form.

A *Gram matrix* of a lattice is the matrix of inner products between a set of basis vectors of the lattice. The *signature* of the lattice K is (m, n) if the Gram matrix of a basis of K has m positive eigenvalues and n negative eigenvalues. Equivalently, m is the dimension of a maximal positive definite subspace of the underlying vector space V and n is the dimension of a maximal negative definite subspace of V . A lattice of rank n is called *Lorentzian* (respectively *negative definite*) if it has signature $(1, n - 1)$ (respectively $(0, n)$). The *dual* K' of a lattice K is defined as $K' = \{v \in V : \langle v, y \rangle \in \mathcal{E} \forall y \in K\}$. Many of the lattices we consider satisfy $L \subseteq \theta L'$. So all the inner products between lattice vectors are divisible by θ . A vector in the lattice K is called *primitive* if it is not of the form αy with $y \in K$ and α a nonunit of \mathcal{E} . The *norm* of a vector v is $|v|^2 = \langle v, v \rangle$.

A *complex reflection* in a vector r is an isometry of the vector space V that fixes r^\perp and takes r to μr , where $\mu \neq 1$ is a root of unity in \mathcal{E} . It is given by the formula

$$\phi_r^\mu(v) = v - r(1 - \mu)\langle r, v \rangle / |r|^2. \tag{1}$$

ϕ_r^μ is called a μ -reflection in the vector r . For any automorphism γ of V we have $\gamma \phi_r^\mu \gamma^{-1} = \phi_{\gamma r}^\mu$. Let $\phi_r := \phi_r^\omega$. By a *root* of a negative definite or Lorentzian lattice K , we mean a primitive lattice vector of negative norm such that a nontrivial reflection in it is an automorphism of K . The *reflection group* $R(K)$ of K is the subgroup of the automorphism group $\text{Aut}(K)$ generated by reflections in roots of K . The lattice E_8 , Λ and L appearing below all satisfy $\theta K' = K$. The complex Leech lattice Λ has no roots. For E_8 and L , the roots are the lattice vectors of norm -3 . The ω -reflections of order 3 in these roots generate their reflection group.

The following notation is used for the lattices K with $K \subseteq \theta K'$. Say that two roots a and b are *adjacent* if $|\langle a, b \rangle| = \sqrt{3}$. By a direct calculation this amounts to requiring that the ω -reflections in a and b braid, that is, $\phi_a \phi_b \phi_a = \phi_b \phi_a \phi_b$. Let Φ be a set of roots; the same symbol also often denotes the graph whose set of vertices is Φ and two vertices are joined if the corresponding roots are adjacent. In such situations roots are often considered only up to units of \mathcal{E} . We call the set of roots *connected* if the corresponding graph is connected. Suppose we have a set of roots Φ such that the inner product of any two distinct roots in that set is zero, $-\omega\theta$ or its conjugate. In such situations we consider a directed graph on the vertex set Φ . We draw an arrow from a vertex a to a vertex b if $\langle b, a \rangle = -\omega\theta$. Both these kind of graphs are called the *root diagrams*. Which kind is being considered will be clear from whether the graph in question is directed or undirected.

Let Φ be a set of roots of L . (Here and in what follows, we often do not distinguish between two roots that differ by an unit of \mathcal{E} . This does not cause any problem because ω -reflection in two such roots are the same.) Define $\Phi^{(n)} = \{\phi_a(b) : a, b \in \Phi^{(n-1)}\}$, and $\Phi_{(n)} = \{\phi_{a_1}^\pm \dots \phi_{a_k}^\pm(b) : a_i, b \in \Phi, k \leq n\}$ where $\Phi^{(0)} = \Phi$. It can be easily seen that $\bigcup_n \Phi^{(n)} = \bigcup_m \Phi_{(m)}$. We call this set the *radical* of Φ and denote it by $\text{rad}(\Phi)$. The radical of Φ consists of

the set of roots obtained from Φ , by repeated reflection in themselves. We make the following elementary observations: if $\Phi \subseteq \Phi'$ then $\text{rad}(\Phi) \subseteq \text{rad}(\Phi')$ and $\text{rad}(\text{rad}(\Phi)) = \text{rad}(\Phi)$.

2.3. Lemma. *If Φ is connected then $\text{rad}(\Phi)$ is also connected.*

Proof. It suffices to show that each $\Phi^{(n)}$ is connected. We use induction on n . Let $c \in \Phi^{(n+1)}$. There are two elements a and b in $\Phi^{(n)}$ such that $c = \phi_a(b)$, and a connected chain, $a = a_0, a_1, \dots, a_n = b$. Then $a, \phi_a(b_1), \dots, \phi_a(b_n) = c$ is a connected chain from a to c in $\Phi^{(n+1)}$. \square

2.4. *The complex Leech lattice*

Below we set up some conventions about the complex lattices that are important for our purpose. A general reference for lattices is the book [10] of Conway and Sloane; we just state the definitions that we need from the book.

The complex Leech lattice Λ consists of the set of vectors in \mathcal{E}^{12} of the form

$$\left\{ (m + \theta c_i + 3z_i)_{i=1, \dots, 12}: m = 0 \text{ or } \pm 1, (c_i) \in \mathcal{C}_{12}, \sum z_i \equiv m \pmod{\theta} \right\}, \tag{2}$$

where \mathcal{C}_{12} is the ternary Golay code in \mathbb{F}_3^{12} . The codes we need are defined explicitly in Appendix A.1. The Hermitian inner product $\langle u, v \rangle = -\frac{1}{3} \sum \bar{u}_i v_i$ is normalized so that the minimal norm of a nonzero lattice vector is -6 . The automorphism group of the complex Leech lattice is $6 \cdot \text{Suz}$, a central extension of Suzuki’s sporadic simple group Suz by $\mathbb{Z}/6\mathbb{Z}$. The set Λ as a \mathbb{Z} -module with the inner product $-\frac{2}{3} \text{Re}\langle, \rangle$ is the usual real Leech lattice with minimal norm 4. The definition of the complex Leech lattice in (2) is similar to the description of the real Leech lattice as the collection of vectors in \mathbb{Z}^{24} of the form

$$\left\{ (m + 2c_i + 4z_i)_{i=1, \dots, 24}: m = 0 \text{ or } 1, (c_i) \in \mathcal{C}_{24}, \sum z_i \equiv m \pmod{2} \right\},$$

where \mathcal{C}_{24} is the binary Golay code in \mathbb{F}_2^{24} .

2.5. *The complex E_8 lattice and the hyperbolic cell*

The root lattice E_8 can also be considered as a negative definite Hermitian lattice over \mathcal{E} . It can be defined as the subset of \mathcal{E}^4 consisting of vectors v such that the image of v modulo θ belong to the ternary tetracode \mathcal{C}_4 in $\mathbb{F}_3^4 \simeq (\mathcal{E}/\theta\mathcal{E})^4$. The inner product $\langle u, v \rangle = -\sum \bar{u}_i v_i$ is normalized to make the minimal norm -3 . The reflection group of this lattice is $6 \cdot \text{PSP}_4(3)$ and is equal to the automorphism group. This group has a minimal set of generators consisting of four ω -reflections, and the corresponding roots form the usual A_4 Dynkin diagram with vertices c, d, e, f . (See Theorem 5.2 in [1].) This will be called the E_8 root diagram or simply, the E_8 diagram. This group is called G_{32} in the list of complex diagrams given on page 185 of [5]. (*Remember:* the vertices of this “complex diagram” correspond to reflections of order 3.)

We can find a unique root b' that we call the “lowest root” such that we have $b' + (2 + \omega)c + 2d + (2 + \omega)e + f = 0$. Then b', c, d, e, f form the affine E_8 diagram (see Fig. 1), that looks like usual A_5 Dynkin diagram (but the vertices correspond to reflections of order 3 instead of 2).

Let \bar{E}_8 be the singular lattice $E_8 \oplus \mathcal{E}_0$, \mathcal{E}_0 being the one-dimensional \mathcal{E} -lattice with zero inner product. Consider E_8 sitting in \bar{E}_8 . Then exactly as for Euclidean root systems one can show that

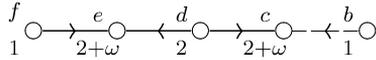


Fig. 1. The affine E_8 diagram with the balanced numbering.

the reflections in the roots $b = b' + (0^4, 1)$, c, d, e, f generate the affine reflection group of \bar{E}_8 . All this is very reminiscent of the theory of Euclidean reflection groups.

Let H be the Lorentzian lattice of signature $(1, 1)$, given by $\mathcal{E} \oplus \mathcal{E}$ with inner product $\langle u, v \rangle = u^* \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} v$, where u^* denotes the conjugate transpose of u . We call H the hyperbolic cell.

The L denote the Lorentzian \mathcal{E} -lattice $\Lambda \oplus H$. The minimum norm of a nonzero vector of L is -3 , $L = \theta L'$ and the discriminant of L is 3^7 . Elements of L are often written in the form $(\lambda; \alpha, \beta)$, where λ is a vector with 12 co-ordinates. $\lambda = (0, \dots, 0)$ is abbreviated as $\lambda = 0^{12}$.

As a \mathbb{Z} -lattice, with the inner product $\frac{2}{3} \text{Re} \langle \cdot, \cdot \rangle$, both L and $3E_8 \oplus H$ are even, unimodular and of signature $(2, 26)$. Hence they are both isomorphic to $II_{2,26}$. The following argument, that I learnt from D. Allcock, show that they are isomorphic over \mathcal{E} too. We only briefly sketch the argument, because we find an explicit isomorphism later anyway.

