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A realization of elliptic Lie algebras of type $F_4^{(2,2)}$ by the Ringel–Hall approach

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ABSTRACT

This article is to study relations between an elliptic Lie algebra $\mathbf{g}$ of type $F_4^{(2,2)}$ and the F-fixed point algebra A^F of a tubular algebra A of type $\mathbb{T}(3,3,3)$ under a Frobenius morphism F. Using the explicit structure of the root category of the F-fixed point algebra A^F, we prove that the elliptic Lie algebra $\mathbf{g}$ of type $F_4^{(2,2)}$ is isomorphic to the Ringel–Hall Lie algebra of the root category of the F-fixed point algebra A^F.

1. Introduction

1.1. In order to describe singularities on surfaces and their deformations, K. Saito in [Sa] introduced one kind of extended affine root systems. In particular, he intensively studied 2-extended affine root systems using Dynkin diagrams with markings (also called elliptic Dynkin diagrams). A 2-extended affine root system is, by definition, a root system belonging to a positive semi-definite quadratic form whose radical has rank 2. It corresponds to the lattice of an elliptic curve. In addition, a rank 1 subspace of the radical, called a marking, corresponds to a choice of a primitive form. So Saito also called it an elliptic root system provided a marking has been given.

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1.2. Various attempts have been made to construct Lie algebras whose non-isotropic roots form elliptic root systems. Among them are intersection matrix Lie algebras, vertex algebras, toroidal Lie algebras, extended affine Lie algebras in general sense, and toral type extended affine Lie algebras.

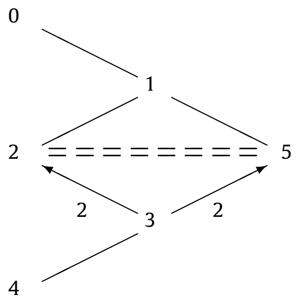
In 2000, K. Saito and D. Yoshii [SY] constructed certain Lie algebras by using the Borcherds lattice vertex, called them *simply-laced elliptic Lie algebras*. They also gave two other equivalent definitions for the Lie algebras. One uses an amalgamation of an affine Kac–Moody algebra and a Heisenberg algebra; this was generalized by D. Yoshii [Yo] in order to define Lie algebras associated with the reduced elliptic root systems, and he called them *elliptic Lie algebras*. The other uses Chevalley generators and generalized Serre relations attached to the elliptic Dynkin diagram; this was generalized by [Ya] and [AYY]. In [Yo] D. Yoshii gave a root space decomposition of any elliptic Lie algebra and pointed out the dimension of any real root space is one (see also [Ya]). Recently, the dimensions of all imaginary root spaces were determined in [AYY].

1.3. The close relation between the Kac–Moody algebras and the representation theory of finite dimensional algebras was discovered in the past twenty years by using the Ringel–Hall algebra approach. Let A be an associative algebra over a finite field and M, N and L finite A-modules. Let $F^L_{M,N}$ be the number of submodules V of L such that $V \simeq N$ and $L/V \simeq M$. By definition in [Rin4], the Ringel–Hall algebra of A is an associative ring with a $\mathbb{Z}$-basis, indexed by the isoclasses $[M]$ of all finite A-modules M, and the multiplication: $[M] \cdot [N] = \sum_{[L]} F^L_{M,N}[L]$. In case A is hereditary of finite type, C.M. Ringel [Rin3,Rin4,Rin5] showed that the subring of the degenerate Ringel–Hall algebra with a $\mathbb{Z}$-basis indexed by isoclasses of all indecomposable A-modules is a Lie subalgebra under the Lie multiplication of commutators, and over complex numbers it is isomorphic to the positive part of the corresponding complex semisimple Lie algebras such that the isoclasses of all indecomposable A-modules correspond to a Chevalley basis. Such Lie subalgebra is called the Ringel–Hall Lie algebra. To realize the whole (not only the positive part) of a Kac–Moody Lie algebra, the Ringel–Hall Lie algebras of 2-period triangulated categories have been constructed in [PX1,PX2]. Here the Ringel–Hall numbers are related to triangles instead of short exact sequences. Then any symmetrizable Kac–Moody Lie algebra can be realized by the Ringel–Hall Lie algebra of the root category of the corresponding hereditary algebra A. Here the root category is the orbit category $\mathcal{R}(A) = \mathcal{D}^b(A)/T^2$, where $\mathcal{D}^b(A)$ is the derived category of A and T is the shift functor (called the translation). On the other hand, a geometric setting of Ringel–Hall algebras is also used to construct Lie algebra (see [Rie,FMV,DXX]). In [XXZ] Xiao, Xu and Zhang gave a geometric realization of the generalized Kac–Moody Lie algebra arising from the 2-period version of the derived category, which is a generalization of the earlier work [PX2].

Furthermore, significant research has been done to realize some simply-laced elliptic Lie algebras by using the Ringel–Hall algebra approach. Lin and Peng [LP] proved that the elliptic Lie algebra of type $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ is isomorphic to the Ringel–Hall Lie algebra of the root category of the tubular algebra with type $\mathbb{T}(2,2,2,2), \mathbb{T}(3,3,3), \mathbb{T}(4,4,2), \mathbb{T}(6,3,2)$. Independently, O. Schiffmann [Sc] considered the quantum version and proved that the Ringel–Hall algebra of the category of coherent sheaves on a weighted projective line of tubular type is isomorphic to the quantized enveloping algebra of the 'half' of the above elliptic Lie algebras.

1.4. In 2006, Deng and Du [DD1] introduced Frobenius morphisms F on algebras A and their modules over the algebraic closure $\overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$ of q elements, and proved that the module category $\operatorname{mod} A^F$ of the F-fixed point algebra A^F over $\mathbb{F}_q$ is equivalent to the subcategory of finite dimensional F-stable A-modules, and the Auslander–Reiten (modulated) quiver of A^F is obtained by 'folding' the Auslander–Reiten quiver of A. Moreover, Deng and Du [DD2] showed that a Frobenius morphism F on an algebra A induces naturally a functor F on the (bounded) derived category $\mathcal{D}^b(A)$ of $\operatorname{mod} A$, and proved that the derived category $\mathcal{D}^b(A^F)$ of $\operatorname{mod} A^F$ is naturally imbedded as the triangulated subcategory $\mathcal{D}^b(A)^F$ of F-stable objects in $\mathcal{D}^b(A)$, and the AR-quiver of $\mathcal{D}^b(A^F)$ can be obtained by 'folding' the AR-quiver of $\mathcal{D}^b(A)$. In [DD2], they also extend this relation to the root categories $\mathcal{R}(A^F)$ of A^F and $\mathcal{R}(A)$ of A, and showed that, when A is hereditary, this folding relation over the indecomposable objects in $\mathcal{R}(A^F)$ and $\mathcal{R}(A)$ results in a folding relation between the root system of a non-simply-laced Kac–Moody algebra and that of the corresponding simply-laced Kac–Moody algebra.

1.5. In this paper, inspired by the results [DD1] and [DD2] of Deng and Du, we consider realizing a non-simply-laced elliptic Lie algebra of type $F_4^{(2,2)}$ via a Frobenius morphism on a tubular algebra. Here the elliptic Lie algebra of type $F_4^{(2,2)}$ is related to the following elliptic Dynkin diagram:



It has been showed in [Sa] that the above elliptic Dynkin diagram can be obtained by a folding of the diagram of type $E_6^{(1,1)}$, and in [LP] that the elliptic Lie algebra of type $E_6^{(1,1)}$ related to the diagram of type $E_6^{(1,1)}$ can be realized by the Ringel–Hall algebra of the root category of the tubular algebra A of type $\mathbb{T}(3,3,3)$. Using a Frobenius morphism F on the above tubular algebra A, we prove that the elliptic Lie algebra of type $F_4^{(2,2)}$ is isomorphic to the Ringel–Hall Lie algebra of the root category of the F-fixed point algebra A^F of A under F.

1.6. Let us give a brief view on the content of this article. In Section 2, we recall a definition of the elliptic Lie algebra of type $F_4^{(2,2)}$ and its grading spaces. In Section 3, we define a Frobenius morphism F on a tubular algebra A of type $\mathbb{T}(3,3,3)$, and characterize the corresponding fixed point subalgebra A^F. In Section 4, we give structures and properties of the root category $\mathcal{R}(A^F)$ of A^F for later use. In Section 5, we state our main theorem. There we define a map Φ from the elliptic Lie algebra $\mathbf{g}$ of type $F_4^{(2,2)}$ to the Ringel–Hall Lie algebra $\mathbf{g}'$ of $\mathcal{R}(A^F)$. The main theorem claims that Φ is an isomorphism. Then we give a simple proof for the first step, that is, such Φ is a well-defined morphism and surjective. In Section 6, we prove that Φ is injective and so Φ is an isomorphism. In Section 7, we point out a Chevalley basis of the elliptic Lie algebra of type $F_4^{(2,2)}$.

2. An elliptic Lie algebra of type $F_4^{(2,2)}$

In this section, we recall some basic facts about definitions and root spaces of elliptic Lie algebras of type $F_4^{(2,2)}$.

2.1. We first recall some concepts about elliptic root systems of type $F_4^{(2,2)}$, which come from [Ya]. Let ε_{af} be an 6-dimensional $\mathbb{C}$-vector space and $\Pi_{af} = \{\alpha_0, \alpha_1, \dots, \alpha_4\}$ be a set of linearly independent 5 elements of ε_{af}. Let $I : \varepsilon_{af} \times \varepsilon_{af} \to \mathbb{C}$ be a non-degenerate symmetric bilinear form satisfying $I(\alpha_i, \alpha_j) = (N_{af})_{ij}$, where

$$N_{af} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 4 \end{pmatrix}.$$

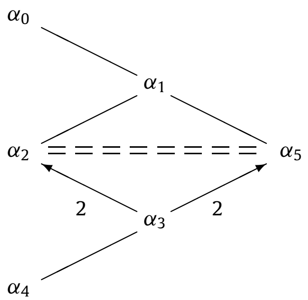
It is easy to see that $\{v \in \mathbb{Z}_+^5 \mid N_{af} v = 0\} = \mathbb{Z}_+ x$, where $x = (x_0, x_1, x_2, x_3, x_4)^T = (1,2,3,2,1)^T$. The pair $(\varepsilon_{af}, \Pi_{af})$ is called the *affine datum* of type $E_6^{(2)}$. The element $\delta := \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$ is called *the lowest positive null root*. Let $\Pi_{fi} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\varepsilon_{fi} := \bigoplus_{i=1}^{4} \mathbb{C}\alpha_i$. Then $\varepsilon_{af} = \varepsilon_{fi} \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_\delta$, where $\Lambda_\delta \in \varepsilon_{af}$ is such that $I(\Lambda_\delta, \varepsilon_{fi}) = \{0\}$, $I(\Lambda_\delta, \delta) = 1$ and $I(\Lambda_\delta, \Lambda_\delta) = 0$.

We define the $\mathbb{C}$-vector space $\varepsilon_{af}^{\natural} := \varepsilon_{af} \oplus \mathbb{C}a \oplus \mathbb{C}\Lambda_a$, and extend the symmetric bilinear form $I(-,-)$ on ε_{af} to the one on $\varepsilon_{af}^{\natural}$ by $I(\mathbb{C}a \oplus \mathbb{C}\Lambda_a, \varepsilon_{af}) = \{0\}$, $I(\Lambda_a, \Lambda_a) = I(a,a) = 0$, $I(a, \Lambda_a) = 1$. Let $k : \Pi_{af} \cup (-\Pi_{af}) \to \{1, 2\}$ be a function such that $k(\pm\alpha_0) = k(\pm\alpha_1) = k(\pm\alpha_2) = 1$, $k(\pm\alpha_3) = k(\pm\alpha_4) = 2$. The triple $(\varepsilon_{af}^{\natural}, \Pi_{af}, k)$ is called a *reduced marked elliptic datum of type* $F_4^{(2,2)}$.

If $\alpha_i \in \Pi_{af}$, let $m_{\alpha_i} := I(\alpha_i, \alpha_i)x_i/k(\alpha_i)$. Let $m_{max} := \max\{m_\alpha \mid \alpha \in \Pi_{af}\}$ and $\Pi_{max} := \{\alpha \in \Pi_{af} \mid m_\alpha = m_{max}\}$. Then $m_{max} = m_{\alpha_2}$, $\Pi_{max} = \{\alpha_2\}$. Let $\alpha_2^* := \alpha_2 + k(\alpha_2)a = \alpha_2 + a$, $\Pi_{max}^* = \{\alpha_2^*\}$. For convenience, we set

$$\alpha_5 = \alpha_2^*, \qquad \Pi := \Pi_{af} \cup \Pi_{max}^* = \{\alpha_0, \ldots, \alpha_4, \alpha_5\}.$$

The *elliptic Dynkin diagram* of the datum $(\varepsilon_{af}^{\natural}, \Pi_{af}, k)$ of type $F_4^{(2,2)}$ is the following:



For $x \in \varepsilon_{af}^{\natural}$ with $I(x,x) \neq 0$, let $x^\vee := \frac{2x}{I(x,x)}$ and define $\omega_x \in GL(\varepsilon_{af}^{\natural})$ by $\omega_x(y) = y - I(x^\vee, y)x$. Denote by W the Weyl group generated by ω_α, $\alpha \in \Pi$. Let $R^{re} = W\Pi$. Let W_{af} be the subgroup of W generated by $\{\omega_\alpha \mid \alpha \in \Pi_{af}\}$. Then, by [Sa, Assertion 6.1], we have

$$R^{re} := \bigcup_{\omega \in W_{af}} \bigcup_{\alpha \in \Pi_{af}} \big(\omega(\alpha) + \mathbb{Z}k(\alpha)a\big),$$

and R^{re} is *an elliptic root system of type* $F_4^{(2,2)}$. Any element in R^{re} is called *a real root*.

2.2. In this subsection, we recall some facts about elliptic Lie algebras of type $F_4^{(2,2)}$.

Inspired by [Yo, Definition 3], Yamane equivalently defined elliptic Lie algebras of all types by using generators and relations, which are expressed by means of the elliptic Dynkin diagrams (see [Ya, Definition 4.1]). Also Yamane gave root space decomposition of the above elliptic Lie algebras and explained that their real roots form an elliptic root system in the sense of [Sa]. Recently, Azam, Yamane and Yousofzadeh determined the dimensions of the imaginary root spaces of all elliptic Lie algebras in [AYY, Theorem 6.1]. The main purpose of this article is to realize the derived algebra of the elliptic Lie algebra of type $F_4^{(2,2)}$ in [Ya, Definition 4.1]. So we will give explicit descriptions about the above derived algebra, including its generators and relations, its root space decomposition and the dimensions of its root spaces.

Definition. Let $\mathbf{g}$ be the Lie algebra presented by Chevalley generators and generalized Serre relations as follows.

