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A reduction theorem for a conjecture on products of two π-decomposable groups

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ABSTRACT

For a set of primes π, a group X is said to be π-decomposable if $X = X_\pi \times X_{\pi'}$ is the direct product of a π-subgroup X_π and a π'-subgroup $X_{\pi'}$, where π' is the complementary of π in the set of all prime numbers. The main result of this paper is a reduction theorem for the following conjecture: "Let π be a set of odd primes. If the finite group $G = AB$ is a product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G." We establish that a minimal counterexample to this conjecture is an almost simple group. The conjecture is then achieved in a forthcoming paper.

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1. Introduction

All groups considered in this paper are finite. In the framework of factorized groups the well-known theorem of Kegel and Wielandt, which states the solubility of a group which is the product of two nilpotent subgroups, has been widely extended from several points of view. For instance, by considering the situation when the factors are π-decomposable groups, for a set of primes π. A group X is said to be *π-decomposable* if $X = X_\pi \times X_{\pi'}$ is the direct product of a π-subgroup X_π

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and a π'-subgroup $X_{\pi'}$, where π' stands for the complementary of π in the set of all prime numbers. X_σ will always denote a Hall σ-subgroup of a group X, for any set of primes σ.

In this paper we take further the study of products of π-decomposable groups carried out in [12] and [13]. Motivated by the previous development, in the second reference we stated the following conjecture:

Conjecture. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.*

This conjecture was also announced in [14] and mentioned in [4]. As a first approach, we had proved in [12] that the conjecture holds in the particular case when one of the factors is a π-group.

Theorem 1. *(See [12, Theorem 1, Lemma 1].) Let π be a set of odd primes. Let the group $G = AB$ be the product of a π-decomposable subgroup $A = A_\pi \times A_{\pi'}$ and a π-subgroup B. Then $A_\pi = O_\pi(A) \leqslant O_\pi(G)$.*

Equivalently, G possesses Hall π-subgroups and $A_\pi B = BA_\pi$ is a Hall π-subgroup of G.

Afterwards, in [13], other progress were achieved and the conjecture was settled when either the factors have coprime orders or they are soluble groups. More concretely, the following results were obtained:

Proposition 1. *(See [13, Proposition 1].) Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume in addition that $(|A_{\pi'}|, |B_{\pi'}|) = 1$. Then $A_\pi B_\pi = B_\pi A_\pi$.*

Theorem 2. *(See [13, Theorem 2].) Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable soluble subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.*

Examples in [12] and [13] show that analogous results to Theorems 1, 2 and Proposition 1 do not hold in general if the set of primes π contains the prime 2. Nevertheless, for this case, related positive results have been obtained in [13].

Our results extend previous ones of Berkovich [5], Arad and Chillag [3], Rowley [17] and Kazarin [10], where products of a 2-decomposable group and a group of odd order, with coprime orders, were considered. Moreover, we obtained some π-separability criteria for products of π-decomposable groups in [12] and [13], which can be seen as extensions of the above mentioned theorem of Kegel and Wielandt.

The purpose of this paper is to establish a reduction theorem which shows that a minimal counterexample for the above conjecture must be an almost simple group. That is, we reduce our study to a question concerning simple groups. Then, in a forthcoming paper [15], a case-by-case analysis will be carried out in order to conclude that no finite almost simple group can be a counterexample, showing that our Conjecture is true.

The layout of the paper is the following. In Section 2 we present some preliminaries that will be necessary in the paper, mainly referring to arithmetical properties of finite simple groups. In Section 3 we will reduce the structure of a minimal counterexample to our conjecture to the case of an almost simple group. Along the paper, if n is an integer and p a prime number, we will denote by n_p the largest power of p dividing n and by $\pi(n)$ the set of prime divisors of n. In particular, for the order $|G|$ of a group G we set $\pi(G) = \pi(|G|)$. Also, $\mathrm{Syl}_p(G)$ will denote the set all Sylow p-subgroups of G.

2. Preliminaries

We need specifically the following results on factorized groups, which will be freely used throughout the paper, usually without further reference.

Lemma 1. *(See [1, Corollary 1.3.3].) Let the group $G = AB$ be the product of the subgroups A and B. Then for each prime p there exist Sylow p-subgroups A_p of A and B_p of B such that A_pB_p is a Sylow p-subgroup of G.*

Lemma 2. *(See [1, Lemma 1.3.1].) Let the group $G = AB$ be the product of two subgroups A and B. If x, y are elements of G, then $G = A^xB^y$. Moreover, there exists an element z of G such that $A^x = A^z$ and $B^y = B^z$.*

Next we gather some arithmetical lemmas, which will be applied later on in the paper. The proof of the following result is straightforward.

Lemma 3. *Let p be an odd prime and $q = p^\alpha$. If $\alpha \equiv 0 \pmod{2^\lambda}$ with $\lambda \geqslant 1$, then $q - 1 \equiv 0 \pmod{2^{\lambda+2}}$. (Note that the last congruence holds also when $\lambda = 0$ and $q \equiv 1 \pmod 4$.) In any case it holds that $(q-1)(q+1) \equiv 0 \pmod{2^3}$.*

The book [7] can be taken as a general source about finite non-abelian simple groups. In particular, in this paper we will make extensive use of the detailed knowledge of the orders of the finite simple groups and of their automorphisms groups. This information can be found in [7] or in [6], and also in [16, Table 2.1] where it is perfectly collected for our purposes.

We will need the following lemmas on groups of Lie type.

Lemma 4. *Let L be a simple group of Lie type defined over a finite field $GF(q)$ of characteristic p. If $|L|_p = p^n$ and $|Out(L)|_p = p^\delta$, then either $n > 3(\delta + 1)$ or one of the following assertions holds:*

(i) $|\pi(L)| < 5$;
(ii) *either $L \cong L_3(q)$ or $L \cong U_3(q)$, with $q = p \geqslant 7$ in both cases;*
(iii) *$L \cong L_2(q)$ with either $q \in \{2^6, 2^8, 3^9, 5^5\}$ or $q = p$ and $|\pi((q-1)(q+1))| \geqslant 4$;*
(iv) *either $L \cong L_3(2^4)$ or $L \cong U_3(2^4)$.*

Note that in all cases (i)–(iv)*, we have that $n \geqslant \delta + 1$.*

Proof. Denote by l the Lie rank of L and $t = \log_p(q) = t_pt_{p'}$. By checking $|L|_p$ for all simple groups of Lie type we can deduce that $n \geqslant lt$. Now by checking $|Out(L)|_p$ we can distinguish two cases:

- Case $|Out(L)|_p = (\log_p(q))_p = t_p$ and so $\delta = \log_p(t_p)$.
  Note that in this case q is odd except for the cases $L \cong L_2(q)$, $L \cong PSp_{2m}(q)$, $m \geqslant 3$, $L \cong {}^2B_2(q)$ or $L \cong {}^2F_4(q)'$.
  It is easy to prove that $t_p \geqslant \log_p(t_p) + 1$. Moreover, equality holds only in the cases $t_p = 1$ and $t_p = 2 = p$. We can consider now the following subcases:
  – $l > 3$. We have $n \geqslant lt > 3t \geqslant 3t_p \geqslant 3(\log_p(t_p) + 1) = 3(\delta + 1)$.
  – $l = 3$. Possible exceptions to the fact $n > 3(\delta + 1)$ could appear when $t_p = 1$ or $t_p = 2 = p$. If $t_p = 1$, then $\delta = 0$, and we can see that $|L|_p = p^n > p^3$ for all groups of Lie type with rank 3, so $n > 3$ and we are done. Now, if $t_p = 2 = p$, the only possibility is $L \cong PSp_6(q)$ and in this case $|L|_p = q^9$, so the inequality $n > 6 = 3(\delta + 1)$ holds again.
  – $l = 2$. In this case it can be proved that $2t > 3(\log_p(t_p) + 1)$ whenever $t \geqslant 8$, and so $n \geqslant 2t > 3(\log_p(t_p) + 1)$. Hence it remains to consider the cases $t < 8$.
  First assume that $t_p = 1$, that is, $\delta = 0$. By checking the orders of the Sylow p-subgroups in the groups of Lie type of rank 2 we can see that the only exception to the fact $n > 3 = 3(\delta + 1)$ appears when $L \cong L_3(q)$ for $q = p \geqslant 7$ (case (ii)).
  Therefore we can assume now that $t_p \geqslant p > 1$ and $t < 8$, which means that $t_p \in \{2, 3, 4, 5, 6, 7\}$. Again by computing $|L|_p$ when L is a simple group of Lie type of rank 2, we can prove that $n > 3(\log_p(t_p) + 1)$ in all possible cases.
  – $l = 1$. If either $L \ncong U_3(q)$ when $q = p$ or $L \ncong L_2(q)$, it can be seen that the inequality $n > 3(\delta + 1)$ holds. Since the case $L \cong U_3(q)$ with $q = p \geqslant 7$ is excluded in (ii), and the case $L \cong U_3(q)$ with $q = p < 7$ is excluded in (i), we may assume that $L \cong L_2(q)$, $q = p^t$. Note that $|L|_p = p^t$

and $t > 3(\log_p(t)+1) \geqslant 3(\log_p(t_p)+1)$ if $t > 10$. Moreover, the cases $5 \leqslant t \leqslant 10$, which do not satisfy $t > 3(\log_p(t_p)+1)$, are excluded by case (i). So we need only to check the cases $t < 5$. Exceptions to the fact that $n > 3(\log_p(t_p)+1)$ with $|\pi(L)| \geqslant 5$ appear when $p = 2$ and $t \in \{6, 8\}$, or $p = 3$ and $t = 9$, or $p = 5 = t$. Also when $t = t_p = 1$, that is, $\delta = 0$, it can occur that $|\pi(L)| \geqslant 5$ when $|\pi((q-1)(q+1))| \geqslant 4$. This provides the exceptions in (iii).

- Case $|Out(L)|_p = p(\log_p(q))_p = pt_p$ and then $\delta = \log_p(t_p)+1$.
  This is the case only when $p = 2$ or $p = 3$. Moreover in all possible cases we have $n \geqslant 3t$, so it is enough to prove $t > \log_p(t_p)+2$.
  If $p = 2$, then this inequality does not hold only when $t = t_2 = 2$ or $t = t_2 = 4$. Moreover, it holds that $n \geqslant 4t$ for $t = t_2 = 4$ and $n > 4t+1$ for $t = t_2 = 2$, except for $L_3(4)$, $U_3(4)$, $L_3(16)$, $U_3(16)$ and $PSp_4(4)$. Hence the possible exceptions to the fact that $n > 3(\delta+1)$ with $|\pi(L)| \geqslant 5$ are those appearing in (iv).
  If $p = 3$, then the inequality $t > \log_p(t_p)+2$ does not hold only when $t = t_3 = 3$. But in all these cases $n > 9 = 3(\log_p(t_p)+2)$.

Note that in all exceptional cases a direct calculation shows that $n \geqslant \delta+1$. Therefore the lemma is proved. □

Lemma 5. *Let L be a simple group of Lie type over a finite field $GF(q)$ of odd characteristic p. If $|L|_2 = 2^n$ and $|Out(L)|_2 = 2^\delta$, then $n \geqslant \delta+1$.*

Proof. Let $\log_p(q)_2 = 2^\lambda$. Clearly, $\lambda \leqslant \delta$. We consider first the following cases:

- $L \cong L_t(q)$, $t \geqslant 2$
  (i) t odd. In this case $\delta = \lambda+1$ and $n \geqslant (t-1)(\lambda+2)$, applying Lemma 3. So $n \geqslant \delta+1$.
  (ii) t even. Let $d := (t, q-1)$ and $k := \log_2(t)$. Here $\delta \leqslant \lambda+1+\log_2(d) \leqslant \lambda+1+k$. Note also that $\log_2(q-1)/\log_2(d) \leqslant 1$. Therefore, by Lemma 3, we can deduce that $n \geqslant (\lambda+k+2)+(t-3)(\lambda+2)+1$. Hence, if $t \geqslant 4$, we get $n \geqslant \delta+1$. If $t = 2$, then $\delta = \lambda+1$ and $n \geqslant \lambda+2$, so we are also done.
- $L \cong U_t(q)$, $t \geqslant 3$
  (i) t odd. Here $\delta = \lambda+1$ and $n \geqslant \frac{t-1}{2}(\lambda+3)-1$. Since $t \geqslant 3$ we get $n \geqslant \delta+1$.
  (ii) t even. Let $d := (t, q+1)$ and $k := \log_2(t)$. Here $\delta \leqslant \lambda+1+\log_2(d) \leqslant \lambda+1+k$. Moreover, $\log_2(q+1)/\log_2(d) \leqslant 1$. Applying Lemma 3 we can deduce that $n \geqslant (\lambda+k+2)+(\frac{t-2}{2})(\lambda+3)$. Since $t \geqslant 4$, we get $n \geqslant \delta+1$, and we are done.

Now, if L is a simple group of Lie type, $L \ncong L_t(q)$, $t \geqslant 2$, and $L \ncong U_t(q)$, $t \geqslant 3$, then a case-by-case checking shows that $n \geqslant \delta+1$ and the result is proved. □

Next we state some arithmetical property of the symmetric groups used later on.

Lemma 6. *Let G be the symmetric group of degree k and let s be a prime. If s^N is the largest power of s dividing $|G| = k!$, then $N \leqslant \frac{k-1}{s-1}$.*

Proof. The order and structure of a Sylow subgroup of the symmetric group is well-known (see, for example, [8, Section 5.9]). If we write k in base s, $k = a_0 + a_1 s + a_2 s^2 + \cdots + a_t s^t$, where $0 \leqslant a_i < s$ and some $a_i \neq 0$, then:

$$\begin{aligned} N &= a_1 + a_2(s+1) + a_3\left(s^2+s+1\right) + \cdots + a_t\left(s^{t-1}+s^{t-2}+\cdots+s+1\right) \\ &= a_1\left(\frac{s-1}{s-1}\right) + a_2\left(\frac{s^2-1}{s-1}\right) + a_3\left(\frac{s^3-1}{s-1}\right) + \cdots + a_t\left(\frac{s^t-1}{s-1}\right) \\ &= \frac{(a_0+a_1 s+a_2 s^2+\cdots+a_t s^t)-(a_0+a_1+\cdots+a_t)}{s-1} \leqslant \frac{k-1}{s-1}. \qquad \square \end{aligned}$$

We end this section with the following particular result on finite simple groups.

