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Canonical double covers of minimal rational surfaces and the non-existence of carpets ☆

Francisco Javier Gallego [a], Miguel González [b], Bangere P. Purnaprajna [c,*]

[a] Departamento de Álgebra and Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, Spain
[b] Departamento de Álgebra, Universidad Complutense de Madrid, Spain
[c] Department of Mathematics, University of Kansas, United States

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ABSTRACT

This article delves into the relation between the deformation theory of finite morphisms to projective space and the existence of ropes, embedded in projective space, with certain invariants. We focus on the case of canonical double covers X of a minimal rational surface Y, embedded in $\mathbf{P}^N$ by a complete linear series, and carpets on Y, canonically embedded in $\mathbf{P}^N$. We prove that these canonical double covers always deform to double covers and that canonically embedded carpets on Y do not exist. This fact parallels the results known for hyperelliptic canonical morphisms of curves and canonical ribbons, and the results for $K3$ double covers of surfaces of minimal degree and Enriques surfaces and $K3$ carpets. That canonical double covers of minimal rational surfaces should deform to double covers is not a priori obvious, for the invariants of most of these surfaces lie on or above the Castelnuovo line; thus, in principle, deformations of such covers could have birational canonical maps. In fact, many canonical double covers of non-minimal rational surfaces do deform to birational canonical morphisms.

We also map the region of the geography of surfaces of general type corresponding to the surfaces X and we compute the dimension of the irreducible moduli component containing $[X]$. In certain cases we exhibit some interesting moduli components

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* Corresponding author.
E-mail addresses: gallego@mat.ucm.es (F.J. Gallego), mgonza@mat.ucm.es (M. González), purna@math.ku.edu (B.P. Purnaprajna).

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parameterizing surfaces S with the same invariants as X but with birational canonical map, unlike X.

Introduction

In this paper we delve into the relation between double covers and multiplicity 2 ropes or, more precisely, into the relation between the deformation theory of double covers and the existence or non-existence of multiplicity 2 ropes. To do so we will focus on the case of surfaces of general type that are canonical double covers of a surface Y in $\mathbf{P}^N$, which is either $\mathbf{P}^2$ or a Hirzebruch surface, and canonically embedded carpets. This relation has been previously studied for the case of the canonical morphism of hyperelliptic curves and canonical ribbons (see [Fon93]) or for the case of $K3$ double covers of surfaces of minimal degree and Enriques surfaces and $K3$ carpets (see [GP97] and [GGP08]). In all of these cases, the following correlation occurs: whenever a double cover X appears as limit of projective embeddings (for example, when X is a hyperelliptic curve of genus $g \geqslant 3$ or a $K3$ double cover of a rational normal scroll or an Enriques surface), then a multiplicity 2 rope with same invariants as X appears (in the previous examples, a canonical ribbon or a $K3$ carpet), so as to bear witness to the event of an embedding degenerating to a degree 2 morphism. On the contrary, when the double cover cannot be obtained as limit of embeddings, as is the case of hyperelliptic morphisms of genus 2 curves or $K3$ double covers of the Veronese surface, no multiplicity 2 ropes come into being.

Deep understanding of the relation between the deformation theory of morphisms to projective space and embedded ropes was gained in [Gon06, Proposition 3.7 and Theorem 3.8]. Because of it, we see that in our case looking at a single cohomology group on Y suffices to understand both the nature of deformations of the canonical double cover and the existence of canonical carpets. As it turns out (see Theorem 1.9), canonical double covers of $\mathbf{P}^2$ or a Hirzebruch surface deform always to another canonical double cover and, giving further evidence of the above-mentioned correlation, there do not exist carpets on $\mathbf{P}^2$ or on a Hirzebruch surface which are canonically embedded (see Definition 1.3 and Theorem 1.12). "Abstract", analytic deformations of double covers were previously studied in general in [Wav68] and [Weh86] and, when the double covers are of Hirzebruch surfaces, in [Kon85] (as a matter of fact, part of Theorem 1.9 can be proved using results in [Kon85] and [Weh86]; see Remark 1.11). In contrast, Theorem 1.9 deals with algebraic deformations of morphisms to projective space and, in this sense, fits in the general framework given by [GGP10, Proposition 1.3 and Theorems 1.4 and 2.6].

Theorem 1.9 is interesting because the invariants of the canonical double covers φ under consideration do not indicate φ should necessarily deform to a double cover. In fact the invariants of many of the surfaces X of Theorem 1.9 lie on or above Castelnuovo's line $c_1^2 = 3p_g - 7$, so the existence of deformations of φ to degree 1 morphisms would be plausible in these cases (for instance, Examples 2.8 and 2.9 show the existence of surfaces of general type with very ample canonical divisors and having the same invariants as certain canonical double covers of minimal rational surfaces). Moreover, if we consider canonical double covers of non-minimal rational surfaces, there exist cases for which the canonical morphism can be deformed to a morphism of degree 1 (see [AK90, 4.5] and [GGP10, Theorem 3.14]). In contrast, Theorem 1.9 can be rephrased in terms of the moduli space in the following way: the irreducible moduli component of $[X]$ parameterizes surfaces whose canonical map is a finite, degree 2 morphism. Therefore Theorem 1.9 generalizes the behavior of the deformations of canonical covers of surfaces of minimal degree (see [Hor76]). This variety of behaviors shows how different and complicated the moduli spaces of these surfaces of general type can be when compared with, for instance, the moduli of curves or moduli spaces of surfaces of lower Kodaira dimension such as a $K3$ or Enriques surfaces.

Finite covers of rational surfaces have interesting implications for the geography and the moduli of surfaces of general type (see e.g. [Cat84] or [Hor76]). In Section 2 we chart the region in the geography covered by the canonical double covers X of Theorem 1.9. This is done in Propositions 2.1 and 2.2 and in Remark 2.4. The Chern quotient $\frac{c_1^2}{c_2}$ of our surfaces approaches $\frac{1}{2}$; on the other hand, our region is not contained but goes well inside the region above Castelnuovo's line. We also explore the moduli space of $[X]$. First, in Proposition 2.6, we compute the dimension of the irreducible moduli component of $[X]$, which is $2d^2+15d+19$ if Y is $\mathbf{P}^2$ embedded by $|\mathcal{O}_{\mathbf{P}^2}(d)|$ and $(2a+5)(2b-ae+5)-7$ if Y is $\mathbf{F}_e$ embedded by an arbitrary very ample linear series $|\mathcal{O}_Y(aC_0+bf)|$. Finally we go a bit further in studying the complexity of some of these moduli spaces. We find examples of moduli spaces having two kind of components: components parameterizing surfaces which can be canonically embedded and components parameterizing surfaces whose canonical map is a degree 2 morphism.

Convention. We will work over an algebraically closed field $\mathbf{k}$ of characteristic 0.

