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# On ternary cubic forms that determine central simple algebras of degree 3

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## ABSTRACT

Fixing a field  $F$  of characteristic different from 2 and 3, we consider pairs  $(A, V)$  consisting of a degree 3 central simple  $F$ -algebra  $A$  and a 3-dimensional vector subspace  $V$  of the reduced trace zero elements of  $A$  which is totally isotropic for the trace quadratic form. Each such pair gives rise to a cubic form mapping an element of  $V$  to its cube; therefore we call it a cubic pair over  $F$ . Using the Okubo product in the case where  $F$  contains a primitive cube root of unity, and extending scalars otherwise, we give an explicit description of all isomorphism classes of such pairs over  $F$ . We deduce that a cubic form associated with an algebra in this manner determines the algebra up to (anti-)isomorphism.

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## Introduction

Consider a field  $F$  of characteristic different from 2. Let  $A$  be a quaternion algebra over  $F$  and let  $A^0$  denote the subspace of reduced trace zero elements of  $A$ . Then for all  $x \in A^0$  we have  $x^2 \in F$ . We thus obtain a quadratic form on  $A^0$  mapping  $x$  to  $x^2$ . Up to the sign, this quadratic form is the norm form of the quaternion algebra restricted to  $A^0$ . By Theorem 2.5, p. 57, in [4], this quadratic form determines the quaternion algebra up to isomorphism.

In this paper we shall generalize this construction for algebras of degree 3. Consider a field  $F$  of characteristic different from 2 and 3 and let  $A$  be a degree 3 central simple algebra over  $F$ . Again let  $A^0$  denote the subspace of reduced trace zero elements of  $A$ . Then the cube of an arbitrary element  $x \in A^0$  need not be in  $F$  in general. In fact, it is in  $F$  if and only if the reduced trace of  $x^2$  is equal to zero. Let  $q: A^0 \rightarrow F$  be the trace quadratic form on  $A^0$  (mapping  $x$  to the reduced trace of  $x^2$ ). Then the Witt index of  $q$  is equal to 4 if  $F$  contains a primitive cube root of unity and is equal

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to 3 otherwise (see Lemma 0.1). In both cases there exist 3-dimensional subspaces of  $A^0$  which are totally isotropic for the trace quadratic form. Each such vector subspace  $V \subset A^0$  gives rise to a cubic form. In this paper we shall prove that this cubic form determines the algebra up to isomorphism or anti-isomorphism.

In the first two sections we shall give an explicit description of the pairs  $(A, V)$  where  $A$  and  $V$  are as above. In the first section we assume that the field  $F$  contains a primitive cube root of unity and we use the fact that we may write  $V$  in terms of the Okubo product. In the second section we assume that  $F$  does not contain a primitive cube root of unity, and we shall minimally extend the field  $F$  to use the results of the previous case. In the last section we use these descriptions to prove that a cubic form associated with a pair  $(A, V)$  determines  $A$  up to (anti-)isomorphism.

Throughout the paper, we denote by  $F$  a field of characteristic different from 2 and 3, by  $F_s$  a separable closure of  $F$ , and by  $\Gamma$  the absolute Galois group  $\text{Gal}(F_s/F)$ . We fix  $\omega \in F_s$  a primitive cube root of unity. We say that a pair  $(A, V)$  is a *cubic pair* over  $F$  if  $A$  is a degree 3 central simple  $F$ -algebra and  $V$  is a 3-dimensional subspace of  $A^0$  (= the subspace of reduced trace zero elements of  $A$ ) which is totally isotropic for the trace quadratic form. For a cubic pair  $(A, V)$  over  $F$  we define a cubic form

$$f_{A,V} : V \rightarrow F : x \mapsto x^3.$$

We say that  $\Theta : (A, V) \rightarrow (B, W)$  is an *isomorphism of cubic pairs* over  $F$  if  $\Theta : A \rightarrow B$  is an  $F$ -algebra isomorphism such that  $\Theta(V) = W$ . Note that if  $(A, V)$  and  $(B, W)$  are isomorphic then  $f_{A,V}$  and  $f_{B,W}$  are isometric (i.e. there exists an  $F$ -vector space isomorphism  $\Theta : V \rightarrow W$  such that  $f_{A,V} = f_{B,W} \circ \Theta$ ). For a field extension  $L$  over  $F$  we write  $A_L$  (resp.  $V_L$ ) for  $A \otimes_F L$  (resp.  $V \otimes_F L$ ). Further we let  $\text{Trd}_A$  denote the reduced trace of  $A$ , and for an  $F$ -algebra  $K$ , we denote by  $\text{Tr}_K$  (resp.  $N_K$ ) the trace (resp. the norm) of  $K$ .

## 0. Some results on quadratic forms

Before we start the classification of cubic pairs, we need preliminary results on quadratic forms.

Let  $A$  be a degree 3 central simple algebra over  $F$ . First we shall compute the Witt index of the trace quadratic form of  $A$ .

**Lemma 0.1.** *Let  $q$  be the trace quadratic form on  $A^0$ . Then the Witt index of  $q$  is equal to 4 if  $F$  contains a primitive cube root of unity and is equal to 3 otherwise.*

**Proof.** There exists a splitting field  $L$  of  $A$  of odd degree over  $F$ . Indeed, we may choose  $L := F$  if  $A$  is split and we choose a splitting field of degree 3 over  $F$  otherwise. Then straightforward computations show that the class of the form  $q_L : A_L^0 \rightarrow L$  is equal to the class of  $2\langle 1, 3 \rangle$  in the Witt ring of  $L$ . Hence, by Springer's Theorem about odd degree extensions (see Theorem 2.7, p. 194, in [4]), the Witt index of  $q$  is greater than or equal to 3 and it is 4 if and only if  $F$  contains a primitive cube root of unity.  $\square$

Suppose that  $F$  contains a primitive cube root of unity.

**Lemma 0.2.** *Let  $V$  be a 3-dimensional totally isotropic subspace of  $A^0$ . Then there exist exactly two maximal totally isotropic subspaces  $W_1, W_2$  of  $A^0$  containing  $V$ ; thus  $V = W_1 \cap W_2$ .*

**Proof.** A more general fact is proved in III.1.11 of [2]: in a quadratic space with a Witt index equal to  $n$ , all  $(n - 1)$ -dimensional totally isotropic subspaces are contained in exactly two maximal totally isotropic subspaces.  $\square$

## 1. Classification of cubic pairs over a field with a primitive cube root of unity

We assume that  $F$  contains a primitive cube root of unity.

### 1.1. Okubo product

Let  $A$  be a degree 3 central simple  $F$ -algebra. In [8] the Okubo product over  $A^0$  is defined as follows:

$$x \star y := \mu xy + (1 - \mu)yx - \frac{1}{3} \text{Trd}_A(xy) = \frac{1}{1 - \omega} (yx - \omega xy) - \frac{1}{3} \text{Trd}_A(xy)$$

where  $\mu := \frac{1-\omega}{3}$ . Let  $q$  denote the trace quadratic form on  $A^0$ . Because  $F$  contains a primitive cube root of unity, the Witt index of  $q$  is equal to 4. In [5], Matzri interprets the results of van der Blij and Springer [9] on triality, in the language of the Okubo product: he gives a description of the 4-dimensional subspaces of  $A$  which are totally isotropic for  $q$ , in terms of the Okubo product.

**Theorem 1.1 (Matzri).** *Let  $u \in A^0 \setminus \{0\}$  be such that  $\text{Trd}_A(u^2) = 0$ . Then  $u \star A^0$  and  $A^0 \star u$  are 4-dimensional totally isotropic subspaces of  $A^0$ . Moreover any 4-dimensional totally isotropic subspace is of this form.*

We may also write the 3-dimensional totally isotropic subspaces of  $A^0$  in terms of the Okubo product. By Lemma 0.2, the 3-dimensional totally isotropic subspaces of  $A^0$  are the intersections of two subspaces of the form  $u \star A^0$  or  $A^0 \star u$ . We can be more precise using the following:

**Theorem 1.2 (Matzri).** *Let  $u, v \in A^0 \setminus \{0\}$  be such that  $\text{Trd}_A(u^2) = 0$  and  $\text{Trd}_A(v^2) = 0$ . Then*

1. *the dimension of  $u \star A^0 \cap v \star A^0$  is even;*
2. *if  $u \star v \neq 0$ , then  $\dim(u \star A^0 \cap A^0 \star v) = 1$ ;*
3. *if  $u \star v = 0$ , then  $\dim(u \star A^0 \cap A^0 \star v) = 3$ .*

Note that the Okubo product depends on the choice of the primitive cube root of unity. If we set, for a primitive cube root of unity  $\rho$ ,

$$x \star_\rho y := \mu_\rho xy + (1 - \mu_\rho)yx - \frac{1}{3} \text{Trd}_A(xy)$$

where  $\mu_\rho := \frac{1-\rho}{3}$ , then  $x \star_\omega y = y \star_{\omega^2} x$ . So by Theorem 1.2, the dimension of  $A^0 \star u \cap A^0 \star v$  is also even.