Lemma 7. *Let L be a non-abelian simple group. Then there exists a prime $s \geqslant 5$ such that $s \in \pi(L)$ and $s \notin \pi(Out(L))$.*

Proof. This follows from an exhaustive and straightforward checking of the orders of all finite simple groups and of their automorphism groups, which can be found in [16, Table 2.1, pp. 18–20], as mentioned before. For simple groups of Lie type see also [16, 2.4. Proposition B]. □

3. The minimal counterexample: Reduction to the almost simple case

We obtain in this section detailed information about the structure of a minimal counterexample to our Conjecture and, in particular, we show that it is an almost simple group.

Hence, from now on we assume that G is a counterexample of minimal order to the Conjecture, that is, we assume the following hypotheses:

(H1) π is a set of odd primes.
(H2) G is a group of minimal order satisfying the following conditions:
  1. $G = AB$ is the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$,
  2. $A_\pi B_\pi \neq B_\pi A_\pi$.

For such a group G the following results hold:

Lemma 8. *(See [13, Proposition 2].) G has a unique minimal normal subgroup $N = N_1 \times \cdots \times N_r$, which is a direct product of isomorphic non-abelian simple groups $N_1, \ldots, N_r$. Moreover, $G = AN = BN = AB$, $(|A_{\pi'}|, |B_{\pi'}|) \neq 1$, $A_{\pi'} \cap B_{\pi'} = 1$ and $A \cap B$ is a π-group. In particular,*

$$|N||A \cap B| = |G/N||N \cap A||N \cap B|$$

and neither A nor B is a π-group or a π'-group.

Lemma 9. *Assume that $S \leqslant X$ and S is an s-group for $X \in \{A, B\}$ and a prime number $s \in \sigma$, with $\sigma \in \{\pi, \pi'\}$. Then $\pi(|X : C_X(S)|) \subseteq \sigma$. In particular, $C_X(S)$ is not an s-group.*

Proof. The first part is clear since $X_{\sigma'} \leqslant C_X(S)$. Consequently, if $C_X(S)$ were an s-group, X would be a σ-group, a contradiction. □

Lemma 10. $\pi(G) = \pi(N)$.

Proof. We have $|N||A \cap B| = |G/N||N \cap A||N \cap B|$. Hence $|G : A \cap B| = |N|^2/|N \cap A||N \cap B|$ and $|G : A \cap B|$ is coprime with any $q \in \pi(G) \setminus \pi(N)$. Since π is a set of odd primes and $A \cap B$ is a π-group, it is a soluble group. Let $\pi_0 = \pi(G) \setminus \pi(N)$. Then $A \cap B$ contains a Hall π_0-subgroup, say Q. Since A_π is a soluble group we can choose some Hall π_0'-subgroup of A_π, say $\tilde{A}_\pi$ such that $A_\pi = \tilde{A}_\pi Q$. Let $\tilde{A} := \tilde{A}_\pi \times A_{\pi'}$ and $\tilde{G} := \tilde{A}N$. Consider now $\tilde{B}_\pi := B_\pi \cap \tilde{G} = B_\pi \cap \tilde{A}_\pi N$ and $\tilde{B} := \tilde{B}_\pi \times B_{\pi'}$. Since $A_\pi N = B_\pi N$ and B_π also contains Q we can deduce that $B_\pi = B_\pi \cap A_\pi N = Q(B_\pi \cap \tilde{A}_\pi N) = \tilde{B}_\pi Q$ and $\tilde{B}_\pi$ is a Hall π_0'-subgroup of B_π. Moreover $\tilde{A}_\pi \cap B_\pi = \tilde{A}_\pi \cap \tilde{B}_\pi$. Since $(|Q|, |N \cap A|) = 1 = (|Q|, |N \cap B|)$ it is easy to see that $|\tilde{G}| = |G|/|Q| = |\tilde{A}\tilde{B}Q|/|Q| = |\tilde{A}||\tilde{B}|/|\tilde{A}_\pi \cap \tilde{B}_\pi| = |\tilde{A}\tilde{B}|$ and so $\tilde{G} = \tilde{A}\tilde{B}$ is a subgroup of G. If $\tilde{G} < G$, then by the choice of G we deduce that $\tilde{A}_\pi \tilde{B}_\pi = \tilde{B}_\pi \tilde{A}_\pi$ is a subgroup. Therefore $A_\pi B_\pi = Q\tilde{A}_\pi \tilde{B}_\pi = Q\tilde{B}_\pi \tilde{A}_\pi = B_\pi A_\pi$ is also a subgroup, which is a contradiction. This implies that $\pi_0 = \emptyset$ and the assertion follows. □

Corollary 1. $|\pi(N)| \geqslant 5$. *In particular, $|\pi(N_i)| \geqslant 5$ for $i = 1, \ldots, r$.*

Proof. By Theorem 2 either $A_{\pi'}$ or $B_{\pi'}$ is non-soluble. Hence $|\pi' \cap \pi(G)| \geqslant 3$. On the other hand, $|\pi \cap \pi(G)| \geqslant 2$ by Lemma 1. So $|\pi(N)| = |\pi(G)| \geqslant 5$ and we are done. □

The remainder of the section is devoted to prove that N is a simple group and G is then almost simple.

We introduce some notation and facts which will be used in this section, related to the action by conjugacy of the subgroups A and B on the set $\Omega = \{N_1, \ldots, N_r\}$. The subsequent results lead to the desired conclusion that $r = 1$.

Notation and facts on the action by conjugacy of A and B on the set $\Omega = \{N_1, \ldots, N_r\}$. The following facts will be often used: *A and B act transitively on Ω, $A = A_\pi \times A_{\pi'}$, $B = B_\pi \times B_{\pi'}$ and $|N||A \cap B| = |G/N||N \cap A||N \cap B|$.*

Set $\{\sigma, \sigma'\} = \{\pi, \pi'\}$.

(i) *The orbits of A_σ and the orbits of B_σ are the same.*
This is clear since $B_\sigma N = A_\sigma N$ and N normalizes each N_i, for $i = 1, \ldots, r$.

(ii) *Let Δ_σ be an orbit of A_σ on Ω of minimal length. Since ${\Delta_\sigma}^{va} = {\Delta_\sigma}^{av} = {\Delta_\sigma}^v$ for any $a \in A_\sigma$ and any $v \in A_{\sigma'}$, we deduce that ${\Delta_\sigma}^v$ and also $\Delta_\sigma \cap {\Delta_\sigma}^v$ are orbits of A_σ. But the choice of Δ_σ having minimal length implies that $\Delta_\sigma \cap {\Delta_\sigma}^v$ is either empty or coincides with Δ_σ, for each $v \in A_{\sigma'}$. Hence there is a partition of Ω of the form*

$$\Omega = \Delta_1 \cup \cdots \cup \Delta_k,$$

where $\Delta_i = {\Delta_\sigma}^{v_i}$ for some $v_i \in A_{\sigma'}$, for $i = 1, \ldots, k$, and $v_1 = 1$.
In particular, $\Delta_1, \ldots, \Delta_k$ are the orbits of A_σ, and the orbits of B_σ, on Ω and they all have the same length.
Note that $A_{\sigma'}$, and also $B_{\sigma'}$, act transitively on the set $\{\Delta_1, \ldots, \Delta_k\}$.

