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# Crystal $B(\lambda)$ as a subset of crystal $B(\infty)$ expressed as tableaux for $A_n$ type ☆

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ABSTRACT

The crystal $B(\infty) \otimes \mathsf{T}_\lambda$ is known to contain a copy of the irreducible highest weight crystal $B(\lambda)$ as a sub-crystal. By explicitly identifying the elements belonging to the mentioned sub-crystal, we present two realizations of $B(\lambda)$ that are based on two realizations of $B(\infty)$, for the special linear Lie algebra type. The first description of $B(\lambda)$ is based on our new realization of $B(\infty)$ as the set of all marginally large reverse tableaux and the second is based on a previous realization of $B(\infty)$ as the set of all marginally large tableaux. We further present two new Nakajima monomial realizations of $B(\lambda)$ that correspond naturally to our reverse tableau and tableau realizations.



## 1. Introduction

Quantum group $U_q(\mathfrak{g})$ is a $q$-deformation of the universal enveloping algebra $U(\mathfrak{g})$ over a Lie algebra $\mathfrak{g}$, and the crystal base $B(\infty)$ reveals the structure of the negative part $U_q^-(\mathfrak{g})$ of the quantum group in a very simplified form [6,7]. This work provides explicit descriptions of the crystal $B(\lambda)$, associated with the irreducible highest weight module of highest weight $\lambda$, in terms of the elements of $B(\infty)$, for the special linear Lie algebra type. This goal is achieved by first describing $B(\infty)$ explicitly and then using this description to express the crystal $B(\lambda)$.

The first part of this article is devoted to introducing an explicit description of $B(\infty)$, for the special linear Lie algebra type. The previous work [3] provided a description of $B(\infty)$ in terms of

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marginally large tableaux, and we follow their approach to present the set of marginally large *reverse* tableaux as another realization for $B(\infty)$. Both of these descriptions are based on the presentation of $B(\infty)$ as a union of highest weight crystals [4]. That is, if certain identifications are made within the union $\bigcup_{\lambda\in P^+} B(\lambda)$, then the resulting set of equivalence classes $\bigcup_{\lambda\in P^+} B(\lambda)/\sim$ has a natural crystal structure induced from those on each $B(\lambda)$ and is crystal isomorphic to $B(\infty)$. A careful combination of this theory with the reverse tableaux description [12] of the crystal $B(\lambda)$ results in our reverse tableau description of $B(\infty)$.

In the second part of this article, we provide two explicit realizations of $B(\lambda)$ (essentially) as subsets of the $B(\infty)$ realizations, i.e., as certain sets of tableaux and reverse tableaux. We also provide two further descriptions of $B(\lambda)$ given in terms of Nakajima monomials, each of which are in natural correspondence with one of our two new tableau descriptions.

The two monomial descriptions of $B(\lambda)$ that were just mentioned are not central to this work, but let us briefly discuss them first. Kashiwara gave a crystal structure [10] to the set of Nakajima monomials [15] and showed that certain connected components of the crystal are isomorphic to the irreducible highest weight crystals $B(\lambda)$. Many descriptions of $B(\lambda)$ for the special linear Lie algebra type that rely on this theorem have already appeared, and our two monomial realizations are no different in that they are connected components containing certain maximal monomials. However, our new monomial realizations have the characteristic that the distance of each element from the maximal monomial, in terms of Kashiwara operator actions, is directly accessible from the expression of the element itself.