**Corollary 1.3.** *Let  $(A, V)$  be a cubic pair over  $F$ . Then there exist nonzero  $u, v \in A^0$  with  $\text{Trd}_A(u^2) = 0$ ,  $\text{Trd}_A(v^2) = 0$  and  $u \star v = 0$  such that  $V = u \star A^0 \cap A^0 \star v$ .*

The vectors  $u$  and  $v$  are in fact uniquely determined up to scalars. We shall prove this as a part of a more general statement:

**Lemma 1.4.** *If  $(B, W)$  is another cubic pair over  $F$  with  $W = r \star B^0 \cap B^0 \star s$  as in Corollary 1.3, then an  $F$ -algebra isomorphism  $\Theta : A \rightarrow B$  induces an isomorphism  $\Theta : (A, V) \rightarrow (B, W)$  of cubic pairs if and only if  $\Theta(u)F = rF$  and  $\Theta(v)F = sF$ .*

**Proof.** Assume that  $\Theta : A \rightarrow B$  is an  $F$ -algebra isomorphism. If  $\Theta(V) = W$ , then  $W = \Theta(u) \star B^0 \cap B^0 \star \Theta(v) = r \star B^0 \cap B^0 \star s$ . By Lemma 0.2, we have

$$\{\Theta(u) \star B^0, B^0 \star \Theta(v)\} = \{r \star B^0, B^0 \star s\}.$$

If  $r \star B^0 = B^0 \star \Theta(v)$ , then  $W = \Theta(u) \star B^0 \cap r \star B^0$  and by Theorem 1.2, the dimension of  $W$  is even. Hence  $\Theta(u) \star B^0 = r \star B^0$  and  $B^0 \star \Theta(v) = B^0 \star s$ . By Theorem 2.10 in [5], we then have  $\Theta(u)F = rF$  and  $\Theta(v)F = sF$ . The converse is obvious.  $\square$

## 1.2. Classification

We shall describe a cubic pair  $(A, V)$  over  $F$  up to isomorphism. By Corollary 1.3, there exist nonzero  $u, v \in A^0$  such that  $\text{Trd}_A(u^2) = 0$ ,  $\text{Trd}_A(v^2) = 0$ ,  $u \star v = 0$  and  $V = u \star A^0 \cap A^0 \star v$ . Since  $\text{Trd}_A(v) = \text{Trd}_A(v^2) = 0$  we have  $v^3 \in F$ , and similarly  $u^3 \in F$ . Note that  $u \star v = 0$  implies that

$$vu = \frac{1-\omega}{3} \text{Trd}_A(uv) + \omega uv.$$

Set  $t := uv - \frac{1}{3} \text{Trd}_A(uv)$ . Then

$$tu = uvu - \frac{1}{3} \text{Trd}_A(uv)u = \frac{1-\omega}{3} \text{Trd}_A(uv)u + \omega u^2v - \frac{1}{3} \text{Trd}_A(uv)u = \omega ut$$

and similarly  $vt = \omega tv$ . We deduce that  $t^3 \in F$ . Indeed,

$$\begin{aligned} t^2 &= t \left( uv - \frac{1}{3} \text{Trd}_A(uv) \right) \\ &= \omega utv - \frac{1}{3} \text{Trd}_A(uv)t \\ &= \omega u^2v^2 + \frac{\omega^2}{3} \text{Trd}_A(uv)uv + \frac{1}{9} \text{Trd}_A(uv)^2 \end{aligned}$$

and thus

$$\begin{aligned} t^3 &= t^2 \left( uv - \frac{1}{3} \text{Trd}_A(uv) \right) \\ &= \omega^2 ut^2v - \frac{1}{3} \text{Trd}_A(uv)t^2 \\ &= u^3v^3 - \frac{1}{27} \text{Trd}_A(uv)^3 \in F. \end{aligned} \tag{1}$$

This implies in particular that  $\text{Trd}_A(t^2) = 0$ , so  $\text{Trd}_A(u^2v^2) = \frac{1}{3} \text{Trd}_A(uv)^2$ .

We shall prove that  $t^2, t^2u, t^2v \in V$ . First we observe that

$$u \star A^0 = \{x \in A^0 \mid x \star u = 0\}.$$

Indeed, by Proposition (34.19) in [8] we have  $(u \star x) \star u = \frac{1}{6} \text{Trd}_A(u^2)x = 0$  for all  $x \in A^0$ . Hence

$$u \star A^0 \subset \{x \in A^0 \mid x \star u = 0\} = \ker(R_u)$$

where  $R_u : A^0 \rightarrow A^0 : x \mapsto x \star u$ . But  $\dim \ker(R_u) + \dim \text{im}(R_u) = 8$ , thus  $\dim \ker(R_u) = \dim(u \star A^0) = 4$ . Similarly,

$$A^0 \star v = \{x \in A^0 \mid v \star x = 0\}.$$

One can see that  $t^2 \in V$  since  $ut^2 = \omega t^2 u$  and  $t^2 v = \omega v t^2$ . Also  $u(ut^2) = \omega(ut^2)u$  and  $(vt^2)v = \omega v(vt^2)$  imply  $(ut^2) \star u = 0$  and  $v \star (vt^2) = 0$ . Now

$$\begin{aligned} v \star (ut^2) &= \frac{1}{1-\omega} (ut^2 v - \omega v ut^2) - \frac{1}{3} \text{Trd}_A(ut^2 v) \\ &= \frac{1}{1-\omega} \left( ut^2 v - \omega \frac{1-\omega}{3} \text{Trd}_A(uv) t^2 - \omega^2 u vt^2 \right) - \frac{1}{3} \text{Trd}_A(ut^2 v) \\ &= ut^2 v - \frac{\omega}{3} \text{Trd}_A(uv) t^2 - \frac{1}{3} \text{Trd}_A(ut^2 v) \\ &= \frac{\omega^2}{27} \text{Trd}_A(uv)^3 - \frac{\omega^2}{9} \text{Trd}_A(uv) \text{Trd}_A(u^2 v^2) \\ &= 0. \end{aligned}$$

Since  $(vt^2) \star u = \omega^2 v \star (ut^2) = 0$ , we obtain that  $ut^2, vt^2 \in V$ . So  $t^2, t^2 u, t^2 v \in V$  because  $vt = \omega tv$  and  $tu = \omega ut$ .

To work out the classification of cubic pairs we shall distinguish different situations:

**First case.** We assume that  $u, v, t \in A^\times$ . Observe that  $t = \frac{1}{1-\omega}(uv - vu) \neq 0$ , hence  $u$  and  $v$  are linearly independent. Since  $\text{Trd}_A(u) = \text{Trd}_A(v) = 0$ , the vectors  $1, u, v$  are also linearly independent. Therefore  $t^2, t^2 u, t^2 v$  span  $V$ . Set  $\xi := t^2$ ,  $\eta := t^2 v$  and  $\lambda := \frac{1}{3t^3} \text{Trd}_A(uv)$ . Because  $t = uv - \frac{1}{3} \text{Trd}_A(uv)$ , we have  $u = tv^{-1} + \frac{1}{3} \text{Trd}_A(uv) v^{-1}$ . One can check that

$$t^2 u = \frac{\omega}{v^3 t^3} (\xi \eta^2 + \lambda \xi^2 \eta^2).$$

Finally  $A$  is the symbol algebra  $(a, b)_{\omega, F}$  generated by  $\xi$  and  $\eta$  such that  $\xi^3 = a$ ,  $\eta^3 = b$ ,  $\xi \eta = \omega \eta \xi$ , and  $V$  is the vector subspace spanned by  $\xi, \eta$  and  $\xi \eta^2 + \lambda \xi^2 \eta^2$  where  $1 + \lambda^3 a \neq 0$  since  $u^3 \neq 0$ . In this basis of  $V$ , the cubic form  $f_{A, V}$  takes the generalized Hesse normal form:

$$f_{A, V}(x\xi + y\eta + z(\xi\eta^2 + \lambda\xi^2\eta^2)) = ax^3 + by^3 + ab^2(1 + \lambda^3 a)z^3 - 3\omega^2 ab\lambda xyz.$$

The form  $f_{A, V}$  is nonsingular.