1. Deformations of canonical double covers of Hirzebruch surfaces and $\mathbf{P}^2$ and the non-existence of carpets

In this section we link two themes: on the one hand, the deformation theory of double covers φ of a smooth variety Y, embedded in the projective space $\mathbf{P}^N$ and, on the other hand, the existence or non-existence of *carpets* supported on Y, also embedded in $\mathbf{P}^N$. We will focus on the case of *canonical double covers* (see Definition 1.2) and *canonically embedded carpets* (see Definition 1.3) and restrict our attention to when Y is either $\mathbf{P}^2$ or a Hirzebruch surface. The study of both themes will be addressed by looking at the same cohomology group on Y, namely $H^0(\mathcal{N}_{Y,\mathbf{P}^N}\otimes\omega_Y(-1))$. To start seeing the reason for this relation between double covers and carpets, consider a morphism φ from a smooth irreducible surface X to $\mathbf{P}^N$ such that φ factors as $\varphi=i\circ\pi$, where π is a finite, double cover of Y and i embeds Y in $\mathbf{P}^N$. The morphism π is flat and its trace-zero module $\mathcal{E}$ is a line bundle. We then focus on the group $H^0(\mathcal{N}_\varphi)$, which parameterizes first order infinitesimal deformations of φ, and on the group $H^0(\mathcal{N}_{i(Y),\mathbf{P}^N}\otimes\mathcal{E})$ which, according to [Gon06, Proposition 2.1], parameterizes pairs $(\widetilde{Y},\tilde{i})$, where $\widetilde{Y}$ is a rope on Y with conormal bundle $\mathcal{E}$ and $\tilde{i}$ is a morphism from Y to $\mathbf{P}^N$ extending i. The relation between these two groups, which is the relation that links the deformation theory of φ with the existence or non-existence of carpets, is given by the following result, which holds in wider generality (it holds for X and Y smooth irreducible projective varieties of arbitrary dimension and π finite morphism of any degree $n\geqslant 2$ with trace-zero module $\mathcal{E}$ of rank $n-1$):

Proposition 1.1. *(See [Gon06, Proposition 3.7].) Let X be a smooth irreducible variety and let φ be a morphism from X to $\mathbf{P}^N$ that factors as $\varphi=i\circ\pi$, where π is a finite cover of smooth variety Y and i embeds Y in $\mathbf{P}^N$. Let $\mathcal{E}$ be the trace-zero module of π and let $\mathcal{I}$ be the ideal sheaf of $i(Y)$ in $\mathbf{P}^N$. There exists a homomorphism*

$$H^0(\mathcal{N}_\varphi)\xrightarrow{\Psi}\operatorname{Hom}\bigl(\pi^*\bigl(\mathcal{I}/\mathcal{I}^2\bigr),\mathcal{O}_X\bigr),$$

that appears when taking cohomology on the commutative diagram [Gon06, (3.3.2)]. Since

$$\operatorname{Hom}\bigl(\pi^*\bigl(\mathcal{I}/\mathcal{I}^2\bigr),\mathcal{O}_X\bigr)=\operatorname{Hom}\bigl(\mathcal{I}/\mathcal{I}^2,\pi_*\mathcal{O}_X\bigr)=\operatorname{Hom}\bigl(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_Y\bigr)\oplus\operatorname{Hom}\bigl(\mathcal{I}/\mathcal{I}^2,\mathcal{E}\bigr)$$

the homomorphism Ψ has two components

$$H^0(\mathcal{N}_\varphi)\xrightarrow{\Psi_1}\operatorname{Hom}\bigl(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_Y\bigr),$$
$$H^0(\mathcal{N}_\varphi)\xrightarrow{\Psi_2}\operatorname{Hom}\bigl(\mathcal{I}/\mathcal{I}^2,\mathcal{E}\bigr).$$

Now we define canonical double covers and canonically embedded carpets and set up the notation for the remaining of the paper:

Definition 1.2. Let Y be a smooth, irreducible, projective surface and let $i : Y \hookrightarrow \mathbf{P}^N$ be an embedding induced by a complete linear series on Y. Let X be a smooth, irreducible surface of general type with ample and base-point-free canonical divisor and let φ be the canonical map of X. If there exists a finite cover $\pi : X \longrightarrow Y$ of degree 2 such that $\varphi = i \circ \pi$, we say that $\pi : X \longrightarrow Y$ (or, for short, X) is a canonical double cover of Y and that the canonical map φ of X factors through the canonical double cover π.

Definition 1.3. Let Y be a smooth, irreducible, projective surface and let $i : Y \hookrightarrow \mathbf{P}^N$ be an embedding of Y in $\mathbf{P}^N$. Let $\widetilde{Y}$ a scheme such that

(a) $(\widetilde{Y})_{\mathrm{red}} = Y$;
(b) $\mathscr{I}^2_{Y,\widetilde{Y}} = 0$; and
(c) $\mathscr{I}_{Y,\widetilde{Y}}$ is a line bundle on Y (i.e., $\widetilde{Y}$ is a *rope* of multiplicity 2 on Y), called the *conormal bundle* of $\widetilde{Y}$.

Let $\tilde{i} : \widetilde{Y} \hookrightarrow \mathbf{P}^N$ be an embedding of $\widetilde{Y}$ in $\mathbf{P}^N$ that extends i. We say that $\widetilde{Y}$ is canonically embedded by $\tilde{i}$ or, for short, that $\tilde{i}(\widetilde{Y})$ is a canonical carpet, if the dualizing sheaf $\omega_{\widetilde{Y}}$ of $\widetilde{Y}$ is very ample and $\tilde{i}$ is induced by the complete linear series of $\omega_{\widetilde{Y}}$.

Throughout the remaining of this article we will use the following

Notation 1.4.

(1) We will denote by $\mathbf{F}_e$ the Hirzebruch surface whose minimal section C_0 has self-intersection $C_0^2 = -e$.
(2) Y will be either $\mathbf{P}^2$ or the Hirzebruch surface $\mathbf{F}_e$, whose minimal section C_0 has self-intersection $C_0^2 = -e$; we will denote by f the fiber of the projection of $\mathbf{F}_e$ onto $\mathbf{P}^1$.
(3) i will denote a projective embedding $i : Y \hookrightarrow \mathbf{P}^N$ induced by a complete linear series on Y. In this case, $\mathscr{I}$ will denote the ideal sheaf of $i(Y)$ in $\mathbf{P}^N$. Likewise, we will often abridge $i^*\mathscr{O}_{\mathbf{P}^N}(1)$ as $\mathscr{O}_Y(1)$.