(1) Generators: $\{h_i, e_i, f_i \mid i = 0, 1, \ldots, 5\}$.
(2) Relations: Let

$$N=\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & -2 & 0 & 2 \\ 0 & 0 & -2 & 4 & -2 & -2 \\ 0 & 0 & 0 & -2 & 4 & 0 \\ 0 & -1 & 2 & -2 & 0 & 2 \end{pmatrix},$$

and $I_R(-,-)$ be a symmetric bilinear form on $H=\bigoplus_{i=0}^{5}\mathbb{C}\alpha_i$ such that $I_R(\alpha_i,\alpha_j)=N_{ij}$. The generators satisfy the following relations:

O.

$$[h_i,h_j]=0, \quad i=0,1,\ldots,5;$$

I.

$$[e_i,f_i]=\frac{2h_i}{I_R(\alpha_i,\alpha_i)}, \quad i=0,1,\ldots,5;$$

II1.

$$[h_i,e_j]=I_R(\alpha_i,\alpha_j)e_j, \qquad [h_i,f_j]=-I_R(\alpha_i,\alpha_j)f_j, \quad \text{for } i,j=0,1,\ldots,5.$$

II2.

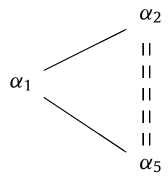
$$(\operatorname{ad} e_i)^{max\{1,1-I_R(\alpha_i^\vee,\alpha_j)\}}e_j=0, \qquad (\operatorname{ad} f_i)^{max\{1,1-I_R(\alpha_i^\vee,\alpha_j)\}}f_j=0, \quad \text{for } i,j=0,1,\ldots,5.$$

III.

$$[[e_2,e_1],e_5]=0$$

$$[[f_2,f_1],f_5]=0$$

for

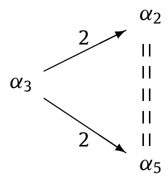


IV.

$$[[e_2,e_3],e_5]=0$$

$$[[f_2,f_3],f_5]=0$$

for

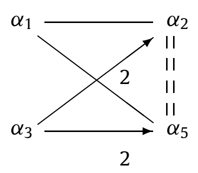


V.

$$[[e_2,e_1],[e_5,[e_5,e_3]]]=0$$

$$[[f_2,f_1],[f_5,[f_5,f_3]]]=0$$

for



It is easy to see that $\mathbf{g}$ is the derived algebra of the elliptic Lie algebra g^{Γ} of type $F_4^{(2,2)}$ defined in [Ya, Definition 4.1] (equivalently, g^{ω} in [AYY, Definition 5.1]). Naturally $\mathbf{g}$ has a grading similar to that of g^{Γ} as follows:

$$\mathbf{g} = \bigoplus_{\alpha \in \mathcal{Q}} \mathbf{g}_{\alpha}$$

such that $\deg(e_i) = \alpha_i$, $\deg(f_i) = -\alpha_i$ and $\deg(h_i) = 0$, $0 \leqslant i \leqslant 5$, where $\mathcal{Q} = \mathbb{Z}\Pi$. It is easy to see that $\mathbf{g}$ has the same root spaces $\mathbf{g}_{\alpha}$ with that of g^{Γ} (or g^{ω}) except $\alpha = 0$, and $\mathbf{g}_0 = \mathbb{C}\Pi$. So

$$\mathbf{g} = \left(\bigoplus_{\alpha \in R^{re}} \mathbf{g}_{\alpha} \right) \oplus \left(\bigoplus_{m,n \in \mathbb{Z}} \mathbf{g}_{m\delta + na} \right).$$

Let

$$\mathcal{M} := \mathbb{Z}\delta \oplus \mathbb{Z}a.$$

Let L_{sh}, L_{lg} be the subsets of $\mathcal{M}$ such that $\alpha_2 + L_{sh} = R^{re} \cap (\alpha_2 + \mathcal{M})$, $\alpha_3 + L_{lg} = R^{re} \cap (\alpha_3 + \mathcal{M})$. Then $L_{sh} = \mathcal{M}$, $L_{lg} = 2\mathcal{M}$. Recalling the dimensions of the real (resp., imaginary) root spaces of g^{Γ} in [Ya, Proposition 3.3] (resp., g^{ω} in [AYY, Theorem 6.1(3)]), we have

$$\dim_{\mathbb{C}} \mathbf{g}_{\alpha} = \begin{cases} 6, & \text{if } \alpha = 0; \\ 1, & \text{if } \alpha \in R^{re}; \\ 5, & \text{if } \alpha \in 2\mathcal{M} \backslash \{0\}; \\ 3, & \text{if } \alpha \in \mathcal{M} \backslash 2\mathcal{M}; \\ 0, & \text{otherwise.} \end{cases}$$

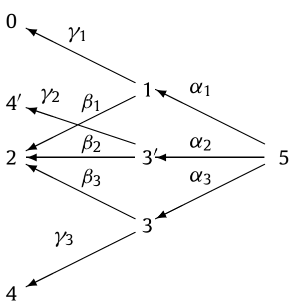
3. The F-fixed point subalgebra A^F of a tubular algebra A under a Frobenius morphism F

Let $\mathbb{F}_q$ be a finite field of q elements, i.e., $q = |\mathbb{F}_q|$, and $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$. Let $\operatorname{char} \mathbb{F}_q = p$, where p is a prime integer. Then $\overline{\mathbb{F}}_q = \bigcup_{i \geqslant 1} \mathbb{F}_{p^i}$. For a finite dimensional algebra B, all B-modules are finite dimensional left modules.

3.1. In this subsection, we introduce a Frobenius morphism on a tubular algebra of type $\mathbb{T}(3,3,3)$, and characterize the fixed point subalgebra under the Frobenius morphism.

In this remainder of this article, we always assume that A is the tubular algebra of type $\mathbb{T}(3,3,3)$ over the field $\overline{\mathbb{F}}_q$ with $A = \overline{\mathbb{F}}_q Q / \langle I \rangle$, where $Q = (Q_0, Q_1)$ is the following quiver, $\overline{\mathbb{F}}_q Q$ is the path algebra of the quiver Q over $\overline{\mathbb{F}}_q$ and $\langle I \rangle$ is the ideal generated by the relations I.

$Q = (Q_0, Q_1)$:



$$I: \quad \begin{aligned} &\alpha_1\beta_1 + \alpha_2\beta_2 = 0, \\ &\alpha_1\beta_1 + \alpha_3\beta_3 = 0. \end{aligned}$$

Note that different tubular algebras of the same type are tilting–cotilting equivalent, so their derived categories are the same [H]. Thus A is derived equivalent to a canonical algebra of type $\mathbb{T}(3,3,3)$. It is easy to see that A has the identity $1=\sum_{i\in Q_0} e_i$, where e_i is the idempotent corresponding to the vertex i. Let σ be an automorphism of the above admissible quiver Q defined by $\sigma(3)=3'$, $\sigma(3')=3$, $\sigma(4)=4'$, $\sigma(4')=4$, $\sigma(\alpha_2)=\alpha_3$, $\sigma(\alpha_3)=\alpha_2$, $\sigma(\beta_2)=\beta_3$, $\sigma(\beta_3)=\beta_2$, $\sigma(\gamma_2)=\gamma_3$, $\sigma(\gamma_3)=\gamma_2$, and σ leaves the other vertices and arrows invariant. So $\sigma^2=\mathrm{id}$. By [DD1,DD2], the relations I are σ-admissible and so σ induces a Frobenius morphism

$$F=F_{Q,\sigma}:A\to A;\quad \sum_s x_s p_s\mapsto \sum_s x_s^q\sigma(p_s),$$

where $\sum_s x_s p_s$ is a $\overline{\mathbb{F}}_q$-linear combination of path p_s, $\sigma(p_s)=\sigma(\rho_t)\cdots\sigma(\rho_2)\sigma(\rho_1)$ if $p_s=\rho_t\cdots\rho_2\rho_1$ for arrows $\rho_1,\rho_2,\dots,\rho_t$ in Q_1, and $\sigma(e_i)=e_{\sigma(i)}$ for $i\in Q_0$. Let

$$A^F=\{a\in A\mid F(a)=a\}$$

be the set of F-fixed elements. Then A^F is an $\mathbb{F}_q$-subalgebra of A and $A=A^F\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_q$. Let $D_1=\{0,1,2,5\}$, $D_2=\{3,4\}$; $G_1=\{\alpha_1,\beta_1,\gamma_1,\alpha_1\beta_1,\alpha_1\gamma_1\}$, $G_2=\{\alpha_2,\beta_2,\gamma_2,\alpha_2\gamma_2\}$. For $i\in D_1$, we define $A_i^F:=\{xe_i\mid x^q=x,\ x\in\overline{\mathbb{F}}_q\}=\mathbb{F}_q e_i$; for $j\in D_2$, $A_j^F:=\{xe_j+x^q e_{\sigma(j)}\mid x^{q^2}=x,\ x\in\overline{\mathbb{F}}_q\}$; for $\xi\in G_1$, $A_\xi^F:=\{x\xi\mid x^q=x,\ x\in\overline{\mathbb{F}}_q\}=\mathbb{F}_q\xi$; for $\xi\in G_2$, $A_\xi^F:=\{x\xi+x^q\sigma(\xi)\mid x^{q^2}=x,\ x\in\overline{\mathbb{F}}_q\}$. Obviously, $A^F_{\alpha_2\gamma_2}=A^F_{\alpha_2}\cdot A^F_{\gamma_2}$, $A^F_{\alpha_1\beta_1}=A^F_{\alpha_1}\cdot A^F_{\beta_1}$, $A^F_{\alpha_1\gamma_1}=A^F_{\alpha_1}\cdot A^F_{\gamma_1}$. Then the subalgebra A^F is the $\mathbb{F}_q$-space $(\bigoplus_{i=0}^5 A_i^F)\oplus(\bigoplus_{\xi\in G_1\cup G_2}A_\xi^F)$. By [CR, Section 8.16], $\mathrm{gl.dim}\,A^F=\mathrm{gl.dim}\,A=2$.

The complete set of primitive orthogonal idempotents of A^F is $\{e_i^F\mid 0\leqslant i\leqslant 5\}$, where $e_i^F=e_i$ for $i=0,1,2,5$, $e_3^F=e_3+e_{3'}$, and $e_4^F=e_4+e_{4'}$. For a A^F-module N, the dimension vector of N is defined by $\underline{\dim}\,N=(\dim_{\mathbb{F}_q}e_i^F N)_{0\leqslant i\leqslant 5}$, where $\dim_{\mathbb{F}_q}e_i^F N$ denotes the dimension of $e_i^F N$ as a $\mathbb{F}_q$-vector space. For simple A^F-module $S(i)$, $0\leqslant i\leqslant 5$, we have $\underline{\dim}\,S(i)=\varepsilon_i$, where ε_i is the standard unit vector.

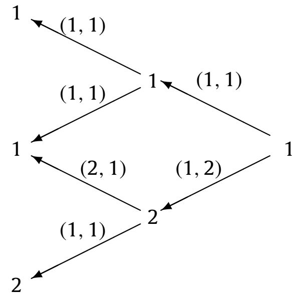
Now we recall the modulated quiver of A^F in the sense of [DD1, Section 6]. Note that there is a small difference from that in [DD1], since we add a relation to the quiver here. Define

$$g_i=\begin{cases}1, & i\in D_1;\\ 2, & i\in D_2,\end{cases}\qquad g_\rho=\begin{cases}1, & \rho\in G_1;\\ 2, & \rho\in G_2,\end{cases}$$
$$d_\rho=\frac{g_\rho}{g_{t(\rho)}},\qquad d'_\rho=\frac{g_\rho}{g_{h(\rho)}},$$
$$\Gamma_0=D_1\cup D_2,\qquad \Gamma_1=G_1\cup G_2,$$

where $h(\rho)$ (resp., $t(\rho)$) is the head (resp., tail) of the arrow ρ. Then the quiver $\Gamma=(\Gamma_0,\Gamma_1)$ together with the valuation $(\{g_i\}_{i\in\Gamma_0},\{(d_\rho,d'_\rho)\}_{\rho\in\Gamma_1})$ defines a valued quiver of A^F, and with the relation I_Γ: $2\alpha_1\beta_1+\alpha_2\beta_2=0$. Denote the quiver $\Gamma=(\Gamma_0,\Gamma_1)$ with the relation I_Γ as follows:

$$\Gamma=(\Gamma_0,\Gamma_1):\quad \begin{tikzcd} 0 & & \\ & 1 \arrow[ul, "\gamma_1"'] \arrow[dl, "\beta_1"'] & \\ 2 & & 5 \arrow[ul, "\alpha_1"'] \arrow[dl, "\alpha_2"] \\ & 3 \arrow[ul, "\beta_2"] \arrow[dl, "\gamma_2"] & \\ 4 & & \end{tikzcd}\qquad I_\Gamma:\quad 2\alpha_1\beta_1+\alpha_2\beta_2=0.$$

And the valuation on Γ is the following:



Let $\mathbb{M} = (\{A_i^F\}_{i\in\Gamma_0}, \{A_\rho^F\}_{\rho\in\Gamma_1})$. Then the $\mathbb{F}_q$-modulated quiver (or $\mathbb{F}_q$-species in [Rin1]) of A^F is $(\Gamma, \mathbb{M})$ with the relation I_Γ. For $i \in D_1$, let $F_i = \{x \mid xe_i \in A_i^F\}$; for $i \in D_2$, let $F_i = \{x \mid xe_i + x^q e_{\sigma(i)} \in A_i^F\}$. Then F_i are field, $0 \leqslant i \leqslant 5$, and $F_i \simeq \mathbb{F}_q$ (resp., $\mathbb{F}_{q^2}$) is a field isomorphism if $i \in D_1$ (resp., D_2). Note that $\dim_{F_{h(\rho)}} A_\rho^F = d'_\rho$, $\dim_{F_{t(\rho)}} A_\rho^F = d_\rho$.

A representation (V_i, φ_ρ) of the $\mathbb{F}_q$-modulated quiver $(\Gamma, \mathbb{M})$ is given by F_i-vector spaces V_i, $i \in \Gamma_0$, and $F_{t(\rho)}$-linear mappings $\varphi_\rho : V_{h(\rho)} \otimes_{F_{h(\rho)}} A_\rho^F \to V_{t(\rho)}$, $\rho \in \Gamma_1$, with the condition that $2(\varphi_{\alpha_1} \otimes A_{\beta_1}^F)\varphi_{\beta_1} + (\varphi_{\alpha_2} \otimes A_{\beta_2}^F)\varphi_{\beta_2} = 0$. Such a representation is called finite dimensional provided that all the V_i are finite dimensional vector spaces. A homomorphism $f = (f_i)_{i\in\Gamma_0} : (V_i, \varphi_\rho) \to (V'_i, \varphi'_\rho)$ is given by F_i-linear mappings $f_i : V_i \to V'_i$, $i \in \Gamma_0$, such that $f_{t(\rho)}\varphi_\rho = \varphi'_\rho(f_{h(\rho)} \otimes 1)$ for any $\rho \in \Gamma_1$. We denote by $(\Gamma, \mathbb{M})$-rep the category of all finite dimensional representations of $(\Gamma, \mathbb{M})$. There is a categorical equivalence

$$\tilde{\Delta} : A^F\text{-mod} \to (\Gamma, \mathbb{M})\text{-rep}.$$

For a finite dimensional representation (V_i, φ_ρ) of A^F, we define its dimension vector $\underline{\dim}(V_i, \varphi_\rho)$ by $(\underline{\dim}(V_i, \varphi_\rho))_k = \dim_{F_k} V_k$ for any $k \in \Gamma_0$.