Conversely, suppose that  $B$  is the symbol algebra  $(a, b)_{\omega, F}$  generated by  $\xi, \eta$  such that  $\xi^3 = a$ ,  $\eta^3 = b$ ,  $\xi \eta = \omega \eta \xi$ , and  $W$  the vector subspace spanned by  $\xi, \eta, \xi \eta^2 + \lambda \xi^2 \eta^2$ , for some  $a, b \in F^\times$  and  $\lambda \in F$  such that  $1 + \lambda^3 a \neq 0$ . Then one can check that  $(B, W)$  is a cubic pair over  $F$  such that  $f_{B, W}$  is nonsingular.

**Second case.** We suppose that  $u, v \in A^\times$  and  $t = 0$ . Then  $uv = \frac{1}{3} \text{Trd}_A(uv) \in F^\times$ . Thus we may assume that  $u = v^2$ . We need the following:

**Lemma 1.5.** Let  $\xi \in A^0$  be such that  $\xi^3 \in F^\times$ . Then there exists  $\eta \in A$  such that  $\eta^3 \in F^\times$  and  $\xi \eta = \omega \eta \xi$ .

**Proof.** Assume that  $\xi^3 \notin F^{\times 3}$ , then  $F(\xi)$  is a subfield of  $A$ . Let  $\sigma : F(\xi) \rightarrow F(\xi)$  be the  $F$ -automorphism defined by  $\sigma(\xi) = \omega^2 \xi$ . By the Skolem-Noether Theorem, there exists  $\eta \in A^\times$  such that  $\eta x \eta^{-1} = \sigma(x)$  for all  $x \in F(\xi)$ . In particular  $\xi \eta = \omega \eta \xi$ . Because  $\eta = \omega \xi^{-1} \eta \xi$  and  $\eta^2 = \omega^2 \xi^{-1} \eta^2 \xi$ , we have  $\text{Trd}_A(\eta) = 0$ ,  $\text{Trd}_A(\eta^2) = 0$ ; so  $\eta^3 \in F^\times$ .

Now we suppose that  $\xi^3 \in F^{\times 3}$ . Then we may assume that  $\xi^3 = 1$  and  $A = M_3(F)$ . The minimal polynomial of  $\xi$  divides  $t^3 - 1$ , so  $\xi$  is diagonalizable and its eigenvalues are cube roots of unity.

Hence we may assume that

$$\xi = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

with  $\lambda_i \in \{1, \omega, \omega^2\}$ . Since  $\text{tr}(\xi) = 0$  we have  $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, \omega, \omega^2\}$ . Conjugating by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

if necessary, we may assume that  $\lambda_1 = 1, \lambda_2 = \omega, \lambda_3 = \omega^2$ . Then

$$\eta := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is such that  $\eta^3 \in F^\times$  and  $\xi\eta = \omega\eta\xi$ .  $\square$

Let  $w \in A$  be such that  $w^3 \in F^\times$  and  $vw = \omega wv$ . Then the subspace of the elements  $x$  in  $A$  such that  $vx = \omega^2 xv$  is spanned by  $w^2, vw^2, v^2w^2$ , and it is contained in  $V$ ; therefore

$$V = \{x \in A \mid vx = \omega^2 xv\}.$$

Set  $\xi := w^2$  and  $\eta := vw^2$ , then  $A$  is the symbol algebra  $(a, b)_{\omega, F}$  generated by  $\xi$  and  $\eta$  such that  $\xi^3 = a, \eta^3 = b$  and  $\xi\eta = \omega\eta\xi$ , and  $V$  is the vector subspace spanned by  $\xi, \eta$  and  $\xi^2\eta^2$ . In this basis of  $V$ , the form  $f_{A,V}$  takes the generalized Hesse normal form:

$$f_{A,V}(x\xi + y\eta + z\xi^2\eta^2) = ax^3 + by^3 + a^2b^2z^3 - 3\omega^2abxyz.$$

The form  $f_{A,V}$  is singular; more precisely, it is *triangular*, i.e., there exist linearly independent forms  $\varphi_1, \varphi_2, \varphi_3 \in (V \otimes_F F_s)^*$  such that  $f_{A,V} = \varphi_1\varphi_2\varphi_3$  as a cubic form over  $V \otimes_F F_s$ .

Conversely, suppose that  $B$  is the symbol algebra  $(a, b)_{\omega, F}$  generated by  $\xi, \eta$  such that  $\xi^3 = a, \eta^3 = b, \xi\eta = \omega\eta\xi$ , and  $W$  is the vector subspace spanned by  $\xi, \eta, \xi^2\eta^2$ , for some  $a, b \in F^\times$ . Then  $(B, W)$  is a cubic pair over  $F$  and  $f_{B,W}$  is triangular.

**Third case.** We suppose that either  $u \notin A^\times$ , or  $v \notin A^\times$ , or  $t \notin A^\times$  and  $t \neq 0$ . Then the algebra  $A$  is split, so we may assume that  $A = M_3(F)$ . This case is less interesting and thus we shall not give an explicit description of the pair  $(A, V)$ , but we shall only prove that the cubic form  $f_{A,V}$  is singular and not triangular (it is possible to describe  $V$  by matrix computations distinguishing several cases; details can be found in [6]).

To show that  $f_{A,V}$  is not triangular, we first prove a more general lemma on triangular forms.

**Lemma 1.6.** Suppose that  $(B, W)$  is any cubic pair over  $F$  such that  $f_{B,W}$  is triangular. Then  $W = s^2 \star B^0 \cap B^0 \star s$  for some  $s \in B^0$  such that  $s$  is invertible.

**Proof.** We may assume that  $F = F_s$  and  $B = M_3(F)$ . Let  $e_1, e_2, e_3 \in W$  be such that

$$f_{B,W}(x_1e_1 + x_2e_2 + x_3e_3) = x_1x_2x_3$$

for all  $x_1, x_2, x_3 \in F$ . Observe that  $x^3 = \frac{1}{3}\text{tr}(x^3)$  for all  $x \in W$ , hence  $f_{B,W}(x_1e_1 + x_2e_2 + x_3e_3)$  is equal to

$$\sum_{i=1}^3 e_i^3 x_i^3 + \sum_{i \neq j} \text{tr}(e_i^2 e_j) x_i^2 x_j + \text{tr}(e_1 e_2 e_3 + e_2 e_1 e_3) x_1 x_2 x_3$$

for all  $x_1, x_2, x_3 \in F$ . We deduce that  $\text{tr}(e_i^2 e_j) = 0$  for all  $i, j$ . Set  $e_2 = (x_{ij})$  and  $e_3 = (y_{ij})$ .

Suppose that  $e_1^2 \neq 0$ . Since  $e_1^3 = 0$ , we may assume that

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Because  $\text{tr}(e_2) = 0$ ,  $\text{tr}(e_1 e_2) = 0$  and  $\text{tr}(e_1^2 e_2) = 0$ , we have

$$x_{33} = -x_{11} - x_{22}, \quad x_{32} = -x_{21}, \quad x_{31} = 0.$$

From  $\text{tr}(e_1 e_2^2) = 0$  we deduce that  $x_{21}(2x_{11} + x_{22}) = 0$ . If  $x_{21} = 0$  then  $\text{tr}(e_2^2) = 0$  and  $e_2^3 = 0$  imply  $x_{11} = x_{22} = 0$ . Then  $e_1 e_2 + e_2 e_1 = (x_{12} + x_{23})e_1^2$  and it contradicts the fact that  $\text{tr}(e_1^2 e_3) = 0$  and  $\text{tr}(e_1 e_2 e_3 + e_2 e_1 e_3) = 1$ . If  $x_{22} = -2x_{11}$  then

$$e_1 e_2 + e_2 e_1 = \begin{pmatrix} x_{21} & -x_{11} & x_{12} + x_{23} \\ 0 & 0 & -x_{11} \\ 0 & 0 & -x_{21} \end{pmatrix}.$$

By symmetry we know that

$$e_3 = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & -2y_{11} & y_{23} \\ 0 & -y_{21} & y_{11} \end{pmatrix},$$

thus  $\text{tr}(e_1 e_2 e_3 + e_2 e_1 e_3) = 1$  is impossible.