Our next goal is to characterize canonical double covers of Y and canonical carpets on Y. For this, we characterize the trace-zero modules of canonical double covers and the conormal bundles of canonical carpets. In doing so we begin to unfold the relation between canonical double covers and canonical carpets:

Lemma 1.5. *Let X be a smooth, irreducible surface of general type and let $\varphi : X \longrightarrow \mathbf{P}^N$ factor through a canonical cover π of Y. Let $\widetilde{Y}$ be a carpet on Y, canonically embedded in $\mathbf{P}^N$ by an embedding $\tilde{i}$ extending i. Let $\mathscr{E}$ be either*

(a) *the trace-zero module of π; or*
(b) *the conormal bundle of $\widetilde{Y}$.*

Then $\mathscr{E}$ is isomorphic to $\omega_Y(-1)$.

Proof. Part (a) follows from relative duality, having in account that, on Y, numerical equivalence is the same as linear equivalence. Part (b) follows from [GGP08, (1.4.2)]. □

Lemma 1.5 allows us to tell exactly for what embeddings i of Y there exists a canonical double cover of $i(Y)$:

Lemma 1.6. *Let i be induced by the complete linear series of a very ample divisor D on Y (if Y is the Hirzebruch surface $\mathbf{F}_e$, then $D = aC_0 + bf$ with $b - ae \geqslant 1$ and $a \geqslant 1$). There exist a smooth surface of general type X and a canonical double cover $\pi : X \longrightarrow Y$ such that the canonical map φ of X factors as $\varphi = i \circ \pi$ if and only if*

(1) *$Y = \mathbf{P}^2$; or*
(2) *Y is a Hirzebruch surface $\mathbf{F}_e$ such that*
  (a) *$e \leqslant 3$, or*
  (b) *$b - ae \geqslant e - 2$ or $b - ae = \frac{1}{2}e - 2$, if $e \geqslant 4$.*

Proof. Assume first that X is a smooth surface of general type such that its canonical map φ factors through a canonical double cover π of Y. Lemma 1.5 tells us that the trace-zero module of π is $\omega_Y(-1)$, so the branch divisor of π belongs to $|\omega_Y^{-2}(2)|$. Since X is smooth, $|\omega_Y^{-2}(2)|$ contains a smooth member. If $Y = \mathbf{P}^2$ and $\mathcal{O}_Y(D) = \mathcal{O}_{\mathbf{P}^2}(d)$, then

$$\omega_Y^{-2}(2) = \mathcal{O}_{\mathbf{P}^2}(2d+6), \tag{1.6.1}$$

so $|\omega_Y^{-2}(2)|$ containing a smooth member implies $2d + 6 \geqslant 0$, which does not impose any extra condition on i, since $d \geqslant 1$. If $Y = \mathbf{F}_e$ and $|\omega_Y^{-2}(2)|$ contains a smooth member but $\omega_Y^{-2}(2)$ is not base-point-free, then C_0 is the fixed part of $|\omega_Y^{-2}(2)|$ and does not intersect the mobile (base-point-free) part of $|\omega_Y^{-2}(2)|$. This happens if and only if $b - ae = \frac{1}{2}e - 2$. On the other hand $\omega_Y^{-2}(2)$ is base-point-free if and only if $b - ae \geqslant e - 2$, which always holds if $0 \leqslant e \leqslant 3$, since $b - ae \geqslant 1$.

Now let us prove that under Conditions (1) and (2) we can construct a canonical double cover of Y. If $Y = \mathbf{P}^2$, since $d \geqslant 1$, (1.6.1) implies that $\omega_Y^{-2}(2)$ is very ample, so $|\omega_Y^{-2}(2)|$ contains smooth members. If $Y = \mathbf{F}_e$ and $b - ae \geqslant e - 2$, as mentioned before, $\omega_Y^{-2}(2)$ is base-point-free and obviously non-trivial, so $|\omega_Y^{-2}(2)|$ contains smooth members. If $b - ae = \frac{1}{2}e - 2$, then C_0 is the fixed part of $|\omega_Y^{-2}(2)|$ and does not intersect the mobile (base-point-free) part of $|\omega_Y^{-2}(2)|$, so $|\omega_Y^{-2}(2)|$ contains smooth (non-connected) members. Thus in all cases we can choose a smooth divisor $B \in |\omega_Y^{-2}(2)|$. Let $\pi : X \longrightarrow Y$ be the double cover of Y branched along B. Since B is smooth, so is X. Since $B \in |\omega_Y^{-2}(2)|$, ramification formula implies that $\omega_X = \pi^*(\mathcal{O}_Y(1))$. Since $p_g(Y) = 0$, $H^0(\omega_X) = \pi^* H^0(\mathcal{O}_Y(1))$, so φ factors through π. $\square$

According to Proposition 1.1 the map Ψ_2 seems central at explaining the relation between deformations of double covers and carpets. Because of that, we go on studying the target of Ψ_2 in the next result:

Proposition 1.7. $\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y(-1)) = 0$.

Proof. First we argue when Y is a Hirzebruch surface. Let $\mathcal{O}_Y(1) = \mathcal{O}_Y(aC_0 + bf)$. Since $\mathcal{O}_Y(aC_0 + bf)$ is very ample, we have

$$b - ae \geqslant 1. \tag{1.7.1}$$

We can apply [GGP10, Lemma 3.9]; indeed, $p_g(Y) = 0$, $h^1(\mathcal{O}_Y(1)) = 0$, $q(Y) = 0$ and $h^2(\mathcal{O}_Y(1)) = 0$. Then, [GGP10, Lemma 3.9] says that it suffices to see that $\operatorname{Ext}^1(\Omega_Y, \omega_Y(-1)) = 0$. For this, apply [Har77, Proposition II.8.11] to the fibration of Y to $\mathbf{P}^1$ (see also the proof of [GP97, Proposition 1.7]) and get the sequence

$$0 \longrightarrow \mathcal{O}_Y(-2f) \longrightarrow \Omega_Y \longrightarrow \mathcal{O}_Y(-2C_0 - ef) \longrightarrow 0. \tag{1.7.2}$$

Then, applying $\mathrm{Hom}(-, \omega_Y(-1))$ we get

$$\mathrm{Ext}^1\big(\mathcal{O}_Y(-2C_0 - ef), \omega_Y(-1)\big) \longrightarrow \mathrm{Ext}^1\big(\Omega_Y, \omega_Y(-1)\big) \longrightarrow \mathrm{Ext}^1\big(\mathcal{O}_Y(-2f), \omega_Y(-1)\big).$$

Now $\mathrm{Ext}^1(\mathcal{O}_Y(-2C_0 - ef), \omega_Y(-1)) = H^1(\mathcal{O}_Y((a-2)C_0 + (b-e)f))^\vee = 0$ (by pushing down to $\mathbf{P}^1$ and because of (1.7.1)). Also $\mathrm{Ext}^1(\mathcal{O}_Y(-2f), \omega_Y(-1)) = H^1(\mathcal{O}_Y(aC_0 + (b-2)f))^\vee = 0$ (again by pushing down to $\mathbf{P}^1$ and because of (1.7.1)), so

$$\mathrm{Ext}^1\big(\Omega_Y, \omega_Y(-1)\big) = 0. \tag{1.7.3}$$

Then, by [GGP10, Lemma 3.9], it follows that $\mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y(-1)) = 0$.