2.6. Lemma. *There is at most one indefinite Eisenstein lattice T in a given signature satisfying $T = \theta T'$.*

Proof. Suppose T is an Eisenstein lattice with $T = \theta T'$. Then the Hermitian inner product defines a symplectic form on the \mathbb{F}_3 vector space T'/T . If we choose a Lagrangian or maximal isotropic subspace of T'/T and take the corresponding enlargement of T under pull-back, then we get a unimodular Eisenstein lattice. Therefore we can get all lattices T with $T = \theta T'$ by starting with a unimodular lattice and reversing the process. In the indefinite case there is only one unimodular lattice denoted by $\mathcal{E}^{1,n}$. (The underlying \mathbb{Z} -lattice has a norm zero vector and hence so does T . From here on one can copy the proof of uniqueness of the indefinite odd unimodular lattice of given signature over \mathbb{Z} , given in Chapter 5 of Serre’s Course in Arithmetic.) So there is no choice about where to start. $\mathcal{E}^{1,n}/\theta\mathcal{E}^{1,n}$ is a vector space over \mathbb{F}_3 , with a bilinear form on it; this bilinear form has isotropic subspace of rank half the dimension of the vector space. Reversing the process above amounts to passing to the sub-lattice of $\mathcal{E}^{1,n}$ corresponding to a maximal isotropic subspace of $\mathcal{E}^{1,n}/\theta\mathcal{E}^{1,n}$. Now, $\text{Aut}(\mathcal{E}^{1,n})$ acts on $\mathcal{E}^{1,n}/\theta\mathcal{E}^{1,n}$ as the full orthogonal group of the form (just take the reflections of $\mathcal{E}^{1,n}/\theta\mathcal{E}^{1,n}$ given by reducing reflections of $\mathcal{E}^{1,n}$). Since this orthogonal group acts transitively on the maximal isotropic subspace, there is essentially only one way to construct T , so T is unique. \square

The above proof is nonconstructive. But we need an explicit isomorphism between $\Lambda \oplus H$ and $3E_8 \oplus H$ to perform calculations involving generators of the automorphism group of the lattice; because the first set of generators we are able to write down are in the coordinate system $\Lambda \oplus H$ (and this uses special properties of the Leech lattice), while the roots of D that are of interest to us are naturally given in the other coordinate system $3E_8 \oplus H$. We give below the steps for this computation.

2.7. Calculation

What we need from the following computation are the bases of $\Lambda \oplus H$ from which we get the change of basis matrix. These are given in Appendix A.2 (see the matrices E_1 and E_2). The computer codes for these computation are contained in **isom.gp** and **isom2.gp**.

- (a) We start by making a list of all the vectors in the first shell of the Eisenstein Leech lattice using coordinates given in Wilson’s paper [14] (except that our coordinates for the ternary Golay code are different).
- (b) We find a regular 23 simplex, all whose 24 vertices are on the first shell of Λ . In other words, we find a set Δ of 24 minimal norm vectors in Λ such that the difference of any two distinct elements of Δ is also a vector of minimal norm.
- (c) We list the 4 tuples of vectors $\delta_i, i = 1, \dots, 4$, inside Δ such that there are 4 roots of L of the form $(\delta_i; 1, *)$ making an E_8 diagram. This amounts to checking a set of linear conditions among the entries of the matrix $([\delta_i, \delta_j])_{i,j=1}^4$: the imaginary parts of the inner products between the elements of Δ . (See the calculation in Lemma 4.1.) Each 4-tuple of vectors generates a sub-lattice of L isomorphic to E_8 .
- (d) In this list, we look for two E_8 lattices that are orthogonal to each other. This step also amount to checking a set of linear conditions among the entries of the matrix $([\delta_i, \delta_j])_{i,j=1}^4$.
- (e) Next, choose one such subspace $K_1 \cong E_8 \oplus E_8$. If z is a primitive norm zero element perpendicular to K_1 then it is of type $3E_8$, because z^\perp/z is a Niemeier lattice containing the root system $E_8 \oplus E_8$ and $3E_8$ is the only such.
- (f) We list all the vectors in the first shell of the Leech lattice such that they are at minimal distance from the 8 distinguished vectors forming the diagram of $K_1 \cong E_8 \oplus E_8$. We find that there are 8 such vectors. It is easy to find an E_8 diagram amongst these eight. So we get 12 vectors forming $3E_8$ diagram inside the Lorentzian Leech lattice L in coordinates given in Wilson’s article [14].
- (g) Now calculate the orthogonal complement of the $3E_8$ inside L , which is a hyperbolic cell H . It remains to find two norm zero vector inside H , that have inner product θ . This gives an explicit isomorphism from $\Lambda \oplus H$ to $3E_8 \oplus H$.

2.8. The complex hyperbolic space

We recall some basic facts about the complex hyperbolic space.

Let $\mathbb{C}^{1,n}$ be the $(n + 1)$ -dimensional complex vector space with inner product $\langle u, v \rangle = -\bar{u}_1 v_1 - \dots - \bar{u}_n v_n + \bar{u}_{n+1} v_{n+1}$. The underlying inner product space of L is isomorphic to $\mathbb{C}^{1,13}$. The complex hyperbolic space $\mathbb{C}H^n$ is defined to be the set of complex lines of positive norm in $\mathbb{C}^{1,n}$. A vector in $\mathbb{C}^{1,n}$ and the point it determines in $\mathbb{C}H^n$ are denoted by the same symbol. Similar convention is adopted for hyperplanes. For two vectors u and v with nonzero norm, let

$$c(u, v)^2 = |\langle u, v \rangle|^2 / |u|^2 |v|^2. \tag{3}$$

The distance $d(u, u')$ between two points u and u' in $\mathbb{C}H^n$ is given by the formula (see [11])

$$\cosh(d(u, u'))^2 = c(u, u')^2. \tag{4}$$

A vector v of negative norm determines a totally geodesic hyperplane v^\perp in $\mathbb{C}H^n$. If r is a root of L , r^\perp or its image in $\mathbb{C}H^{13}$ is called the mirror of a reflection in r .

If u is a point in $\mathbb{C}H^n$ and v is a vector of negative norm in $\mathbb{C}^{1,n}$ then the distance between u and the hyperplane v^\perp is given by

$$\sinh^2(d(u, v^\perp)) = -c(u, v)^2. \tag{5}$$

We sketch a proof of (5) because we could not find a reference. Scale v so that $|v|^2 = -1$. Then $a = u + v\langle v, u \rangle$ is the intersection of the line joining u and v and the hyperplane v^\perp in $\mathbb{C}H^n$. We claim that a minimizes the distance from u to v^\perp . Then (5) follows from (4).

Proof of the claim. We note that $|a|^2 = \langle a, u \rangle = |u|^2 + |\langle u, v \rangle|^2$. Let a' be any point in $\mathbb{C}H^n$ lying on v^\perp . We need to minimize $|\langle u, a' \rangle|^2 / |a'|^2$. There is a basis of v^\perp consisting of vectors a, w_2, \dots, w_n where the span of the w_i 's is negative definite. So we can write $a' = a + w$ with w a vector of negative norm in v^\perp . We find that $|a'|^2 = |a|^2 + 2\operatorname{Re}\langle u, w \rangle + |w|^2$ and

$$\begin{aligned} \frac{|\langle u, a' \rangle|^2}{|a'|^2} &= \frac{(|a|^2 + \langle u, w \rangle)^2}{|a'|^2} = \frac{|a|^4 + 2|a|^2 \operatorname{Re}\langle u, w \rangle + |\langle u, w \rangle|^2}{|a'|^2} \\ &= |a|^2 + \frac{|\langle u, w \rangle|^2 - |w|^2|a|^2}{|a'|^2}. \end{aligned}$$

The second term in the last expression is positive since $|a'|^2 > 0$ and $|w|^2 < 0$. So it is minimized when $w = 0$, that is $a' = a$.

The distance between two hyperplanes v^\perp and v'^\perp is zero if $c(v, v')^2 \leq 1$ and otherwise is given by

$$\cosh^2(d(v^\perp, v'^\perp)) = c(v, v')^2. \tag{6}$$

The reflection group of L acts by isometries on the complex hyperbolic space $\mathbb{C}H^{13}$.

3. The root diagram of L

3.1. The diagram of 26 roots

As above, let L be the direct sum of the Leech lattice and hyperbolic cell. In view of the isomorphism in Lemma 2.6 this is the same as $3E_8 \oplus H$. In this section we use this later coordinate system. There are 16 roots of L called $a, b_i, c_i, d_i, e_i, f_i, i = 1, 2, 3$, that form the M_{666} diagram (see Fig. 2).

The three hands correspond to the three copies of E_8 . The reflections in the roots c_i, d_i, e_i, f_i generate the reflection group of the three copies of E_8 . Along with b_i they form the “affine E_8 diagrams.” The remaining node a is called the “hyperbolizing node.” The reflection ϕ_a fixes $3E_8$ and acts on a hyperbolic cell.

The M_{666} diagram can be extended to a diagram D of 26 roots. Given the 16 roots of the M_{666} diagram there is only one way to find the remaining roots because their inner product with the 16 roots are specified, and the 16 roots of the M_{666} diagram contain a basis of the underlying vector space. D is the incidence graph of the projective plane $P^2(\mathbb{F}_3)$ (see Fig. 3). $P^2(\mathbb{F}_3)$ has 13 points and 13 lines. There is one vertex in D for each point of $P^2(\mathbb{F}_3)$ and one for each line of $P^2(\mathbb{F}_3)$. Two vertices are joined if and only if the point is on the line.

We use the following shorthand borrowed from [9], to draw the 26 node diagram (see Fig. 3). The index i runs from 1 to 3. A single vertex labeled a_i stands for three vertices a_1, a_2 and a_3 . A single edge between a_i and b_i means that a_i and b_j are connected only if the indices i and j are equal. A double edge between a_i and f_i mean that a_i and f_j are connected if and only if the indices i and j are not equal. Note that the double edges in this diagram does not have their usual

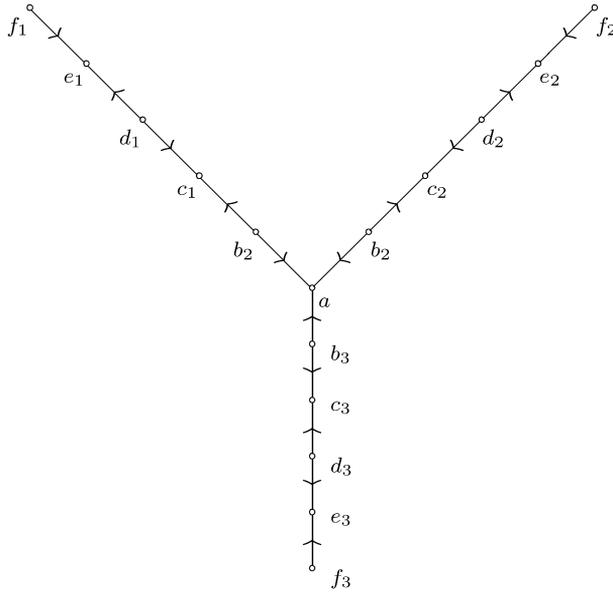


Fig. 2. The M_{666} diagram.