Let us return to the discussion of our main result, which is the expression of $B(\lambda)$ in terms of elements from a realization of $B(\infty)$. The statement $B(\infty) \cong \bigcup_{\lambda\in P^+} B(\lambda)/\sim$, when interpreted correctly, implies that a copy of $B(\lambda)$ exists within $B(\infty)$. Hence, to reach our goal, it suffices to specify the correct elements and make this subset of $B(\infty)$, corresponding to $B(\lambda)$, explicit. The approach we took was to consider the distance, in terms of Kashiwara operator actions, of each element from the highest weight element of $B(\infty)$, and to gather just the elements that are within the correct distance bound. The two monomial realizations that were discussed above allowed us to see how far the elements of $B(\lambda)$ were from the highest weight element and this gave us the crucial insight as to how our distance bounds had to be defined. Once the appropriate subsets of our two $B(\infty)$ realizations were explicitly written down, we could verify their correctness as two realizations of $B(\lambda)$ through a separate method. In technically more correct terms, our realizations of $B(\lambda)$ are subsets of $B(\infty)\otimes \mathsf{T}_\lambda$, and the verification could be based on the fact, introduced by Nakashima [16], that the connected component in the crystal $B(\infty)\otimes \mathsf{T}_\lambda$ containing the element $b_\infty \otimes \mathsf{t}_\lambda$ is isomorphic to $B(\lambda)$.

Let us mention two results that give explicit descriptions of $B(\lambda)$ as subsets of $B(\infty)\otimes \mathsf{T}_\lambda$, for the special liner Lie algebra type. The first is a polyhedral realization [16] given by Nakashima. We will not provide any details, but anyone with an understanding of both the polyhedral realization and our marginally large reverse tableau realization of $B(\lambda)$ will be able to write down a natural correspondence between the two sets. Kashiwara and Saito[1] also gave a matrix form description of $B(\infty)$ and expressed the crystal $B(\lambda)$ in terms of these matrices. Their expression involved certain bounds on the values of $\varepsilon_i^*$.

The reader may have found it peculiar that we are presenting a (reverse) tableau realization of $B(\lambda)$ by treating it as a subset of $B(\infty)$, with $B(\infty)$ itself realized as a union of $B(\lambda)$'s, which are again written as tableaux. Our two realizations of $B(\lambda)$, consisting of marginally large (reverse) tableaux satisfying certain conditions, are different from the original realizations of $B(\lambda)$, consisting of semi-standard (reverse) tableaux, so that our work is clearly not circular.

Additional value of this work lies in its position as the first step to analogous results for the finite simple Lie algebra types other than the special linear case treated here. The realizations of $B(\infty)$ in terms of marginally large tableaux already exist for all finite simple Lie algebra types [3,4]. Because these were based on only the simplest $B(\lambda)$ realizations available for each type, none of these involve half-boxes or configurations, and are almost as simple as the $B(\infty)$ realization for the special linear Lie algebra type. The results of this work allow us to expect such an approach for the remaining

[1] Private communication with Y. Saito (August, 2010).

finite simple Lie algebra types to return realizations of $B(\lambda)$ that are simpler than the ones that are available and to even create the first realizations for the currently nonexistent cases.

The rest of this paper is organized as follows. In Section 2, we recall how the crystal $B(\infty)$ may be expressed in the form $\bigcup_{\lambda\in P^+} B(\lambda)/\sim$, and in Section 3, we recall the basic theory of Nakajima monomials and the reverse tableau description of $B(\lambda)$. In Section 4, we transfer the expression $B(\infty)=\bigcup_{\lambda\in P^+} B(\lambda)/\sim$ to an analogous statement for the reverse tableaux and collect an appropriate set of representatives to be used as an explicit realization for $B(\infty)$. The section also reviews the previous explicit (non-reverse) tableau realization of $B(\infty)$ from [3]. Our two monomial descriptions of $B(\lambda)$ are presented in Section 5. The final section contains our two descriptions of $B(\lambda)$ as certain sets of marginally large tableaux and marginally large reverse tableaux, and also illustrates how these are related to our two monomial descriptions.

## 2. Crystal $B(\infty)$ as a union of crystals $B(\lambda)$

In this section, we recall how the crystal $B(\infty)$ may be seen as a union of the highest weight crystals $B(\lambda)$. The contents of this section are valid for all symmetrizable Kac–Moody algebras, but we will restrict our discussion to the special linear Lie algebras.