Now we prove the proposition for $Y = \mathbf{P}^2$ embedded by $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbf{P}^2}(d)$. Also in this occasion $p_g(Y) = 0$, $h^1(\mathcal{O}_Y(1)) = 0$, $q(Y) = 0$ and $h^2(\mathcal{O}_Y(1)) = 0$, so by [GGP10, Lemma 3.9] it suffices to see that $\mathrm{Ext}^1(\Omega_Y, \omega_Y(-1)) = 0$. For this we use the Euler sequence of $\mathbf{P}^2$

$$0 \longrightarrow \Omega_{\mathbf{P}^2} \longrightarrow H^0\big(\mathcal{O}_{\mathbf{P}^2}(1)\big) \otimes \mathcal{O}_{\mathbf{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}^2} \longrightarrow 0. \tag{1.7.4}$$

To the sequence (1.7.4) we apply the functor $\mathrm{Hom}(-, \omega_Y(-1))$ to obtain the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1\big(\Omega_{\mathbf{P}^2}, \omega_Y(-1)\big) \longrightarrow \mathrm{Ext}^2\big(\mathcal{O}_{\mathbf{P}^2}, \omega_Y(-1)\big) \longrightarrow \mathrm{Ext}^2\big(H^0\big(\mathcal{O}_{\mathbf{P}^2}(1)\big) \otimes \mathcal{O}_{\mathbf{P}^2}(-1), \omega_Y(-1)\big).$$

Dualizing we get

$$H^0\big(\mathcal{O}_{\mathbf{P}^2}(d-1)\big) \otimes H^0\big(\mathcal{O}_{\mathbf{P}^2}(1)\big) \xrightarrow{\alpha} H^0\big(\mathcal{O}_{\mathbf{P}^2}(d)\big) \longrightarrow \mathrm{Ext}^1\big(\Omega_{\mathbf{P}^2}, \omega_Y(-1)\big)^\vee \longrightarrow 0.$$

Now the multiplication map α is surjective if $d \geqslant 1$, so

$$\mathrm{Ext}^1\big(\Omega_{\mathbf{P}^2}, \omega_Y(-1)\big) = 0 \tag{1.7.5}$$

and, by [GGP10, Lemma 3.9], so does $\mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y(-1))$. □

In the remaining of this section we will use Proposition 1.7 to extract consequences for both the deformations of canonical double covers (Theorem 1.9) and the existence or non-existence of canonical carpets (Theorem 1.12). In order to prove part of Theorem 1.9 we need first the following:

Lemma 1.8. *$H^1(\mathcal{N}_{i(Y)/\mathbf{P}^N}) = 0$; in particular, $i(Y)$ is unobstructed in $\mathbf{P}^N$.*

Proof. Recall that $H^1(\mathcal{N}_{i(Y)/\mathbf{P}^N}) = \mathrm{Ext}^1(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$, which fits into the exact sequence

$$\mathrm{Ext}^1(\Omega_{\mathbf{P}^N}|_Y, \mathcal{O}_Y) \longrightarrow \mathrm{Ext}^1\big(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y\big) \longrightarrow \mathrm{Ext}^2(\Omega_Y, \mathcal{O}_Y). \tag{1.8.1}$$

We want to see that both $\mathrm{Ext}^1(\Omega_{\mathbf{P}^N}|_Y, \mathcal{O}_Y)$ and $\mathrm{Ext}^2(\Omega_Y, \mathcal{O}_Y)$ vanish. To handle the vanishing of $\mathrm{Ext}^1(\Omega_{\mathbf{P}^N}|_Y, \mathcal{O}_Y)$, consider the sequence

$$\mathrm{Ext}^1\big(\mathcal{O}_Y^{N+1}(-1), \mathcal{O}_Y\big) \longrightarrow \mathrm{Ext}^1(\Omega_{\mathbf{P}^N}|_Y, \mathcal{O}_Y) \longrightarrow \mathrm{Ext}^2(\mathcal{O}_Y, \mathcal{O}_Y). \tag{1.8.2}$$

It is clear that $\mathrm{Ext}^1(\mathcal{O}_Y^{N+1}(-1), \mathcal{O}_Y)$ and $\mathrm{Ext}^2(\mathcal{O}_Y, \mathcal{O}_Y)$ both vanish because $h^1(\mathcal{O}_Y(1)) = 0$ and $p_g(Y) = 0$. To prove the vanishing of $\mathrm{Ext}^2(\Omega_Y, \mathcal{O}_Y)$ we argue for $\mathbf{P}^2$ and for Hirzebruch surfaces separately. If $Y = \mathbf{P}^2$, after applying $\mathrm{Hom}(-, \mathcal{O}_Y)$ to (1.7.4) we see that $\mathrm{Ext}^2(\Omega_{\mathbf{P}^2}, \mathcal{O}_{\mathbf{P}^2})$ fits into the exact sequence

$$\mathrm{Ext}^2\big(\mathcal{O}_{\mathbf{P}^2}(-1), \mathcal{O}_{\mathbf{P}^2}\big)^{\oplus 3} \longrightarrow \mathrm{Ext}^2(\Omega_{\mathbf{P}^2}, \mathcal{O}_{\mathbf{P}^2}) \longrightarrow \mathrm{Ext}^3(\mathcal{O}_{\mathbf{P}^2}, \mathcal{O}_{\mathbf{P}^2}),$$

so $\mathrm{Ext}^2(\Omega_Y, \mathcal{O}_Y) = 0$. Now, if Y is a Hirzebruch surface we apply $\mathrm{Hom}(-, \mathcal{O}_Y)$ to (1.7.2) and get

$$\mathrm{Ext}^2\big(\mathcal{O}_Y(-2C_0 - ef), \mathcal{O}_Y\big) \longrightarrow \mathrm{Ext}^2(\Omega_Y, \mathcal{O}_Y) \longrightarrow \mathrm{Ext}^2\big(\mathcal{O}_Y(-2f), \mathcal{O}_Y\big).$$

Then $\mathrm{Ext}^2(\Omega_Y, \mathcal{O}_Y) = 0$ because $H^2(\mathcal{O}_Y(2C_0 + ef)) = H^2(\mathcal{O}_Y(2f)) = 0$. Thus $H^1(\mathcal{N}_{i(Y)/\mathbf{P}^N}) = 0$, and by [Ser06, Corollary 3.2.7], $i(Y)$ is unobstructed in $\mathbf{P}^N$. □