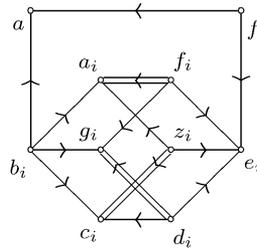


Fig. 3. The 26 node diagram $D \cong \text{Inc}(P^2(\mathbb{F}_3))$.

meaning in Dynkin diagrams. Both these root diagrams are directed, where the arrows always go from the “lines” to the “points” of $P^2(\mathbb{F}_3)$.

It was a surprise for me to find that these 26 roots with specified inner product fit into a 14-dimensional lattice L . The actual coordinates used for the calculation are given below:

$$\begin{aligned}
 r[1] &= a = [\quad \quad \quad , 1 \quad \quad , \omega^2], \\
 r[1+i] &= c_i = [(-\theta\omega^2, \quad , \quad)_i \quad , \quad \quad , \quad], \\
 r[4+i] &= e_i = [(, -\theta\omega^2, \quad , \quad)_i \quad , \quad \quad , \quad], \\
 r[7+i] &= a_i = [\quad \quad \quad , (\quad , \quad , -\theta\omega^2)_{jk} \quad , \omega \quad , \omega^2], \\
 r[10+i] &= g_i = [(, \quad , -\theta\omega^2)_i \quad , (\quad , \theta\omega^2, -\theta\omega^2)_{jk} \quad , 2\omega \quad , 2\omega^2], \\
 r[14] &= f = [(, 1, 1, -2)^3 \quad \quad \quad , -2 + \omega \quad , -\theta], \\
 r[14+i] &= f_i = [(, 1, 1, 1)_i \quad \quad \quad , \quad \quad \quad , \quad], \\
 r[17+i] &= b_i = [(1, \quad , 1, -1)_i \quad \quad \quad , \quad \quad \quad , -1 \quad , \quad], \\
 r[20+i] &= z_i = [(, 1, 1, -2)_i \quad , (1, \quad , 1, -1)_{jk} \quad , -\theta\omega^2 \quad , -\theta], \\
 r[23+i] &= d_i = [(1, 1, -1, \quad)_i \quad \quad \quad , \quad \quad \quad , \quad \quad \quad , \quad].
 \end{aligned}$$

The numbering $r[i]$ refers to the names we use for these vectors for computer calculation. An E_8 vector with subscript i means we put it in place of the i th E_8 , while the subscript jk means that we put it at the j th and k th place. The indices i, j and k are in cyclic permutation of $(1, 2, 3)$. Blank spaces are to be filled with zero. Let $R_1 \subseteq R_2$ be the subgroups of $R(L)$ generated by ω -reflections in these 16 and the 26 roots, respectively.

3.2. Lemma. *We have $\text{rad}(M_{666}) = \text{rad}(D)$ and hence $R_1 = R_2$.*

Proof. Take any A_{11} sub-diagram in M_{666} and add an extra vertex to make an affine A_{11} diagram. For example, the A_{11} Dynkin diagram $(f_1, e_1, d_1, c_1, b_1, a, b_2, c_2, d_2, e_2, f_2)$ is completed to a 12-gon by adding the vertex a_3 . The graph D is closed under this operation of completing 12-gons (see [9]). Let $y = (y_1, \dots, y_{12})$ be any “free” 12-gon in D . In the Conway–Simons presentation of the bimonster on 26 generators, there are the relations called “deflating a 12-gon” (see [9]). For any 12-gon y in D let $\text{deflate}(y)$ be the relation

$$(y_1 y_2 \cdots y_{10}) y_{11} (y_1 y_2 \cdots y_{10})^{-1} = y_{12}.$$

To prove the lemma we just check that the reflections making the diagram D also satisfy the above relations. For the 12-gon mentioned above the relation $\text{deflate}(y)$ follows from

$$\phi_{f_1} \phi_{e_1} \phi_{d_1} \phi_{c_1} \phi_{b_1} \phi_a \phi_{b_2} \phi_{c_2} \phi_{d_2} \phi_{e_2} (f_2) = \omega^2 a_3.$$

Using the diagram automorphisms (see Section 5) we see that the relation $\text{deflate}(y)$ holds for the other 12-gons in D too. \square

3.3. Linear relations among the roots of D

Let x_1, \dots, x_{13} denote the thirteen points of $P^2(\mathbb{F}_3)$ or the roots of L corresponding to them and l_1, \dots, l_{13} denote the 13 lines. These roots will often be referred to simply as *points* and *lines*. Let $\xi = e^{-\pi i/6}$ and $(\rho_1, \dots, \rho_{26}) = (x_1, \dots, x_{13}, \xi l_1, \dots, \xi l_{13})$. We say ρ_i and ρ_j are connected if the corresponding roots are. For each i let Σ_i be the sum of the ρ_j ’s that are connected to ρ_i . For any line ρ_i the vector $\sqrt{3}\rho_i + \Sigma_i$ is an element of L that has norm 3 and is orthogonal to the points. But there is only one such vector. (This vector, called $w_{\mathcal{P}}$ in later sections is fixed by the diagram automorphisms $PGL_3(\mathbb{F}_3)$.) Thus for any two lines ρ_i and ρ_j we have

$$\sqrt{3}\rho_i + \Sigma_i = \sqrt{3}\rho_j + \Sigma_j. \tag{7}$$

These give 12 independent relations that generate all the relations among the roots of D . We shall see later that there is an automorphism σ of L interchanging ρ_i with $\xi\rho_{13+i}$. So Eqs. (7) hold also for every pair of points ρ_i and ρ_j . Equation (7) for a pair of points ρ_i and ρ_j is a linear combination of the equations for pairs of lines and vice versa.

3.4. Remark. One can define the lattice L using the relations (7) as follows. Define a singular \mathcal{E} -lattice spanned by 26 linearly independent vectors of norm -3 corresponding to the vertices of D . The inner product of two vectors is zero if there is no edge between the corresponding vertices of D and $\sqrt{3}$ if there is an edge. Then L can be obtained by taking the quotient of this singular lattice by imposing the relations (7). The inner product is induced from the singular lattice.

4. Preliminary results about the reflection group of L

In this section we give an elementary proof of the fact that the reflection group $R(L)$ acts transitively on the root system Φ_L and we describe how to write down generators for the reflection group of L .

4.1. Lemma. *We choose the coordinate system $\Lambda \oplus H$ for L . Consider the set of roots Ψ of the form $r = r_{\lambda, \beta} = (\lambda; 1, \theta \frac{(-3-|\lambda|^2)}{6} + \beta)$, where $\lambda \in \Lambda$ and $\beta \in \frac{1}{2}\mathbb{Z}$ is chosen satisfying $2\beta + 1 \equiv |\lambda|^2 \pmod{2}$, so that the last entry of r is in \mathcal{E} . Then Ψ is a connected set.*

Proof. Recall that $[a, b] = \frac{\text{Im}(a, b)}{\theta}$. By direct computation we have

$$\langle r, r' \rangle = \left(-3 - \frac{1}{2}|\lambda - \lambda'|^2 \right) + \theta(\beta - \beta' + [\lambda, \lambda']).$$

We find that r and r' are adjacent if and only if $|\langle r, r' \rangle| = \sqrt{3}$. This can happen in two ways:

Case (I). $\langle r, r' \rangle = \pm\theta$, that is, $|\lambda - \lambda'|^2 = -6$ and $\beta' = \beta + [\lambda, \lambda'] \pm 1$;

or

Case (II). $\langle r, r' \rangle = \pm\theta\omega$ or $\pm\theta\omega^2$, that is, $|\lambda - \lambda'|^2 = -3$ or -9 and $\beta' = \beta + [\lambda, \lambda'] \pm 1/2$ (note that the case $|\lambda - \lambda'|^2 = -3$ does not occur here).

Given a root $r_{\lambda, \beta} \in \Psi$ choose λ' satisfying $|\lambda - \lambda'|^2 = -9$, and choose $\beta' = \beta + [\lambda, \lambda'] + 1/2$. $r' = r_{\lambda', \beta'}$ belongs to Ψ . By Case (II) above, both $r_{\lambda, \beta}$ and $r_{\lambda, \beta+1}$ are adjacent to r' , hence are connected by a chain of length 2. It follows that any two roots $r_{\lambda, \beta}$ and $r_{\lambda, \alpha}$ in Ψ are connected by a chain, because $\beta - \alpha \in \mathbb{Z}$.

Given a root $r_{\lambda, \beta}$ in Ψ and λ' in Λ , with $|\lambda - \lambda'|^2 = -6$, it follows from Case (I) above, that there is some β' such that $r_{\lambda', \beta'}$ is a root connected to $r_{\lambda, \beta}$. Hence two roots of the form $r_{\lambda, \alpha}$ and $r_{\lambda', \alpha'}$ are connected for all α and α' , whenever $|\lambda - \lambda'|^2 = -6$.

Now given any two roots $r_{\lambda, \beta}$ and $r_{\lambda', \beta'}$ in Ψ there is a connected chain $\lambda = \lambda_0, \lambda_1, \dots, \lambda_n = \lambda'$ such that $|\lambda_i - \lambda_{i+1}|^2 = -6$, because the vectors of Λ of norm -6 generate Λ . Hence $r_{\lambda, \beta}$ and $r_{\lambda', \beta'}$ are connected. \square

4.2. Proposition. *Let $L = \Lambda \oplus H$. Let Ψ be the roots of the form $r = (\lambda; 1, \theta \frac{(-3-|\lambda|^2)}{6} + \beta + n)$, where β is chosen so that the last entry is in \mathcal{E} , $\lambda \in \Lambda$, and n is an integer. Then $\text{rad}(\Psi) = \Phi_L$. In particular Ψ generates the reflection group of L .*

Proof. This is exactly like Conway’s calculation in [6] of the reflection group of $II_{1,25}$ as adapted by Allcock. Let $\rho = (0^{12}; 0, 1)$ and define $h(r) = |\langle r, \rho \rangle|/\theta$, so that Ψ is the set of roots r with $h(r) = 1$ (h is only used in this proof). We show that if $h(\mu) > 1$ then there exists a reflection ϕ_r^\pm in a root $r \in \Psi$ such that $h(\phi_r^\pm(\mu)) < h(\mu)$. This proof uses the fact that the covering radius of the Leech lattice is $\sqrt{2}$ and would not have worked if the radius were any bigger.