The reader is assumed to be familiar with the basic theory of crystal bases. Standard notation, such as those found in the textbook [2], will be used. In particular, we assume familiarity with the following notions and notation: index set $I=\{1,\dots,n\}$, simple root $\alpha_i$, coroot $h_i$, fundamental weight $\Lambda_i$, set of dominant integral weights $P^+$, quantum group $U_q(A_n)$, abstract crystal with associated Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ and maps wt, $\varepsilon_i$, $\varphi_i$, irreducible highest weight crystal $B(\lambda)$, tensor product rule, negative part $U_q^-(A_n)$ of $U_q(A_n)$, and crystal basis $B(\infty)$ of $U_q^-(A_n)$.

Theorem 5 of the work [7] states that, for every dominant integral weight $\lambda\in P^+$, there is a surjective map $\bar{\pi}_\lambda$ from $B(\infty)$ to $B(\lambda)\cup\{0\}$ satisfying

$$\bar{\pi}_\lambda(\tilde{f}_{i_k}\cdots\tilde{f}_{i_2}\tilde{f}_{i_1}b_\infty)=\tilde{f}_{i_k}\cdots\tilde{f}_{i_2}\tilde{f}_{i_1}b_\lambda,$$

where $b_\infty$ and $b_\lambda$ are the highest weight elements, and that each of the mappings $\bar{\pi}_\lambda$ is a bijection between $\{b\in B(\infty)\mid \bar{\pi}_\lambda(b)\neq 0\}$ and $B(\lambda)$. That is, for every element of $B(\lambda)$, there is a single naturally corresponding element of $B(\infty)$.

**Definition 2.1.** Two elements from the disjoint union $\bigcup_{\lambda\in P^+} B(\lambda)$ are defined to be equivalent ($\overset{\alpha}{\sim}$) to each other if they correspond to the same element of $B(\infty)$. This is clearly an equivalence relation. We fix the notation $B(\cup)=\bigcup_{\lambda\in P^+} B(\lambda)/\overset{\alpha}{\sim}$ for the set of all such equivalence classes.

Let us now recall the crystal structure on $B(\cup)$ given in [4]. Note that any element of $B(\cup)$ can be expressed in the form $\overline{\bar{\pi}_\lambda(b)}$, for some $\lambda\in P^+$ and $b\in B(\infty)$.

- $\tilde{f}_i(\overline{\bar{\pi}_\lambda(b)})=\overline{\tilde{f}_i(\bar{\pi}_\mu(b))}$, using any $\mu\in P^+$ such that $\tilde{f}_i(\bar{\pi}_\mu(b))$ is nonzero.
- $\tilde{e}_i(\overline{\bar{\pi}_\lambda(b)})=0$, if $\tilde{e}_i\bar{\pi}_\lambda(b)=0$, and $\tilde{e}_i(\overline{\bar{\pi}_\lambda(b)})=\overline{\tilde{e}_i(\bar{\pi}_\lambda(b))}$, if otherwise.
- $\mathrm{wt}(\overline{\bar{\pi}_\lambda(b)})=\mathrm{wt}(\bar{\pi}_\lambda(b))-\lambda$, for any choice of $\lambda\in P^+$ such that $\bar{\pi}_\lambda(b)\neq 0$.
- $\varepsilon_i(\overline{\bar{\pi}_\lambda(b)})=\varepsilon_i(\bar{\pi}_\lambda(b))$, using any choice of $\lambda\in P^+$ such that $\bar{\pi}_\lambda(b)\neq 0$.
- $\varphi_i(\overline{\bar{\pi}_\lambda(b)})=\varepsilon_i(\overline{\bar{\pi}_\lambda(b)})+\mathrm{wt}(\overline{\bar{\pi}_\lambda(b)})(h_i)$.

This crystal structure is referred to as the crystal structure on $B(\cup)$ that has been *induced* from those on $B(\lambda)$. The next result may be found in [4,9].