3.2. In this subsection, we recall some facts about Frobenius maps on mod A shown in [DD1].

Let V be a $\overline{\mathbb{F}}_q$-space. If there is an $\mathbb{F}_q$-linear isomorphism $F : V \to V$ satisfying

(a) $F(\lambda v) = \lambda^q F(v)$ for all $v \in V$ and $\lambda \in \overline{\mathbb{F}}_q$;
(b) for any $v \in V$, $F^n(v) = v$ for some $n > 0$,

then F is called a *Frobenius map*. Let $V^F = \{v \in V \mid F(v) = v\}$. Then $V = V^F \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.

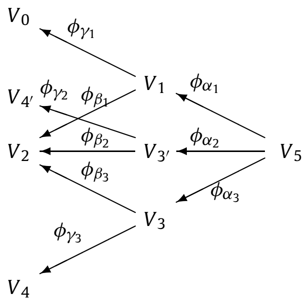
Assume that $F = F_{Q,\sigma}$ be the Frobenius morphism on the $\overline{\mathbb{F}}_q$-algebra A defined in Section 3.1. For any given A-module M together with a Frobenius map F_M on M, we define a new A-module $M^{[1]}$ such that $M^{[1]} = M$ as vector spaces with F-twist action:

$$a * m := F_M\big(F_A^{-1}(a)F_M^{-1}(m)\big), \quad \forall a \in A,\ m \in M.$$

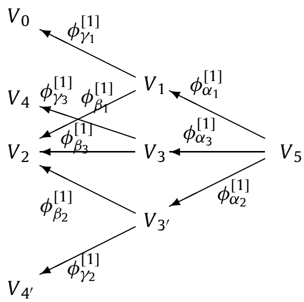
We call $M^{[1]}$ the F_M-twist of M. If M is isomorphic to $M^{[1]}$ as A-modules, then M is said to be F-stable. Let $\mathrm{Rep}(Q, I)$ be the representation category of Q satisfying the relations I, and $\Theta : \mathrm{Rep}(Q, I) \to \operatorname{mod} A$ be the categorical equivalent. A representation $V = (V_i, \phi_\rho) \in \mathrm{Rep}(Q, I)$ is called *F-stable representation*, if $\Phi(V)$ is an F-stable A-module. Let $(\mathrm{Rep}(Q, I))^F$ (resp., $(A\text{-mod})^F$) be the category of F-stable representations (resp., F-stable modules). So there is an induced categorical equivalence

$$\Theta : \big(\mathrm{Rep}(Q, I)\big)^F \to (A\text{-mod})^F.$$

Let $V=(V_i,\phi_\alpha)\in \operatorname{Rep}(Q,I)$ as follows,



where $\phi_\alpha=(a_{ij})$ denotes a $\dim V_{h(\alpha)}\times \dim V_{t(\alpha)}$ matrix for any arrow $\alpha\in Q_1$. The twist representation $V^{[1]}$ of V is defined to be the following representation:



where $\phi^{[1]}:=(a_{ij}^q)$ for $\phi=(a_{ij})$. If V is F-stable, then V is isomorphic to $V^{[1]}$. So for any $i\in\Gamma_0$, there is an isomorphism ψ_i: $V_i\cong V_{\sigma(i)}$ between the $\overline{\mathbb{F}}_q$-linear spaces, and so $\dim_{\overline{\mathbb{F}}_q} V_i=\dim_{\overline{\mathbb{F}}_q} V_{\sigma(i)}$.

4. The root category $\mathcal{R}(A^F)$ of A^F

4.1. In this subsection, we recall some facts about the root category $\mathcal{R}(A)$ of the tubular algebra A shown in Section 3.1.

Let $\mathcal{C}$ be an abelian category, and let $\mathcal{K}^b(\mathcal{C})$ be the homotopy category associated with the category of bounded complexes over $\mathcal{C}$ and $\mathcal{D}^b(\mathcal{C})$ the bounded derived category of $\mathcal{K}^b(\mathcal{C})$ by localization with quasi-isomorphisms. Denote by T the shift functor of complexes in $\mathcal{D}^b(\mathcal{C})$. If $\mathcal{C}$ is the module category $\operatorname{mod} B$ of a finite dimensional algebra B, we denote $D^b(\operatorname{mod} B)$ by $D^b(B)$. Given a finite dimensional algebra B, the orbit category $\mathcal{R}(B)=\mathcal{D}^b(B)/T^2$ is called the root category of B, and the Galois covering functor $\mathcal{F}:\mathcal{D}^b(B)\to\mathcal{R}(B)$ is dense. Furthermore, if $D^b(B)\simeq D^b(\mathcal{A})$ for some hereditary abelian category $\mathcal{A}$, then using a proof similar to [LP, Theorem 3.3], we can prove that $\mathcal{R}(B)$ is a triangulated category.

Denote by $K_0(\mathcal{C})$ the Grothendieck group of an abelian category $\mathcal{C}$, that is the free abelian group on isomorphism classes $[M]$ of objects in $\mathcal{C}$ modulo the relations $[M]=[N]+[L]$ for any exact sequence $0\to N\to M\to L\to 0$. Similarly, $K_0(\mathcal{D}^b(\mathcal{C}))$ is defined as the free abelian group on isomorphism classes $[X]$ of complexes in $\mathcal{D}^b(\mathcal{C})$ modulo the relations $[X]=[Y]+[Z]$ for any distinguished triangle $Y\to X\to Z\to TY$. It has been shown in [H, Lemma III.1.2] that we can identify $K_0(\operatorname{mod} B)$ and the Grothendieck group $K_0(\mathcal{D}^b(B))$. Given an object X in $\mathcal{D}^b(B)$, we denote by $\underline{\dim}\, X$ the corresponding element in $K_0(\mathcal{D}^b(B))=K_0(\operatorname{mod} B)$. Note that there is a canonical embedding of $\operatorname{mod} B$ into $\mathcal{D}^b(B)$ (as the full subcategory of complexes concentrated in degree zero), and the restriction of $\underline{\dim}$ to this full subcategory $\operatorname{mod} B$ coincides with the usual dimension vector function. Also, for any complex

$X = (X^i, d^i)$ in $\mathcal{D}^b(B)$, we have

$$\underline{\dim}\, X := \sum_{i\in\mathbb{Z}} (-1)^i \underline{\dim}\, X^i.$$

Let $K_0(\mathcal{R}(B))$ be the Grothendieck group of $\mathcal{R}(B)$. Then $K_0(\mathcal{R}(B)) = K_0(\mathcal{D}^b(B))$, and this identification is made so that for any $X \in \mathcal{D}^b(B)$, $\underline{\dim}\, X$ in $K_0(\mathcal{D}^b(B))$ coincides with $\underline{\dim}\, \mathcal{F}X$ in $K_0(\mathcal{R}(B))$.

For an object $\tilde{X}$ in the root category $\mathcal{R}(A)$ of the tubular algebra A, define a function $\chi : K_0(\mathcal{R}(A)) \to \mathbb{Z}$ by

$$\chi(\underline{\dim}\, \tilde{X}) = \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\mathcal{R}(A)}(\tilde{X}, \tilde{X}) - \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\mathcal{R}(A)}(\tilde{X}, T\tilde{X}).$$

By [LP, Theorem 8.6], for an indecomposable object $\tilde{X}$ in $\mathcal{R}(A)$, $\chi(\underline{\dim}\, \tilde{X}) = 0$ or 1, and $\underline{\dim}\, \tilde{X} = k_0\delta'_0 + k_1\delta'_1$ or $\dot{\alpha} + k_0\delta'_0 + k_1\delta'_1$, where $k_0, k_1 \in \mathbb{Z}$,

$$\delta'_0 = (1, 2, 3, 2, 1, 2, 1, 0), \qquad \delta'_1 = (0, 0, -1, 0, 0, 0, 0, 1),$$

$\dot{\alpha} = (0, c_1, c_2, c_3, c_4, c_{3'}, c_{4'}, 0)$ with $\chi(\dot{\alpha}) = 1$. We call $\tilde{X}$ *a real* (resp., *an imaginary*) *object* if $\chi(\underline{\dim}\, \tilde{X}) = 1$ (resp., 0), and $\underline{\dim}\, \tilde{X}$ *a real* (resp., *an imaginary*) *root* correspondingly. Call an imaginary root δ' minimal if for any imaginary root $\delta'' \in \mathbb{Q}\delta'$, there exists an integer s such that $\delta'' = s\delta'$. Let Υ' be the set of the minimal imaginary roots of $\mathcal{R}(A)$.

From [LP, Section 9.2], the AR-quiver $\Gamma_{\mathcal{R}(A)}$ of the root category $\mathcal{R}(A)$ can be described as

$$\Gamma_{\mathcal{R}(A)} = \bigcup_{\delta'\in\Upsilon'} \mathcal{T}(\delta'),$$

where each $\mathcal{T}(\delta')$ is a stable tubular family in which there are three non-homogeneous tubes with rank 3, and the others are homogeneous tubes. According to [LP, Section 9.2], each tubular family $\mathcal{T}(\delta')$ is determined by a minimal imaginary root, and for any indecomposable imaginary object $\tilde{X}$ in $\mathcal{T}(\delta')$, there is a positive integer s such that $\underline{\dim}\, \tilde{X} = s\delta'$. Let τ be the AR-translation of the AR-quiver of $\mathcal{R}(A)$.

4.2. In this subsection, using some results in [DD2] about the Frobenius map F on the root category, we give explicit actions of a Frobenius map F on the indecomposable objects of $\mathcal{R}(A)$ of the above tubular algebra A.

For an bounded complex $X = (X^i, d^i)$ in the derived category $\mathcal{D}^b(A)$, there is a Frobenius twist complex $X^{[1]}$ defined by

$$X^{[1]} := \big(X^{i[1]}, d^{i[1]}\big),$$

where $X^{i[1]}$, $d^{i[1]}$ are defined in Section 3.2. By construction, the Frobenius functor $(\ )^{[1]} : \mathcal{D}^b(A) \to \mathcal{D}^b(A)$; $X \mapsto X^{[1]}$, commutes with the shift functor T, and so $(\ )^{[1]}$ induces a functor $(\ )^{[1]}$ on $\mathcal{R}(A)$, $\tilde{X} \mapsto \tilde{X}^{[1]}$, where $\tilde{X}^{[1]} = \widetilde{X^{[1]}}$ for each $X \in \mathcal{D}^b(A)$. An object $\tilde{M}$ in $\mathcal{R}(A)$ is said to be *F-stable* if $\tilde{M} \cong (\tilde{M})^{[1]}$, and is said to be *F-periodic* if $\tilde{M} \cong (\tilde{M})^{[r]}$ for some integer $r \geqslant 1$. Call the minimal r with $\tilde{M} \cong (\tilde{M})^{[r]}$ the F-period of $\tilde{M}$.

For abbreviation, an object $\tilde{X}$ in $\mathcal{R}(A)$ is denoted by X in the remainder of this article. The action of σ on Q_0 induces an isomorphism $\hat{\sigma}$ of order 2 on the additive group $K_0(\mathcal{R}(A))$:

$$\hat{\sigma}\left(\sum_{i\in Q_0} c_i\varepsilon_i\right) = \sum_{i\in Q_0} c_i\varepsilon_{\sigma(i)}.$$

Then $\underline{\dim}\, X^{[1]} = \sum_{i\in\mathbb{Z}}(-1)^i \underline{\dim}\, X^{i[1]} = \sum_{i\in\mathbb{Z}}(-1)^i \hat{\sigma}(\underline{\dim}\, X^i) = \hat{\sigma}(\underline{\dim}\, X)$. If X is an F-stable object in $\mathcal{R}(A)$, $\underline{\dim}\, X = \sum_{i\in Q_0} c_i\varepsilon_i$, then $\hat{\sigma}(\underline{\dim}\, X) = \underline{\dim}\, X$, and so $c_3 = c_{3'}$, $c_4 = c_{4'}$.

Lemma.

(1) *Let X be an indecomposable real object in $\mathcal{R}(A)$. Then X is of F-period 1 or 2. Moreover, if $\hat{\sigma}(\underline{\dim}\, X) = \underline{\dim}\, X$, then X is F-stable; if $\hat{\sigma}(\underline{\dim}\, X) \neq \underline{\dim}\, X$, then X is of F-period 2, and $X \oplus X^{[1]}$ is F-stable.*
(2) *Let X be an indecomposable imaginary object of F-period r in $\mathcal{R}(A)$, $r \geqslant 1$. Then $\bigoplus_{i=0}^{r-1} X^{[i]}$ is F-stable, where $X^{[0]} = X$.*

Proof. (1) $X^{[2]}$ is a real object satisfying that $\underline{\dim}\, X^{[2]} = \hat{\sigma}^2(\underline{\dim}\, X) = \underline{\dim}\, X$. By [LP, Theorem 8.6(2)], $X^{[2]} \cong X$. Therefore X is of F-period 1 or 2.

If $\hat{\sigma}(\underline{\dim}\, X) = \underline{\dim}\, X$, then $\underline{\dim}\, X^{[1]} = \underline{\dim}\, X$, and so $X^{[1]} \cong X$. Or equivalently, X is F-stable. If $\hat{\sigma}(\underline{\dim}\, X) \neq \underline{\dim}\, X$, then $X^{[1]} \ncong X$, and so X is of F-period 2 and $X \oplus X^{[1]}$ is F-stable.