Suppose μ is a root such that $h(\mu) > 1$. Let $w = (l; 1, \alpha - \frac{\theta|l|^2}{6})$ be a scalar multiple of μ such that $h(w) = 1$. Let $\alpha = \alpha_1 + \theta\alpha_2$. We have $-3 < w^2 < 0$; it follows that $-\frac{1}{2} < \alpha_2 < 0$.

We try to find an ϵ -reflection in a root r of the form $(\lambda; 1, \theta \frac{(-3-|\lambda|^2)}{6} + \beta + n)$ so that $h(\phi_r^\epsilon(w)) < h(w)$. We have

$$\langle r, w \rangle = -3(a + b),$$

where a is real

$$a = \frac{1}{2} + \frac{|l - \lambda|^2}{6} - \alpha_2$$

and b is imaginary

$$b = n\theta^{-1} + \alpha_1\bar{\theta}^{-1} + \beta\theta^{-1} - \frac{\text{Im}\langle \lambda, l \rangle}{3}.$$

Then $h(\phi_r^\epsilon(w))$ is equal to $|1 - (1 - \epsilon)(a + b)|$.

By choice of n , can make $b \in [-\frac{\theta}{6}, \frac{\theta}{6}]$, and by choice of λ can make $0 \geq \frac{|l-\lambda|^2}{6} \geq -\frac{1}{2}$. (Note: covering norm of Leech lattice in our scaling is -3 .) Hence, using $-\frac{1}{2} < \alpha_2 < 0$, we get that $1 > a > 0$. If $b \in [-\frac{\theta}{6}, 0]$, choose $\epsilon = \omega^2$, and if $b \in [0, \frac{\theta}{6}]$ choose $\epsilon = \omega$. Then we get that $h(\phi_r^\epsilon(w))$ is strictly less than 1. \square

4.3. Proposition. Φ_L is connected. So $R(L)$ acts transitively on Φ_L .

Proof. By 4.1 Ψ is connected and by 4.2 $\text{rad}(\Psi) = \Phi_L$. So by 2.3 Φ_L is connected. Hence any two roots can be connected by a chain of adjacent roots. If a and b are adjacent roots then $\phi_a\phi_b a$ is an unit multiple of b , so a and b are in the same orbit under $R(L)$. \square

4.4. Generators for the reflection group of L

Let \mathbb{T} be the subgroup of $\text{Aut}(L)$ consisting of the automorphisms of L that fix the vector $\rho = (0^{12}; 0, 1)$ and act trivially on ρ^\perp/ρ . \mathbb{T} is called the group of *translations*:

$$\mathbb{T} = \{ T_{\lambda, \theta\alpha/2} \mid \lambda \in \Lambda, \alpha \in \mathbb{Z}, \alpha = |\lambda|^2 \pmod{2} \},$$

where $T_{\lambda, z}$ is given by

$$T_{\lambda, z} = \begin{pmatrix} I & \lambda & 0 \\ 0 & 1 & 0 \\ \theta^{-1}\lambda^* & \bar{\theta}^{-1}(z - |\lambda|^2/2) & 1 \end{pmatrix},$$

where I is the identity acting on Λ and λ^* is the linear map on Λ given by $x \mapsto \langle \lambda, x \rangle$. We call $T_{\lambda, z}$, a translation by λ .

4.5. Lemma. Let ϕ_{r_1} and ϕ_{r_2} be the ω -reflections in the roots $r_1 = (0^{12}; 1, \omega^2)$ and $r_2 = (0^{12}; 1, -\omega)$. Let $\lambda_1, \dots, \lambda_{24}$ be a basis of Λ over \mathbb{Z} . Choose z_j such that $T_{\lambda_j, z_j} \in \mathbb{T}$. Then reflections in the roots $T_{\lambda_j, z_j}(r_1)$ and $T_{\lambda_j, z_j}(r_2)$ for $j = 1, \dots, 24$, together with r_1 and r_2 generate the reflection group of L .

Proof. The lemma is deduced from the work done in Section 3 of [1] as follows. From the equation $T_{\lambda_1, z_1} \circ T_{\lambda_2, z_2} = T_{\lambda_1 + \lambda_2, z_1 + z_2 + \text{Im}(\lambda_1, \lambda_2)}$, we see that the group generated by the 24 translations of the form $T_{\lambda_i, *}$ contains a translation by every vector of the Leech lattice. It also contains all the central translations $T_{0, z}$ because if we take λ and λ' in Λ with $\langle \lambda, \lambda' \rangle = -\theta\omega$ then $T_{\lambda, z}^{-1} T_{\lambda', z'}^{-1} T_{\lambda, z} T_{\lambda', z'} = T_{0, \theta}$. Hence the 24 translations $T_{\lambda_i, *}$ generate whole of \mathbb{T} .

The orbit of the root $r_1 = (0^{12}; 1, \omega^2)$ under \mathbb{T} is all the roots of the form $(\lambda; 1, *)$, and by 4.2 they generate $R(L)$. So \mathbb{T} together with ϕ_{r_1} generate it too. Now from the equation $T_{\lambda, z} \phi_{r_1} \phi_{r_2} T_{\lambda, z}^{-1} (\phi_{r_1} \phi_{r_2})^{-1} = T_{-\omega\lambda, \theta|\lambda|^2/2}$ it follows that reflections in the roots $T_{\lambda_j, z_j}(r_1)$ and $T_{\lambda_j, z_j}(r_2)$ for $j = 1, \dots, 24$, together with r_1 and r_2 generates \mathbb{T} . So these 50 roots generate the whole reflection group $R(L)$. \square

5. The diagram automorphisms

5.1. The action of the automorphism group of the diagram D

The group $G \cong PGL_3(\mathbb{F}_3)$ acts on the 26 node graph D . We fix coordinates where $\mathcal{P} = \{a, c_i, e_i, a_i, g_i\}$ are the points and $\mathcal{L} = \{f, b_i, d_i, f_i, z_i\}$ are the lines of $P^2(\mathbb{F}_3)$. For computations on $P_2(\mathbb{F}_3)$, the points $x \in \mathcal{P}$ are represented as column vectors of length 3 and the lines $l \in \mathcal{L}$ as row vectors of length 3 such that, x lies on l if and only if $l \cdot x = 0$. (See the pari code **P2F3.gp** for an actual coordinates used.) An element g of G can be represented by a 3×3 matrix which acts on the points by $x \mapsto gx$ and on the lines by $l \mapsto lg^{-1}$, so that it preserves the incidence relations.

\mathcal{P} and \mathcal{L} will also denote the set of roots of L given in 3.1. An element of \mathcal{P} will be simply referred to as a *point* and an element of \mathcal{L} as a *line*.

5.2. Lemma. The action of G on D induces a linear action of G on the lattice L .

Proof. The lemma follows from the construction of the lattice L given in Remark 3.4 since the relations (7) are invariant under G . It can also be proved directly by verifying that the generators of G acting on L satisfy the relations of the following presentation of $PGL_3(\mathbb{F}_3)$:

$$G = \langle x, y \mid x^2 = y^3 = (xy)^{13} = ((xy)^4 xy^{-1})^2 xyxyxy^{-1} xy^{-1} xyxy^{-1} xy^{-1} xyxyxy^{-1} = 1 \rangle.$$

\square

5.3. The automorphism σ

Actually a group slightly larger than G acts naturally on mirrors orthogonal to the roots of D . To see this note that the automorphism $\bar{\sigma}$ that takes a point $x = [a : b : c]$ to the line $l = \{aX + bY + cZ = 0\}$ lifts to an automorphism σ of L of order 12. σ takes a point x to $-\omega l$ and l to x and its square is equal to $-\omega$. Let Q be the group generated by G and σ . The group Q acts transitively and faithfully on the unit multiples of the roots of D . For $g \in G$, $\sigma g \sigma^{-1}$ preserves the set of points and hence is in G . So we have an exact sequence

$$1 \rightarrow PGL_3(\mathbb{F}_3) \rightarrow Q \rightarrow \mathbb{Z}/12\mathbb{Z} \rightarrow 1. \tag{8}$$

Recall $(\rho_1, \dots, \rho_{26}) = (x_1, \dots, x_{13}, \xi l_1, \dots, \xi l_{13})$. Then σ interchanges ρ_i with $\xi \rho_{13+i}$.

5.4. The fixed vectors of the action of G on L

The group G fixes a two-dimensional sub-lattice F of signature $(1, 1)$. F is spanned by two norm 3 vectors $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$, where $w_{\mathcal{P}} = [(0, 0, \theta, -2\theta)^3; 4\omega^2, 4]$ and $w_{\mathcal{L}} = [(\omega, \omega, 2\omega, -3\omega)^3; -2 - 5\omega, -2\theta\omega]$. These two vectors can be expressed by

$$w_{\mathcal{P}} = \omega^2\theta l + \sum_{x \in \mathcal{P}} x, \quad w_{\mathcal{L}} = -\omega\theta x + \sum_{x \in \mathcal{L}} l, \tag{9}$$

where x is any point and l is any line (also see (7)). $w_{\mathcal{P}}^\perp$ is the 13-dimensional negative definite subspaces containing the points \mathcal{P} , and $w_{\mathcal{L}}^\perp$ is the 13-dimensional negative definite subspaces containing the lines \mathcal{L} . We note $\langle w_{\mathcal{P}}, w_{\mathcal{L}} \rangle = -4\theta\omega$ and so the discriminant of F is 39. From the explicit coordinates for $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$ it can be checked that F is primitive, and thus is equal to the sub-lattice fixed by G .

The vectors corresponding to the sum of points,

$$\Sigma_{\mathcal{P}} = \sum_{x \in \mathcal{P}} x = [-\theta\omega^2(1, 1, -2, 5)^3; 1 + 9\omega, 10\omega^2]$$

and the vector corresponding to the sum of lines,

$$\Sigma_{\mathcal{L}} = \sum_{l \in \mathcal{L}} l = [(4, 4, 5, -6)^3; -8 + 4\omega, -4\theta]$$

are also fixed by the action of G . They span a finite index sub-lattice of F .