Theorem 1.9. *Let X be a smooth surface of general type with ample and base-point-free canonical line bundle and let φ be its canonical map. Let i be induced by the complete linear series of a very ample divisor D on Y (if Y is a Hirzebruch surface, then $D = aC_0 + bf$ with $b - ae \geqslant 1$ and $a \geqslant 1$). Assume furthermore that φ factors through a canonical double cover π of Y. Then:*

(a) *Any deformation of φ is a canonical $2:1$ morphism; therefore, the canonical map of a surface corresponding to a general point of the moduli component of X is a finite morphism of degree 2.*
(b) *If, in the case when Y is a Hirzebruch surface, we assume in addition that $\omega_Y^{-2}(2)$ is base-point-free (that is, $b - ae \geqslant e - 2$), then X and φ are unobstructed.*

Proof. To prove Part (a), suppose the contrary, that is, suppose there exists a deformation of φ which is not $2:1$ onto its image. Then, in particular, there would exist an analytic deformation of X which is not a double cover of any analytic deformation of Y. Lemma 1.5(a), (1.7.3) and (1.7.5) yield $H^1(\mathcal{T}_Y \otimes \mathcal{E}) = 0$, but this contradicts [Weh86, Corollary 1.11].

To prove Part (b) we will use [GGP10, Theorem 2.6]. Its hypotheses hold because Y is regular, $p_g(Y) = 0$, $h^1(\mathcal{O}_Y(1)) = 0$, $h^1(\omega_Y^{-2}(2)) = 0$ (this is clear in the case when $Y = \mathbf{P}^2$ and, in the case when Y is a Hirzebruch surface, true by the assumption that $\omega_Y^{-2}(2)$ is base-point-free), Lemmas 1.5(a) and 1.8 and Propositions 1.1 and 1.7. □

The assumption made in Theorem 1.9(b) asking $\omega_Y^{-2}(2)$ to be base-point-free if Y is a Hirzebruch surface is only needed if $e \geqslant 6$ and even, in which case it is not a very strong assumption:

Remark 1.10. Let $Y = \mathbf{F}_e$ and assume that X, Y, π and φ are as in Theorem 1.9. Then $\omega_Y^{-2}(2)$ is base-point-free unless e is even, $e \geqslant 6$ and $b - ae = \frac{1}{2}e - 2$.

Proof. Lemma 1.5 implies that the branch locus of π is a divisor in $|\omega_Y^{-2}(2)|$. If $\omega_Y^{-2}(2)$ is not base-point-free, as we argued in the proof of Lemma 1.6, $Y = \mathbf{P}^2$ and $b - ae < e - 2$. Since $b - ae \geqslant 1$, this only may happen when $e \geqslant 4$. However, since X is smooth, should have a smooth member. This is only possible the fixed part of $|\omega_Y^{-2}(2)|$ is C_0 and does not meet the mobile part of $|\omega_Y^{-2}(2)|$. As seen in the proof of Lemma 1.6 this only happens if

$$b - ae = \frac{1}{2}e - 2. \tag{1.10.1}$$

Obviously, (1.10.1) is only possible if e is even and, since $b - ae \geqslant 1$, if $e \geqslant 6$. □

Remark 1.11. Theorem 1.9 can be partly proved using alternate arguments, that use [GGP10, Theorem 2.6] or [Kon85, Theorems 4.6 and 4.7]. Indeed, if $Y = \mathbf{P}^2$ or if Y is a Hirzebruch surface with $\omega_Y^{-2}(2)$ base-point-free, Part (a) can be also deduced from Proposition 1.7 and [GGP10, Theorem 2.6].

On the other hand, if $Y = \mathbf{F}_e$ and $b - ae \geqslant e - 1$, Theorem 1.9 can be proved using [Kon85, Theorems 4.6 and 4.7] (be aware of the differences between the notation of [Kon85] and ours!). Note also that Theorem 1.9(b) covers infinitely many cases that cannot be proved using Konno's results.

Finally we apply Proposition 1.7 to prove the non-existence of canonical double structures on either $\mathbf{P}^2$ or a Hirzebruch surface. This result contrasts with the results of [GP97] where the existence of double structures on smooth rational normal scrolls having the same invariants of smooth $K3$ surfaces is shown.

Theorem 1.12. *Let Y and i be as in Notation 1.4. There are no double structures inside $\mathbf{P}^N$, supported on $i(Y)$, canonically embedded in $\mathbf{P}^N$.*

Proof. The result follows Lemma 1.5(b) and [Gon06, Proposition 2.1]. □

Remark 1.13. Let X be a surface of general type with ample and base-point-free canonical divisor, and let φ be the canonical map of X. Assume that φ factors as $\varphi = i \circ \pi$, where π finite cover of Y, non-necessarily of degree n. Let $\mathscr{E}$ be the trace-zero module of π. It is easy to see that $\omega_Y(-1)$ is a direct summand of $\mathscr{E}$. Then, the same arguments used in the proof of Theorem 1.12 show that

(1) $\operatorname{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$ does not contain surjective homomorphisms;
(2) there are no multiple structures inside $\mathbf{P}^N$, supported on $i(Y)$ with conormal bundle $\mathscr{E}$; and
(3) there are no double structures inside $\mathbf{P}^N$, supported on $i(Y)$ whose conormal bundle is a subsheaf of $\mathscr{E}$.

2. Consequences for geography and moduli

In this section we compute the invariants of the surfaces of general type that appear in Section 1 (see Lemma 1.6). In this way we find the region of the geography of surfaces of general type they reside in. In addition, we compute the dimension of the moduli components parameterizing our surfaces. Finally we show two examples of moduli spaces having components of different nature: components parameterizing canonically embedded surfaces and components parameterizing surfaces whose canonical map is a finite morphism of degree 2.

Proposition 2.1. *Let $Y = \mathbf{P}^2$ embedded by $|\mathscr{O}_{\mathbf{P}^2}(d)|$ and let X be a canonical double cover of Y as the ones appearing in Lemma 1.6. The surface X has the following invariants*:

$$\begin{aligned} p_g &= \frac{1}{2}d^2 + \frac{3}{2}d + 1, \\ q &= 0, \\ \chi &= \frac{1}{2}d^2 + \frac{3}{2}d + 2, \\ c_1^2 &= 2d^2, \\ \frac{c_1^2}{c_2} &= \frac{d^2}{2d^2 + 9d + 12}. \end{aligned} \tag{2.1.1}$$

Proof. In our situation $p_g = h^0(\mathscr{O}_{\mathbf{P}^2}(d))$. Since $h^1(\mathscr{O}_X) = h^1(\mathscr{O}_Y) + h^1(\omega_Y(-1))$ and $q(Y) = h^1(\mathscr{O}_Y(1)) = 0$, $q(X) = 0$. The values of c_1^2 are obvious, since φ has degree 2 onto Y. Finally, the values of $\frac{c_1^2}{c_2}$ follow from Noether's formula. □