(2) Obviously. □

Proposition. *Let A be a tubular algebra of type $\mathbb{T}(3,3,3)$ and F the Frobenius morphism shown in Section 3.1, $(\ )^{[1]}$ is the functor on the root category of $\mathcal{R}(A)$ induced by F. Let $\mathcal{T}(\delta')$ be a tubular family determined by a minimal imaginary root δ'.*

(1) *For any indecomposable object X in the tubular family $\mathcal{T}(\delta')$, $X^{[1]}$ is still in the same tubular family $\mathcal{T}(\delta')$.*
(2) *The functor $(\ )^{[1]}$ commutes with the AR-translation τ.*
(3) *For an indecomposable object X on a tube $\mathcal{J}$ of $\mathcal{T}(\delta')$, X and $X^{[1]}$ have the same quasi-length and τ-period. Moreover, either X is F-stable, or $X^{[1]}$ lies on some different tube $\mathcal{J}'$.*
(4) *For any tube $\mathcal{J}$ of $\mathcal{T}(\delta')$, all indecomposable objects Y on $\mathcal{J}$ have a common F-period r. Therefore, for any homogeneous (resp., non-homogeneous) tube $\mathcal{J}$ of $\mathcal{T}(\delta')$, the image*

$$\mathcal{J}^{[1]} = \left\{X^{[1]} \;\middle|\; X \text{ is an indecomposable object on } \mathcal{J}\right\}$$

is some homogeneous (resp., non-homogeneous) tube.
(5) *There exists a unique non-homogeneous tube $\mathcal{J}_1$ such that any indecomposable object X in $\mathcal{J}_1$ is F-stable, and for the other two non-homogeneous tubes $\mathcal{J}_2$, $\mathcal{J}_3$, we have $\mathcal{J}_2^{[1]} = \mathcal{J}_3$, $\mathcal{J}_3^{[1]} = \mathcal{J}_2$.*
(6) *There exists at least one homogeneous tube whose mouth object is an F-stable object with dimension vector δ'.*

Proof. (1) Let X be an imaginary object with the dimension vector $\underline{\dim}\, X = \delta'$. We can write it as $\underline{\dim}\, X = b_1\delta'_0 + b_2\delta'_1$, where $b_1, b_2 \in \mathbb{Z}$. Then $\underline{\dim}\, X^{[1]} = \hat{\sigma}(\underline{\dim}\, X) = b_1\hat{\sigma}(\delta'_0) + b_2\hat{\sigma}(\delta'_1) = b_1\delta'_0 + b_2\delta'_1 = \underline{\dim}\, X$. Since any stable tubular family is determined by a unique minimal imaginary root, then $X^{[1]}$ is still in the tubular family $\mathcal{T}(\delta')$. So (1) holds.

(2) By [DD2, Lemma 6.5], a distinguished triangle $N \xrightarrow{f} L \xrightarrow{g} M \xrightarrow{h} TN$ is an AR-triangle if and only if so is $N^{[1]} \xrightarrow{f^{[1]}} L^{[1]} \xrightarrow{g^{[1]}} M^{[1]} \xrightarrow{h^{[1]}} T(N^{[1]})$. Thus $\tau(N^{[1]}) \cong (\tau N)^{[1]}$, i.e., the functor $(\ )^{[1]}$ commutes with τ.

(3) By (2), X and $X^{[1]}$ have the same τ-period. It is easy to see that $f: X \to Y$ is an irreducible map if and only if $f^{[1]}: X^{[1]} \to Y^{[1]}$ is an irreducible map. Thus X and $X^{[1]}$ have the same quasi-length.

Assume that X belongs to a homogeneous tube. Since X and $X^{[1]}$ have the same quasi-length, then $X \simeq X^{[1]}$, or $X^{[1]}$ belongs to another tube. So (3) holds in this case. Next, in case that X is not F-stable and lying on a non-homogeneous tube $\mathcal{J}$, we prove that $X^{[1]}$ does not lie on $\mathcal{J}$ by induction on the quasi-length of X.

Assume that X is of quasi-length 1, i.e., a mouth object on the tube $\mathcal{J}$. Assume that X and $X^{[1]}$ lie on the same tube $\mathcal{J}$. By (2), $X^{[1]}$ is also a mouth object on $\mathcal{J}$. Since $\mathcal{J}$ is a non-homogeneous tube of rank 3, then $X^{[1]} \cong \tau X$ or $X^{[1]} \cong \tau^- X$. For each case, we can obtain a contradiction. We only prove it in the case that $X^{[1]} \cong \tau X$. Assume that $X^{[1]} \cong \tau X$. There is an AR-triangle $X^{[1]} \to Z \to X \to TX^{[1]}$, and so $X^{[2]} \to Z^{[1]} \to X^{[1]} \to TX^{[2]}$ is also an AR-triangle. Since X is a real object, then $X \cong X^{[2]} \cong \tau(X^{[1]}) \cong \tau(\tau X) \cong \tau^2(X)$, which contradicts the fact that X is of τ-period 3. Therefore X and $X^{[1]}$ do not lie on a same tube.

Let X be an indecomposable object of quasi-length l on the tube $\mathcal{J}$, and $X_1 \overset{f_1}{\hookrightarrow} X_2 \overset{f_2}{\hookrightarrow} \cdots \overset{f_{l-1}}{\hookrightarrow} X_l = X$ is a series of irreducible maps such that $f_i, 1 \leqslant i \leqslant l-1$, are monomorphisms on the tube $\mathcal{J}$ and $X_1, X_2/X_1, \ldots, X_l/X_{l-1}$ are isomorphic to mouth objects on $\mathcal{J}$. Assume that X_1 is F-stable. Then $(\tau X_1)^{[1]} \cong \tau(X_1^{[1]}) \cong \tau X_1$, and so τX_1 is also F-stable. Moreover, since τ and $(\ )^{[1]}$ commute, then all mouth objects on $\mathcal{J}$ are F-stable, and so all indecomposable objects on $\mathcal{J}$ are F-stable, a contradiction to that X is not F-stable. So we obtain that X_1 is not F-stable. By the above proof, $X_1^{[1]}$ lies on some tube $\mathcal{J}'$ different from $\mathcal{J}$. Since $X_1^{[1]} \overset{f_1^{[1]}}{\hookrightarrow} X_2^{[1]} \overset{f_2^{[1]}}{\hookrightarrow} \cdots \overset{f_{l-1}^{[1]}}{\hookrightarrow} X_l^{[1]} = X^{[1]}$ is also a series of irreducible maps, then $X^{[1]}$ also lies on the tube $\mathcal{J}'$. Thus X and $X^{[1]}$ lie on different tubes, and so (3) holds.

(4) We only need to prove that if a mouth object S on $\mathcal{J}$ has F-period r, then any indecomposable object Y on $\mathcal{J}$ has the same F-period r, where $r \geqslant 1$. We prove it by induction on the quasi-length l of Y. Since $(\tau Y)^{[s]} = \tau(Y^{[s]})$ for any $s \geqslant 1$, then $Y^{[s]} \simeq Y$ if and only if $(\tau Y)^{[s]} \simeq \tau Y$ for any $s \geqslant 1$. Therefore any mouth object on $\mathcal{J}$ has the F-period r, i.e., it holds when $l = 1$. Assume that any indecomposable object with quasi-length $l \leqslant k-1$ on $\mathcal{J}$ is of F-period r. We will prove that any indecomposable object Y with quasi-length $l = k$ on $\mathcal{J}$ is also of F-period r. There is an AR-triangle $Y' \to Y \to X \to TY'$ such that Y' is of quasi-length $k-1$ and X is a mouth object. By induction, Y' and X are of F-period r, so Y is of F-period r. Thus (4) holds.

(5) Let $\mathcal{J}_j$, $j = 1, 2, 3$, be the three non-homogeneous tubes in the tubular family $\mathcal{T}(\delta')$, $\mathcal{V}$ the sublattice of $K_0(\mathcal{R}(A))$ generated by the dimension vectors $\underline{\dim}\, M$ with $M \in \mathcal{T}(\delta')$. Since $K_0(\mathcal{R}(A)) = K_0(A)$, we have the rank of $\mathcal{V}$ is greater than 7 by [Rin2, Corollary 5.3(2′)]. Assume that the dimension vector $\underline{\dim}\, X = \sum_{i \in Q_0} c_i \varepsilon_i$ of any indecomposable object X on the non-homogeneous tubes satisfies $c_0 = c_4 = c_{4'}$ and $c_1 = c_3 = c_{3'}$. Then the rank of $\mathcal{V}$ is less than 5, a contradiction. Therefore there exists at least one indecomposable object X which is not F-stable on some non-homogeneous tube, denoted by $\mathcal{J}_2$. Then we can find at least one mouth object Y which is not F-stable on $\mathcal{J}_2$. By (3), $Y^{[1]}$ is a mouth object on a different non-homogeneous tube, denoted by $\mathcal{J}_3$. By (4), $\mathcal{J}_2^{[1]} = \mathcal{J}_3$. Since Y is of F-period 2, then any indecomposable object on $\mathcal{J}_2$ is of F-period 2 by (4). Therefore, $\mathcal{J}_2^{[2]} = \mathcal{J}_2$ and $\mathcal{J}_3^{[1]} = \mathcal{J}_2$.

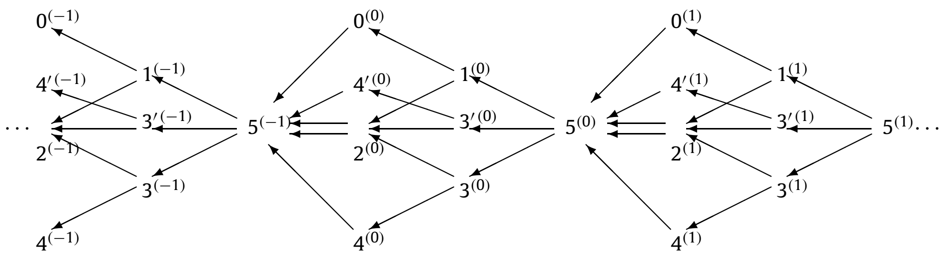
We denote by $\mathcal{J}_1$ the unique non-homogeneous tube different from $\mathcal{J}_2$ and $\mathcal{J}_3$. Then $\mathcal{J}_1^{[1]}$ is also a non-homogeneous tube different from $\mathcal{J}_2$ and $\mathcal{J}_3$, and so $\mathcal{J}_1^{[1]} = \mathcal{J}_1$. For any mouth object X in $\mathcal{J}_1$, $X^{[1]}$ must be F-stable by (3). Then any indecomposable object in $\mathcal{J}_1$ is F-stable by (4).

(6) We recall some facts about the repetitive algebra $\hat{A}$ and the Frobenius morphism F on $\hat{A}$ induced by the Frobenius morphism F on A.

Let $D(-) = \mathrm{Hom}_{\overline{\mathbb{F}}_q}(-, \overline{\mathbb{F}}_q)$. The repetitive algebra $\hat{A}$, introduced by Hughes and Waschbüsch [HW], is the doubly infinite matrix algebra, without identity,

$$\hat{A} = \begin{pmatrix} \cdots & \cdots & & & & 0 \\ \cdots & A(i-1) & DA(i) & & & \\ & & A(i) & DA(i+1) & & \\ & & & A(i+1) & \cdots & \\ 0 & & & \cdots & \cdots & \end{pmatrix}$$

in which matrices have only finitely non-zero entries, $A(i) = A$ are placed on the main diagonal, $DA(i) = DA$ on the upper next diagonal, for $i \in \mathbb{Z}$, all the remaining entries are zero, and the multiplication is induced from the canonical maps $A \otimes_A DA \to DA$, $DA \otimes_A A \to DA$ and the zero map $DA \otimes_A DA \to 0$. There is a triangle-equivalence $\varXi : \underline{\dim}\, \hat{A} \simeq D^b(A)$ (see [HR] and [H]). The vertices of the ordinary quiver Δ of $\hat{A}$ can be denoted by $i^{(j)}$, $i \in Q_0$, $j \in \mathbb{Z}$, such that for each j the full subquiver of Δ consisting of $\{i^{(j)} \mid i \in Q_0\}$ coincides with the ordinary quiver of A with the same numbering vertices as in Section 3.1. Ignoring the relations of arrows, we denote the quiver Δ of $\hat{A}$ as follows:



Let A_{-1}, A_0, A_1 be the convex projective idempotent subalgebras (see [LP, Section 8.5]) determined by the set of vertices $\{5^{(-1)}, i^{(0)} \mid i \in Q_0 \backslash \{5\}\}$, $\{i^{(0)} \mid i \in Q_0\}$, $\{1^{(0)}, 3'^{(0)}, 3^{(0)}, 5^{(0)}, 0^{(1)}, 4^{(1)}, 2^{(1)}, 4'^{(1)}\}$.

By [DW], the Frobenius morphism F on A induces a Frobenius morphism on $\hat{A}$, and the Frobenius twist functor $(\ )^{[1]} : X \mapsto X^{[1]}$ on $\operatorname{mod} A$ can be lifted to a Frobenius twist functor $(\ )^{[1]}$ on the module category $\operatorname{mod} \hat{A}$ and the stable module category $\underline{\operatorname{mod}}\, \hat{A}$. The elements of $\hat{A}$ will be denoted by $(a_i, \varphi_i)_i$, where $a_i \in A$, $\varphi_i \in DA$ with almost all a_i, φ_i being zero. We recall some main facts in [DW]. Define $F((a_i, \varphi_i)_i) = (F(a_i), \varphi_i^{[1]})_i$ for all $(a_i, \varphi_i)_i \in \hat{A}$. Then F is a Frobenius morphism on $\hat{A}$. It is easy to see that $F(e_{i^{(j)}}) = e_{(\sigma(i))^{(j)}}$ for any $i \in Q_0$, $j \in \mathbb{Z}$, where $e_{i^{(j)}}$ is the orthogonal idempotent element in $\hat{A}$ related to the vertex $i^{(j)}$ in the quiver of $\hat{A}$. Moreover, the finite dimensional $\hat{A}$-modules identify with $M = (M_i, f_i)$, where M_i are A-modules, all but finitely many being zero, and $f_i : M_i \to \operatorname{Hom}_A(DA, M_{i+1})$ are A-module homomorphisms satisfying $\operatorname{Hom}_A(DA, f_{i+1}) \cdot f_i = 0$ for all $i \in \mathbb{Z}$. For an $\hat{A}$-module $M = (M_i, f_i)$, define $M^{[1]} = (M_i^{[1]}, \hat{f}_i^{[1]})$, where $\hat{f}_i^{[1]}$ is shown in [DW, p. 173]. Then $(\ )^{[1]} : M \mapsto M^{[1]}$ is a Frobenius map on $\operatorname{mod} \hat{A}$. Since an $\hat{A}$-module M is injective if and only if its Frobenius twist $M^{[1]}$ is injective, then the Frobenius twist functor $(\ )^{[1]}$ on $\operatorname{mod} \hat{A}$ induces a Frobenius map F on $\underline{\operatorname{mod}}\, \hat{A} : \underline{\operatorname{mod}}\, \hat{A} \to \underline{\operatorname{mod}}\, \hat{A}$, $M \mapsto M^{[1]}$.

By [LP, Lemma 8.6], for any tubular family $\mathcal{T}$ in $\mathcal{R}(A)$, there is a convex projective idempotent subalgebra $A' \in \{A_{-1}, A_0, A_1\}$ such that $\hat{A}' = \hat{A}$ and there is a tubular family $\mathcal{T}'$ in $\operatorname{mod} A'$ such that all homogeneous tubes in $\mathcal{T}'$ coincide with all those in $\mathcal{T}$. By the symmetry of $\operatorname{mod} \hat{A}'$, there is an F-stable $\hat{A}'$-module M on some homogeneous tube $\mathcal{J}$ of $\mathcal{T}'$. Correspondingly, there is an F-stable indecomposable object $\Xi(M)$ on some homogeneous tube $\Xi(\mathcal{J})$ of the derived categories $D^b(A)$, and so there is an F-stable indecomposable object $\widetilde{\Xi(M)}$ on some homogeneous tube $\widetilde{\Xi(\mathcal{J})}$ of $\mathcal{R}(A)$. $\square$

4.3. In this subsection, we give an explicit description of the structure of the root category $\mathcal{R}(A^F)$ of A^F.