Therefore  $e_1^2 = 0$  (by symmetry we also have  $e_2^2 = 0$ ,  $e_3^2 = 0$ ) and we may assume that

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since  $\text{tr}(e_2) = 0$  and  $\text{tr}(e_1 e_2) = 0$ , we have  $x_{33} = -x_{11} - x_{22}$  and  $x_{31} = 0$ . Then  $e_2^2 = 0$  implies  $x_{21} x_{32} = 0$ . Observe that

$$e_1 e_2 + e_2 e_1 = \begin{pmatrix} 0 & x_{32} & -x_{22} \\ 0 & 0 & x_{21} \\ 0 & 0 & 0 \end{pmatrix}.$$

Because  $\text{tr}(e_1 e_2 e_3 + e_2 e_1 e_3) = 1$  we have either  $x_{21} = 0 = y_{32}$  and  $x_{32}, y_{21} \neq 0$  or  $x_{21}, y_{32} \neq 0$  and  $x_{32} = 0 = y_{21}$ . Thus we may assume that  $x_{21} = 0$  and  $x_{32} = 1$ . From  $e_2^2 = 0$  we deduce that

$$e_2 = \begin{pmatrix} 0 & \alpha & -\alpha\beta \\ 0 & \beta & -\beta^2 \\ 0 & 1 & -\beta \end{pmatrix}$$

for some  $\alpha, \beta \in F$ . We may assume that  $\alpha = \beta = 0$  since the invertible matrix

$$m = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}$$

is such that  $me_1m^{-1} = e_1$  and

$$me_2m^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly we see that

$$e_3 = \begin{pmatrix} \alpha & -\alpha^2 & 0 \\ 1 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some  $\alpha \in F$ . Again we may assume that  $\alpha = 0$  conjugating by

$$\begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

if necessary.

Then one can check that  $W = s^2 \star B^0 \cap B^0 \star s$  with

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \quad \square$$

In our case, the subspace  $V$  is equal to  $u \star A^0 \cap A^0 \star v$  where either  $u \notin A^\times$ , or  $v \notin A^\times$ , or  $t \notin A^\times$  and  $t \neq 0$ . Observe that if  $u, v \in A^\times$ , then  $uF = v^2F$  if and only if  $t = 0$ . Thus, by the previous lemma, the form  $f_{A,V}$  is not triangular.

To prove that  $f_{A,V}$  is singular we shall distinguish different cases.

### 1. Suppose that $v \notin A^\times$ .

If  $\text{tr}(uv) \neq 0$  then, by the relation (1),  $t$  is invertible; hence  $V$  is spanned by  $t^2$ ,  $t^2u$  and  $t^2v$ . Since  $\text{tr}(x(t^2v)^2) = 0$  for all  $x \in V$ , the point  $t^2vF_s$  of the projective plane  $\mathbb{P}_V(F_s)$  is a singular zero of  $f_{A,V}$ .

If  $v^2 \neq 0$  and  $\text{tr}(uv) = 0$ , then  $v^2 \in V$  and  $v^2F_s$  is a singular zero of  $f_{A,V}$ .

If  $v^2 = 0$ , then we may assume that

$$v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{so} \quad u = \begin{pmatrix} \omega^2\alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \omega\alpha_1 & \alpha_4 \\ 0 & 0 & \alpha_1 \end{pmatrix}$$

for some  $\alpha_i \in F$ . If  $\alpha_1 = 0$  then  $vF_s$  is a singular zero of  $f_{A,V}$ . If  $\alpha_1 \neq 0$  then  $V$  is spanned by

$$\begin{pmatrix} \alpha_1 & 0 & -\alpha_3 \\ 0 & \omega\alpha_1 & 0 \\ 0 & 0 & \omega^2\alpha_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & (\omega-1)\alpha_1 & \alpha_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & (\omega-\omega^2)\alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the cubic curve associated with  $f_{A,V}$  is a triple line.



2. Suppose that  $u \notin A^\times$ , then, by symmetry, we deduce that  $f_{A,V}$  is also singular.

3. Suppose that  $u, v \in A^\times$ ,  $t \notin A^\times$  and  $t \neq 0$ .

If  $t^2 \neq 0$  then  $t^2 F_s$  is a singular zero of  $f_{A,V}$ .

If  $t^2 = 0$ , we shall prove that there exists a nonzero  $s \in A$  such that  $s^2 = 0$ ,  $vs = \omega^2 sv$ ,  $s(tv^{-1}) = (tv^{-1})s = 0$ . Since  $u = tv^{-1} + \frac{1}{3}\text{tr}(uv)v^{-1}$ , we then have  $s \in V$  and so  $sF_s$  is a singular zero of  $f_{A,V}$ . Let  $w \in A$  be such that  $w^3 \in F^\times$  and  $vw = \omega wv$ . Since  $v(tv^{-1}) = \omega(tv^{-1})v$  and  $tv^{-1} \neq 0$ , there exist  $\alpha_i \in F$  not all zero such that  $tv^{-1} = \alpha_0 w + \alpha_1 vw + \alpha_2 v^2 w$ . But  $(tv^{-1})^2 = \omega^2 t^2 v^{-2} = 0$ , so  $\alpha_0 \neq 0$ ,  $\alpha_2 = \alpha_1^2 \alpha_0^{-1}$  and  $\alpha_0^3 = v^3 \alpha_1^3$ . Hence  $v^3 \in F^{\times 3}$  and we may assume that  $v^3 = 1$ . Replacing  $w$  by  $\alpha_0^{-1} w$  if necessary, we may assume that  $\alpha_0 = 1$ . Then  $\alpha_1 = 1$ ,  $\omega$  or  $\omega^2$ . Conjugating by  $w$  or  $w^{-1}$  if necessary, we may assume that  $tv^{-1} = w + vw + v^2 w$ . Then we may choose  $s = w^2 + vw^2 + v^2 w^2$ .

We summarize the above classification in the following theorem.

**Theorem 1.7.** Suppose that  $F$  contains a primitive cube root of unity. Let  $(A, V)$  be a cubic pair over  $F$ .

1. If  $f_{A,V}$  is nonsingular, then

$$(A, V) \cong ((a, b)_{\omega, F}, \text{span}_F\{\xi, \eta, \xi\eta^2 + \lambda\xi^2\eta^2\})$$

for some  $a, b \in F^\times$ ,  $\lambda \in F$  such that  $1 + \lambda^3 a \neq 0$ , where  $\xi, \eta$  are generators of the symbol algebra such that  $\xi^3 = a$ ,  $\eta^3 = b$ ,  $\xi\eta = \omega\eta\xi$ . Conversely, let  $a, b \in F^\times$ ,  $\lambda \in F$  be such that  $1 + \lambda^3 a \neq 0$ . Let  $B$  be the symbol algebra  $(a, b)_{\omega, F}$  generated by  $\xi, \eta$  such that  $\xi^3 = a$ ,  $\eta^3 = b$ ,  $\xi\eta = \omega\eta\xi$ , and  $W$  the subspace spanned by  $\xi, \eta, \xi\eta^2 + \lambda\xi^2\eta^2$ . Then  $(B, W)$  is a cubic pair over  $F$  and  $f_{B,W}$  is nonsingular. In the basis  $(\xi, \eta, \xi\eta^2 + \lambda\xi^2\eta^2)$ , the form  $f_{B,W}$  takes the generalized Hesse normal form:

$$(x\xi + y\eta + z(\xi\eta^2 + \lambda\xi^2\eta^2))^3 = ax^3 + by^3 + ab^2(1 + \lambda^3 a)z^3 - 3\omega^2 ab\lambda xyz.$$