Proposition 2.2. *Let Y be a Hirzebruch surface $Y = \mathbf{F}_e$ embedded by a very ample linear system $|aC_0 + bf|$ and let X be a canonical double cover of Y as the ones appearing in Lemma 1.6. Then X has the following invariants*:

$$\begin{aligned} p_g &= (a+1)\left(b+1-\frac{ae}{2}\right), \\ q &= 0, \\ \chi &= (a+1)\left(b+1-\frac{ae}{2}\right)+1, \\ c_1^2 &= 2a(2b-ae), \\ \frac{c_1^2}{c_2} &= \frac{2ab-a^2e}{4ab-2a^2e+6a-3ae+6b+12}. \end{aligned} \tag{2.2.1}$$

Proof. Since φ factors through the canonical double cover π, $p_g(X)$ is the dimension of $H^0(\mathcal{O}_Y(aC_0+bf))$, which is well known. By the construction of X, $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y) + h^1(\omega_Y(-1))$. Again we know that $q(Y)$ and $h^1(\mathcal{O}_Y(1))$ are both 0, so $q(X) = 0$ and then the values stated for χ are obvious. The values of c_1^2 follow also from the construction and properties of X and φ; indeed, $\omega_X = \pi^*\mathcal{O}_Y(1)$ and π has degree 2. Finally, the values of $\frac{c_1^2}{c_2}$ follow from Noether's formula. □

Remark 2.3. For any of the surfaces X in Propositions 2.1 and 2.2, $\frac{c_1^2}{c_2} < \frac{1}{2}$ but there exist surfaces X as in Proposition 2.1 and surfaces X as in Proposition 2.2 for which $\frac{c_1^2}{c_2}$ is arbitrarily close to $\frac{1}{2}$.

Proof. If X is as in Propositions 2.1 and 2.2, the claim is clear. If X is as in Proposition 2.2, note that

$$\frac{c_1^2}{c_2} = \frac{2ab-a^2e}{4ab-2a^2e+6a-3ae+6b+12} = \frac{a(2b-ae)}{2a(2b-ae)+6a+3b+3(b-ae)+12}.$$

The very ampleness of $\mathcal{O}_Y(aC_0+bf)$ implies $b - ae \geqslant 1$, so $6a+3b+3(b-ae)+12 > 0$ and the claim is also clear in this case. □

Remark 2.4. We now present the information given in Proposition 2.2 more graphically, by displaying on a plane the pairs $(x, y) = (\chi, c_1^2)$ of the covers of ruled surfaces appearing in Lemma 1.6. If we fix an integer $a \geqslant 1$, then the points (x, y) corresponding to the covers of ruled surfaces appearing in Lemma 1.6 are points (with integer coordinates) lying on the line l_a passing through the point $(a+2, 0)$ with slope $\frac{4a}{a+1}$, i.e., the line of equation

$$y = \frac{4a}{a+1}(x-a-2). \tag{2.4.1}$$

More precisely, for each $a \geqslant 1$, the invariants form an unbounded set consisting of all the integer points on the semiline of l_a including and up and to the right of the point $(2a+3, 4a)$. Note that each two distinct lines l_a and $l_{a'}$ described above meet at the point with $x = aa' + a + a' + 2$. Note also that l_1 is obviously the Noether's line $y - 2x + 6 = 0$; the reason for this is that if $a = 1$, Y is embedded as a surface of minimal degree. Note also that the limiting points of the semilines described above lie on Noether's line $y - 2x + 6 = 0$ (as should be, since in this case the limiting point $(2a+3, 4a)$ is obtained when considering canonical double covers of $\mathbf{F}_0$ embedded by $|aC_0 + f|$, which are surfaces of minimal degree).

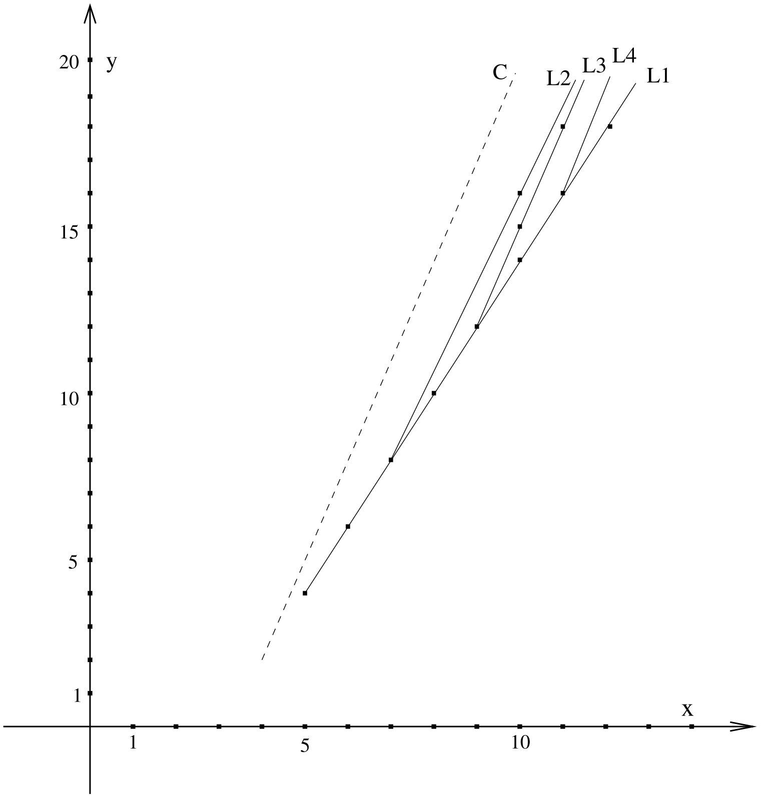

Fig. 1. Solid lines, from less steep to more steep, are l_1 (which is also Noether's line) to l_4. The dashed line is Castelnuovo's line. The points marked are the integer points lying on l_1 with first coordinate $x \geqslant 5$ and the integer points lying on l_2, l_3 and l_4 and above l_1.

Note that as a goes to infinity, the slopes of the lines l_a approaches 4. This means that Chern ratio $\frac{c_1^2}{c_2}$ approaches $\frac{1}{2}$. Note also that, if $a \geqslant 4$, when x is sufficiently large, the line l_a goes into the region $y \geqslant 3x - 10$, bounded by the Castelnuovo line $y = 3x - 10$.

We will illustrate Remark 2.4 with Figs. 1 and 2. In Fig. 1 low values of (χ, c_1^2) appear while Fig. 2 zooms out in order to display larger values (in these figures, we will keep denoting χ by x and c_1^2 by y).

Next we remark that the surfaces constructed in Theorem 1.9(b) are not only regular, but also simply connected:

Remark 2.5. The surfaces of general type X that appear in Theorem 1.9(b) are simply connected.