5.5. The “Weyl vector”

There is a unique fixed point $\bar{\rho}$ in $\mathbb{C}H^{13}$ under the action of the group Q that we call the *Weyl vector* (note that $\bar{\rho}$ has nothing to do with the vector ρ used in 4.2). Since $\sigma(\Sigma_{\mathcal{P}}) = -\omega\Sigma_{\mathcal{L}}$ and $\sigma(\Sigma_{\mathcal{L}}) = \Sigma_{\mathcal{P}}$, the lines containing the vectors

$$\bar{\rho}_{\pm} = (\Sigma_{\mathcal{P}} \pm \xi \Sigma_{\mathcal{L}})/26 \tag{10}$$

are fixed by Q , where $\xi = e^{-\pi i/6}$. Of these two vectors $\bar{\rho} = \bar{\rho}_+$ has positive norm, and $\bar{\rho}_-$ have negative norm. So $\bar{\rho}$ given by

$$\bar{\rho} = (\Sigma_{\mathcal{P}} + \xi \Sigma_{\mathcal{L}})/26 = (w_{\mathcal{P}} + \xi w_{\mathcal{L}})/2(4 + \sqrt{3}) \tag{11}$$

is the unique point of $\mathbb{C}H^{13}$ fixed by Q . We note some inner products:

$$\langle \rho_i, \rho_j \rangle = \sqrt{3} \text{ or } 0, \tag{12}$$

$$\langle w_{\mathcal{P}}, \bar{\rho} \rangle = \sqrt{3}/2, \tag{13}$$

$$\langle \bar{\rho}, \rho_i \rangle = |\bar{\rho}|^2 = (4\sqrt{3} - 3)/26, \tag{14}$$

$$|\bar{\rho}_-|^2 = (-3 - 4\sqrt{3})/26 = \pm \langle \bar{\rho}_-, \rho_j \rangle. \tag{15}$$

In (15) the positive sign holds if ρ_j corresponds to a point and negative sign holds otherwise.

The fixed vector $\bar{\rho}$ can be expressed more canonically as follows. Let χ be the character of the group $\mathbb{Z}/12\mathbb{Z}$ generated by σ , given by $\sigma \mapsto \xi^{-1}$ extended to a character of the group Q . Define the sum

$$\tau = \frac{1}{m} \sum_{g \in Q} \chi(g)g$$

and $\tau_1 = \sum_{g \in G} g$, where m is the order of the group Q . Then

$$\begin{aligned} m\tau &= \sum_{j=1}^{12} \sum_{g \in G} \chi(\sigma^j g)\sigma^j g = \sum_{j=1}^6 \sum_{g \in Q} \xi^{-2j} \sigma^{2j} g + \xi^{-2j+1} \sigma^{2j-1} g = 6 \sum_{g \in G} g + \xi^{-1} \sigma g \\ &= 6(1 + \xi^{-1} \sigma)\tau_1. \end{aligned}$$

Let x be any point in \mathcal{P} . If $n_x = m/12.13$ is the size of the stabilizer of x in G we have $\tau_1 x = n_x \Sigma_{\mathcal{P}}$, $\sigma \tau_1 x = \xi n_x (\xi \Sigma_{\mathcal{L}})$, $\tau_1(\xi l) = n_x (\xi \Sigma_{\mathcal{L}})$ and $\sigma \tau_1(\xi l) = \xi n_x \Sigma_{\mathcal{P}}$. It follows that $\tau \rho_i = 6n_x.26\bar{\rho}/m = \bar{\rho}$. We also have $\langle \tau u, v \rangle = \langle u, \tau v \rangle$ for all u and v . Thus τ is a self adjoint rank one idempotent and its image gives the unique point in $\mathbb{C}H^{13}$ fixed by Q .

5.6. Height of a root

Let $r = \sum \lambda_i \rho_i$. Then

$$\langle \bar{\rho}, r \rangle = \left\langle \bar{\rho}, \sum \rho_i \lambda_i \right\rangle = |\bar{\rho}|^2 \sum \lambda_i.$$

The vector $\bar{\rho}$ has some of the formal properties of a Weyl vector in a reflection group, while the 26 roots of D play the role of simple roots. Define the *height* of a root r by

$$\text{ht}(r) = |\langle \bar{\rho}, r \rangle| / |\bar{\rho}|^2. \tag{16}$$

The 26 roots of D have height 1. We will see later that this is the minimum height of a root. This suggests the strategy for the proof of the following theorem.

5.7. Theorem. The 26 ω -reflections in the roots of D generate the reflection group of L .

Proof. Let R_1 be the group generated by the ω -reflections in the 26 roots of D . Take the 50 roots g_1, \dots, g_{50} as in Lemma 4.5 generating the reflection group of L . We convert them to co-ordinate system $3E_8 \oplus H$ using the explicit isomorphism found by the method described in 2.7. One can prove the theorem by showing that each ϕ_{g_i} is in R_1 . The algorithm for this is the following.

Start with a root $y_0 = g_i$ and find a $r \in D$ and $\epsilon \in \{\omega, \bar{\omega}\}$ such that $y_1 = \phi_r^\epsilon(y_0)$ has height less than y_0 . Then repeat the process with y_1 instead of y_0 to get an y_2 such that $\text{ht}(y_2) < \text{ht}(y_1)$, etc. Continue until some y_n is an unit multiple of an element of D .

In many cases, such as for $y_0 = g_3$, this algorithm works. That is, we find that some y_n is an unit multiple of D , showing that the reflection ϕ_{g_3} is in R_1 .

But this does not work for all the generating roots g_i . For example, when the above algorithm is implemented starting with $y_0 = g_1$ it gets stuck at a root y_k whose height cannot be decreased by a single reflection in any root of D . In such a situation we perturb y_k by reflecting it using one of the reflections ϕ_{g_j} that has already been shown in R_1 . For example, we take $y_{k+1} = \phi_{g_3}(y_k)$ and then run the algorithm again on y_{k+1} . If the algorithm works now then too it follows that ϕ_{g_1} is in R_1 . Or else one could repeat this process by perturbing again. It was verified that the algorithm works with at-most one perturbation for all the generating roots g_1, \dots, g_{50} .

The details of the computer programs that are needed for the validation of the above claims are given in Appendix B. \square

5.8. Theorem. *The automorphism group of L is equal to the reflection group. So the 26 reflections of D generate the full automorphism group of L .*

Proof. The idea of the proof is due to D. Allcock. He showed in [1] that the group $6 \cdot \text{Suz}$ of automorphisms of Λ maps onto the quotient $\text{Aut}(L)/R(L)$. We find a permutation group S_3 in the intersection of $R(L)$ and $6 \cdot \text{Suz}$ showing that $6 \cdot \text{Suz} \subseteq R(L)$, because a normal subgroup of $6 \cdot \text{Suz}$ containing an S_3 has to be the whole group.

Working in the coordinates $\Lambda \oplus H$ we can find a M_{666} diagram inside L such that all the 12 roots of the three E_8 hands, that is, c_i, d_i, e_i, f_i have the form $(\lambda; 1, \eta)$ up-to units, where λ is in the first shell of the Leech lattice. Such a set of vectors is written down in Appendix A.2. See the matrix E'_1 and the vectors a', b'_i, \dots, f'_i .

Consider the automorphisms φ_{12} and φ_{23} of order two that fix, respectively the third and the first hand of the M_{666} diagram and flips the other two. This corresponds to interchanging two E_8 components fixing the third E_8 . They generate a subgroup S_3 of the group Q of diagram automorphisms.

The automorphism φ_{12} flips the A_{11} braid diagram of the first and the second hand formed by $f_1, e_1, d_1, c_1, b_1, a, b_2, c_2, d_2, e_2, f_2$ and fixes the A_4 disjoint from it in the third hand formed by f_3, e_3, d_3, c_3 . This determines φ_{12} because these 15 roots contain a basis of L .

The automorphism of flipping the braid diagram in the 12 strand braid group is inner, so φ_{12} and φ_{23} are in the reflection group. But they also fix the vector $\rho = (0^{12}; 0, 1) \in \Lambda \oplus H$ which is a norm zero vector of Leech type. In fact one can directly check that the automorphisms φ_{12} and φ_{23} fix the hyperbolic cell spanned by $(0^{12}; 0, 1)$ and $(-2, \omega^2, \omega^2, 1, \omega, -2\omega, 1, \omega^2, 1, 1, 1, \omega, 1, \theta)$ in $\Lambda \oplus H$. So they are in $6 \cdot \text{Suz}$ too. \square

6. The minimum height of a root

Let $\xi = e^{-\pi i/6}$ and $(\rho_1, \dots, \rho_{26}) = (x_1, \dots, x_{13}, \xi l_1, \dots, \xi l_{13})$, as before. If y is equal to a multiple of a root r then $\phi_y = \phi_r$. We see that $\phi_{\rho_i}^\pm(\rho_i)$ is equal to $\rho_i + \xi^\pm \rho_j$ or ρ_i according to whether the two nodes ρ_i and ρ_j are connected in D or not. Let $r = \sum \alpha_j \rho_j$. Then

$$\begin{aligned} \phi_{\rho_i}^\pm(r) &= \omega^\pm \alpha_i \rho_i + \sum_{j: \langle \rho_i, \rho_j \rangle \neq 0} \alpha_j (\rho_j + \xi^\pm \rho_i) + \sum_{j: \langle \rho_i, \rho_j \rangle = 0} \alpha_j \rho_j \\ &= \left(\omega^\pm \alpha_i + \xi^\pm \sum_{j: \langle \rho_i, \rho_j \rangle \neq 0} \alpha_j \right) \rho_i + \sum_{j \neq i} \alpha_j \rho_j. \end{aligned}$$

By induction we can prove that all elements r in $\text{rad}(D)$ can be written as $\sum \alpha_i \rho_i$ with $\alpha_i \in \mathbb{Z}[\xi]$. Thus $\langle r, \bar{\rho} \rangle / |\bar{\rho}|^2 = \sum \alpha_i$ is an algebraic integer. Note that the norm of the algebraic integer $\langle r, \bar{\rho} \rangle / |\bar{\rho}|^2$ is equal to $\text{Nm}(r) = \frac{|\langle r, \bar{\rho} \rangle|^2}{|\bar{\rho}|^4} \frac{|\langle r, \bar{\rho} \rangle|^2}{|\bar{\rho} \rangle|^4} \in \mathbb{Z}$. Experimentally we find that a lot of roots have $\text{Nm}(r) = 1$. We tried defining the height of r by $\text{Nm}(r)$ in the proof of 5.7 but failed to make the height reduction algorithm converge even with a lot of computer time. We also tried other definitions of height, for example, absolute value of inner product with some other norm zero or norm 3 vectors of the lattice L , with the same effect.

The following proposition also seem to justify the definition of height we use.