2. If  $f_{A,V}$  is triangular, then

$$(A, V) \cong ((a, b)_{\omega, F}, \text{span}_F\{\xi, \eta, \xi^2\eta^2\})$$

for some  $a, b \in F^\times$ , where  $\xi, \eta$  are generators of the symbol algebra such that  $\xi^3 = a$ ,  $\eta^3 = b$  and  $\xi\eta = \omega\eta\xi$ . Conversely, let  $a, b \in F^\times$ , let  $B$  be the symbol algebra  $(a, b)_{\omega, F}$  generated by  $\xi, \eta$  such that  $\xi^3 = a$ ,  $\eta^3 = b$ ,  $\xi\eta = \omega\eta\xi$ , and  $W$  the subspace spanned by  $\xi, \eta, \xi^2\eta^2$ . Then  $(B, W)$  is a cubic pair over  $F$  and  $f_{B,W}$  is triangular. In the basis  $(\xi, \eta, \xi^2\eta^2)$ , the form  $f_{B,W}$  takes the generalized Hesse normal form:

$$(x\xi + y\eta + z\xi^2\eta^2)^3 = ax^3 + by^3 + a^2b^2z^3 - 3\omega^2 abxyz.$$

3. If  $f_{A,V}$  is singular and not triangular, then  $A$  is split.

## 2. Classification of cubic pairs over a field without primitive cube root of unity

Suppose that  $F$  does not contain a primitive cube root of unity. We shall give the classification of cubic pairs over  $F$  in the case where the associated cubic form is nonsingular or triangular. For the remaining cases we know by Theorem 1.7 that the algebra is split and it is just a matter of matrix computations to describe all the possible subspaces up to conjugacy (see [6] for details). We shall extend the scalars to  $F(\omega)$  to use the previous classification. To simplify notations, let  $T$  denote the reduced trace of  $A_{F(\omega)}$ . Throughout this section, we denote by  $\sigma$  the  $F$ -automorphism of  $F(\omega)$  such that  $\sigma(\omega) = \omega^2$ .

## 2.1. Nonsingular form

Let  $(A, V)$  be a cubic pair over  $F$  such that  $f_{A,V}$  is nonsingular. By Section 1.2, there exist nonzero  $u, v \in A_{F(\omega)}^0$  such that  $T(u^2) = 0$ ,  $T(v^2) = 0$ ,  $u \star v = 0$  and

$$V_{F(\omega)} = (u \star A_{F(\omega)}^0) \cap (A_{F(\omega)}^0 \star v)$$

where  $u, v, t := uv - \frac{1}{3}T(uv) \in A_{F(\omega)}^\times$ . Then  $V_{F(\omega)}$  is spanned by  $t^2$ ,  $t^2v$  and  $t^2u$ .

We extend  $\sigma$  to an  $F$ -automorphism of  $A_{F(\omega)}$ : for  $x \in A_{F(\omega)}$  and  $\lambda \in F(\omega)$ , define  $\sigma(x \otimes \lambda) = x \otimes \sigma(\lambda)$ . Then  $A$  (resp.  $V$ ) consists of the elements of  $A_{F(\omega)}$  (resp.  $V_{F(\omega)}$ ) which are fixed under  $\sigma$ . Note that  $\sigma(x \star y) = \sigma(y) \star \sigma(x)$  for all  $x, y \in A_{F(\omega)}$ . Since  $\sigma(V_{F(\omega)}) = V_{F(\omega)}$ , we have

$$(\sigma(v) \star A_{F(\omega)}^0) \cap (A_{F(\omega)}^0 \star \sigma(u)) = (u \star A_{F(\omega)}^0) \cap (A_{F(\omega)}^0 \star v).$$

Hence there exists  $\lambda \in F(\omega)^\times$  such that  $\sigma(u) = \lambda v$ . Replacing  $v$  by  $\lambda v$  if necessary we may assume that  $\sigma(u) = v$  and thus  $\sigma(v) = u$ . Recall that

$$vu = \frac{1-\omega}{3}T(uv) + \omega uv,$$

hence we have

$$\begin{aligned} \sigma(t) &= \sigma(uv) - \frac{1}{3}\sigma(T(uv)) \\ &= \sigma(u)\sigma(v) - \frac{1}{3}T(\sigma(uv)) \\ &= vu - \frac{1}{3}T(vu) \\ &= \omega t. \end{aligned}$$

Therefore  $\omega^2 t \in A$  and  $\omega t^2 \in V$ . Set  $e := \omega t^2$ , then  $V$  is spanned by  $e$ ,  $e(v+u)$  and  $e(\omega v + \omega^2 u)$ .

We shall find a Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra  $(L, \rho)$  such that  $L \subset A$ , the vector  $e$  and  $L$  generate  $A$ , and  $ex = \rho(x)e$  for all  $x \in L$ . To do this we first construct an element  $\eta \in A_{F(\omega)}$  such that  $e\eta = \omega\eta e$ ,  $\eta^3 \in F(\omega)^\times$  and  $\sigma(\eta)\eta \in F^\times$ . Recall that  $t^3 = u^3v^3 - \frac{1}{27}T(uv)^3$ ,  $vt = \omega tv$ ,  $tu = \omega ut$  and

$$t^2 = \omega u^2 v^2 + \frac{\omega^2}{3}T(uv)uv + \frac{1}{9}T(uv)^2.$$

Set

$$\eta := \frac{T(uv)}{3}v + \frac{1}{t^3}e^2v,$$

then  $e\eta = \omega\eta e$ ,  $\eta^3 = u^3v^6$  and

$$\begin{aligned} \sigma(\eta)\eta &= \frac{T(uv)^2}{9}uv - \frac{\omega T(uv)}{3}utv + \omega^2 ut^2v \\ &= u^3v^3. \end{aligned}$$

We set  $\lambda := u^3v^3$ , so  $\eta + \lambda\eta^{-1}$  is fixed under  $\sigma$ .

If  $\eta^3 \notin F(\omega)^{\times 3}$ , then set  $L := F(\eta + \lambda\eta^{-1})$  and let  $\rho : L \rightarrow L$  be the  $F$ -automorphism defined by  $\rho(\eta + \lambda\eta^{-1}) = \omega\eta + \omega^2\lambda\eta^{-1}$ . Then  $L$  is a cyclic extension of degree 3 over  $F$  which is contained in  $A$  and with a Galois group generated by  $\rho$ . Moreover  $ex = \rho(x)e$  for all  $x \in L$ .

If  $\eta^3 \in F(\omega)^{\times 3}$ , then  $\eta^3 = v^3$  for some  $v \in F(\omega)^{\times}$ . Replacing  $\eta$  by  $v^{-1}\eta$  if necessary, we may assume that  $\eta^3 = 1$  and  $\sigma(\eta)\eta = 1$ . Set

$$L := F \cdot 1 + F \cdot (\eta + \eta^{-1}) + F \cdot (\omega\eta + \omega^2\eta^{-1})$$

and define  $\rho$  as the  $F$ -automorphism of  $L$  such that  $\rho(\eta + \eta^{-1}) = \omega\eta + \omega^2\eta^{-1}$  and  $\rho(\omega\eta + \omega^2\eta^{-1}) = \omega^2\eta + \omega\eta^{-1}$ . Since  $(\eta + \eta^{-1})^2 = \eta + \eta^{-1} + 2$ , the algebra  $L$  is isomorphic to  $F \times F \times F$ . Again  $(L, \rho)$  is a Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra such that  $L \subset A$  and  $ex = \rho(x)e$  for all  $x \in L$ .

In both cases we obtain that  $A = \bigoplus_{i=0}^2 Le^i$  where the multiplication in  $A$  is determined by  $e^3 = a \in F^{\times}$  and  $ex = \rho(x)e$  for all  $x \in L$ .