**Theorem 2.2.** *The set of equivalence classes $B(\cup)=\bigcup_{\lambda\in P^+} B(\lambda)/\overset{\alpha}{\sim}$ can be given a crystal structure induced from those on each $B(\lambda)$. When $B(\cup)$ is given this crystal structure, $B(\cup)\cong B(\infty)$ as crystals.*

The first goal of this paper is to express $B(\infty)$ in terms of reverse tableaux. This will be achieved by combining the above result with the reverse tableau description of the crystal $B(\lambda)$.

## 3. Nakajima monomials, reverse tableaux, and crystal $B(\lambda)$

In this section, we recall basic knowledge concerning the Nakajima monomials and the reverse tableaux description of crystal $B(\lambda)$, for the special linear Lie algebras.

Our exposition of the crystal structure on Nakajima monomials follows that of Kashiwara [10]. We denote by $\mathcal{M}$ the set of *Nakajima monomials* in the variables $Y_i(m)$, where $i \in I$ and $m \in \mathbf{Z}$. Each monomial is of the form $\prod_{(i,m)} Y_i(m)^{y_i(m)}$, with nonzero exponent $y_i(m) \in \mathbf{Z}$ appearing at only finitely many $(i,m) \in I \times \mathbf{Z}$. For each $i \in I$ and $m \in \mathbf{Z}$, the notation

$$U_i(m) = Y_i(m)Y_i(m+1)Y_{i-1}(m+1)^{-1}Y_{i+1}(m)^{-1}$$

is used, where we are setting $Y_0(k)^{\pm 1} = Y_{n+1}(k)^{\pm 1} = 1$.

Recall that the crystal structure on the set $\mathcal{M}$ is defined as follows. For every monomial $M = \prod_{(i,m)} Y_i(m)^{y_i(m)} \in \mathcal{M}$ and $i \in I$, we set

- $\mathrm{wt}(M) = \sum_{i\in I}(\sum_{m\in\mathbf{Z}} y_i(m))\Lambda_i$,
- $\varphi_i(M) = \max\{\sum_{k\leqslant m} y_i(k) \mid m \in \mathbf{Z}\}$,
- $\varepsilon_i(M) = \max\{-\sum_{k>m} y_i(k) \mid m \in \mathbf{Z}\}$.

To prepare for the definition of the Kashiwara operator applications, we introduce the values

- $m_f = m_f(M,i) = \min\{m \mid \varphi_i(M) = \sum_{k\leqslant m} y_i(k)\}$,
- $m_e = m_e(M,i) = \max\{m \mid \varepsilon_i(M) = -\sum_{k>m} y_i(k)\}$.

The Kashiwara operator actions are given by

- $\tilde{f}_i(M) = 0$ if $\varphi_i(M) = 0$, and $\tilde{f}_i(M) = U_i(m_f)^{-1}M$ if $\varphi_i(M) > 0$,
- $\tilde{e}_i(M) = 0$ if $\varepsilon_i(M) = 0$, and $\tilde{e}_i(M) = U_i(m_e)M$ if $\varepsilon_i(M) > 0$.

The following theorem from [10] gives a realization of the irreducible highest weight crystal.

**Theorem 3.1.** *For a highest weight element $M \in \mathcal{M}$, the connected component of the crystal $\mathcal{M}$ containing $M$ is isomorphic to $B(\mathrm{wt}(M))$.*

Kashiwara [10] gave multiple crystal structures on the set $\mathcal{M}$ and the above realization theorem holds true for each of these crystal structures, but we will deal only with the crystal structure explained above.

In the rest of this section, we will recall a reverse tableau description [12] of the irreducible highest weight crystal $B(\lambda)$. Let us use $R(\lambda)$ to denote the set of all semi-standard reverse tableaux of shape $\lambda \in P^+$, with entries taken from the set $\{1,\dots,n+1\}$. A semi-standard reverse tableau consists of a finite number of identically sized square boxes arranged in rows and columns. The rows are right-justified and the count of boxes in each row must be weakly decreasing from bottom to top. Also, the entries written to its boxes should be weakly increasing from left to right within each row and strictly increasing from top to bottom within each column. A semi-standard reverse tableau of shape $\lambda = \sum_{i\in I}\lambda(h_i)\Lambda_i$ is a semi-standard reverse tableau consisting of $\lambda(h_i)$-many columns of height $i$, for all $i \in I$.