6.1. Proposition. *All the roots of L have height greater than or equal to one and the unit multiples of the 26 roots of D are the only ones with minimal height.*

Proof. The proof uses the facts about the complex hyperbolic space given in Section 2.8. The proof does not use the fact that the reflection group is generated by the 26 reflections of D .

Suppose if possible r is a root of L with $\text{ht}(r) = |\langle \bar{\rho}, r \rangle| / |\bar{\rho}|^2 \leq 1$. We will show that r must be an unit multiple of an element of D .

From (3) and (16) we have

$$c(\bar{\rho}, r)^2 = -\text{ht}(r)^2 |\bar{\rho}|^2 / 3. \tag{17}$$

From the triangle inequality we get $d(r^\perp, y^\perp) \leq d(r^\perp, \bar{\rho}) + d(y^\perp, \bar{\rho})$ for any other root y . Using the distance formulae (5) and (6) the inequality takes the form

$$\begin{aligned} c(r, y)^2 &= \cosh^2(d(r^\perp, y^\perp)) \leq \cosh^2(\sinh^{-1}(|c(\bar{\rho}, r)|) + \sinh^{-1}(|c(\bar{\rho}, y)|)) \\ &\leq \cosh^2\left(\sinh^{-1}\left(\frac{|\bar{\rho}|}{\sqrt{3}}\right) + \sinh^{-1}\left(\frac{\text{ht}(y)|\bar{\rho}|}{\sqrt{3}}\right)\right) \end{aligned} \tag{18}$$

if $c(y, r)^2 > 1$. From this we get the following bound: either $|\langle r, y \rangle| \leq 3$ or

$$|\langle r, y \rangle| \leq 3 \cosh\left(\sinh^{-1}\left(\frac{|\bar{\rho}|}{\sqrt{3}}\right) + \sinh^{-1}\left(\frac{\text{ht}(y)|\bar{\rho}|}{\sqrt{3}}\right)\right). \tag{19}$$

Using the inequality (19) with $y = \rho_i$ gives

$$|\langle r, \rho_i \rangle|^2 \leq 9 \cosh^2\left(2 \sinh^{-1}\left(\frac{|\bar{\rho}|}{\sqrt{3}}\right)\right) = 9\left(2 \frac{|\bar{\rho}|^2}{3} + 1\right)^2 \approx 10.9. \tag{20}$$

But the inner product between lattice vectors is in $\theta\mathcal{E}$, hence we have

$$|\langle r, \rho_i \rangle| \leq 3. \tag{21}$$

Similarly using (4), (5), (13), (17) and the triangle inequality $d(w_p, r^\perp) \leq d(w_p, \bar{\rho}) + d(\bar{\rho}, r^\perp)$ we get the bound

$$|\langle w_p, r \rangle|^2 \leq 9 \sinh^2(\sinh^{-1}(|c(\bar{\rho}, r)|) + \cosh^{-1}(|c(\bar{\rho}, w_p)|)) \approx 11.2 \tag{22}$$

and thus

$$|\langle r, w_{\mathcal{P}} \rangle| \leq 3. \tag{23}$$

Write r in terms of the basis $x_1, \dots, x_{13}, w_{\mathcal{P}}$ as

$$r = \sum -\langle x_i, r \rangle x_i / 3 + \langle w_{\mathcal{P}}, r \rangle w_{\mathcal{P}} / 3. \tag{24}$$

Taking norm we get

$$-3 = -\sum |\langle x_i, r \rangle|^2 / 3 + |\langle w_{\mathcal{P}}, r \rangle|^2 / 3. \tag{25}$$

Changing r up-to units we may assume that $\langle r, w_{\mathcal{P}} \rangle$ is either 0 or θ or 3 and it follows from (25) that $\sum |\langle x_i, r \rangle|^2$ is equal to 9, 12 and 18, respectively. In each of these three cases we will check that there is no root of height less than one, thus finishing the proof. In the following let u_1, u_2, \dots , denote units in \mathcal{E}^* .

If $\langle r, w_{\mathcal{P}} \rangle = 0$ and $\sum |\langle x_i, r \rangle|^2 = 9$, the unordered tuple $(\langle x_1, r \rangle, \dots, \langle x_{13}, r \rangle)$ is equal to $(3u_1, 0^{12})$ or $(\theta u_1, \theta u_2, \theta u_3, 0^{10})$. So either r is an unit multiple of x_i (in which case it has height equal to one) or $r = (\theta u_1 x_1 + \theta u_2 x_2 + \theta u_3 x_3) / (-3)$. Using diagram automorphisms we can assume that $x_1 = a$ and $x_2 = c_1$ and then check that there is no such root r .

If $\langle r, w_{\mathcal{P}} \rangle = \theta$ and $\sum |\langle x_i, r \rangle|^2 = 12$, the unordered tuple $(\langle x_1, r \rangle, \dots, \langle x_{13}, r \rangle)$ is equal to $(3u_1, \theta u_2, 0^{11})$ (Case (I)) or $(\theta u_1, \theta u_2, \theta u_3, \theta u_4, 0^9)$ (Case (II)). In (Case (I)) we get $r = \theta u_1 x_1 / (-3) + 3u_2 x_2 / (-3) + \theta w_{\mathcal{P}} / 3$. Taking inner product with $\bar{\rho}$ and using $\langle \bar{\rho}, w_{\mathcal{P}} \rangle / |\bar{\rho}|^2 = 4 + \sqrt{3}$ we get $\langle \bar{\rho}, r \rangle / |\bar{\rho}|^2 = (u_1 - u_2 \theta - 4 - \sqrt{3}) / \theta$ which clearly has norm greater than one. In (Case (II)) we get $r = \theta w_{\mathcal{P}} / 3 + \sum_{i=1}^4 \theta u_i x_i / (-3)$ which give $\langle \bar{\rho}, r \rangle / |\bar{\rho}|^2 = (-4 - \sqrt{3} + \sum_{i=1}^4 u_i) / \theta$, again this quantity has norm at least one. Note that the lines l_1, \dots, l_{13} fall in this case and they have height one. Now we prove that in this case they are the only ones.

In the above paragraph we can have $\text{ht}(r) = 1$ only if r has inner product θ with four of the points x_1, \dots, x_4 and orthogonal to others. If x_1, \dots, x_4 do not all lie on a line then there is a line l that avoids all these four points. (To see this, take a point x_5 not among x_1, \dots, x_4 on the line joining x_1 and x_2 . There are four lines though x_5 and one of them already contain x_1 and x_2 . So there is one that does not contain any of x_1, \dots, x_4 .) Taking inner products with r in the equation $w_{\mathcal{P}} = \omega^2 \theta l + \sum_{x \in l} x$ gives $\theta = \omega^2 \theta \langle l, r \rangle$ contradicting $\theta L' = L$. So x_1, \dots, x_4 are points on a line l_1 . So r and ωl_1 has the same inner product with $w_{\mathcal{P}}$ and the elements of \mathcal{P} . So $r = \omega l_1$.

If $\langle r, w_{\mathcal{P}} \rangle = 3$, and $\sum |\langle x_i, r \rangle|^2 = 18$, the unordered tuple $(\langle x_1, r \rangle, \dots, \langle x_{13}, r \rangle)$ is equal to $(3u_1, 3u_2, 0^{11})$ or $(3u_1, \theta u_2, \theta u_3, \theta u_4, 0^9)$ or $(\theta u_1, \dots, \theta u_6, 0^7)$. Using similar calculation as above, we get $\langle r, \bar{\rho} \rangle / |\bar{\rho}|^2$ is equal to $(-u_1 - u_2 + 4 + \sqrt{3})$ or $(-u_1 + (u_2 + u_3 + u_4) / \theta + 4 + \sqrt{3})$ or $((u_1 + \dots + u_6) / \theta + 4 + \sqrt{3})$, respectively. Again each of these quantities are clearly seen to have norm strictly bigger than one. \square

Let \mathcal{M} be the set of mirrors of reflections in $R(L)$. Let $d_{\mathcal{M}}(x)$ be the distance of a point x in $\mathbb{C}H^{13}$ from \mathcal{M} . Then we have

6.2. Proposition. *The function $d_{\mathcal{M}}$ attains a local maximum at the point $\bar{\rho}$.*

Proof. Let $N(x)$ be the set of mirrors that are at minimum distance from a point x in $\mathbb{C}H^{13}$. By Proposition 6.1 we know $N(\bar{\rho})$ consists of the 26 mirrors ρ_j^{\perp} corresponding to the vertices

of D . If x is close enough to $\bar{\rho}$ then $N(x) \subseteq N(\bar{\rho})$. Thus it suffices to show that for any vector v , moving the point $\bar{\rho}$ a little in v direction decreases its distance from at-least one of $\rho_1^\perp, \dots, \rho_{26}^\perp$. Let ϵ be a small positive real number. Using the formula (5) for the distance between a point and hyperplane and ignoring the terms of order ϵ^2 we have

$$\begin{aligned} d(\bar{\rho} + \epsilon v, \rho_j^\perp) < d(\bar{\rho}, \rho_j^\perp) &\Leftrightarrow -\frac{|\langle \bar{\rho} + \epsilon v, \rho_j \rangle|^2}{|\bar{\rho} + \epsilon v|^2 |\rho_j|^2} < -\frac{|\langle \bar{\rho}, \rho_j \rangle|^2}{|\bar{\rho}|^2 |\rho_j|^2} \Leftrightarrow \frac{(|\bar{\rho}|^2 + \epsilon \langle v, \rho_j \rangle)^2}{|\bar{\rho}|^2 + 2 \operatorname{Re}\langle v, \bar{\rho} \rangle \epsilon} < |\bar{\rho}|^2 \\ &\Leftrightarrow \frac{|\bar{\rho}|^4 + 2|\bar{\rho}|^2 \operatorname{Re}\langle v, \rho_j \rangle \epsilon}{|\bar{\rho}|^2 + 2 \operatorname{Re}\langle v, \bar{\rho} \rangle \epsilon} < |\bar{\rho}|^2 \Leftrightarrow \frac{|\bar{\rho}|^2 + 2 \operatorname{Re}\langle v, \rho_j \rangle \epsilon}{(|\bar{\rho}|^2 + 2 \operatorname{Re}\langle v, \bar{\rho} \rangle \epsilon)} < 1 \\ &\Leftrightarrow \operatorname{Re}\langle v, \rho_j \rangle \epsilon < \operatorname{Re}\langle v, \bar{\rho} \rangle \epsilon. \end{aligned}$$

For any v such that $\operatorname{Re}\langle v, \rho_j \rangle, j = 1, \dots, 26$, are not all equal, there is an j_0 such that $\operatorname{Re}\langle v, \rho_{j_0} \rangle < \operatorname{Re}\langle v, \bar{\rho} \rangle$ since $\operatorname{Re}\langle v, \bar{\rho} \rangle = \frac{1}{26} \sum_{j=1}^{26} \operatorname{Re}\langle v, \rho_j \rangle$. So moving the point $\bar{\rho}$ a little in v direction will decrease its distance from $\rho_{j_0}^\perp$ and hence decrease the value of $d_{\mathcal{M}}$.