To finish the description of the pair  $(A, V)$ , we shall write  $ev$  in function of  $e$  and  $\eta$ :

$$\begin{aligned} ev &= e \left( \frac{T(uv)}{3} + \frac{1}{t^3} e^2 \right)^{-1} \eta \\ &= \frac{1}{u^3 v^3} (\alpha + a^{-1} \alpha^2 e + e^2) \eta \end{aligned}$$

where  $\alpha := -T(uv)t^3/3$ . Hence  $V_{F(\omega)}$  is spanned by  $e$ ,  $(\alpha + a^{-1} \alpha^2 e + e^2)\eta$  and  $(\alpha + a^{-1} \alpha^2 e + e^2)\lambda\eta^{-1}$ . Note that  $\alpha^3 \neq a^2$  since  $ev$  is invertible. We obtain that

$$V = \text{span}_F \{ e, (\alpha + a^{-1} \alpha^2 e + e^2)\theta, (\alpha + a^{-1} \alpha^2 e + e^2)\rho(\theta) \}$$

where  $\theta = \eta + \lambda\eta^{-1} \in L \setminus \{0\}$  is such that  $\theta + \rho(\theta) + \rho^2(\theta) = 0$ .

We shall prove that the cubic form  $f_{A,V}$  is isometric to the form

$$\text{Tr}_{K,\beta} - 3bN_K : K \rightarrow F : x \mapsto \text{Tr}_K(\beta x^3) - 3bN_K(x)$$

for  $K = F \times F(\omega)$  and, for some  $\beta \in K^{\times}$  and  $b \in F$ . We set

$$v_1 := e, \quad v_2 := (\alpha + a^{-1} \alpha^2 e + e^2)\theta, \quad v_3 := (\alpha + a^{-1} \alpha^2 e + e^2)\rho(\theta)$$

so that  $V$  is the vector space spanned by  $v_1, v_2, v_3$ . Then  $f_{A,V}(xv_1 + yv_2 + zv_3)$  is equal to

$$(xu_1 + (y + \omega z)u_2 + (y + \omega^2 z)u_3)^3$$

where  $u_1 := e$ ,  $u_2 := (\alpha + a^{-1} \alpha^2 e + e^2)\eta$ ,  $u_3 := (\alpha + a^{-1} \alpha^2 e + e^2)\lambda\eta^{-1}$ . But  $(xu_1 + yu_2 + zu_3)^3$  is equal to

$$ax^3 + \eta^3(a^{-1}\alpha^3 - a)^2y^3 + \sigma(\eta^3)(a^{-1}\alpha^3 - a)^2z^3 - 3\lambda\alpha(a^{-1}\alpha^3 - a)xyz.$$

Therefore  $f_{A,V}$  is isometric to the form  $\text{Tr}_{K,\beta} - 3bN_K$  where  $K := F \times F(\omega)$ ,  $\beta := (a, \eta^3(a^{-1}\alpha^3 - a)^2)$  and  $b := \lambda\alpha(a^{-1}\alpha^3 - a)$ .

Observe that if  $(L, \rho)$  is a Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra, then there exist  $\lambda \in F^{\times}$  and  $\phi \in L \otimes_F F(\omega)$  such that  $\phi^3 \in F(\omega)^{\times}$ ,  $\phi \notin F(\omega)$ ,  $\sigma(\phi^3) = \lambda^3\phi^{-3}$  and

$$L = F \cdot 1 + F \cdot (\phi + \lambda\phi^{-1}) + F \cdot (\omega\phi + \omega^2\lambda\phi^{-1}).$$

Indeed, suppose that  $(L, \rho)$  is not split, then  $L(\omega) \cong L \otimes_F F(\omega)$  is a Galois extension of  $F$  with a Galois group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ . Let  $\tilde{\sigma}, \tau$  be  $F$ -automorphisms of  $L(\omega)$  such that  $L$  (resp.  $F(\omega)$ ) is the subfield of  $L(\omega)$  which is fixed under  $\tilde{\sigma}$  (resp.  $\tau$ ). Let  $\phi \in L(\omega)$  be such that  $\phi^3 \in F(\omega)^\times$  and  $\phi \notin F(\omega)$ . Replacing  $\phi$  by  $\phi^{-1}$  if necessary, we may assume that  $\tau(\phi) = \omega\phi$ . Hence

$$\tau\tilde{\sigma}(\phi) = \tilde{\sigma}\tau(\phi) = \tilde{\sigma}(\omega\phi) = \omega^2\tilde{\sigma}(\phi).$$

Thus  $\tilde{\sigma}(\phi) = \lambda\phi^{-1}$  for some  $\lambda \in F(\omega)^\times$ . But

$$\phi = \tilde{\sigma}(\lambda\phi^{-1}) = \tilde{\sigma}(\lambda)\lambda^{-1}\phi,$$

so  $\tilde{\sigma}(\lambda) = \lambda$  and  $\lambda \in F^\times$ . Then we have  $L = F(\phi + \lambda\phi^{-1})$  with

$$\sigma(\phi^3) = (\tilde{\sigma}(\phi))^3 = (\lambda\phi^{-1})^3 = \lambda^3\phi^{-3}.$$

Suppose that  $L = F \times F \times F$ , then we may choose  $\lambda = 1$  and  $\phi = (1, \omega, \omega^2)$ .

Now let  $(L, \rho)$  be a Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra,  $a \in F^\times$  and  $\alpha \in F$  such that  $\alpha^3 \neq a^2$ . Set

$$(B, W) := \left( \bigoplus_{i=0}^2 Le^i, \text{span}_F[e, (\alpha + a^{-1}\alpha^2e + e^2)\theta, (\alpha + a^{-1}\alpha^2e + e^2)\rho(\theta)] \right),$$

where the multiplication in  $B$  is defined by  $e^3 = a$ ,  $ex = \rho(x)e$  for all  $x \in L$ , and  $\theta \in L \setminus \{0\}$  is such that  $\theta + \rho(\theta) + \rho^2(\theta) = 0$ . Then one can check that  $(B, W)$  is a cubic pair over  $F$  and  $f_{B,W}$  is nonsingular.

We summarize this subsection by the following theorem:

**Theorem 2.1.** *Suppose that  $F$  does not contain a primitive cube root of unity. Let  $(A, V)$  be a cubic pair over  $F$  such that  $f_{A,V}$  is nonsingular. Then  $(A, V)$  is isomorphic to*

$$\left( \bigoplus_{i=0}^2 Le^i, \text{span}_F[e, (\alpha + a^{-1}\alpha^2e + e^2)\theta, (\alpha + a^{-1}\alpha^2e + e^2)\rho(\theta)] \right)$$

for some Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra  $(L, \rho)$ ,  $a \in F^\times$ ,  $\alpha \in F$  such that  $\alpha^3 \neq a^2$ , where  $e^3 = a$ ,  $ex = \rho(x)e$  for all  $x \in L$  and  $\theta \in L \setminus \{0\}$  is such that  $\theta + \rho(\theta) + \rho^2(\theta) = 0$ . Conversely, let  $(L, \rho)$  be a Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra,  $a \in F^\times$ ,  $\alpha \in F$  such that  $\alpha^3 \neq a^2$ . Let  $B = \bigoplus_{i=0}^2 Le^i$  be the algebra with multiplication defined by  $e^3 = a$ ,  $ex = \rho(x)e$  for all  $x \in L$ , and let  $W$  be the subspace spanned by  $e, (\alpha + a^{-1}\alpha^2e + e^2)\theta, (\alpha + a^{-1}\alpha^2e + e^2)\rho(\theta)$ , where  $\theta \in L \setminus \{0\}$  is such that  $\theta + \rho(\theta) + \rho^2(\theta) = 0$ . Then  $(B, W)$  is a cubic pair over  $F$  and  $f_{B,W}$  is nonsingular. Let  $\phi \in L \otimes_F F(\omega)$  and  $\lambda \in F^\times$  be such that  $\phi^3 \in F(\omega)^\times$ ,  $\phi \notin F(\omega)$ ,  $\sigma(\phi^3) = \lambda\phi^{-3}$  and  $1, \phi + \lambda\phi^{-1}, \omega\phi + \omega^2\lambda\phi^{-1}$  span  $L$ , where  $\sigma$  is the nontrivial  $F$ -automorphism of  $F(\omega)$ . Then  $f_{B,W}$  is isometric to  $\text{Tr}_{K,\beta} - 3b\mathbf{N}_K$  where  $K = F \times F(\omega)$ ,  $\beta = (a, \phi^3(a^{-1}\alpha^3 - a)^2)$  and  $b = \lambda\alpha(a^{-1}\alpha^3 - a)$ .