**Example 3.2.** The followings four objects are semi-standard reverse tableaux of shapes $\lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$ and $\lambda' = 3\Lambda_1 + 2\Lambda_2 + \Lambda_3$.

$$R_\lambda = \begin{array}{ccc} & & \boxed{1} \\ & \boxed{1} & \boxed{2} \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array}, \quad \begin{array}{ccc} & & \boxed{1} \\ & \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{3} & \boxed{4} \end{array}, \qquad R_{\lambda'} = \begin{array}{cccccc} & & & & & \boxed{1} \\ & & & \boxed{1} & \boxed{1} & \boxed{2} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{3} \end{array}, \quad \begin{array}{cccccc} & & & & & \boxed{2} \\ & & & \boxed{2} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{4} \end{array}.$$

The reverse tableaux $R_\lambda$ and $R_{\lambda'}$ are the highest weight elements of the crystals $R(\lambda)$ and $R(\lambda')$, respectively.

The maps defining the crystal structure on the set $R(\lambda)$ are very similar to those given to the crystal $T(\lambda)$ consisting of semi-standard tableaux, which is a very well known description of $B(\lambda)$ introduced by Kashiwara and Nakashima in [11]. Let us only briefly recall the Kashiwara operator actions on the reverse tableaux. One first expands a reverse tableau into its *tensor product form* through the *far eastern reading*, as exemplified below:

$$\begin{array}{ccc} & & \boxed{1} \\ & \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{3} & \boxed{4} \end{array} = \boxed{1} \otimes \boxed{2} \otimes \boxed{4} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{1}$$

Then, the *tensor product rule* is used to apply $\tilde{f}_i$ or $\tilde{e}_i$ to one of the boxes, after which the resulting tensor product form is reconstructed into the shape of the original tableau.

The tensor product rule on multiple tensors can be applied through calculation of the *i-signature*. This is done though the following steps.

1. Under each tensor component $x$, write down $\varepsilon_i(x)$-many 1's followed by $\varphi_i(x)$-many 0's. This sequence is called the *i-signature of x*.
2. Then, from the long sequence of mixed 0's and 1's, successively cancel out every occurrence of $(0, 1)$ pairs until we arrive at a sequence of 1's followed by 0's, reading from left to right. This is called the *i-signature* of the whole tensor product form.
3. To apply $\tilde{f}_i$ to the whole product, apply it to the single tensor component corresponding to the leftmost 0 remaining in the $i$-signature. If no 0 remains, the result of the $\tilde{f}_i$ action is set to zero.
4. Similarly, for $\tilde{e}_i$, apply it to the component corresponding to the rightmost 1, or set it to zero when no 1 remains.

In the next section, we obtain a description of $B(\infty)$ by combining the result of the previous section with the reverse tableau description of $B(\lambda)$.

## 4. Crystal $R(\cup)$ and the reverse tableau description of $B(\infty)$

In this section, we construct a realization of $B(\infty)$ using reverse tableaux, for the special linear Lie algebra types. The realization will be given as a subset of the union of the $R(\lambda)$'s. We will follow the approach taken by [3], which provided a description of $B(\infty)$ in terms of tableaux. The approach will rely on the identifications $B(\cup) \cong B(\infty)$ and $R(\lambda) \cong B(\lambda)$, reviewed in Sections 2 and 3, respectively. At the end of this section, we will briefly recall the (usual) tableau description of $B(\infty)$ [3] for use in the final section.

### *4.1. Crystal $R(\cup)$*

We will first fix a certain set of reverse tableaux and then identify some of its elements. The resulting set of equivalence classes will become a realization for $B(\infty)$.

The bottom row of a reverse tableau shall be referred to as the *first* row throughout this work.