For an object X in $\mathcal{R}(A^F)$, we define a function $\chi : K_0(\mathcal{R}(A^F)) \to \mathbb{Z}$ by

$$\chi(\underline{\dim}\, X) = \dim_{\mathbb{F}_q} \operatorname{Hom}_{\mathcal{R}(A^F)}(X, X) - \dim_{\mathbb{F}_q} \operatorname{Hom}_{\mathcal{R}(A^F)}(X, TX).$$

We call X a *real* (resp., *an imaginary*) *object* if $\chi(\underline{\dim}\, X) \neq 0$ (resp., $= 0$), and $\underline{\dim}\, X$ *a real* (resp., *an imaginary*) *root* correspondingly. We denote by R^{re} (resp., R^{im}) the set of all real (resp., imaginary) roots of $\mathcal{R}(A^F)$. Let Υ be the set of the minimal imaginary roots of $\mathcal{R}(A^F)$.

We define the category $\mathcal{R}(A)^F$ consisting of F-stable objects in $\mathcal{R}(A)$ and Hom-spaces $\operatorname{Hom}_{\mathcal{R}(A)^F}(M, N) := \operatorname{Hom}_{\mathcal{R}(A)}(M, N)^F$. By [DD2, Theorem 8.5], there is a triangulated category equivalence

$$\tilde{F}: \quad \mathcal{R}(A)^F \cong \mathcal{R}\big(A^F\big).$$

Under the functor $\tilde{F}$, simple A-module $S(i) \in \mathcal{R}(A)^F$ is mapped to simple A^F-module $S(i)$ for $i = 0, 1, 2, 5$, and the indecomposable object $S(i) \oplus S(i') \in \mathcal{R}(A)^F$ is mapped to simple A^F-module $S(i)$

for $i = 3, 4$. So for $X \in \mathcal{R}(A)^F$, $\underline{\dim} X = \sum_{i \in Q_0} c_i \varepsilon_i$, we have $\underline{\dim} \tilde{F}(X) = \sum_{i=0}^{5} c_i \varepsilon_i$. By Lemma 4.2, we have the following lemma.

Lemma. *An F-stable object Y in $\mathcal{R}(A)^F$ is indecomposable if and only if Y is of one form in the following three forms*:

(1) *Y is an indecomposable real object of F-period 1 in $\mathcal{R}(A)$;*
(2) *$Y = X \oplus X^{[1]}$, where X is an indecomposable real object of F-period 2 in $\mathcal{R}(A)$;*
(3) *$Y = \bigoplus_{i=0}^{r-1} X^{[i]}$, where X is an indecomposable imaginary object of F-period r in $\mathcal{R}(A)$.*

By [DD2, Theorem 7.2], the AR-quiver of the root category $\mathcal{R}(A^F)$ can be obtained by folding the AR-quiver of the root category $\mathcal{R}(A)$. So we have the following proposition:

Proposition.

(1)

$$\Gamma_{\mathcal{R}(A^F)} = \bigcup_{\delta' \in \Upsilon} \mathcal{T}(\delta'),$$

where each $\mathcal{T}(\delta')$ is a stable tubular family in which there are two non-homogeneous tubes and the others are homogeneous.

(2) *Given a tubular family $\mathcal{T}$ in $\mathcal{R}(A^F)$, we put*

$$\mathcal{C} = \langle \mathcal{T}' \mid \mathcal{T}' \text{ a tubular family such that } \mathcal{T}' \neq T\mathcal{T} \text{ and } \operatorname{Hom}_{\mathcal{R}(A^F)}(\mathcal{T}', \mathcal{T}) \neq 0 \rangle.$$

Then

(i) *$\mathcal{R}(A^F) = \mathcal{C} \cup T\mathcal{C}$ and $\operatorname{Hom}_{\mathcal{R}(A^F)}(\mathcal{C}, T\mathcal{C}) = 0$;*
(ii) *$\mathcal{C}$ is a hereditary abelian $\mathbb{F}_q$-category with finite dimensional Hom-spaces and Ext-spaces*;
(iii) *$\mathcal{C}$ has a Serre duality, in other words, there is an equivalence $\tau : \mathcal{C} \to \mathcal{C}$ such that $D\operatorname{Ext}^1_{\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{C}}(Y, \tau X)$ for $X, Y \in \mathcal{C}$;*
(iv) *for $X, Y, Z \in \mathcal{C}$, $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact in $\mathcal{C}$ if and only if there is a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ in $\mathcal{R}(A^F)$;*
(v) *$\mathcal{C}$ is closed under extension in $\mathcal{R}(A^F)$, that is, for a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$, if $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$;*
(vi) *$\mathcal{R}(A^F) \cong \mathcal{D}^b(\mathcal{C})/T^2$.*

4.4. In this subsection, we show that $D^b(A^F)$ is in fact equivalent to a derived category of coherent sheaves of an exceptional curve over $\mathbb{F}_q$.

Denote by $\mathcal{T}_0$ the tubular family containing the indecomposable object $TS(5)$ in the root category $\mathcal{R}(A^F)$. Let

$$\mathcal{C}_0 = \langle \mathcal{T}' \mid \mathcal{T}' \text{ a tubular family such that } \mathcal{T}' \neq T\mathcal{T}_0 \text{ and } \operatorname{Hom}_{\mathcal{R}(A^F)}(\mathcal{T}', \mathcal{T}_0) \neq 0 \rangle.$$

By Proposition 4.3, $\mathcal{C}_0$ is a hereditary abelian $\mathbb{F}_q$-category. It is easy to know that $S(i) \in \mathcal{C}_0$ for any $0 \leqslant i \leqslant 4$.

An object X in $\mathcal{C}_0$ is called *exceptional* if X is indecomposable and $\operatorname{Ext}^1_{\mathcal{C}_0}(X, X) = 0$. A pair (X, Y) of exceptional objects in $\mathcal{C}_0$ is called *exceptional* if $\operatorname{Hom}_{\mathcal{C}_0}(Y, X) = 0$ and $\operatorname{Ext}^1_{\mathcal{C}_0}(Y, X) = 0$. A sequence $\mathcal{X} = (E_1, E_2, \dots, E_r)$ of exceptional objects in $\mathcal{C}_0$ is called an *exceptional sequence of length r* provided that each pair (E_i, E_j) with $i < j$ is an exceptional pair. An exceptional sequence $\mathcal{X} = (E_1, E_2, \dots, E_r)$

in $\mathcal{C}_0$ is called *complete* if in the derived category $\mathcal{D}^b(\mathcal{C}_0)$ the minimal full triangulated subcategory containing the objects $E_1, E_2, \ldots, E_r$ coincides with $\mathcal{D}^b(\mathcal{C}_0)$.

Lemma. *A sequence*

$$\mathcal{S} = \big(TS(5), S(1), S(3), S(0), S(4), S(2)\big)$$

is a complete exceptional sequence in $\mathcal{C}_0$.

Proof. It is easily checked that all objects $TS(5)$, $S(i)$, $0 \leqslant i \leqslant 4$, are exceptional objects and $\mathcal{S}$ is an exceptional sequence in $\mathcal{C}_0$. In $\mathcal{D}^b(\mathcal{C}_0)$, we consider the minimal triangulated subcategory $\mathcal{D}'$ containing $TS(5)$, $S(i)$, $0 \leqslant i \leqslant 4$. For any $i \in \mathbb{Z}$, $T^i(\operatorname{mod} A^F) \subseteq \mathcal{D}'$, which implies that $\mathcal{D}^b(A^F) \subseteq \mathcal{D}'$. Since $\mathcal{D}' \subseteq \mathcal{D}^b(\mathcal{C}_0)$ and $\mathcal{D}^b(A^F) = \mathcal{D}^b(\mathcal{C}_0)$, then $\mathcal{D}^b(\mathcal{C}_0) = \mathcal{D}'$. Thus the lemma holds. $\square$

Following [L, Section 2.5], an *exceptional curve* $\mathbb{X}$ over a field k is defined by the following requests on its associated category $\operatorname{coh}(\mathbb{X})$ of coherent sheaves:

(1) $\operatorname{coh}(\mathbb{X})$ is a connected small abelian k-category with morphism spaces that are finite dimensional over k;
(2) $\operatorname{coh}(\mathbb{X})$ is hereditary and noetherian and there exists an equivalence $\tau : \operatorname{coh}(\mathbb{X}) \to \operatorname{coh}(\mathbb{X})$ such that Serre duality $D\operatorname{Ext}^1(X, Y) \simeq \operatorname{Hom}(Y, \tau X)$ holds;
(3) $\operatorname{coh}(\mathbb{X})$ admits a complete exceptional sequence.

By Proposition 4.3 and Lemma 4.4, we have the following important proposition.

Proposition.

(1) *The category $\mathcal{C}_0$ is a category of coherent sheaves corresponding to an exceptional curve, denoted by $\mathbb{X}_0$, over a finite field $\mathbb{F}_q$.*
(2) *The derived category $\mathcal{D}^b(A^F)$ is isomorphic to the derived category of the category of coherent sheaves corresponding to $\mathbb{X}_0$ over $\mathbb{F}_q$.*

Remark. It is easy to compute that the *genus*, defined in [L, Section 2.5], of the above exceptional curve $\mathbb{X}_0$ is 1, i.e., $\mathbb{X}_0$ is of *tubular* type.

4.5. In this subsection, we give an explicit description of the real and imaginary roots of the root category $\mathcal{R}(A^F)$.

Theorem.

(1) *For any indecomposable object Z in $\mathcal{R}(A^F)$, $\chi(\underline{\dim}\, Z) = 0$, 1 or 2. Moreover, set*

$$\delta = (1, 2, 3, 2, 1, 0) \quad \textit{and} \quad a = (0, 0, -1, 0, 0, 1),$$

we have that $\chi(\underline{\dim}\, Z) = 0$ if and only if $\underline{\dim}\, Z = k_1\delta + k_2 a$, where $k_1, k_2 \in \mathbb{Z}$ and at least one of k_1, k_2 is not equal to zero; and $\chi(\underline{\dim}\, Z) = 1$ or 2 if and only if $\underline{\dim}\, Z = \dot{\alpha} + \chi(\dot{\alpha})k_1\delta + \chi(\dot{\alpha})k_2 a$, where $\dot{\alpha} = (0, c_1, c_2, c_3, c_4, 0)$ with $\chi(\dot{\alpha}) = 1$ or 2.
(2) *Let $0 \neq \alpha \in K_0(\mathcal{R}(A^F))$ with $\chi(\alpha) = 0$. Then there exists at least one homogeneous tube $\mathcal{J}$ which contains an object Z with $\underline{\dim}\, Z = \alpha$ in some tubular family $\mathcal{T}$ of the AR-quiver of $\mathcal{R}(A^F)$. Moreover, if $\alpha \in 2R^{im}\backslash\{0\}$, then any one of the two non-homogeneous tubes in the above tubular family $\mathcal{T}$ contains an indecomposable object Z such that $\underline{\dim}\, Z = \alpha$, and if $\alpha \in R^{im}\backslash 2R^{im}$, then only one of the two non-homogeneous tubes in $\mathcal{T}$ contains an object Z with $\underline{\dim}\, Z = \alpha$.*

Proof. (1) By Lemma 4.3, the indecomposable F-stable object $\tilde{F}^{-1}(Z)$ in $\mathcal{R}(A)^F$ belongs to one of the following three cases.

Case 1. $\tilde{F}^{-1}(Z)$ is an indecomposable real object X of F-period 1 in $\mathcal{R}(A)$.

In this case, $\chi(\underline{\dim}\, Z) = \dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{R}(A^F)}(Z, Z) - \dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{R}(A^F)}(Z, TZ) = \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\mathcal{R}(A)}(X, X) - \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\mathcal{R}(A)}(X, TX) = 1$. Set $\underline{\dim}\, X = (c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3 + c_4\varepsilon_4 + c_{3'}\varepsilon_{3'} + c_{4'}\varepsilon_{4'}) + k_1\delta'_0 + k_2\delta'_1$. Then $\underline{\dim}\, Z = \underline{\dim}\, \tilde{F}(X) = \sum_{i=1}^4 c_i\varepsilon_i + k_1\delta + k_2 a$, where $\chi(\sum_{i=1}^4 c_i\varepsilon_i) = \chi(\dim Z) = 1$.

Case 2. $\tilde{F}^{-1}(Z) = X \oplus X^{[1]}$, where X is an indecomposable real object of F-period 2 in $\mathcal{R}(A)$.

By Proposition 4.2(3), X and $X^{[1]}$ lie on two different tubes in a same tubular family, then $\chi(\underline{\dim}\, Z) = \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\mathcal{R}(A)}(X \oplus X^{[1]}, X \oplus X^{[1]}) - \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\mathcal{R}(A)}(X \oplus X^{[1]}, T(X \oplus X^{[1]})) = \chi(\underline{\dim}\, X) + \chi(\underline{\dim}\, X^{[1]}) = 2$. Set $\underline{\dim}\, X = c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3 + c_4\varepsilon_4 + c_{3'}\varepsilon_{3'} + c_{4'}\varepsilon_{4'} + k_1\delta'_0 + k_2\delta'_1$. Then $\underline{\dim}\, Z = \underline{\dim}\, \tilde{F}(X \oplus X^{[1]}) = (2c_1\varepsilon_1 + 2c_2\varepsilon_2 + (c_3 + c_{3'})\varepsilon_3 + (c_4 + c_{4'})\varepsilon_4) + 2k_1\delta + 2k_2 a$, where $\chi(2c_1\varepsilon_1 + 2c_2\varepsilon_2 + (c_3 + c_{3'})\varepsilon_3 + (c_4 + c_{4'})\varepsilon_4) = \chi(\underline{\dim}\, Z) = 2$.

Case 3. $\tilde{F}^{-1}(Z) = \bigoplus_{i=0}^{r-1} X^{[i]}$, where X is an indecomposable imaginary object of F-period r in $\mathcal{R}(A)$.

By Proposition 4.2(3), for $i \neq j$, $X^{[i]}$ and $X^{[j]}$ lie on two different tubes. Then $\chi(\underline{\dim}\, Z) = \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\mathcal{R}(A)}(\bigoplus_{i=0}^{r-1} X^{[i]}, \bigoplus_{i=0}^{r-1} X^{[i]}) - \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\mathcal{R}(A)}(\bigoplus_{i=0}^{r-1} X^{[i]}, \bigoplus_{i=0}^{r-1} TX^{[i]}) = 0$. Set $\underline{\dim}\, X = k_1\delta'_0 + k_2\delta'_1$, where $k_1, k_2 \in \mathbb{Z}$. Then $\underline{\dim}\, X^{[i]} = \underline{\dim}\, X = k_1\delta'_0 + k_2\delta'_1$, and so $\underline{\dim}\, Z = \underline{\dim}\, \tilde{F}(\bigoplus_{i=0}^{r-1} X^{[i]}) = r(k_1\delta + k_2 a)$.