## 2.2. Triangular form

Let  $(A, V)$  be a cubic pair over  $F$  such that  $f_{A,V}$  is triangular. By Section 1.2, there exists  $v \in A_{F(\omega)}^0$  such that  $T(v^2) = 0$ ,  $v \in A_{F(\omega)}^\times$  and

$$V = (v^2 \star A_{F(\omega)}^0) \cap (A_{F(\omega)}^0 \star v);$$

then  $V = \{x \in A_{F(\omega)} \mid vx = \omega^2 xv\}$ . Fix  $e \in V$ , then  $ev = \omega ve$ . We extend the  $F$ -automorphism  $\sigma$  of  $F(\omega)$  to  $A_{F(\omega)}$ . Then  $\sigma(v) = \lambda v^2$  for some  $\lambda \in F(\omega)^\times$ . Since

$$v = \sigma(\lambda v^2) = \sigma(\lambda) \lambda^2 v^4,$$

we deduce that  $\sigma(\lambda) \lambda^2 v^3 = 1$ . Hence

$$\sigma(\lambda v) = \sigma(\lambda) \sigma(v) = \lambda^{-2} v^{-3} \lambda v^2 = (\lambda v)^{-1};$$

so we may assume that  $\sigma(v) = v^{-1}$ . Set

$$L := F \cdot 1 + F \cdot (v + v^{-1}) + F \cdot (\omega v + \omega^2 v^{-1})$$

and define  $\rho$  as the  $F$ -automorphism of  $L$  such that  $\rho(v + v^{-1}) = \omega v + \omega^2 v^{-1}$  and  $\rho(\omega v + \omega^2 v^{-1}) = \omega^2 v + \omega v^{-1}$ . Then  $(L, \rho)$  is a Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra (note that  $(L, \rho)$  is split if and only if  $v^3 \in F(\omega)^{\times 3}$ ). Moreover  $L \subset A$ ,  $ex = \rho(x)e$  for all  $x \in L$ ,  $A = \bigoplus_{i=0}^2 Le^i$  and  $V = eL$ . It is easy to check that  $f_{A,V}$  is isometric to  $aN_L$ , where  $a := e^3$ .

Conversely, let  $(L, \rho)$  be a Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra and  $a \in F^\times$ . Set

$$(B, W) := \left( \bigoplus_{i=0}^2 Le^i, eL \right)$$

where the multiplication in  $B$  is defined by  $e^3 = a$  and  $ex = \rho(x)e$  for all  $x \in L$ . Then  $(B, W)$  is a cubic pair over  $F$  and  $f_{B,W}$  is triangular.

Thus we obtain:

**Theorem 2.2.** Suppose that  $F$  does not contain a primitive cube root of unity. Let  $(A, V)$  be a cubic pair over  $F$  such that  $f_{A,V}$  is triangular. Then  $(A, V)$  is isomorphic to

$$\left( \bigoplus_{i=0}^2 Le^i, eL \right)$$

for some Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra  $(L, \rho)$  and  $a \in F^\times$ , where  $e^3 = a$  and  $ex = \rho(x)e$  for all  $x \in L$ . Conversely, let  $(L, \rho)$  be a Galois  $\mathbb{Z}/3\mathbb{Z}$ -algebra and  $a \in F^\times$ . Let  $B = \bigoplus_{i=0}^2 Le^i$  be the algebra with multiplication defined by  $e^3 = a$ ,  $ex = \rho(x)e$  for all  $x \in L$ , and set  $W := eL$ . Then  $(B, W)$  is a cubic pair over  $F$  and  $f_{B,W}$  is triangular. Moreover  $f_{B,W}$  is isometric to  $aN_L$ .

### 3. The form determines the algebra

Let  $(A, V)$  and  $(A', V')$  be cubic pairs over  $F$  and suppose that  $f_{A,V}$  and  $f_{A',V'}$  are isometric. In this section we shall prove that  $A$  and  $A'$  are either isomorphic or anti-isomorphic.

We may assume that  $F$  contains a primitive cube root of unity. Indeed, if  $A \otimes_F F(\omega) \cong A' \otimes_F F(\omega)$ , then  $A \cong A'$  since  $A$  and  $A'$  are central simple algebras of degree 3 and  $F(\omega)/F$  is an extension of degree at most 2. We may also assume that  $A$  is division, because there is nothing to prove if  $A$  and  $A'$  are split. Therefore, by Theorem 1.7, the cubic form  $f_{A,V}$  is either nonsingular or triangular.

**First case.** Suppose that  $f_{A,V}$  is nonsingular, then so is  $f_{A',V'}$ . By Theorem 1.7, there exist  $a_i, a'_i \in F^\times$ ,  $\lambda, \lambda' \in F$  such that  $1 + \lambda^3 a_1, 1 + \lambda'^3 a'_1 \neq 0$  and

$$A = (a_1, a_2)_{\omega, F}, \quad V = \text{span}_F \langle \xi_1, \xi_2, \xi_1 \xi_2^2 + \lambda \xi_1^2 \xi_2^2 \rangle,$$

$$A' = (a'_1, a'_2)_{\omega, F}, \quad V' = \text{span}_F \langle \xi'_1, \xi'_2, \xi'_1 \xi'^2_2 + \lambda' \xi'^2_1 \xi'^2_2 \rangle$$

where  $A$  (resp.  $A'$ ) is generated by  $\xi_1, \xi_2$  such that  $\xi_i^3 = a_i$  and  $\xi_1 \xi_2 = \omega \xi_2 \xi_1$  (resp.  $\xi'_1, \xi'_2$  such that  $\xi'^3_i = a'_i$  and  $\xi'_1 \xi'_2 = \omega \xi'_2 \xi'_1$ ). Set

$$\xi_3 := \xi_1 \xi_2^2 + \lambda \xi_1^2 \xi_2^2, \quad a_3 := \xi_3^3, \quad \xi'_3 := \xi'_1 \xi'^2_2 + \lambda' \xi'^2_1 \xi'^2_2, \quad a'_3 := \xi'^3_3.$$

We recall properties of nonsingular cubic forms and we refer to [1] or [3] for more details. A nonsingular cubic form  $f$  on a 3-dimensional vector space  $V$  has 9 inflexion points in the projective plane  $\mathbb{P}_V(F_s)$ . There are four triangles (i.e. cubic curves associated with triangular cubic forms) in  $\mathbb{P}_V(F_s)$  with the property that each inflexion point is incident with one and only one line of the triangle and each line of the triangle passes through exactly 3 inflexion points. These triangles are called *inflexional triangles* of  $f$ . For a triangular cubic form  $g = \varphi_1 \varphi_2 \varphi_3$  over  $V$ , we denote by  $g = 0$  the triangle formed by the zeros of the linear forms  $\varphi_i$  in  $\mathbb{P}_V(F_s)$ .

The map  $V \rightarrow K := F \times F \times F$  which sends  $\xi_1, \xi_2, \xi_3$  on the canonical basis of  $K$  is an  $F$ -vector space isomorphism. Under this isomorphism,  $f_{A,V}$  is isometric to the form

$$\text{Tr}_{K,\alpha} - 3bN_K : K \rightarrow F : x \mapsto \text{Tr}_K(\alpha x^3) - 3bN_K(x)$$

where  $\alpha = (a_1, a_2, a_3)$  and  $b = \omega^2 a_1 a_2 \lambda$ . The inflexional triangles of the form  $\text{Tr}_{K,\alpha} - 3bN_K$  are  $N_K = 0$  and  $\text{Tr}_{K,\alpha} - 3\theta N_K = 0$  for all  $\theta \in F_s$  such that  $\theta^3 = N_K(\alpha)$ . Let  $(\varphi_1, \varphi_2, \varphi_3)$  denote the dual basis of  $(\xi_1, \xi_2, \xi_3)$ , then under the previous isomorphism  $V \rightarrow K$ , the form  $\varphi_1 \varphi_2 \varphi_3$  is isometric to  $N_K$ . Hence  $\Gamma$  acts trivially on the lines of the corresponding inflexional triangle. In fact, we have the following:

**Lemma 3.1.** *There exists a unique inflexional triangle of  $f_{A,V}$  whose lines are defined over  $F$ .*

**Proof.** Suppose that  $\Gamma$  acts trivially on the lines of  $\text{Tr}_{K,\alpha} - 3\theta N_K = 0$ , for some  $\theta \in F_s$  such that  $\theta^3 = N_K(\alpha)$ . For  $x = (x_1, x_2, x_3) \in K$ ,  $\text{Tr}_K(\alpha x^3) - 3\theta N_K(x)$  is equal to

$$(\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3)(\theta_1 x_1 + \omega \theta_2 x_2 + \omega^2 \theta_3 x_3)(\theta_1 x_1 + \omega^2 \theta_2 x_2 + \omega \theta_3 x_3)$$

for some  $\theta_i \in F_s$  such that  $\theta_i^3 = a_i$  and  $\theta_1 \theta_2 \theta_3 = \theta$ . Since  $\Gamma$  acts trivially on the line  $\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 = 0$ , there exists a nonzero  $u \in F$  such that  $\theta_2 = u \theta_1$ . This implies that  $a_2 = u^3 a_1$ , which contradicts the assumption that  $A$  is division.  $\square$

Let  $(\varphi'_1, \varphi'_2, \varphi'_3)$  be the dual basis of  $(\xi'_1, \xi'_2, \xi'_3)$ . Then  $\varphi'_1 \varphi'_2 \varphi'_3 = 0$  is an inflexional triangle of  $f_{A',V'}$  whose lines are defined over  $F$ . Let  $\Theta : V \rightarrow V'$  be an  $F$ -vector space isomorphism such that  $f_{A,V} = f_{A',V'} \circ \Theta$ , then

$$\varphi_1 \varphi_2 \varphi_3 F = (\varphi'_1 \varphi'_2 \varphi'_3 \circ \Theta) F.$$

Thus there exist  $\lambda_i \in F^\times$  and a permutation  $\pi$  of  $\{1, 2, 3\}$  such that

$$\varphi'_i \circ \Theta = \lambda_{\pi(i)} \varphi_{\pi(i)} \quad \text{for all } i \in \{1, 2, 3\}.$$

For all  $i, j \in \{1, 2, 3\}$ , we have

$$\varphi'_i \circ \Theta(\xi_{\pi(j)}) = \lambda_{\pi(i)} \varphi_{\pi(i)}(\xi_{\pi(j)}) = \delta_{ij} \lambda_{\pi(i)},$$

hence  $\Theta(\xi_{\pi(j)}) = \lambda_{\pi(j)} \xi'_j$ . We obtain that  $a_{\pi(j)} = \lambda_{\pi(j)}^3 a'_j$  and  $b = \lambda_1 \lambda_2 \lambda_3 b'$  where  $b' = \omega^2 a'_1 a'_2 \lambda'$ . But  $a_1$  is the only scalar among the  $a_i$ 's such that

$$\frac{a_1 a_2 a_3 - b^3}{a_i^2} \in F^{\times 3}.$$

Indeed  $a_1 a_2 a_3 - b^3 = a_1^2 a_2^3$ , thus  $a_1^{-2}(a_1 a_2 a_3 - b^3) \in F^{\times 3}$ ; if  $a_2^{-2}(a_1 a_2 a_3 - b^3) \in F^{\times 3}$ , then  $a_1 F^{\times 3} = a_2 F^{\times 3}$  and it contradicts the assumption that  $A$  is division; similarly,  $a_3^{-2}(a_1 a_2 a_3 - b^3) \notin F^{\times 3}$ . On the other hand, we have  $a_1'^{-2}(a'_1 a'_2 a'_3 - b'^3) \in F^{\times 3}$  and

$$\frac{a_1 a_2 a_3 - b^3}{a_{\pi(1)}^2} = \left( \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_{\pi(1)}^2} \right)^3 \frac{a'_1 a'_2 a'_3 - b'^3}{a_1'^2} \in F^{\times 3};$$

therefore  $\pi(1) = 1$ . If  $\pi(2) = 2$ , then

$$(\lambda_i \xi'_i)^3 = a_i \quad \text{and} \quad (\lambda_1 \xi'_1)(\lambda_2 \xi'_2) = \omega(\lambda_2 \xi'_2)(\lambda_1 \xi'_1),$$

thus  $A' \cong A$ . If  $\pi(2) = 3$ , then

$$(\lambda_1 \xi'_1)^3 = a_1, \quad (\lambda_2 \xi'_3)^3 = a_2 \quad \text{and} \quad (\lambda_1 \xi'_1)(\lambda_2 \xi'_3) = \omega^2(\lambda_2 \xi'_3)(\lambda_1 \xi'_1),$$

thus  $A' \cong A^{\text{op}}$ .

**Second case.** Suppose that  $f_{A,V}$  is triangular. Then there exist  $a_i, a'_i \in F^\times$  such that

$$\begin{aligned} A &= (a_1, a_2)_{\omega, F}, & V &= \text{span}_F(\xi_1, \xi_2, \xi_1^2 \xi_2^2), \\ A' &= (a'_1, a'_2)_{\omega, F}, & V' &= \text{span}_F(\xi'_1, \xi'_2, \xi_1'^2 \xi_2'^2) \end{aligned}$$

where  $A$  (resp.  $A'$ ) is generated by  $\xi_1, \xi_2$  such that  $\xi_i^3 = a_i$  and  $\xi_1 \xi_2 = \omega \xi_2 \xi_1$  (resp.  $\xi'_1, \xi'_2$  such that  $\xi_i'^3 = a'_i$  and  $\xi'_1 \xi'_2 = \omega \xi'_2 \xi'_1$ ). Let  $\theta \in F_s$  be a cube root of  $a_1^{-1} a_2$ . Since  $A$  is division,  $\theta \notin F$ . We have

$$f_{A,V}(x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_1^2 \xi_2^2) = a_1 N_{F(\theta)}(x_1 + x_2 \theta + \omega^2 a_1 x_3 \theta^2).$$

Let  $\theta' \in F_s$  be a cube root of  $a_1'^{-1} a'_2$ . Similarly,  $f_{A',V'}(x_1 \xi'_1 + x_2 \xi'_2 + x_3 \xi_1'^2 \xi_2'^2)$  is equal to

$$a'_1 (x_1 + x_2 \theta' + \omega^2 a'_1 x_3 \theta'^2) (x_1 + \omega x_2 \theta' + \omega a'_1 x_3 \theta'^2) (x_1 + \omega^2 x_2 \theta' + a'_1 x_3 \theta'^2).$$

Since  $f_{A,V} \cong f_{A',V'}$ , we have  $\theta' \notin F$  and, by Proposition 8 in [7], the fields  $F(\theta)$  and  $F(\theta')$  are isomorphic. We deduce that either  $\theta' F = \theta F$  or  $\theta' F = \theta^2 F$ . Identifying  $\theta$  with  $\xi_1^{-1} \xi_2$  (resp.  $\theta'$  with  $\xi_1'^{-1} \xi_2'$ ), we have

$$A = \bigoplus_{i=0}^2 F(\theta) \xi_1^i \quad \text{and} \quad A' = \bigoplus_{i=0}^2 F(\theta') \xi_1'^i$$

where  $\xi_1 \theta = \omega \theta \xi_1$  and  $\xi_1' \theta' = \omega \theta' \xi_1'$ . Because  $f_{A,V}$  is isometric to  $f_{A',V'}$ , there exist  $u_i \in F$  such that

$$a_1 N_{F(\theta)}(u_1 + u_2 \theta + u_3 \theta^2) = a'_1.$$

Set  $\eta_1 := \xi_1(u_1 + u_2\theta + u_3\theta^2)$ , then

$$\eta_1^3 = a_1 N_{F(\theta)}(u_1 + u_2\theta + u_3\theta^2) = a'_1 \quad \text{and} \quad \eta_1\theta = \omega\theta\eta_1.$$

Hence  $A = \bigoplus_{i=0}^2 F(\theta)\eta_1^i$  with  $\eta_1^3 = a'_1$  and  $\eta_1\theta = \omega\theta\eta_1$ . So

$$A \cong \begin{cases} A' & \text{if } \theta'F = \theta F, \\ A'^{\text{op}} & \text{if } \theta'F = \theta^2 F. \end{cases}$$

We thus obtain the following:

**Theorem 3.2.** *Let  $(A, V)$  and  $(A', V')$  be cubic pairs over  $F$ . Suppose that  $f_{A,V}$  and  $f_{A',V'}$  are isometric, then the algebras  $A$  and  $A'$  are either isomorphic or anti-isomorphic.*

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