It remains to check the directions v for which $\operatorname{Re}\langle v, \rho_j \rangle = \operatorname{Re}\langle v, \rho_k \rangle$ for $j, k = 1, \dots, 26$. Such vectors form a real vector space of dimension three spanned by $\{\bar{\rho}_+, i\bar{\rho}_+, i\bar{\rho}_-\}$. So we only need to check for $v = i\bar{\rho}_-$. To calculate $d(\bar{\rho}_+ + i\epsilon\bar{\rho}_-, \rho_j^\perp)$ we recall the formulae (10), (14) and (15). Let $\alpha = -|\bar{\rho}_-|^2/|\bar{\rho}_+|^2$. Then $\alpha > 1$, and

$$\begin{aligned} \sinh^2(d(\bar{\rho}_+ + i\epsilon\bar{\rho}_-, \rho_j^\perp)) &= -\frac{|\langle \bar{\rho}_+ + i\epsilon\bar{\rho}_-, \rho_j \rangle|^2}{|\bar{\rho}_+ + i\epsilon\bar{\rho}_-|^2 |\rho_j|^2} = \frac{(|\bar{\rho}_+|^2 \pm i\epsilon|\bar{\rho}_-|^2)^2}{3(|\bar{\rho}_+|^2 + i\epsilon|\bar{\rho}_-|^2)} \\ &= \frac{|\bar{\rho}_+|^4 - \epsilon^2|\bar{\rho}_-|^4}{3(|\bar{\rho}_+|^2 + \epsilon^2|\bar{\rho}_-|^2)} = \frac{|\bar{\rho}_+|^2}{3} \left(\frac{1 - \epsilon^2|\bar{\rho}_-|^4/|\bar{\rho}_+|^4}{1 + \epsilon^2|\bar{\rho}_-|^2/|\bar{\rho}_+|^2} \right) \\ &= \frac{|\bar{\rho}_+|^2}{3} \left(\frac{1 - \epsilon^2\alpha^2}{1 - \epsilon^2\alpha} \right) < \frac{|\bar{\rho}_+|^2}{3}. \end{aligned}$$

So $d(\bar{\rho}_+ + i\epsilon\bar{\rho}_-, \rho_j^\perp) < d(\bar{\rho}_+, \rho_j^\perp)$. \square

7. Relations in the reflection group of L

7.1. The presentation of the bimonster

Let $M \wr 2$ be the wreath product of the monster simple group M with the group of order two. It is defined as the semi-direct product of $M \times M$ with the group of order 2 acting on it by interchanging the two copies of M . This group, called the bimonster, has two simple presentations, see [7–9, 13]. The first presentation is on 16 generators of order 2 corresponding to the vertices of the M_{666} diagram, satisfying the Coxeter relations of this diagram and the one extra relation called the “spider relation” given by $S^{10} = 1$, where $S = ab_1c_1ab_2c_2ab_3c_3$.

The other presentation of $M \wr 2$ is on the 26 node diagram D (see [9]). Each generator has order 2, they satisfy the Coxeter relations of the diagram D and for every sub-diagram y of D isomorphic to a 12-gon we need the extra relation $\text{deflate}(y)$ given in Lemma 3.2, so that y generates the symmetric group S_{12} . (A sub-diagram of D consists of a subset of vertices of D and all the edges between them that were present in D .)

There are some other relators for $M \wr 2$ that are particularly nice. Let Δ be any spherical Dynkin diagram sitting inside M_{666} or D . A Coxeter element $w(\Delta)$ of Δ is the product (in any order) of the generators corresponding to the vertices of the diagram Δ . The order of $w(\Delta)$ in the Euclidean reflection group determined by Δ is called the Coxeter number and denoted by $h(\Delta)$. The relation $w(\Delta)^{h(\Delta)/2} = 1$, is satisfied in $M \wr 2$. We find analogs of all the above relations in the reflection group of L .

7.2. Conjectured relation of $\text{Aut}(L)$ with the bimonster

D. Allcock conjectured the following connection between the automorphism group of L and the bimonster, in the late 90’s on the basis of the appearance of the M_{666} diagram as the “hole diagram” for E_8^3 and the idea that the orbifold construction is the natural way to turn triflections into biflections.

Conjecture. (D. Allcock, personal communication) *Let L be the unique Eisenstein lattice of signature $(1, 13)$ with $\theta L^* = L$. Let X be the quotient $\mathbb{C}H^{13} / \text{Aut}(L)$ and \mathfrak{D} be the image therein of the mirrors for the triflections in $\text{Aut}(L)$. Then the bimonster is a quotient of the orbifold π_1 of $X \setminus \mathfrak{D}$, namely the quotient by the normal subgroup generated by the squares of the meridians around \mathfrak{D} .*

This conjecture was made long before the 26 generators of $\text{Aut}(L)$ were found. The results in this paper can thus be taken as supporting evidence for the suggested connection. Allcock also pointed out to me the possible connection with the “Monster Manifold” conjectured by Hirzebruch et al. (see [12]). Recently Allcock has written a preprint [3] where he discusses the conjecture and the evidence for it in more detail.

7.3. The relations in the reflection group of L

All the Coxeter relations of the diagram D hold in $R(L)$, except that now the nodes have order three instead of two. In addition the following relations hold:

- (1) the spider: $S^{20} = 1$, where $S = ab_1c_1ab_2c_2ab_3c_3$. (Here a means ω reflection in a , etc.)
- (2) The relation deflate(y) (see Lemma 3.2) present in the Conway–Simons presentation of the bimonster holds in $\text{Aut}(L)$ too. If $A = ab_2c_2d_2e_2f_2a_3f_1e_1d_1c_1b_1$ then $A^{11} = 1$.
- (3) Let $w = w(\Delta)$ be the Coxeter element of a free spherical Dynkin diagram Δ inside the sixteen node diagram, then in all the cases except $\Delta = D_4$ and A_5 , the element w has finite order in $R(L)$ and the order is a simple multiple of half-Coxeter number in all these cases.

The two exceptional cases are the “affine diagrams” for the Eisenstein version of E_6 and E_8 , and in these two cases the corresponding Coxeter elements in $R(L)$ have infinite order:

Δ :	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11} ;
order $w(\Delta)$:	3	6	12	30	∞	42	24	18	30	66	12;
Δ :	D_4	D_5	D_6	D_7	D_8	E_6	E_7	E_8 ;			
order $w(\Delta)$:	∞	24	15	12	21	12	9	15.			

- (4) More relators can be found using the action of Q on $R(L)$, induced from its action on L . If $r = 1$ is a relation valid in $R(L)$ and σ is in M then $\sigma(r) = 1$ is also a relator.

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Appendix A. Coordinates for the codes and lattices

A.1. The codes

A *code* is, for us, a linear subspace of a vector space over a finite field. A *generator matrix* for a code is a matrix whose rows give a basis for the subspace. We give generator matrices for the codes that are used in the computer calculation with the lattice L . The *ternary tetracode* C_4 is a two-dimensional subspace of \mathbb{F}_3^4 with generator matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

The *ternary Golay code* C_{12} is a six-dimensional subspace of \mathbb{F}_3^{12} with generator matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

The binary Golay code C_{24} (which is a 12-dimensional subspace of \mathbb{F}_2^{24}) and the ternary Golay code C_{12} are obtained from the Golay codes C_{23} and C_{11} by appending “zero sum check digit.” C_{11} and C_{23} are both “cyclic quadratic residue codes.” They can be generated as follows:

Let q be equal to 11 or 23. Label the entries of a vector of length $q + 1$ by elements of $\Omega = \mathbb{F}_q \cup \{\infty\}$. Let $N = \Omega \setminus \{x^2: x \in \mathbb{F}_q\}$. Let v be a vector of length 12 with -1 in the places of N and the 1 in the places of $\Omega \setminus N$, or vectors of length 24 with 1 in the places of N and 0 in the places of $\Omega \setminus N$. Then the vectors obtained by cyclic permutation of the co-ordinates of v span the code C_{12} and C_{24} , respectively. The above are all taken from chapter 10 of the book [10], where much more can be found. The coordinates for the ternary Golay code C_{12} used to list the vectors in the first shell of the Leech lattice is taken from page 85 of the book [10]. Note that this gives different coordinates on the Leech lattice than those used in [14] or in Chapter 11 of the book [10].