(iii) *It follows from* (ii) *that $r = km$, where $k \geqslant 1$ and $m \geqslant 1$ are divisors of r, and m is the length of any orbit of A_σ on Ω.*

(iv) *The length of an orbit of $A_{\sigma'}$ on Ω is k. In particular, m is the number of different orbits of $A_{\sigma'}$ on Ω, and $(k, m) = 1$.*
Denote by Θ an orbit of $A_{\sigma'}$ on Ω. Clearly $|\Theta| \geqslant k$ and $|\Theta|$ divides $r = km$. Now, since $\sigma \cap \sigma' = \emptyset$ and the length of an orbit of A_σ on Ω divides $|A_\sigma|$ it follows that $|\Theta|$ divides k. But then the equality holds.

(v) *Without loss of generality we may set $M_{\Delta_\sigma} = \prod_{N_i \in \Delta_\sigma, i=1,\ldots,m} N_i$. Then M_{Δ_σ} is a minimal normal subgroup of NA_σ.*
Moreover, if $R \leqslant N$, $R \trianglelefteq NA_\sigma$, there exists a subset $\{v_{i_1}, \ldots, v_{i_d}\} \subseteq \{v_1, \ldots, v_k\}$ such that $R = M_{\Delta_\sigma}^{v_{i_1}} \times \cdots \times M_{\Delta_\sigma}^{v_{i_d}}$.
The same assertion is true for B_σ instead of A_σ.

(vi) *With $M_{\Delta_\sigma} = N_1 \times \cdots \times N_m$, if $m > 1$, define the subgroups $F_1 = N_2 \times \cdots \times N_m$ and $F_i = F_1^{v_i}$ for $i = 2, \ldots, k$. Note that, in this case, the subgroup $F_{\Delta_\sigma} := F_1 \times \cdots \times F_k$ does not contain any $M_{\Delta_\sigma}^{v_i}$ for $i = 1, \ldots, k$.*

(vii) *If $m > 1$, then $F_{\Delta_\sigma} \cap A_{\sigma'} = 1 = F_{\Delta_\sigma} \cap B_{\sigma'}$.*
Observe that $F_{\Delta_\sigma} \cap A_{\sigma'} \leqslant E := \bigcap_{a \in A_\sigma} (F_{\Delta_\sigma})^a$ and $E \leqslant F_{\Delta_\sigma}$ is a normal subgroup of N normalized by A_σ. Hence $F_{\Delta_\sigma} \cap A_{\sigma'} = E = 1$ by (v) and (vi).
Analogously it follows that $F_{\Delta_\sigma} \cap B_{\sigma'} = 1$.

(viii) *If $k > 1$, then $A_\sigma \cap M_{\Delta_\sigma} = 1 = C_{A_\sigma}(M_{\Delta_\sigma})$.*
If $A_\sigma \cap M_{\Delta_\sigma} \neq 1$, since this is an $A_{\sigma'}$-invariant subgroup, we have that for any v_i, $i = 2, \ldots, k$, $A_\sigma \cap M_{\Delta_\sigma} = (A_\sigma \cap M_{\Delta_\sigma})^{v_i} \leqslant M_{\Delta_\sigma} \cap M_{\Delta_\sigma}^{v_i} = 1$, a contradiction.
Now, since $C_{A_\sigma}(M_{\Delta_\sigma}) = C_{A_\sigma}((M_{\Delta_\sigma})^a)$ for every $a \in A$, we deduce that $C_{A_\sigma}(M_{\Delta_\sigma}) \leqslant C_G(N) = 1$. □

Lemma 11. *Let $s \in \pi(N) \cap \sigma'$ and let m be the length of an orbit of A_σ on Ω. Suppose that $|N_1|_s = s^n$ and $|Out(N_1)|_s = s^\delta$. Then*

$$n(m-2) \leqslant \delta + \frac{k-1}{k(s-1)}.$$

In particular, $n(m-2) < \delta + 1$.

Proof. We recall that $r = km$, with the previous notation. If $m = 1$, then the assertion holds. Assume that $m > 1$. Let A_s be a Sylow s-subgroup of A and B_s a Sylow s-subgroup of B, $A_s \leqslant A_{\sigma'}$ and $B_s \leqslant B_{\sigma'}$. Recall that the subgroup F_{Δ_σ} defined in (vii) above is a normal subgroup of N and has trivial intersection with $N \cap A_{\sigma'}$. Hence $|N \cap A_s| \leqslant |N : F_{\Delta_\sigma}|_s = s^{kn}$. Analogously, $|N \cap B_s| \leqslant s^{kn}$.

On the other hand, from (v) and (viii) above, and replacing σ by σ', we have that the subgroup $M := M_{\Delta_{\sigma'}}$ is a normal subgroup of N normalized by $A_{\sigma'}$ and $A_{\sigma'} \cap M = 1 = C_{A_{\sigma'}}(M)$. Hence $A_{\sigma'} \cong A_{\sigma'}C_G(M)/C_G(M) \lesssim Aut(M)$. We may assume that $M = N_1 \times \cdots \times N_k$ and so $Aut(M) \cong [Aut(N_1) \times \cdots \times Aut(N_k)]S_k \cong Aut(N_1) \wr S_k$, the natural wreath product of $Aut(N_1)$ with S_k, the symmetric group of degree k. Since $s \in \sigma'$ we deduce that $|A_s|$ divides $|Aut(M)|_s$ and so $s^{(\delta+n)k} s^{\frac{k-1}{s-1}}$ by Lemma 6.

Now denote $|G/N|_s = s^\gamma$ and recall that $|G/N|_s = |A_sN/N| = |B_sN/N|$. We have that $|G|_s = |N|_s|G/N|_s = s^{nr}s^\gamma$. On the other hand, $|B_s| = |N \cap B_s||B_s/N \cap B_s|$ divides $s^{kn}s^\gamma$. Since $|G|_s$ divides $|A|_s|B|_s$ we deduce

$$s^{nr+\gamma} \leqslant s^{(\delta+n)k} s^{\frac{k-1}{s-1}} s^{kn+\gamma}.$$

Consequently, $rn \leqslant \delta k + 2kn + \frac{k-1}{s-1}$. Since $r = km$ we get $kn(m-2) \leqslant \delta k + \frac{k-1}{s-1}$, that is:

$$n(m-2) \leqslant \delta + \frac{k-1}{k(s-1)}.$$

In particular, $n(m-2) < \delta + 1$. □

Lemma 12. *Let $s \in \pi(N) \cap \sigma'$. Suppose that $|N_1|_s = s^n$, $|Out(N_1)|_s = s^\delta$ and assume that $n \geqslant \delta + 1$. Then the length of an orbit of A_σ on Ω is at most 2.*

Proof. Let m be the length of an orbit of A_σ on Ω. From Lemma 11, if $m \geqslant 3$, then $\delta + 1 > n(m-2) \geqslant n$. So the assertion holds. □

Corollary 2. *Let $|N_1|_s = s^n$, where $s \in \pi(N_1)$ does not divide $|Out(N_1)|$. If $s \in \sigma'$, then the length of an orbit of A_σ on Ω is at most 2.*

Proof. Note that such a prime exists by Lemma 7. Now the result follows from Lemma 12. □

Lemma 13. *If there exist primes $s_1 \in \pi \cap \pi(N_1)$ and $s_2 \in \pi' \cap \pi(N_1)$ such that $(s_1s_2, |Out(N_1)|) = 1$, then either $r = 1$ or $r = 2$.*

In particular, this is the case when N_1 is either a sporadic group or an alternating group.