(2) By conditions, α is an imaginary root of $K_0(\mathcal{R}(A^F))$. There exists a unique minimal imaginary root $\delta' \in K_0(\mathcal{R}(A^F))$ such that $\mathbb{N}\delta' = \mathbb{Q}^+\alpha \cap \mathbb{Z}^6$. We set $\alpha = s\delta'$, $s \in \mathbb{N}$, and $\delta' = k_1\delta + k_2 a$, where $k_1, k_2 \in \mathbb{Z}$. So $\alpha \in 2R^{im}\backslash\{0\}$ (resp., $\alpha \in R^{im}\backslash 2R^{im}$) if and only if s is a positive even (resp., odd) integer. There exists a tubular family $\mathcal{T}$ determined by the imaginary root $k_1\delta + k_2 a$ in the AR-quiver of $\mathcal{R}(A^F)$. By Proposition 4.2(6), in the AR-quiver of $\mathcal{R}(A)$, there exists a homogeneous tube of $\mathcal{T}' = \tilde{F}^{-1}(\mathcal{T})$ containing an F-stable indecomposable object X with the dimension vector $s(k_1\delta'_0 + k_2\delta'_1)$. Then $\tilde{F}(X)$ is an indecomposable object with dimension vector α and on some homogeneous tube of the tubular family $\mathcal{T}$ in the AR-quiver of $\mathcal{R}(A^F)$.

In the tubular family $\mathcal{T}'$ of the AR-quiver $\Gamma_{\mathcal{R}(A)}$, there exists a non-homogeneous tube $\mathcal{J}_1$ whose indecomposable objects are F-stable and for the other two homogeneous tubes $\mathcal{J}_2$ and $\mathcal{J}_3$, $\mathcal{J}_2^{[1]} = \mathcal{J}_3$ and $\mathcal{J}_3^{[1]} = \mathcal{J}_2$, where $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ are the same to those in Proposition 4.2(5). For any positive integer s, there exists an F-stable indecomposable object X_1 with dimension vector $s(k_1\delta'_0 + k_2\delta'_1)$ on $\mathcal{J}_1$, and so there exists an indecomposable object $\tilde{F}(X_1)$ with dimension vector α on the non-homogeneous tube $\tilde{F}(\mathcal{J}_1)$ of the above tubular family $\mathcal{T}$. Moreover, if s is even, then there exists another indecomposable F-stable object $X_2 \oplus X_2^{[1]}$ with dimension vector $s(k_1\delta'_0 + k_2\delta'_1)$, where X_2 is an indecomposable object on $\mathcal{J}_2$, and so there exists an indecomposable object $\tilde{F}(X_2 \oplus X_2^{[1]})$ with dimension vector α on a non-homogeneous tube different from $\tilde{F}(\mathcal{J}_1)$. Thus (2) holds. $\square$

4.6. In this subsection, we describe basic facts about exceptional sequences in $\mathcal{R}(A^F)$, which will be used in Section 6.

An indecomposable object in $\mathcal{R}(A^F)$ is called *exceptional*, if $\mathrm{Hom}_{\mathcal{R}(A^F)}(X, TX) = 0$. A pair (X, Y) of exceptional objects in $\mathcal{R}(A^F)$ is called *exceptional* if $\mathrm{Hom}_{\mathcal{R}(A^F)}(Y, X) = 0$ and $\mathrm{Hom}_{\mathcal{R}(A^F)}(Y, TX) = 0$. By [KM, Lemma 3.2], we have the following proposition, which is similar to [LP, Proposition 7.2].

Lemma. *Let (X, Y) be an exceptional pair in $\mathcal{R}(A^F)$, $\mathrm{Hom}_{\mathcal{R}(A^F)}(X, Y) \neq 0$. Put $m = \dim_{\mathrm{End}(X)} \mathrm{Hom}_{\mathcal{R}(A^F)}(X, Y)$. Let*

$$TY \xrightarrow{f} Z \xrightarrow{g} X^{(m)} \xrightarrow{can.} Y$$

be the triangle induced by the canonical morphism $X^{(m)} \xrightarrow{can.} Y$. Then

(1) *Z is an exceptional object and (Z, X) is an exceptional pair*;
(2) *if there exists a triangle $TY \xrightarrow{f'} Z' \xrightarrow{g'} X^{(m)} \xrightarrow{h'} Y$ such that Z' is indecomposable, then $Z' \cong Z$.*

Dually, put $n = \dim_{\mathrm{End}(Y)} \mathrm{Hom}_{\mathcal{R}(A^F)}(X, Y)$. Let

$$X \xrightarrow{can.} Y^{(n)} \xrightarrow{g} W \xrightarrow{h} TX$$

be the triangle induced by the canonical morphism $X \xrightarrow{can.} Y^{(n)}$. Then

(1′) *W is an exceptional object and (Y, W) is an exceptional pair*;
(2′) *if there exists a triangle $X \xrightarrow{f'} Y^{(n)} \xrightarrow{g'} W' \xrightarrow{h'} TX$ such that W' is indecomposable, then $W' \cong W$.*

In the above proposition, Z is said to be the *left mutation* of Y by X, denoted by $L_X Y$; W is said to be the *right mutation* of X by Y, denoted by $R_X Y$.

A sequence $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ in $\mathcal{R}(A^F)$ is called an *exceptional sequence of length r*, if any pair (X_i, X_j) is exceptional for any $i < j$. An exceptional sequence $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ in $\mathcal{R}(A^F)$ is called *complete* if the minimal full triangulated subcategory containing the objects $X_1, X_2, \ldots, X_r$ coincides with $\mathcal{R}(A^F)$. Since $K_0(\mathcal{R}(A^F)) \cong \mathbb{Z}^6$, then by Lemma 4.4, an exceptional sequence $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ is complete if and only if $r = 6$.

Given an exceptional pair (X, Y) in $\mathcal{R}(A^F)$, we can define two exceptional pairs $(L_X Y, X)$, $(Y, R_Y X)$ as follows.

(1) If $\mathrm{Hom}_{\mathcal{R}(A^F)}(X, Y) = 0$, then $L_X Y = TY$ and $R_Y X = TX$.
(2) If $\mathrm{Hom}_{\mathcal{R}(A^F)}(X, Y) \neq 0$, then $L_X Y$ and $R_X Y$ are determined by the above triangles as in Lemma 4.6.

Recall the braid group B_6 is generated by generators $\sigma_1, \sigma_2, \ldots, \sigma_5$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, 3, 4$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $j \geqslant i + 2$. Denote by $\mathbb{Z}_2^6$ the free module over $\mathbb{Z}_2$ with basis $\gamma_1, \gamma_2, \ldots, \gamma_6$. Let $\mathbb{Z}_2^6 \ltimes B_6$ be the semidirect product of groups $\mathbb{Z}_2^6$ and B_6. For a given complete exceptional sequence $\mathcal{X} = (X_1, X_2, \ldots, X_6)$, we define

$$\sigma_i(X_1, X_2, \ldots, X_6) = (X_1, \ldots, X_{i-1}, X_{i+1}, R_{X_{i+1}} X_i, X_{i+2}, \ldots, X_6),$$
$$\sigma_i^{-1}(X_1, X_2, \ldots, X_6) = (X_1, \ldots, X_{i-1}, L_{X_i} X_{i+1}, X_i, X_{i+2}, \ldots, X_6),$$

for $i = 1, 2, \ldots, 5$ and

$$\gamma_i(X_1, X_2, \ldots, X_6) = (X_1, \ldots, X_{i-1}, TX_i, X_{i+1}, \ldots, X_6)$$

for $i = 1, 2, \ldots, 6$. In this way, we obtain an action of $\mathbb{Z}_2^6 \ltimes B_6$ on the set of the complete exceptional sequences in $\mathcal{R}(A^F)$.

Since $\mathcal{R}(A^F)$ is the root category of the category of coherent sheaves on an exceptional curve $\mathbb{X}_0$ by Proposition 4.4, we have the following result by a proof similar to that of [LP, Proposition 8.2].

Proposition.

(1) *Any exceptional object in $\mathcal{R}(A^F)$ can be extended to a complete exceptional sequence in $\mathcal{R}(A^F)$.*
(2) *The action of $\mathbb{Z}_2^6 \ltimes B_6$ on the set of the complete exceptional sequences in $\mathcal{R}(A^F)$ is transitive.*

5. The Ringel–Hall Lie algebra of the root category $\mathcal{R}(A^F)$

5.1. In this subsection, following [PX2], we recall the definition of the Ringel–Hall Lie algebras of the root category of A^F. The definition is also seen in [LP, Section 5].

Given $X, Y, L \in \mathcal{R}(A^F)$, consider

$$W(X, Y; L) = \big\{(f, g, h) \in \mathrm{Hom}(X, L) \times \mathrm{Hom}(L, Y) \times \mathrm{Hom}(Y, TX) \bigm| X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} TX \text{ is a triangle}\big\}.$$

Applying $\mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ for acting on $W(X, Y; L)$ defined by

$$(a, c) \circ (f, g, h) = \big(af, gc^{-1}, ch(Ta)^{-1}\big)$$

for $(a, c) \in \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$, $(f, g, h) \in W(X, Y; L)$, we get the Ringel–Hall number $F^L_{YX} = |W(X, Y; L)/\mathrm{Aut}(X) \times \mathrm{Aut}(Y)|$.

For any $M \in \mathcal{R}(A^F)$, we denote by $h_M := \underline{\dim}\, M$ the canonical image of $[M]$ in $K_0(\mathcal{R}(A^F))$. Denote by $\mathrm{ind}\,\mathcal{R}(A^F)$ the set of representatives of isoclasses of the indecomposable objects in $\mathcal{R}(A^F)$. And denote by $\mathbf{h}'$ the subgroup of $K_0(\mathcal{R}(A^F)) \otimes_{\mathbb{Z}} \mathbb{C}$ generated by $\frac{h_M}{d(M)}$, $M \in \mathrm{ind}\,\mathcal{R}(A^F)$, where $d(M) = \dim_{\mathbb{F}_q}(\mathrm{End}\, M/\mathrm{Rad}(\mathrm{End}\, M))$.

We define a symmetric Euler bilinear function $I_{\mathcal{R}(A^F)}(-,-)$ on $\mathbf{h}' \times \mathbf{h}'$ determined by

$$\begin{aligned} I_{\mathcal{R}(A^F)}(h_X, h_Y) = {} & \dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{R}(A^F)}(X, Y) - \dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{R}(A^F)}(X, TY) \\ & + \dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{R}(A^F)}(Y, X) - \dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{R}(A^F)}(Y, TX) \end{aligned}$$

for any $X, Y \in \mathcal{R}(A^F)$.

Let $\mathbf{n}$ be the free abelian group with a basis $\{u_X \mid X \in \mathrm{ind}\,\mathcal{R}(A^F)\}$. Let

$$g\big(\mathcal{R}\big(A^F\big)\big) = \mathbf{h}' \oplus \mathbf{n},$$

a direct sum of $\mathbb{Z}$-modules. We shall consider the quotient group

$$g\big(\mathcal{R}\big(A^F\big)\big)_{(q-1)} = g\big(\mathcal{R}\big(A^F\big)\big)/(q-1)g\big(\mathcal{R}\big(A^F\big)\big).$$

We still use u_M, h_M to denote the corresponding residue classes for $M \in \mathcal{R}(A^F)$.

Then by [PX2], $g(\mathcal{R}(A^F))_{(q-1)}$ is a Lie algebra over $\mathbb{Z}/(q-1)\mathbb{Z}$ with the Lie operation $[-,-]$ as follows.

(1) For any two indecomposable objects $X, Y \in \mathcal{R}(A^F)$,

$$[u_X, u_Y] = \begin{cases} \sum_{L \in \mathrm{ind}\,\mathcal{R}(A^F)} (F^L_{YX} - F^L_{XY}) u_L, & \text{if } Y \ncong TX; \\ -\frac{h_X}{d(X)}, & \text{if } Y \cong TX. \end{cases}$$

(2) For any objects $X, Y \in \mathcal{R}(A^F)$ with Y indecomposable,

$$[h_X, u_Y] = I_{\mathcal{R}(A^F)}(h_X, h_Y) u_Y \quad \text{and} \quad [u_Y, h_X] = -[h_X, u_Y].$$

(3) $[\mathbf{h}', \mathbf{h}'] = 0$.

Obviously, $g(\mathcal{R}(A^F))_{(q-1)}$ has the canonical decomposition

$$g(\mathcal{R}(A^F))_{(q-1)} = \mathbf{h}' \oplus \bigoplus_{\alpha=\underline{\dim} X,\, X\in \operatorname{ind}\mathcal{R}(A^F)} \big(g(\mathcal{R}(A^F))_{(q-1)}\big)_\alpha,$$

where $(g(\mathcal{R}(A^F))_{(q-1)})_\alpha$ is the $\mathbb{Z}/(q-1)\mathbb{Z}$-submodule spanned by all u_X with $X \in \operatorname{ind}\mathcal{R}(A^F)$ and $\underline{\dim} X = \alpha$.

As in [LP, Section 5.2], we define the direct product $\prod_{E\in\Omega} g(\mathcal{R}(A^F)^E)_{(|E|-1)}$ of Lie algebras and let $\mathcal{LC}(\mathcal{R}(A^F))_1$ be the Lie subalgebra of $\prod_{E\in\Omega} g(\mathcal{R}(A^F)^E)_{(|E|-1)}$ generated by $u_{S_i} = (u_{S_i^E})_{E\in\Omega}$ and $u_{TS_i} = (u_{TS_i^E})_{E\in\Omega}$, $i = 0, 1, \ldots, 5$. Write

$$\mathbf{g}' = \mathcal{LC}(\mathcal{R}(A^F))_1 \otimes_{\mathbb{Z}} \mathbb{C},$$

then $\mathbf{g}'$ is a Lie algebra over $\mathbb{C}$, called the *Ringel–Hall Lie algebra* of the root category $\mathcal{R}(A^F)$. Naturally $\mathbf{g}'$ has the following grading

$$\mathbf{g}' = \bigoplus_{\alpha\in K_0(\mathcal{R}(A^F))} \mathbf{g}'_\alpha$$

such that $\deg(u_{S_i}) = \underline{\dim}\, S_i$ and $\deg(u_{TS_i}) = \underline{\dim}\, TS_i$, where $\mathbf{g}'_0$ is just $\mathbf{h}'$.

5.2. The following is the main theorem in this article, which gives a realization of the elliptic Lie algebra of type $F_4^{(2,2)}$.