Proof. Let Y be either a minimal rational surface or F_1. The fundamental group of Y is well known to be 0. The morphism π is a double cover of Y branched along a divisor in $|\omega_Y^{-2}(2)|$. If Y is $\mathbf{P}^2$ this branch divisor is obviously base-point-free and if Y is a Hirzebruch surface, the branch divisor is also base-point-free by the hypothesis of Theorem 1.9(b). Then the fundamental group of X is the same as the fundamental group of Y by [Nor83, Corollary 2.7] (note that the ampleness hypothesis required there can be relaxed to big and nefness), so X is simply connected. Then, consider the families of surfaces associated to the deformations of X given in Theorem 1.9. All the smooth fibers in such

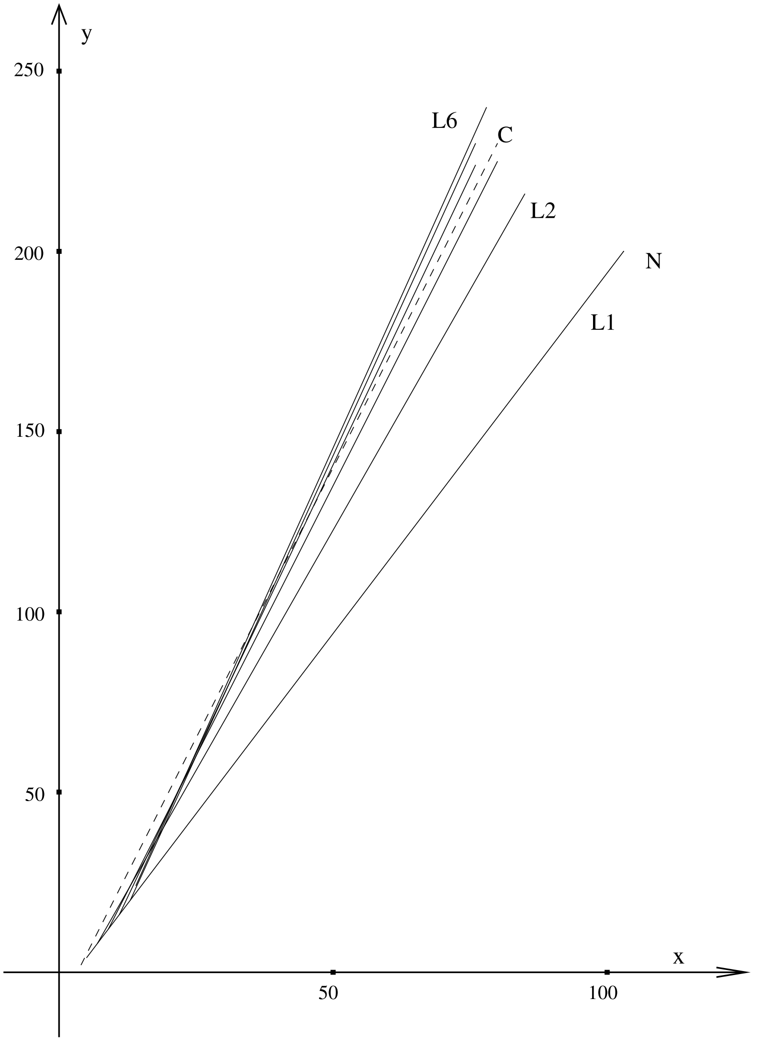

Fig. 2. Solid lines, from less steep to more steep, are l_1 (which is also Noether's line) to l_6; the dashed line is Castelnuovo's line.

families are diffeomorphic to each other, hence they are also simply connected. Thus the surfaces appearing in Theorem 1.9(b) are simply connected. □

In the second part of this section we compute the dimension of the components of the moduli parameterizing the surfaces of general type appearing in Theorem 1.9(b):

Proposition 2.6. *Let X be a surface of general type as in Theorem* 1.9(b)*. Then there is only one irreducible component of the moduli containing* $[X]$ *and its dimension is*

(1) $\mu = 2d^2 + 15d + 19$, *if* $Y = \mathbf{P}^2$;
(2) $\mu = (2a+5)(2b-ae+5) - 7$, *if Y is a Hirzebruch surface.*

Proof. Part (b) of Theorem 1.9 implies that the base of the formal semiuniversal deformation space of X is smooth, so in particular $[X]$ belongs to a unique irreducible component of the moduli. Note that [GGP10, Lemma 4.4] holds also under our hypothesis. Then

$$\mu = h^0(\mathcal{N}_\pi) - h^1(\mathcal{N}_\pi) + h^1(\mathcal{T}_Y) - h^0(\mathcal{T}_Y) + \dim \operatorname{Ext}^1\bigl(\Omega_Y, \omega_Y(-1)\bigr). \tag{2.6.1}$$

Thus, we will compute now the dimensions of the cohomology groups that appear in (2.6.1). First recall that in (1.7.3) and (1.7.5) we obtained $\operatorname{Ext}^1(\Omega_Y, \omega_Y(-1)) = 0$.

Now we prove that $h^1(\mathcal{N}_\pi) = 0$. Recall (see e.g. [Hor75, Lemma 10]) that $h^1(\mathcal{N}_\pi) = h^1(\mathcal{O}_B(B))$, where B is the branch divisor of π. The divisor B is a smooth member of $|\omega_Y^{-2}(2)|$. Recall also that we showed in the proof of Theorem 1.9 that $H^1(\omega_Y^{-2}(2)) = 0$. Then the sequence

$$H^1(\mathcal{O}_Y) \longrightarrow H^1\bigl(\mathcal{O}_Y(B)\bigr) \longrightarrow H^1\bigl(\mathcal{O}_B(B)\bigr) \longrightarrow H^2(\mathcal{O}_Y) \tag{2.6.2}$$

and the fact that $p_g(Y) = 0$ imply the vanishing of $H^1(\mathcal{N}_\pi)$.

Next we compute the number $h^0(\mathcal{T}_Y) - h^1(\mathcal{T}_Y)$. If $Y = \mathbf{P}^2$, since $h^2(\mathcal{T}_Y) = 0$, then

$$h^0(\mathcal{T}_Y) - h^1(\mathcal{T}_Y) = \chi(\mathcal{T}_Y) = 8. \tag{2.6.3}$$

Now if Y is a Hirzebruch surface, dualizing and taking global sections on (1.7.2) yields

$$\begin{aligned} 0 &\longrightarrow H^0\bigl(\mathcal{O}_Y(2C_0 + ef)\bigr) \longrightarrow H^0(\mathcal{T}_Y) \longrightarrow H^0\bigl(\mathcal{O}_Y(2f)\bigr) \\ &\longrightarrow H^1\bigl(\mathcal{O}_Y(2C_0 + ef)\bigr) \longrightarrow H^1(\mathcal{T}_Y) \longrightarrow 0. \end{aligned} \tag{2.6.4}$$

Then $h^0(\mathcal{T}_Y) - h^1(\mathcal{T}_Y) = 6$.