Proof. Let m be the length of an A_π-orbit on Ω and k be the length of an $A_{\pi'}$-orbit on Ω. Since $s_1 \in \pi$, $s_2 \in \pi'$ and $(s_1s_2, |Out(N_1)|) = 1$, applying Corollary 2 we have that $m \leqslant 2$ and $k \leqslant 2$. But the length of an A_π-orbit must be odd since $|A_\pi|$ is odd. Therefore, $m = 1$ and $r = km = k \leqslant 2$, by (iii) and (iv) in the notation above. □

Lemma 14. *The case $r = 2$ is not possible.*

Proof. Suppose that $r=2$. Then the length of an A_π-orbit must be 1 since $|A_\pi|$ is odd. This means that A_π normalizes each $N_i \in \Omega$. Denote by L the normalizer in G of N_1. It is clear that $A_\pi, B_\pi \leqslant L$, $|A_{\pi'} : L \cap A_{\pi'}| \leqslant 2$, $|B_{\pi'} : L \cap B_{\pi'}| \leqslant 2$ and $|G : L| = 2$, because $r=2$. Clearly $G = AL = BL = AB$, since $N \leqslant L$. Hence

$$2 = |G/L| = \frac{|L||A \cap B|}{|L \cap A||L \cap B|}.$$

But, since $A \cap B = A_\pi \cap B_\pi$ by Lemma 8, we have that $A \cap B \cap L = A \cap B$. Therefore:

$$2 = |G/L| = \frac{|L|}{|(L \cap A)(L \cap B)|}.$$

Take now any $g \in G$ and write $g = ba$ with $a \in A$, $b \in B$. We have $(L \cap A) \cap (L \cap B)^g = (L \cap A) \cap (L \cap B)^a = (L \cap A \cap B)^a$ and so $|(L \cap A) \cap (L \cap B)^g| = |L \cap A \cap B|$, for any $g \in G$. Hence $|(L \cap A)(L \cap B)| = |(L \cap A)(L \cap B)^g| = |(L \cap A)g(L \cap B)|$ for any $g \in G$, and consequently the number of $(L \cap A, L \cap B)$-double cosets in L should be 2. If $N \subseteq (L \cap A)(L \cap B)$, then $(L \cap A)(L \cap B) = (L \cap A)N(L \cap B) = (L \cap AN)(L \cap B) = L$, a contradiction.

Hence we may assume that $(L \cap A)N_1(L \cap B) \neq (L \cap A)(L \cap B)$ and $(L \cap A)N_1(L \cap B) = L$. Now, since N_1 is normal in L, we can consider $L/N_1 = ((L \cap A)N_1/N_1)((L \cap B)N_1/N_1)$ which is a product of two π-decomposable groups. By the choice of G we can deduce that $K := A_\pi B_\pi N_1 = (L \cap A_\pi)(L \cap B_\pi)N_1$ is a subgroup of G. Set $H = \langle A_\pi, B_\pi \rangle \leqslant K$. By [1, Lemma 1.2.2], $N_G(H) = N_A(H)N_B(H)$ and hence, if $N_G(H)$ were a proper subgroup of G, we could deduce that $A_\pi B_\pi$ is a subgroup, a contradiction. So $1 \neq H \trianglelefteqslant G$ and $N \leqslant H$. But this means that $K = A_\pi B_\pi N_1 = H \trianglelefteqslant G$. Therefore, the soluble residual $K^{\mathcal{S}}$ of K is a normal subgroup of G contained in N_1, which implies that $K^{\mathcal{S}} = 1$, that is, K is soluble. But this is a contradiction since $N_1 \leqslant K$. □

Corollary 3. *If N_1 is either sporadic or an alternating group, then $r=1$ and G is an almost simple group.*

Lemma 15. *If N_1 is a simple group of Lie type defined over the field $GF(q)$ of characteristic $p \in \sigma'$, then the length of an orbit of A_σ on Ω is at most 2.*

Proof. Let $|N_1|_p = p^n$ and $|Out(N_1)|_p = p^\delta$. By Lemma 4, it holds that $n \geqslant \delta + 1$. Then by Lemma 12 and Corollary 1 we have that the length of an A_σ-orbit on Ω is at most 2. □

Lemma 16. *Let N_1 be a simple group of Lie type. Then the length of an A_π-orbit on Ω equals 1.*

Proof. Let m be the length of an A_π-orbit. We will prove that $m \leqslant 2$ and since m divides $|A_\pi|$ and $|A_\pi|$ is odd, we may assume that $m=1$.

Let p be the characteristic of the group N_1 of Lie type. If $p \in \pi'$, then we get the conclusion from Lemma 15. So we may assume that $p \in \pi$ and, in particular, p is odd.

Let $|N_1|_2 = 2^n$ and $|Out(N_1)|_2 = 2^\delta$. Then, by Lemma 5, it holds that $n \geqslant \delta + 1$. Therefore, we get $m \leqslant 2$, by using Lemma 12 for the prime $s=2$, and we are done. □

Lemma 17. *Assume that N_1 is a simple group of Lie type of characteristic p. If $p \in \pi$, then $r=1$. If $p \notin \pi$, then $A \cap B = 1$.*

Proof. Assume that $p \in \pi$. By Lemma 15 the length of an orbit of $A_{\pi'}$ on Ω is at most 2. But since the case $r=2$ is not possible and the length of an A_π-orbit on Ω equals 1 by Lemma 16, we get that $A_{\pi'}$ has orbits of length 1 on Ω and then $N = N_1$, that is, $r=1$.

Assume now that $p \in \pi'$. There exists a Sylow p-subgroup $P = A_pB_p$ of G which is a product of some $A_p \in \mathrm{Syl}_p(A)$ and some $B_p \in \mathrm{Syl}_p(B)$. Since $A \cap B$ is a π-group by Lemma 8, it centralizes each

Sylow p-subgroup of both A and B, and so it centralizes also P. Consequently, $A \cap B$ centralizes $1 \neq P \cap N_1 \in \mathrm{Syl}_p(N_1)$. But $A \cap B \leqslant A_\pi$ normalizes N_1 by Lemma 16, which implies that $[A \cap B, N_1] = 1$, since a Sylow p-subgroup of N_1 is self-centralizing in $Aut(N_1)$ by [11, 1.17]. Hence $[A \cap B, N_1^a] = 1$ for each $a \in A_{\pi'}$, and then $A \cap B \leqslant C_G(N) = 1$ because $A_{\pi'}$ acts transitively on Ω; i.e., $A \cap B = 1$. □

Lemma 18. *Assume that N_1 is a simple group of Lie type. If $r \leqslant 3$, then $r = 1$.*

Proof. Assume that $r = 3$. Hence $N = N_1 \times N_2 \times N_3$ and A and B act transitively on the set $\Omega = \{N_1, N_2, N_3\}$. Let $R := \bigcap_{i=1}^{3} N_G(N_i)$ the subgroup of G normalizing every N_i. By Lemma 16 the subgroups A_π and B_π are in R. Clearly, G/R is isomorphic to a transitive subgroup of S_3, the symmetric group of degree 3, and hence isomorphic either to S_3 or C_3.