Theorem. *Let A^F be the F-fixed point subalgebra of the tubular algebra of type $\mathbb{T}(3,3,3)$ under the Frobenius morphism $F = F_{Q,\sigma}$ defined in Section 3.1, $\mathcal{R}(A^F) = \mathcal{D}^b(A^F)/T^2$ the root category of A^F, $\mathbf{g}'$ the Ringel–Hall Lie algebra over $\mathbb{C}$ of the root category $\mathcal{R}(A^F)$ in the sense of [PX2], $\mathbf{g}$ the elliptic Lie algebra of type $F_4^{(2,2)}$ over $\mathbb{C}$ defined in Section 2.2. Then there is a Lie algebra isomorphism*

$$\Phi : \mathbf{g} \to \mathbf{g}'$$

defined by $\Phi(h_i) = h_{S_i}$, $\Phi(e_i) = u_{S_i}$, $\Phi(f_i) = -u_{TS_i}$ for any $i = 0, 1, \ldots, 5$.

The remainder of this article aims to prove it.

5.3.

Lemma. *There exists a group isomorphism*

$$\Psi : K_0(\mathcal{R}(A^F)) \to \mathcal{Q}$$

defined by $h_{S_i} \mapsto h_i$, $0 \leqslant i \leqslant 5$. And under Ψ we have $I_{\mathcal{R}(A^F)}(-,-) = I_R(-,-)$.

Proof. Since $\{h_{S_i} \mid 0 \leqslant i \leqslant 5\}$ is a basis of $K_0(\mathcal{R}(A^F))$ and $\{h_i \mid 0 \leqslant i \leqslant 5\}$ is a basis of $\mathcal{Q}$, then Ψ is a group isomorphism. Set $a_{ij} = I_{\mathcal{R}(A^F)}(h_{S_i}, h_{S_j})$, $0 \leqslant i, j \leqslant 5$. Then $a_{ij} = \dim_{\mathbb{F}_q} \operatorname{Hom}_{\mathcal{R}(A^F)}(S_i, S_j) - \dim_{\mathbb{F}_q} \operatorname{Hom}_{\mathcal{R}(A^F)}(S_i, TS_j) + \dim_{\mathbb{F}_q} \operatorname{Hom}_{\mathcal{R}(A^F)}(S_j, S_i) - \dim_{\mathbb{F}_q} \operatorname{Hom}_{\mathcal{R}(A^F)}(S_j, TS_i)$. By computation, the matrix $A = (a_{ij})_{6\times 6}$ is equal to the matrix N in Section 2.2. Thus $I_{\mathcal{R}(A^F)}(-,-) = I_R(-,-)$. $\square$

From now on, we identify $K_0(\mathcal{R}(A^F))$ with $\mathcal{Q}$, and we can think that $\mathbf{g}'$ is also graded by $\mathcal{Q}$ such that $\deg(u_{S_i}) = \alpha_i$ and $\deg(u_{TS_i}) = -\alpha_i$ for $i = 0, 1, \ldots, 5$.

5.4.

Lemma.

(1) *Φ defined in Theorem 5.2 is a well-defined Lie morphism which is surjective and keeps the gradations.*
(2) *There is an isomorphism $\Phi|_{\mathbf{g}_0} : \mathbf{g}_0 \to \mathbf{g}'_0$.*

Proof. (1) To prove that Φ is a well-defined map, we only need to check that the generators u_{S_i}, u_{TS_i} and h_{S_i}, $i = 0, 1, \ldots, 5$, satisfy all relations in Section 2.2.

By definition, the relation O holds.

By easy computation, $d(S_i) = \frac{1}{2} I_R(\alpha_i, \alpha_i)$ for any $i = 0, 1, \ldots, 5$, and so the relation I holds.

By Lemma 5.3, the relation II1 obviously holds.

To check the relation II2, for any $i = 0, 1, \ldots, 5$, we set $\alpha = \alpha_i + \max\{1, 1 - I_R(\alpha_i^\vee, \alpha_j)\}\alpha_j$. If $I_R(\alpha_i^\vee, \alpha_j) > 0$, $i \neq j$, then $\alpha_i, \alpha_j \in \{\alpha_2, \alpha_5\}$, and $\alpha = \alpha_2 + \alpha_5$, which implies that $I_R(\alpha, \alpha) = 8$. Thus $\chi(\alpha) = \frac{1}{2} I_R(\alpha, \alpha) = 4$. By Theorem 4.5(1), α is not a real root of $\mathcal{R}(A^F)$, then there is no indecomposable object X in $\mathcal{R}((A^F)^E)$ for any $E \in \Omega$ such that $\underline{\dim} X = \alpha$. Similar to [LP, Proposition 5.2], we have $\mathbf{g}'_\alpha = 0$. If $I_R(\alpha_i^\vee, \alpha_j) \leqslant 0$, $i \neq j$, then $\alpha = \alpha_i + (1 - I_R(\alpha_i^\vee, \alpha_j))\alpha_j$. Since $S(i)$, $S(j)$ lie in a representation-finite hereditary algebra, then following [Rin3], we have the Serre relations: $(\mathrm{ad}\, u_{S_i})^{1 - I_R(\alpha_i^\vee, \alpha_j)} u_{S_j} = 0$ and $(\mathrm{ad}\, u_{TS_i})^{1 - I_R(\alpha_i^\vee, \alpha_j)} u_{TS_j} = 0$. Thus the relation II2 holds.

For $\alpha = \pm(\alpha_2 + \alpha_1 + \alpha_5)$ or $\pm(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_5)$, it is easily computed that $\chi(\alpha) = \frac{1}{2} I_R(\alpha, \alpha) = 3$, and then by Theorem 4.5(1), α is not a real root of $\mathcal{R}(A^F)$. Thus there is no indecomposable object X in $\mathcal{R}((A^F)^E)$ for any $E \in \Omega$ such that $\underline{\dim} X = \alpha$, which implies that the relations III and V hold.

For $\alpha = \pm(\alpha_2 + \alpha_3 + \alpha_5)$, write it as $\alpha = \pm[(2\alpha_2 + \alpha_3) + a]$, where $\chi(2\alpha_2 + \alpha_3) = \chi(\alpha) = 2$. By Theorem 4.5(1), there is no indecomposable object X in $\mathcal{R}((A^F)^E)$ for any $E \in \Omega$ such that $\underline{\dim} X = \alpha$. So we have checked the relation IV.

Thus Φ is well defined, and obviously an epimorphism.

(2) By the definition of $\mathbf{g}'_0$, $h_{S_0}, h_{S_1}, \ldots, h_{S_5}$ are linearly independent in $\mathbf{g}'_0$, and so $\dim_{\mathbb{C}} \mathbf{g}'_0 \geqslant 6$. By Section 2.2, $\dim_{\mathbb{C}} \mathbf{g}_0 = 6$. Thus $\Phi|_{\mathbf{g}_0} : \mathbf{g}_0 \to \mathbf{g}'_0$ is isomorphic. □

6. Isomorphisms between root spaces

For any $\alpha \in R^{re} \cup R^{im}$, by Lemma 5.4, the map Φ defined in Section 5.2 is an epimorphism and induces the surjective map $\Phi|_{\mathbf{g}_\alpha} : \mathbf{g}_\alpha \to \mathbf{g}'_\alpha$. Moreover, the following Propositions 6.1 and 6.5 show that $\Phi|_{\mathbf{g}_\alpha}$ is in fact an isomorphism.

6.1.

Proposition. *For any $\alpha \in R^{re}$, there is an isomorphism $\Phi|_{\mathbf{g}_\alpha} : \mathbf{g}_\alpha \to \mathbf{g}'_\alpha$ of linear spaces.*

Proof. For $i = 0, 1, \ldots, 5$, $u_{S_i} \in \mathbf{g}'_{\alpha_i}$, and so $1 \leqslant \dim_{\mathbb{C}} \mathbf{g}'_{\alpha_i} \leqslant \dim_{\mathbb{C}} \mathbf{g}_{\alpha_i} = 1$ by Lemma 5.4, then $\dim_{\mathbb{C}} \mathbf{g}'_{\alpha_i} = 1$. Obviously, $\mathrm{ad}\, u_{S_i}$, $\mathrm{ad}\, u_{TS_i}$ are locally nilpotent. So the automorphism $\exp(\mathrm{ad}(-u_{TS_i})) \times \exp(-\mathrm{ad}\, u_{S_i}) \exp(\mathrm{ad}(-u_{TS_i}))$ induces an isomorphism $\mathbf{g}'_\alpha \cong \mathbf{g}'_{\omega_{\alpha_i}(\alpha)}$ for any $\alpha \in R^{re}$. Since $R^{re} = W\Pi$, then for any $\alpha \in R^{re}$, there is some $i \in \{0, 1, \ldots, 5\}$ such that $\dim_{\mathbb{C}} \mathbf{g}'_\alpha = \dim_{\mathbb{C}} \mathbf{g}'_{\alpha_i}$, and so $\dim_{\mathbb{C}} \mathbf{g}'_\alpha = 1$. By Section 2.2, $\dim_{\mathbb{C}} \mathbf{g}_\alpha = 1$. Thus $\Phi|_{\mathbf{g}_\alpha} : \mathbf{g}_\alpha \to \mathbf{g}'_\alpha$ is an isomorphism. □

The following Sections 6.2–6.5 aim to prove that $\Phi|_{\mathbf{g}_\alpha}$ is an isomorphism for any $\alpha \in R^{im}$.

6.2. In this subsection, we show that the Ringel–Hall algebra $\mathbf{g}'$ can be generated by the objects and their shifts in any complete exceptional sequence of $\mathcal{R}(A^F)$.

Lemma. *Let (X, Y) be an exceptional pair in $\mathcal{R}(A^F)$, and $\mathrm{Hom}_{\mathcal{R}(A^F)}(X, Y) \neq 0$. Assume that $m = \dim_{\mathrm{End}(X)} \mathrm{Hom}_{\mathcal{R}(A^F)}(X, Y)$. If $TY \xrightarrow{f} Z \xrightarrow{g} X^{(m)} \xrightarrow{can.} Y$ is the triangle induced by the canonical morphism,* *then in* $g(\mathcal{R}(A^F))_{(q-1)}$, $(-1)^m(m!)u_Z = (\operatorname{ad} u_X)^m u_{TY}$. *Dually, put* $n = \dim_{\operatorname{End}(Y)} \operatorname{Hom}_{\mathcal{R}(A^F)}(X, Y)$. *If* $X \xrightarrow{can.} Y^{(n)} \xrightarrow{g} W \xrightarrow{h} TY$ *is the triangle induced by the canonical morphism. Then* $(n!)u_W = (\operatorname{ad} u_Y)^n u_{TX}$.

Proof. By Proposition 4.3, $\mathcal{R}(A^F)$ has properties similar to those of $\mathcal{R}(A)$, then we can take a process similar to that in the proof of [LP, Proposition 7.3]. □

Let $\mathcal{X} = (X_1, X_2, \ldots, X_6)$ be a complete exceptional sequence in $\mathcal{R}(A^F)$, $\mathcal{L}(\mathcal{X}, T\mathcal{X})$ is the Lie subalgebra of $\prod_{E\in\Omega} g(\mathcal{R}(A^F)^E)_{(|E|-1)}$ generated by $u_{X_i} := (u_{X_i^E})_{E\in\Omega}$ and $u_{TX_i} := (u_{TX_i^E})_{E\in\Omega}$, $1 \leqslant i \leqslant 6$.

Proposition. *Let* $\mathcal{X}$ *be a complete exceptional sequence in* $\mathcal{R}(A^F)$. *Then*

(1) $\mathcal{L}(\mathcal{X}, T\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{L}(\sigma_i\mathcal{X}, T\sigma_i\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{L}(\gamma_j\mathcal{X}, T\gamma_j\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C}$, *for any* $i = 1, 2, \ldots, 5$, $j = 1, 2, \ldots, 6$.
(2) $\mathcal{L}(\mathcal{X}, T\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbf{g}'$.

Proof. (1) Let $m = \dim_{\operatorname{End}(X_i)} \operatorname{Hom}_{\mathcal{R}(A^F)}(X_i, X_{i+1})$, $n = \dim_{\operatorname{End}(X_{i+1})} \operatorname{Hom}_{\mathcal{R}(A^F)}(X_i, X_{i+1})$. By Lemma 6.2, $(-1)^m(m!)u_{L_{X_i}X_{i+1}} = (\operatorname{ad} u_{X_i})^m u_{TX_{i+1}}$, $(n!)u_{R_{X_{i+1}}X_i} = (\operatorname{ad} u_{X_{i+1}})^n u_{TX_i}$, then $\mathcal{L}(\mathcal{X}, T\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{L}(\sigma_i\mathcal{X}, T\sigma_i\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C}$. The equality $\mathcal{L}(\mathcal{X}, T\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{L}(\gamma_j\mathcal{X}, T\gamma_j\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C}$ is obvious.

(2) By Lemma 4.4, $\mathcal{S} = (TS(5), S(1), S(3), S(0), S(4), S(2))$ is a complete exceptional sequence in $\mathcal{R}(A^F)$. By the definition of $\mathbf{g}'$, $\mathcal{L}(\mathcal{S}, T\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbf{g}'$. By the above (1) and Proposition 4.6(2), we have $\mathcal{L}(\mathcal{X}, T\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbf{g}'$. □

6.3. The Ringel–Hall Lie algebra of $\mathcal{R}(A^F)$ is in nature a Lie algebra generated by u_{S_i}, u_{TS_i}, $i = 0, 1, \ldots, 5$. The following proposition shows that the real root objects lie in the Ringel–Hall Lie algebra $\mathbf{g}'$.

Proposition. *Let* X *be an indecomposable object in* $\mathcal{R}(A^F)$. *If* $\underline{\dim}\, X \in R^{re}$, *then* $u_X \in \mathbf{g}'$.

Proof. By the structure of the AR-quiver of $\mathcal{R}(A^F)$, X is an indecomposable object on some non-homogeneous tube $\mathcal{J}$ in the AR-quiver of $\mathcal{R}(A^F)$. If X is an exceptional object, then by Proposition 4.6(1), X can be extended to a complete exceptional sequence $\mathcal{X}$. By Proposition 6.2(2), $u_X \in \mathcal{L}(\mathcal{X}, T\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbf{g}'$. Assume that X is not an exceptional object. Since the mouth objects T_1, T_2, T_3 lying on the non-homogeneous tube $\mathcal{J}$ are exceptional, then $u_X = [\cdots[[u_{T_{i_1}}, u_{T_{i_2}}], u_{T_{i_3}}], \ldots, u_{T_{i_r}}] \in \mathbf{g}'$, where $T_{i_k} \in \{T_1, T_2, T_3\}$, $1 \leqslant k \leqslant r$. □

6.4.

Proposition. *Given a tubular family* $\mathcal{T}$ *in* $\mathcal{R}(A^F)$, *there is a Kronecker algebra* K *with the corresponding imbedding functor* $G : \operatorname{mod} K \to \mathcal{R}(A^F)$ *which satisfies the following two conditions*:

(1) $0 \to X \to Y \to Z \to 0$ *is exact in* mod K *if and only if* $GX \to GY \to GZ \to TGX$ *is a triangle in* $\mathcal{R}(A^F)$.
(2) *Any homogeneous tube in* $\mathcal{T}$ *is an image of some homogeneous tube in* mod K *under the functor* G.