A.2. Change of basis between two preferred coordinates

Let E_2 be the matrix whose rows are given by the vectors $f_i, \omega e_i, d_i, \omega c_i, i = 1, 2, 3, (0^{12}, 0, 1)$ and $(0^{12}, 1, 0)$, respectively (see 3.1). Clearly E_2 forms a basis for $3E_8 \oplus H$. The

calculation procedure described in 2.7 produces the following matrix E_1 (notation: the matrix entry a, b stands for the number $a + \omega b$ and \bar{x} stands for $-x$):

$$E_1 = \begin{pmatrix} \bar{3},0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 2,\bar{1} & 0,1 & \bar{1},1 & 0,1 & \bar{1},1 & 0,1 & \bar{1},1 & \bar{1},1 & 0,1 & 0,1 & 0,1 & \bar{1},1 & \bar{1},0 & \bar{1},1 \\ \bar{3},0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 3,1 & 0,1 & 0,1 & 0,1 & \bar{1},1 & \bar{1},1 & \bar{1},1 & 0,1 & \bar{1},1 & 0,1 & \bar{1},1 & \bar{1},1 & \bar{1},0 & 0,\bar{1} \\ \bar{3},0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 3,1 & \bar{1},1 & 0,1 & \bar{1},1 & \bar{1},1 & 0,1 & 0,1 & \bar{1},1 & \bar{1},1 & 0,1 & 0,1 & \bar{1},1 & \bar{1},0 & 0,\bar{1} \\ \bar{3},0 & 0,0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 2,\bar{1} & 0,1 & 0,1 & \bar{1},1 & \bar{1},1 & \bar{1},1 & 0,1 & 0,1 & 0,1 & \bar{1},1 & 0,1 & \bar{1},1 & \bar{1},0 & \bar{1},1 \\ \bar{3},0 & 0,0 & 0,0 & 0,0 & 0,3 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 2,0 & \bar{1},0 & \bar{1},0 & \bar{1},0 & 2,0 & \bar{1},0 & 0,\bar{1} \\ \bar{3},\bar{2} & 1,0 & 1,0 & 1,0 & 0,1 & \bar{1},1 & 1,0 & \bar{1},1 & 0,1 & 1,0 & 1,0 & \bar{1},1 & 1,1 & 0,1 \\ \bar{2},\bar{2} & 1,1 & 1,1 & 1,1 & \bar{1},0 & 0,2 & 1,1 & \bar{1},0 & 1,1 & 0,\bar{1} & 0,\bar{1} & \bar{1},0 & 1,1 & \bar{1},0 \\ 12,\bar{2} & \bar{3},1 & \bar{3},1 & \bar{3},1 & \bar{1},4 & \bar{2},0 & \bar{3},1 & \bar{1},1 & \bar{3},\bar{2} & \bar{2},0 & \bar{2},0 & \bar{1},4 & \bar{5},0 & \bar{2},6 \\ 12,16 & \bar{4},4 & \bar{4},4 & \bar{4},4 & 4,0 & 0,\bar{2} & \bar{4},4 & 2,\bar{1} & 0,\bar{2} & \bar{2},3 & \bar{2},3 & 2,\bar{1} & \bar{4},6 & 6,0 \end{pmatrix}.$$

The rows of E_1 belong to $\Lambda \oplus H$. The rows of E_1 and the rows of E_2 have the same Gram matrix. Hence E_1 is a basis for $\Lambda \oplus H$ and so left multiplication of a column vector by $C = E_2 E_1^{-1}$ gives an explicit isomorphism from $\Lambda \oplus H$ to $3E_8 \oplus H$. This is the isomorphism we used to transfer the generators for the reflection group found in 4.5 to the coordinate system $3E_8 \oplus H$ because the 26 nodes of the diagram D are naturally given in coordinate system $3E_8 \oplus H$.

The proof of equality of the automorphism group and the reflection group of L in 5.8 depends on the existence of a M_{666} diagram in $\Lambda \oplus H$ such the 12 roots in the three hands of the diagram spanning $3E_8$ all have the form $(*, 1, *)$ up to units; so that the diagram automorphisms permuting the hands of the M_{666} diagram will fix the vector $(0^{12}; 0, 1)$ in $\Lambda \oplus H$. For this purpose the matrix E'_1 given below was found, by basically the same calculation scheme as described in 2.7:

$$E'_1 = \begin{pmatrix} \bar{3},0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 2,\bar{1} & 0,1 & \bar{1},1 & 0,1 & \bar{1},1 & 0,1 & \bar{1},1 & \bar{1},1 & 0,1 & 0,1 & 0,1 & \bar{1},1 & \bar{1},0 & \bar{1},1 \\ \bar{3},0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 3,1 & 0,1 & 0,1 & 0,1 & \bar{1},1 & \bar{1},1 & \bar{1},1 & 0,1 & \bar{1},1 & 0,1 & \bar{1},1 & \bar{1},1 & \bar{1},0 & 0,\bar{1} \\ \bar{3},0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 3,1 & \bar{1},1 & 0,1 & \bar{1},1 & \bar{1},1 & 0,1 & 0,1 & \bar{1},1 & \bar{1},1 & 0,1 & 0,1 & \bar{1},1 & \bar{1},0 & 0,\bar{1} \\ \bar{3},0 & 0,0 & 0,0 & 3,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 1,0 & 0,1 \\ 2,\bar{1} & 0,1 & 0,1 & \bar{1},1 & \bar{1},1 & \bar{1},1 & 0,1 & 0,1 & 0,1 & \bar{1},1 & 0,1 & \bar{1},1 & \bar{1},0 & \bar{1},1 \\ \bar{2},0 & 1,0 & 1,0 & 1,0 & 0,1 & 2,2 & 1,0 & 0,1 & 1,0 & \bar{1},1 & \bar{1},1 & 0,1 & 1,0 & 0,1 \\ 2,\bar{1} & 0,1 & 0,1 & 0,1 & \bar{1},1 & 1,0 & 0,1 & 1,0 & \bar{1},1 & 0,1 & 0,1 & 1,0 & \bar{1},0 & \bar{1},1 \\ \bar{3},0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,0 & 0,3 & 1,0 & 0,1 \\ 2,0 & \bar{1},0 & 2,0 & \bar{1},0 & 0,\bar{1} \\ \bar{5},\bar{18} & 2,5 & 2,5 & 2,5 & \bar{4},1 & \bar{1},2 & 2,5 & \bar{2},0 & \bar{2},0 & 1,3 & 1,3 & \bar{5},3 & 1,6 & \bar{8},4 \\ 12,\bar{2} & \bar{3},1 & \bar{3},1 & \bar{3},1 & \bar{1},4 & \bar{2},0 & \bar{3},1 & \bar{1},1 & \bar{3},\bar{2} & \bar{2},0 & \bar{2},0 & \bar{1},4 & \bar{5},0 & \bar{2},6 \end{pmatrix}.$$

Let the rows of E'_1 be $f'_i, \omega e'_i, d'_i, \omega c'_i, i = 1, 2, 3, n'_1, n'_2$, respectively. Then the fourteen vectors $f'_i, e'_i, d'_i, c'_i, n'_1, n'_2$ have the same inner product matrix as that of the vectors $f_i, e_i, d_i, c_i, n_1, n_2$ of $3E_8 \oplus H$, where $n_1 = (0^{12}, 0, 1)$ and $n_2 = (0^{12}, 1, 0)$. Let $a' = n'_2 + \omega^2 n'_1$ and $b'_i = -n'_2 - (f'_i + (2 + \omega)e'_i + 2d'_i + (2 + \omega)c'_i)$. Then $a', b'_i, c'_i, d'_i, e'_i, f'_i$ give 16 roots of $\Lambda \oplus H$ that

have the same inner products as the vectors $a, b_i, c_i, d_i, e_i, f_i$ in $3E_8 \oplus H$. These 16 vectors form the M_{666} diagram as required in the proof of Theorem 5.8. In fact one can directly check that the diagram automorphism φ_{12} and φ_{23} fix the hyperbolic cell spanned by $(0^{12}; 0, 1)$ and $(-2, \omega^2, \omega^2, 1, \omega, -2\omega, 1, \omega^2, 1, 1, 1, \omega, 1, \theta)$ in $\Lambda \oplus H$.

Appendix B. Some computer programs

We describe some of the computer programs required for the validation of the proof of Theorem 5.7. They are available at <http://www.math.berkeley.edu/~tathagat>. The calculations were done using the **GP** calculator. All the calculations that are needed for verification of Theorem 5.7 are exact. We used some floating point calculation to find the proof, because $ht(r)$ was defined as a floating point function, but once we had the answer, its validity was checked by exact arithmetic.

We give the names of some of the files that contains the relevant codes, and indicate how to execute them to verify our result. Often there are brief comments above the codes explaining their function.

We represent $a + \omega b \in \mathcal{E}$ as **[a, b]**. So elements of \mathcal{E}^{12} are represented as 12×2 matrices. In the gp calculator, we first need to read the file **inp.gp**. The command for this is

```
\r inp.gp.
```

This file contains the codes for the basic linear algebra operations for \mathcal{E} -lattices. The other codes used and vectors mentioned (and many other things) are contained in the files **isom.gp**, **c12.gp** and **ht.gp** that we also want to read.

The code **C12** generates the codewords of \mathcal{C}_{12} Golay code. The output is saved in the file **c12.gp**. The code **checkleech(v)** returns one if the vector **v** is in Λ and zero otherwise. The vectors **bb[1], bb[2], ..., bb[12]** given in **isom.gp** form a basis for Λ . Using **checkleech** one can check that they are in Λ and one can calculate the discriminant to see that they indeed form a basis. The commands are

```
for(i=1, 12, print(checkleech(bb[i])))
mdet(checkipd(bb)).
```

The code **mdet** calculates determinant and the code **checkipd** calculates the inner product matrix using the inner product **ipd**. Note that the inner product of Λ is one third of **ipd**.

Next, the generators g_1, \dots, g_{50} are generated as described in Lemma 4.5 using the basis **bb**. The code **Trans(l, z)** generates the matrix of the translation $T_{l,z}$. The generators, g_1, \dots, g_{50} are obtained in the coordinates $\Lambda \oplus H$ using the code **genge**. The matrix $C = E_2 E_1^{-1}$ given in Appendix A.2 gives the isomorphism from $\Lambda \oplus H$ to $3E_8 \oplus H$. Multiplying by C we get the generators in the $3E_8 \oplus H$ -coordinate system. They are called **ge[1], ..., ge[50]**. The vectors **ge[i]** were generated by:

```
for(i=1, 50, ge[i]=mv(C, genge[i])).
```

They are already stored in the file **ht.gp**.

Now we can use the code **decreasehtnew(x, m)** to run the height reduction algorithm described in 5.7, with the arguments **x=ge[i]**, **m = i** where $i = 1, \dots, 50$. (First check this for $i = 3, 4, 6$; in these cases the algorithm leads to one of the simple roots showing these generators

are in $\text{rad}(D)$.) For some of the vectors **ge[i]** this algorithm gets stuck and then we perturbed it by reflecting in either **ge[3]**, **ge[4]** or **ge[6]**. Which vector to perturb by is given in the list **perturb**. These are also contained in **ht.gp**.

In fact the file **path.gp** contains the output obtained by running **decreasehtnew(x,m)** with the given values of **perturb**. Read in this file **path.gp**. To validate the proof of 5.7 one only needs to run the code **checktrack(i)** for $i = 1, \dots, 50$. All that the program does is to start with the vector **ge[i]** and reflect by the simple roots in the order given by the vectors **trac1(i)** and **trac2(i)** that are contained in **path.gp** (or one of **ge[3]**, **ge[4]** or **ge[6]** when perturbation is needed). We use the command

```
for(i=1,50,checktrack(i)).
```

In every case one arrives at a unit multiple of one of the simple roots. This verifies the proof.

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