Let $A_0 := R \cap A$ and $B_0 := R \cap B$. Recall that by Lemma 17 we have that $|A \cap B| = 1 = |A^y \cap B|$ for every $y \in G$. Then, since $G = RA = RB = AB$, we have that

$$\frac{|R||A \cap B|}{|R \cap A||R \cap B|} = \frac{|R|}{|A_0||B_0|} = |G/R|.$$

On the other hand, the size of a double coset $A_0 y B_0$, for any $y \in G$, is equal to $|A_0||B_0|/|A_0^y \cap B_0| = |A_0||B_0|$. Hence $|G/R|$ is equal to the number of different double cosets in R with respect to the pair (A_0, B_0).

We claim that there exists a subgroup $X = N_i N_j$, $i, j \in \{1, 2, 3\}$ (eventually, $i = j$ and $X = N_i$), such that $A_0 X B_0$ is a subgroup of G. Assume this is not true. In particular, $A_0 N_i N_j B_0 \neq R$ for each choice of $1 \leqslant i, j \leqslant 3$.

We will now count the number of different double cosets with respect to (A_0, B_0) in R. We will prove first that for each $i \neq j$ we have $A_0 N_i B_0 \neq A_0 N_i N_j B_0$ and $A_0 N_t B_0 \nsubseteq A_0 N_i N_j B_0$, for $t \in \{1, 2, 3\} \setminus \{i, j\}$.

Indeed, if $A_0 N_i B_0 = A_0 N_i N_j B_0$, then for $t \in \{1, 2, 3\} \setminus \{i, j\}$ we have $A_0 N_t N_i B_0 = N_t A_0 N_i B_0 = N_t A_0 N_i N_j B_0 = A_0 N_t N_i N_j B_0 = A_0 N B_0 = R$. This is a contradiction. Hence $A_0 N_i B_0 \neq A_0 N_i N_j B_0$ for each $i \neq j$.

Suppose now that $A_0 N_t B_0 \subseteq A_0 N_i N_j B_0$, for $t \in \{1, 2, 3\} \setminus \{i, j\}$. Then $A_0 N_i N_j B_0 = A_0 (N_i N_j)^2 B_0 = N_i N_j A_0 N_i N_j B_0 \supseteq N_i N_j A_0 N_t B_0 = A_0 N B_0 = R$. This is also a contradiction.

It follows that $A_0 N_i B_0$ contains at least two different (A_0, B_0)-cosets, including $A_0 B_0$.

We will prove now that $A_0 N_i N_j B_0$ contains at least 4 different (A_0, B_0)-cosets. Indeed, if $n_1 \in N_1$, $n_1 \notin A_0 N_2 B_0$ and $n_2 \in N_2$, $n_2 \notin A_0 N_1 B_0$, then $n_1 n_2 \notin A_0 N_1 B_0 \cup A_0 N_2 B_0$. Hence $A_0 N_1 N_2 B_0 \neq A_0 N_1 B_0 \cup A_0 N_2 B_0$. Since $A_0 N_1 B_0 \cup A_0 N_2 B_0$ contains at least 3 different (A_0, B_0)-cosets, it follows that $A_0 N_1 N_2 B_0$ contains at least 4 different (A_0, B_0)-cosets. Note that the sets $A_0 N_1 N_2 B_0$ and $A_0 N_1 N_3 B_0$ are different and do not contain each other. Hence the number of double cosets contained in $A_0 N_1 N_2 B_0 \cup A_0 N_1 N_3 B_0$ is at least 5.

Moreover, $A_0 N_2 B_0$ is not contained in $A_0 N_1 N_3 B_0$ and $A_0 N_3 B_0$ is not contained in $A_0 N_1 N_2 B_0$. Hence we can choose elements $n_2' \in N_2$ and $n_3' \in N_3$ such that $n_2' \notin A_0 N_1 N_3 B_0$ and $n_3' \notin A_0 N_1 N_2 B_0$. We claim that $n_2' n_3' \notin A_0 N_1 N_3 B_0 \cup A_0 N_1 N_2 B_0$. Indeed, if $n_2' n_3' \in A_0 N_1 N_3 B_0$, then $n_2' \in A_0 N_1 N_3 B_0 (n_3')^{-1} \subseteq A_0 N_1 N_3 B_0 N_3 = A_0 N_1 N_3 B_0$ which is not the case. By the same reason $n_2' n_3' \notin A_0 N_1 N_2 B_0$.

Hence the set $A_0 N_1 N_3 B_0 \cup A_0 N_1 N_2 B_0 \cup A_0 N_2 N_3 B_0$ consists of at least 6 different (A_0, B_0)-cosets. Now we choose elements $n_i'' \in N_i$ such that $n_i'' \notin A_0 N_j N_t B_0$ with $\{i, j, t\} = \{1, 2, 3\}$. As above it is easy to see that $n_1'' n_2'' n_3'' \notin A_0 N_1 N_2 B_0 \cup A_0 N_1 N_3 B_0 \cup A_0 N_2 N_3 B_0$. This means that the number of different (A_0, B_0)-cosets in R is at least 7, a contradiction (recall that the number of (A_0, B_0)-cosets in R is $|G/R| \leqslant 6$). The claim is proved.

Now, if $T = A_0 X B_0$ is a subgroup of R for some proper normal subgroup X of N, then $T/X = (A_0 X/X)(B_0 X/X)$ is a product of two π-decomposable groups and, by minimality, we have that $A_\pi B_\pi X/X$ is a Hall π-subgroup of T/X. In particular, $A_\pi B_\pi X$ is a subgroup of G. Consider now the subgroup $U := \langle A_\pi, B_\pi \rangle \leqslant T$. If $N_G(U) = N_A(U) N_B(U)$ is a proper subgroup of G, then $A_\pi B_\pi$ is a subgroup, a contradiction. So $1 \neq U \trianglelefteq G$ and $N \leqslant U$. But this means that $A_\pi B_\pi X = U \trianglelefteq G$ and the soluble residual $U^{\mathcal{S}}$ of U is a normal subgroup of G with $U^{\mathcal{S}} \leqslant X$. Then $U^{\mathcal{S}} = 1$ and U is soluble. But this is a contradiction since $X \leqslant U$.