Now, to complete the computation we find $h^0(\mathcal{N}_\pi)$. Again by [Hor75, Lemma 10] we have $h^0(\mathcal{N}_\pi) = h^0(\mathcal{O}_B(B)) = h^0(\mathcal{O}_Y(B)) - 1$, because Y is regular. If $Y = \mathbf{P}^2$, embedded by $|\mathcal{O}_{\mathbf{P}^2}(d)|$, then $h^0(\mathcal{N}_\pi) = 2d^2 + 15d + 27$ and if Y is a Hirzebruch surface $\mathbf{F}_e$ as in Theorem 1.9, then by Riemann–Roch $h^0(\mathcal{N}_\pi) = (2a+5)(2b-ae+5) - 1$ (recall that $\omega_Y^{-2}(2)$ is base-point-free and that for a Hirzebruch surface Y this implies the vanishing of $H^1(\omega_Y^{-2}(2))$). Plugging all this in (2.6.1) yields the result. □

Remark 2.7. If X is a surface of general type as in Theorem 1.9(b), $Y = \mathbf{F}_e$, i is induced by the complete linear series of a very ample divisor $aC_0 + bf$ and we require in addition that $b - ae \geqslant e - 1$, then Proposition 2.6 can be proved using [Kon85, Proposition 2.7] (note however the misprint in [Kon85, (4.6.4)] where $i = 2$ should be written instead of $i = 0$; note also that our Proposition 2.6 covers infinitely many cases that cannot be proved using [Kon85, Proposition 2.7]).

Some of the surfaces constructed in Theorem 1.9 provide examples of moduli spaces with interesting properties:

Example 2.8. The moduli space $\mathcal{M}_{(39,0,110)}$ parameterizing surfaces of general type with $p_g = 39$, $q = 0$ and $c_1^2 = 110$ has one component $\mathcal{M}_1$ whose general point corresponds to a surface S as in [GGP10, Lemma 4.10] and another component $\mathcal{M}_2$ whose general point corresponds to a surface X as in Lemma 1.6. In particular, a general point of $\mathcal{M}_1$ corresponds to a surface that can be canonically embedded whereas a general point of $\mathcal{M}_2$ corresponds to a surface whose canonical map is a degree 2, finite morphism.

Proof. For instance, the linear system $|4H + 10F|$ of $S(1,1,1)$ has smooth members S which are surfaces like those of [GGP10, Lemma 4.10] and with $(p_g(S), c_1^2(S)) = (39, 110)$ (H is the tautological divisor of $S(1,1,1)$ and F is a fiber over $\mathbf{P}^1$). On the other hand, canonical double covers X of $\mathbf{F}_1$

embedded by $|5C_0 + 8f|$ are surfaces like those in Lemma 1.6 and have $(p_g(X), c_1^2(X)) = (39, 110)$. Since by Theorem 1.9(a) the canonical map of X deforms to a finite morphism of degree 2, S and X belong to different components of $\mathscr{M}_{(39,0,110)}$. $\square$

Example 2.9. The moduli space $\mathscr{M}_{(45,0,128)}$ parameterizing surfaces of general type with $p_g = 45$, $q = 0$ and $c_1^2 = 128$ has at least three components $\mathscr{M}_1$, $\mathscr{M}_2$ and $\mathscr{M}_3$. A general point of $\mathscr{M}_1$ corresponds to a surface S as in [GGP10, Lemma 4.10], while general points of $\mathscr{M}_2$ and $\mathscr{M}_3$ correspond to surfaces X as in Lemma 1.6. In particular, a general point of $\mathscr{M}_1$ corresponds to a surface that can be canonically embedded whereas general points of $\mathscr{M}_2$ and $\mathscr{M}_3$ correspond to surfaces whose canonical map is a degree 2, finite morphism.

Proof. For instance, the linear system $|4H + 12F|$ of $S(1, 1, 1)$ has smooth members S which are surfaces like those of [GGP10, Lemma 4.10] and with $(p_g(S), c_1^2(S)) = (45, 128)$ (recall that H is the tautological divisor of $S(1, 1, 1)$ and F is a fiber over $\mathbf{P}^1$). On the other hand, canonical double covers of $\mathbf{P}^2$ embedded by octics are surfaces X_2 as in Lemma 1.6 having $(p_g(X_2), c_1^2(X_2)) = (45, 128)$ (see (2.1.1)). In addition, canonical double covers of $\mathbf{F}_0$ embedded by $|4C_0 + 8f|$ are also surfaces X_3 as in Lemma 1.6 having $(p_g(X_3), c_1^2(X_3)) = (45, 128)$ (see (2.2.1)). Now recall that the point in the moduli space corresponding to X_2 belongs to only one component of $\mathscr{M}_{(45,0,128)}$ (see Proposition 2.6), which we will call $\mathscr{M}_2$. Likewise, the point in the moduli space corresponding to X_3 belongs to only one component of $\mathscr{M}_{(45,0,128)}$, which we will call $\mathscr{M}_3$. Then $\mathscr{M}_2$ and $\mathscr{M}_3$ are different for their dimensions are: indeed, applying Proposition 2.6 we get that the dimension of $\mathscr{M}_2$ is 267 whereas the dimension of $\mathscr{M}_3$ is 266. $\square$

Remark 2.10. For any integer m, $m \geqslant 4$, let Ξ_m be the set of values (x', y) for which there exist a smooth surface X as in Lemma 1.6 with $(p_g(X), c_1^2(X)) = (x', y)$ and a smooth surface S as in [GGP10, Lemma 4.10] with $(p_g(S), c_1^2(S)) = (x', y)$.

(1) The set Ξ_m is finite (and possibly empty) for every $m \geqslant 4$. In particular, $\Xi_4 = \{(39, 110), (45, 128)\}$ and $\Xi_5 = \Xi_6 = \emptyset$.
(2) There are no surfaces X with $Y = \mathbf{P}^2$ such that $(p_g(X), c_1^2(X)) \in \Xi_4 \cup \Xi_5 \cup \cdots$, except the surfaces X_2 appearing in Example 2.9.

Proof. The remark follows from elementary although somehow involved computations, once we take in account [GGP10, (3.17.1)], the hypothesis of [GGP10, Lemma 4.10], (2.1.1), (2.2.1), (2.4.1) and the fact that if S is as in [GGP10, Lemma 4.10], then $(p_g(S), c_1^2(S)) = (x', y)$ satisfies the equation

$$y = 6\frac{m-3}{m-2}x' - (m-3)(m+3) \tag{2.10.1}$$

(see [GGP10, (3.17.2)]). $\square$

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