Proof. By [DD2, Theorem 8.5], the AR-quiver of $\mathcal{R}(A^F)$ is 'folded' by the AR-quiver of $\mathcal{R}(A)$. Let the tubular family $\mathcal{T}$ in $\mathcal{R}(A^F)$ be 'folded' by the tubular family $\mathcal{T}_A$ in the AR-quiver of $\mathcal{R}(A)$. By [LP, Lemma 8.6], there is a convex projective idempotent subalgebra $A' \in \{A_{-1}, A_0, A_1\}$ and a tubular family $\mathcal{T}'_A$ over A' satisfying that $\mathcal{D}^b(A') \cong \mathcal{D}^b(A)$ and all homogeneous tubes in $\mathcal{T}'_A$ coincide with all those in $\mathcal{T}_A$. So we can see $\mathcal{T}_A$ as a full subcategory of mod A'. The Frobenius morphism on $\hat{A}$, which is defined in the proof of Proposition 4.2(6), induces a Frobenius morphism F on its subalgebra A'. Let A'^F be the F-fixed point subalgebra of A'. Then $\mathcal{D}^b(A'^F) \cong \mathcal{D}^b(A')^F \cong \mathcal{D}^b(A)^F \cong \mathcal{D}^b(A^F)$, and so there is a tubular family $\mathcal{T}'$ in the AR-quiver of mod A'^F whose homogeneous tubes coincide with the homogeneous tubes of $\mathcal{T}$. We choose a partial tilting module M in mod A'^F which contains four pairwise different indecomposable direct summands lying on two non-homogeneous tubes of $\mathcal{T}'$. By Bongartz's Lemma, there is a tilting A'^F-module $M_1 \oplus M_2 \oplus M$ such that M_i is indecomposable and either projective or satisfies $\mathrm{Hom}_{A'^F}(M_i, M) \neq 0$ for $i = 1, 2$. For any module X on a homogeneous tube of $\mathcal{T}'$, $\mathrm{Ext}^1_{A'^F}(M_i, X) = D\,\underline{\mathrm{Hom}}_{A'^F}(\tau^- X, M_i) = 0$ for $i = 1, 2$. So X is generated by $M_1 \oplus M_2$, i.e., all homogeneous tubes of $\mathcal{T}'$ are generated by $M_1 \oplus M_2$. Let $K = \mathrm{End}_{A'^F}(M_1 \oplus M_2)$. Since $\mathcal{D}^b(\mathrm{End}(M_1 \oplus M_2 \oplus M)) \cong \mathcal{D}^b(A^F)$, it is easy to see that K is the Kronecker algebra with two simple modules and $\mathrm{mod}\, K$ contains all homogeneous tubes in $\mathcal{T}'$. We have naturally an embedding functor $L : \mathrm{mod}\, K \to \mathcal{R}(A^F)$, as required. □

6.5.

Proposition. *Let $\alpha \in R^{im}$.*

(1) *If $\alpha \in R^{im} \backslash 2R^{im}$, then $\dim_{\mathbb{C}} \mathbf{g}'_\alpha = 3$.*
(2) *If $\alpha \in 2R^{im} \backslash \{0\}$, then $\dim_{\mathbb{C}} \mathbf{g}'_\alpha = 5$.*

As a consequence, $\Phi|_{\mathbf{g}_\alpha}$ is isomorphic.

Proof. (1) If $\alpha \in R^{im} \backslash 2R^{im}$, then by Theorem 4.5(2), there exists one tubular family $\mathcal{T}$ of $\mathcal{R}(A^F)$ such that there is only one non-homogeneous tube $\mathcal{J}_1$ in $\mathcal{T}$ containing indecomposable objects X_1, X_2, X_3 with the same dimension vector α such that $\tau^- X_1 = X_2$, $\tau^- X_2 = X_3$, $\tau^- X_3 = X_1$, and there is one homogeneous tube containing one indecomposable object Y with the dimension vector α. Note that the mouth objects T_1, T_2, T_3 on the non-homogeneous tube $\mathcal{J}_1$ are exceptional. By Proposition 6.3, $u_{T_i} \in \mathbf{g}'$, $i = 1, 2, 3$. Then $u_{X_1} - u_{X_2}$, $u_{X_2} - u_{X_3}$ lie in $\mathbf{g}'_\alpha$.

Let E be a finite field extension in Ω with $|E|$ large enough and such that Y^E is still indecomposable. By Proposition 6.4, there is a Kronecker algebra K such that $\mathrm{mod}\, K^E \subseteq \mathcal{R}(A^F)^E$ and $Y^E \in \mathrm{mod}\, K^E$. By [LP, Lemma 8.7], there are exceptional modules E_1, E_2 in $\mathrm{mod}\, K$ such that $F^{Y^E}_{E_1^E, E_2^E} = 0$ and $|E| - 1$ does not divide $F^{Y^E}_{E_2^E, E_1^E}$. By Proposition 6.3, $u_{E_1^E}, u_{E_2^E} \in \mathbf{g}'$, and so $[u_{E_1^E}, u_{E_2^E}] \in \mathbf{g}'_\alpha$, and $[u_{E_1^E}, u_{E_2^E}] = a u_{Y^E} + \sum_s b_s u_{Z_s}$ such that $a \not\equiv 0 \pmod{|E| - 1}$, and u_{Y^E}, u_{Z_s} is pairwise different in $g(\mathcal{R}(A^F)^E)_{(|E|-1)}$. Thus $[u_{E_1^E}, u_{E_2^E}] \neq 0$. By the definition of $g(\mathcal{R}(A^F)^E)_{(|E|-1)}$, we know that $[u_{E_1^E}, u_{E_2^E}]$, $u_{X_1^E} - u_{X_2^E}$, $u_{X_2^E} - u_{X_3^E}$ are linearly independent in $g(\mathcal{R}(A^F)^E)_{(|E|-1)}$, and so $[u_{E_1}, u_{E_2}]$, $u_{X_1} - u_{X_2}$, $u_{X_2} - u_{X_3}$ are linearly independent in $\mathbf{g}'$. Therefore, $\dim_{\mathbb{C}} \mathbf{g}'_\alpha \geqslant 3$. On the other hand, by Section 2.2, $\dim_{\mathbb{C}} \mathbf{g}_\alpha = 3$, and so $\dim_{\mathbb{C}} \mathbf{g}'_\alpha \leqslant \dim_{\mathbb{C}} \mathbf{g}_\alpha = 3$ by Lemma 5.4. Therefore, $\dim_{\mathbb{C}} \mathbf{g}'_\alpha = \dim_{\mathbb{C}} \mathbf{g}_\alpha = 3$. As a consequence, $\Phi|_{\mathbf{g}_\alpha}$ is isomorphic.

(2) If $\alpha \in 2R^{im} \backslash \{0\}$, then by Theorem 4.5(2), there exists a tubular family $\mathcal{T}$ such that each one of the two non-homogeneous tubes $\mathcal{J}_i$, $i = 1, 2$, of $\mathcal{T}$ contains three indecomposable objects with the same dimension vector α. Similarly, we can prove that $\dim_{\mathbb{C}} \mathbf{g}'_\alpha = 5$, and $\Phi|_{\mathbf{g}_\alpha}$ is isomorphic. □

Up to now, Theorem 5.2 is followed from Lemma 5.4, Propositions 6.1 and 6.5.

7. A Chevalley basis

In this section, we give a Chevalley basis of the elliptic Lie algebra of type $F_4^{(2,2)}$ via the indecomposable objects of the root category $\mathcal{R}(A^F)$.

By Proposition 4.3,

$$\Gamma_{\mathcal{R}(A^F)} = \bigcup_{\delta' \in \Upsilon} \mathcal{T}(\delta'),$$

where each $\mathcal{T}(\delta')$ contains two non-homogeneous tubes, denoted by $\mathcal{J}(\delta', 1), \mathcal{J}(\delta', 2)$ with rank 3. Assume that any mouth object X on $\mathcal{J}(\delta', 1)$ (resp., $\mathcal{J}(\delta', 2)$) satisfies that $\chi(\underline{\dim}\, X) = 1$ (resp., 2).

For the above non-homogeneous tube $\mathcal{J}(\delta', i)$, $i = 1, 2$, its mouth contains three indecomposable objects $X_{(\delta',i,1,1)}, X_{(\delta',i,2,1)}, X_{(\delta',i,3,1)}$ such that $\tau X_{(\delta',i,j,1)} = X_{(\delta',i,j-1,1)}$, $j = 1, 2, 3$, where $X_{(\delta',i,0,1)} = X_{(\delta',i,3,1)}$. Denote by $X_{(\delta',i,j,l)}$ the unique indecomposable object in $\mathcal{J}(\delta', i)$ which has the filtration of the form

$$0 \subseteq X_{(\delta',i,j,1)} \subseteq X_{(\delta',i,j,2)} \subseteq \cdots \subseteq X_{(\delta',i,j,l)}$$

such that $X_{(\delta',i,j,h)}/X_{(\delta',i,j,h-1)} \cong X_{(\delta',i,j',1)}$, where $j' = 1, 2, 3$, and $j' \equiv j + h - 1 \pmod 3$. Furthermore,

$$\left\{X_{(\delta',i,j,l)} \;\middle|\; \delta' \in \Upsilon,\ i = 1, 2,\ j = 1, 2, 3,\ l \in \mathbb{N}\right\}$$

is the complete set of the indecomposable non-homogeneous objects of $\mathcal{R}(A^F)$. In addition, from Proposition 6.4 we know that there is a relatively simple injective object $E_{(\delta',0)}$ and a relatively preprojective object $E_{(\delta',s)}$ in the Kronecker subcategory $\mathcal{K}(\delta')$ associated to the tubular family $\mathcal{T}(\delta')$ such that $\underline{\dim}\, E_{(\delta',0)} + \underline{\dim}\, E_{(\delta',s)} = s\delta'$. By the proof of Theorem 5.2, we can see easily that the set

$$\begin{aligned}\mathcal{B} = {} & \left\{u_{X_{(\delta',i,j,l+3s)}} \;\middle|\; \delta' \in \Upsilon,\ i = 1, 2,\ j = 1, 2, 3,\ l = 1, 2,\ s \in \mathbb{N} \cup \{0\}\right\} \\ & \cup \left\{u_{X_{(\delta',i,j,3s)}} - u_{X_{(\delta',i,j+1,3s)}} \;\middle|\; \delta' \in \Upsilon,\ i = 1, 2,\ j = 1, 2,\ s \in \mathbb{N}\right\} \\ & \cup \left\{[u_{E_{(\delta',0)}}, u_{E_{(\delta',s)}}] \;\middle|\; \delta' \in \Upsilon,\ s \in \mathbb{N}\right\} \cup \{h_{S_i} \mid 0 \leqslant i \leqslant 5\}\end{aligned}$$

is a Chevalley basis.

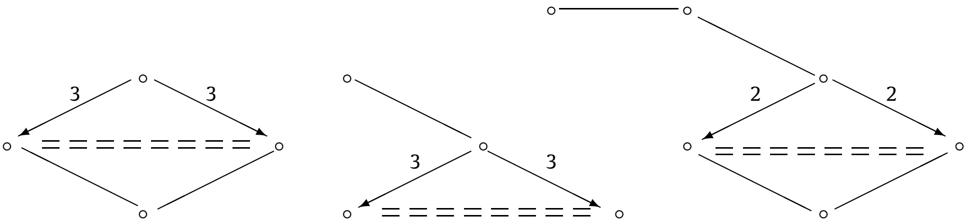
7.1.

Proposition. *$\mathcal{B}$ is a Chevalley basis of the elliptic Lie algebra $\mathbf{g}'$ of type $F_4^{(2,2)}$.*

Remark. Elliptic Lie algebras of type $G_2^{(3,3)}$, $G_2^{(1,3)}$, or $F_4^{(1,2)}$ can be realized by the Ringel–Hall approach in a way similar to that in this article. Our idea to realize an elliptic Lie algebra $\mathbf{g}$ of type $F_4^{(2,2)}$ is based on the observation that the elliptic Dynkin diagram of $\mathbf{g}$ can be obtained by 'folding' the quiver Q of the tubular algebra of type $\mathbb{T}(3, 3, 3)$ via an admissible automorphism σ of Q. Let Q be the quiver, shown in [LP, Section 1.3], of the tubular algebra of type $\mathbb{T}(2, 2, 2, 2)$, $\mathbb{T}(3, 3, 3)$ or $\mathbb{T}(4, 4, 2)$. It is easy to see that there is an admissible automorphism σ of $Q = (Q_0, Q_1)$ as follows:

(1) for type $\mathbb{T}(2, 2, 2, 2)$, $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(4) = 2$, $\sigma(\alpha_1) = \alpha_2$, $\sigma(\alpha_2) = \alpha_3$, $\sigma(\alpha_3) = \alpha_1$, $\sigma(\beta_1) = \beta_2$, $\sigma(\beta_2) = \beta_3$, $\sigma(\beta_3) = \beta_1$, and σ leaves the other vertices and arrows invariant;
(2) for type $\mathbb{T}(3, 3, 3)$, $\sigma(2) = 4$, $\sigma(4) = 6$, $\sigma(6) = 2$, $\sigma(3) = 5$, $\sigma(5) = 7$, $\sigma(7) = 3$, $\sigma(\alpha_1) = \alpha_2$, $\sigma(\alpha_2) = \alpha_3$, $\sigma(\alpha_3) = \alpha_1$, $\sigma(\beta_1) = \beta_2$, $\sigma(\beta_2) = \beta_3$, $\sigma(\beta_3) = \beta_1$, $\sigma(\gamma_1) = \gamma_2$, $\sigma(\gamma_2) = \gamma_3$, $\sigma(\gamma_3) = \gamma_1$, and σ leaves the other vertices invariant;
(3) for type $\mathbb{T}(4, 4, 2)$, $\sigma(3) = 6$, $\sigma(4) = 7$, $\sigma(5) = 8$, $\sigma(6) = 3$, $\sigma(7) = 4$, $\sigma(8) = 5$, $\sigma(\alpha_1) = \alpha_3$, $\sigma(\alpha_3) = \alpha_1$, $\sigma(\gamma_1) = \gamma_3$, $\sigma(\gamma_2) = \gamma_4$, $\sigma(\gamma_3) = \gamma_1$, $\sigma(\gamma_4) = \gamma_2$, and σ leaves the other vertices and arrows invariant.

So the above quiver Q can be 'folded' to be the following elliptic Dynkin diagram of the elliptic Lie algebra of type $G_2^{(3,3)}$, $G_2^{(1,3)}$, or $F_4^{(1,2)}$:



$G_2^{(3,3)}$ $G_2^{(1,3)}$ $F_4^{(1,2)}$

Thus elliptic Lie algebras of type $G_2^{(3,3)}$, $G_2^{(1,3)}$, or $F_4^{(1,2)}$ can be realized by the Ringel–Hall approach similarly.

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