Hence $r < 3$ and applying Lemma 14 we deduce that $r = 1$. □

Lemma 19. *Assume that $r > 1$ and let $\hat{N}_i = \prod_{j=1,\, j\neq i}^{r} N_j$, for $i = 1, 2, \ldots, r$. If A_σ has an orbit on Ω of length 1, then $A_\sigma \cap \hat{N}_i = 1$, for each $i \leqslant r$.*

Proof. If A_σ has an orbit on Ω of length 1, then $A_{\sigma'}$ acts transitively on Ω. If $A_\sigma \cap \hat{N}_i \neq 1$, then $\hat{N}_i$ contains an $A_{\sigma'}$-invariant subgroup, which is a contradiction. □

Lemma 20. *Assume that N_1 is a simple group of Lie type. If $r > 1$, then $\pi \cap \pi(N_1) \subseteq \pi(Out(N_1))$.*

Proof. Let $s \in \pi \cap \pi(N_1)$ and assume that $|N_1|_s = s^n$ and $|Out(N_1)|_s = s^\delta$. In particular, $n \geqslant 1$. From Lemma 16 we have that the length of an A_π-orbit on Ω is $m = 1$ and so the length of an $A_{\pi'}$-orbit on Ω is $k = r$. Hence by Lemma 11 it follows that $(r-2)n \leqslant \delta$. Now if $\delta = 0$, that is, s does not divide $|Out(N_1)|$, we get a contradiction, since we are assuming $r > 1$ and so $r > 2$ by Lemma 14. □

Lemma 21. *If N_1 is a non-abelian simple group of Lie type of characteristic p, then $r = 1$.*

Proof. Assume that N_1 is a non-abelian simple group of Lie type of characteristic p and $r > 1$. We recall that $2 \notin \pi$ and $|\pi \cap \pi(G)| \geqslant 2$ by Lemma 1. Consequently, there exists $s \in \pi \cap \pi(G)$ such that $s \geqslant 5$. Let P be a Sylow s-subgroup of G. We may write $P = A_s B_s$, for some $A_s \in \mathrm{Syl}_s(A)$ and some $B_s \in \mathrm{Syl}_s(B)$. Since $P \cap N \in \mathrm{Syl}_s(N)$, $P \cap N \trianglelefteq P$ and $A_\pi N = B_\pi N$ it follows easily that $P = A_s B_s = A_s(P \cap N) = B_s(P \cap N)$.

We know from Lemma 16 that $G = RA_{\pi'}$, where $NA_\pi \leqslant R = \bigcap_{i=1}^{r} N_G(N_i)$. In particular, $A_{\pi'}$ acts transitively on Ω. Then G/N is isomorphic to a subgroup, say $\bar{G}$, of $Out(N_1) \wr S_r$, the natural wreath product of $Out(N_1)$ with S_r, the symmetric group of degree r. We denote by $\bar{A}_\pi$ and $\bar{A}_{\pi'}$, the images of $A_\pi N/N$ and $A_{\pi'}N/N$ in $Out(N_1) \wr S_r$, respectively, and $F := L_1 \times L_2 \times \cdots \times L_r$, where $L_i = Out(N_i)$ for every $i = 1, \ldots, r$, the base group of the wreath product. Set $E = \bar{G} \cap F$. Then $\bar{A}_\pi \leqslant E$ and $F\bar{A}_{\pi'} \cap S_r$ acts transitively on $\{L_1, L_2, \ldots, L_r\}$. In particular, for each $i = 2, \ldots, r$, there exists an element $a_i \in \bar{A}_{\pi'}$ such that $L_1^{a_i} = L_i$. We claim that $C_E(\bar{A}_{\pi'}) \leqslant \{y_1 y_1^{a_2} \cdots y_1^{a_r} \mid y_1 \in L_1\} \cong L_1$. Let $z \in C_E(\bar{A}_{\pi'})$. We have $z = y_1 y_2 \cdots y_r \in F$, where $y_i \in L_i$, for every $i = 1, \ldots, r$, and this expression is unique. Then $z = y_1 y_2 \cdots y_r = z^{a_i} = y_1^{a_i} y_2^{a_i} \cdots y_r^{a_i}$, which implies that $y_i = y_1^{a_i}$, for every $i = 2, \ldots, r$. Consequently, $z = y_1 y_1^{a_2} \cdots y_1^{a_r}$, with $y_1 \in L_1$, and the claim follows.

Therefore, $\bar{A}_\pi \leqslant C_E(\bar{A}_{\pi'})$, which implies that $A_\pi N/N$ is isomorphic to a subgroup of $Out(N_1)$. In particular, a Sylow s-subgroup of G/N is isomorphic to an s-subgroup of $Out(N_1)$ and has order dividing $|Out(N_1)|_s$.

Let $L_n^\epsilon(q)$, where $\epsilon = \pm$, as follows: $L_n^+(q) = L_n(q)$, whereas $L_n^-(q) = U_n(q)$. Similarly, let $GL_n^\epsilon(q)$, for $\epsilon = \pm$, as follows: $GL_n^+(q) = GL_n(q)$, $GL_n^-(q) = GU_n(q)$.

By checking the structure of $Out(N_1)$, we distinguish two possibilities:

(i) $A_s N/N = B_s N/N$ is cyclic, or
(ii) $A_s N/N = B_s N/N$ is metacyclic (non-cyclic). This is the case only when $N_1 \cong L_n^\epsilon(q)$, $n \geqslant 5$, with s dividing $(q - \epsilon 1, n, \log_p(q))$.

Note that $P \cap N = (P \cap N_1) \times \cdots \times (P \cap N_r)$ and $\Phi(P \cap N) = \Phi(P \cap N_1) \times \cdots \times \Phi(P \cap N_r)$ char $P \cap N \trianglelefteq P$, where $\Phi(X)$ denotes the Frattini subgroup of any group X. We also denote by $\widetilde{\ }$ the corresponding factor subgroups of P over $\Phi(P \cap N)$. In particular, the group $U := \widetilde{P \cap N} = (P \cap N)/\Phi(P \cap N)$ is an elementary abelian s-group. We consider the group

$$\widetilde{P} = P/\Phi(P \cap N) = U\widetilde{A}_s = U\widetilde{B}_s = \widetilde{A}_s \widetilde{B}_s.$$

If we let

$$|\widetilde{P \cap N_i}| = \left|(P \cap N_i)/\Phi(P \cap N_i)\right| = s^t, \quad t \geqslant 1,$$

it is clear that $|U| = s^{rt}$. Moreover, we claim that $|\widetilde{A}_s \cap U| \leqslant s^t$ and, analogously, $|\widetilde{B}_s \cap U| \leqslant s^t$. Since $A_{\pi'}$ acts transitively on Ω, we may assume that for each $i = 2, \dots, r$, there exists an element $x_i \in A_{\pi'}$ such that $(P \cap N_1)^{x_i} = P \cap N_i$ and $\Phi(P \cap N_1)^{x_i} = \Phi(P \cap N_i)$. Now, since $s \in \pi$ and $[A_\pi, A_{\pi'}] = 1$, if we let $\hat{N}_i = \prod_{j=1,\, j\neq i}^{r} N_j$ for each $i = 1, 2, \dots, r$, we have that:

$$\begin{aligned} A_s \cap \Phi(P \cap N_1)(P \cap \hat{N}_1) &= \left(A_s \cap \Phi(P \cap N_1)(P \cap \hat{N}_1)\right)^{x_i} \\ &\leqslant \Phi(P \cap N_1)^{x_i}(P \cap \hat{N}_1)^{x_i} \leqslant \Phi(P \cap N_i)\hat{N}_i \end{aligned}$$

for each $i = 2, \dots, r$. Therefore:

$$\begin{aligned} A_s \cap \Phi(P \cap N_1)(P \cap \hat{N}_1) &\leqslant \bigcap_{i=1}^{r} \left(\Phi(P \cap N_i)\hat{N}_i\right) \\ &= \Phi(P \cap N_1) \cdots \Phi(P \cap N_r)\left(\bigcap_{i=1}^{r} \hat{N}_i\right) = \Phi(P \cap N) \end{aligned}$$

and so, $A_s \cap \Phi(P \cap N_1)(P \cap \hat{N}_1) = A_s \cap \Phi(P \cap N)$. Using this fact, it follows that

$$\widetilde{A}_s \cap U = \frac{(A_s \cap N)\Phi(P \cap N)}{\Phi(P \cap N)} \cong \frac{(A_s \cap N)\Phi(P \cap N_1)(P \cap \hat{N}_1)}{\Phi(P \cap N_1)(P \cap \hat{N}_1)},$$

which is isomorphic to a subgroup of

$$\frac{\frac{P \cap N}{P \cap \hat{N}_1}}{\frac{\Phi(P \cap N_1)(P \cap \hat{N}_1)}{P \cap \hat{N}_1}} \cong \frac{P \cap N_1}{\Phi(P \cap N_1)}$$

where $|(P \cap N_1)/\Phi(P \cap N_1)| = s^t$. So the claim follows.

Observe that $\widetilde{A}_s \cap U$ is a normal subgroup both of $\widetilde{A}_s$ and U, so it is normal in $\widetilde{P} = U\widetilde{A}_s$. Analogously, $\widetilde{B}_s \cap U$ is normal in $\widetilde{P}$. Hence the subgroup $V := (\widetilde{A}_s \cap U)(\widetilde{B}_s \cap U)$ is normal in $\widetilde{P}$ and $|V|$ divides $s^t s^t = s^{2t}$.

Consider now the group

$$\widetilde{P}/V = (\widetilde{A}_s V/V)(\widetilde{B}_s V/V) = (U/V)(\widetilde{A}_s V/V) = (U/V)(\widetilde{B}_s V/V).$$

It follows from [9, Theorem III.11.5] and [2, Theorem 1.3] that:

(i) If $A_s N/N = B_s N/N$ is cyclic, then the Prüfer rank of $\widetilde{P}/V$ is at most 2.
(ii) If $A_s N/N = B_s N/N$ is metacyclic (non-cyclic), then the Prüfer rank of $\widetilde{P}/V$ is at most 4. This is the case only when $N_1 \cong L_n^{\epsilon}(q)$, $n \geqslant 5$, and s divides $(q - \epsilon 1, n, \log_p(q))$.

On the other hand, in any case the Prüfer rank of $\widetilde{P}/V$ is at least $rt - 2t = (r - 2)t$, since $|U/V| \geqslant s^{rt-2t}$. Hence we deduce that:

(i) If $A_s N/N = B_s N/N$ is cyclic, then $(r - 2)t \leqslant 2$.

(ii) If $A_sN/N = B_sN/N$ is metacyclic, then $(r-2)t \leqslant 4$.

From now on we will study each case separately:

(i) $A_sN/N = B_sN/N$ is cyclic.
First observe that the case $(r-2)t = 2$ is not possible, because the cyclic subgroups $\widetilde{A}_sV/V$ and $\widetilde{B}_sV/V$ intersect trivially with the normal subgroup U/V of the metacyclic group $\widetilde{P}/V$. Hence we deduce:

$$(r-2)t \leqslant 1$$

where $t \geqslant 1$. Then $r \leqslant 3$, and so $r = 1$ by Lemma 18.

(ii) $A_sN/N = B_sN/N$ is metacyclic (non-cyclic).
Recall that this case can happen only if $N_1 \cong L_n^{\epsilon}(q)$, $n \geqslant 5$, with s dividing $(q-\epsilon 1, n, \log_p(q))$. Assume that $r > 3$. We have that U/V is an elementary abelian group of order at most s^4. On the other hand, $\widetilde{A}_sV/V \cap U/V$ is trivial and $C_{\widetilde{A}_sV/V}(U/V)$ is also trivial. Hence $|\widetilde{A}_sV/V| \leqslant |Aut(U/V)|_s \leqslant |GL_4(s)|_s$. In particular $|\widetilde{A}_sV/V| \leqslant s^6$. Note also that $\widetilde{A}_sV/V \cong A_sN/N$. Hence we have:

$$|A_s| = |A_sN/N||A_s \cap N| \leqslant s^6|N_i|_s$$

since $A_s \cap \hat{N}_i = 1$ by Lemma 19, and so $A_s \cap N \cong (A_s \cap N)\hat{N}_i/\hat{N}_i \leqslant N/\hat{N}_i \cong N_i$.
If we denote by s^u the order of a Sylow subgroup of N_i, it follows that:

$$s^{ru} \leqslant |P| \leqslant |A_s||B_s| \leqslant s^{12}s^{2u}.$$

This implies $4u \leqslant ru \leqslant 12 + 2u$ and $u \leqslant 6$.
Now recall that there exists a non-cyclic abelian s-subgroup of $GL_n^{\epsilon}(q)$ of rank at least n with elements of the form $\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, where $\lambda_i \in GF(q)$ for $\epsilon = +$ and $\lambda_i \in GF(q^2)$ for $\epsilon = -$. Since s divides $q - \epsilon 1$ this implies that a Sylow s-subgroup of $L_n^{\epsilon}(q)$ has an abelian s-subgroup of rank at least $n-2$. Hence it follows that $n - 2 \leqslant u \leqslant 6$, that is $n \leqslant 8$.
Therefore in this case we can deduce that either $r \leqslant 3$ or $n \leqslant 8$. This latter case can be discarded since we may choose $s \in \pi \cap \pi(Out(N_1))$ (by Lemma 20), $s \geqslant 5$, such that s does not divide $(n, q - \epsilon 1, \log_p(q))$. Hence $r \leqslant 3$ and we deduce that $r = 1$, applying Lemma 18. □

From Corollary 3 and Lemma 21 we conclude that $r = 1$ and then N is simple and G is an almost simple group, as desired.

We gather in the next result the gained information about the structure of our minimal counterexample G.

Theorem 3. *Assume that G is a counterexample of minimal order to our Conjecture, that is*:

(H1) *π is a set of odd primes.*
(H2) *G is a group of minimal order satisfying the following conditions*:
1. *$G = AB$ is the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$*,
2. *$A_\pi B_\pi \neq B_\pi A_\pi$.*

Then G is an almost simple group, i.e., G has a unique minimal normal subgroup N, which is a non-abelian simple group; *in particular, $N \trianglelefteq G \leqslant Aut(N)$.*

Moreover, the following properties hold:

(i) *$G = AN = BN = AB$*; *in particular, $|N||A \cap B| = |G/N||N \cap A||N \cap B|$.*

(ii) $(|A_{\pi'}|, |B_{\pi'}|) \neq 1$, $A_{\pi'} \cap B_{\pi'} = 1$ *and* $A \cap B$ *is a* π*-group.*
(iii) *Neither* A *nor* B *is a* π*-group or a* π'*-group.*
(iv) $\pi(G) = \pi(N) \geqslant 5$.
(v) *If, in addition,* N *is a simple group of Lie type of characteristic* p *and* $p \notin \pi$*, then* $A \cap B = 1$.

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