**Definition 4.1.**

1. For $i \in I$, a *basic i-column* is a single column of $i$-many boxes, with its $k$-th row box occupied by the entry $(i + 1 - k)$, for each $1 \leqslant k \leqslant i$.

2. A semi-standard reverse tableau of shape $\lambda \in P^+$ is *large* if it contains at least one basic $i$-column, for every $i \in I$.
3. The basic columns create a division of any large reverse tableau into $n$ separate groups of boxes. The boxes within each of these groups may further be separated in accordance to their row positions. This combination gives us a grouping of the non-basic column boxes into $\frac{n(n+1)}{2}$ separate collections of boxes, some of which could be empty. Given a large reverse tableau, the collection of boxes that is right-adjacent to an $i$-box from the basic column and is located on the $m$-th row is defined to be the $(i, m)$-*collection of boxes* for the reverse tableau. We will mostly refer to these simply as the $(i, m)$-collection. Note that, for each $i \in I$, we must have $1 \leqslant m \leqslant n + 1 - i$.
4. We denote by $R(\lambda)^L$, the set of all large reverse tableaux of shape $\lambda$. The collection of all large reverse tableaux is written as $R^L = \bigcup_{\lambda \in P^+} R(\lambda)^L$, where the right-hand side $R(\lambda)^L$ could be empty for some $\lambda$.

**Remark 4.2.** An $(i, m)$-collection of a large reverse tableaux cannot contain boxes other than $i$-boxes and $(i + 1)$-boxes. Furthermore, each $(i, 1)$-collection consists only of $(i + 1)$-boxes.

**Example 4.3.** Any $A_3$-type large reverse tableau is of the following form.

| | | | | | |
|---|---|---|---|---|---|
| | | | | $1 \cdots 1$ | $1 \quad \cdots \quad 1 \ 2 \cdots 2$ |
| | | $1 \cdots 1$ | $1 \cdots 1 \ 2 \cdots 2$ | $2 \cdots 2$ | $2 \cdots 2 \ 3 \quad \cdots \quad 3$ |
| $1 \cdots 1$ | $2 \cdots 2$ | $2 \cdots 2$ | $3 \quad \cdots \quad 3$ | $3 \cdots 3$ | $4 \quad \cdots \quad 4$ |

The parts without any shading are the basic columns. All three groups of boxes, each of nonzero width, must be present. On the other hand, the shaded parts are optional, and each part may be of arbitrary width. The group of shaded boxes $\boxed{1 \cdots 1 \ 2 \cdots 2}$, appearing in the second row, is the $(1, 2)$-collection of boxes, and the group of shaded boxes $\boxed{2 \cdots 2 \ 3 \quad \cdots \quad 3}$ is the $(2, 2)$-collection for this large reverse tableau.

We now provide an equivalence relation among the large reverse tableaux.

**Definition 4.4.** Two reverse tableaux $R_1$ and $R_2$ are said to be *equivalent*, written as $R_1 \overset{\beta}{\sim} R_2$, if what remains of the two are identical after removal of all basic $i$-columns from them, for every $i \in I$. This is clearly an equivalence relation. We fix the notation $R(\cup) := R^L / \overset{\beta}{\sim}$ for the set of equivalence classes of large reverse tableaux.

In Section 2, we reviewed the equivalence relation on $\bigcup_{\lambda \in P^+} B(\lambda)$ that relied on the correspondences between $B(\infty)$ and $B(\lambda)$. The equivalence relation $\overset{\alpha}{\sim}$, defined on the set $\bigcup_{\lambda \in P^+} B(\lambda)$, can be carried over to that on $R^L \subset \bigcup_{\lambda \in P^+} R(\lambda)$ through the identification $B(\lambda) \cong R(\lambda)$. That is, we have two equivalence relations $\overset{\alpha}{\sim}$ and $\overset{\beta}{\sim}$ on $R^L$. We wish to show that the two equivalence relations $\overset{\alpha}{\sim}$ and $\overset{\beta}{\sim}$ on $R^L$ are identical.

**Lemma 4.5.** *Fix an $i \in I$ and let $R$, $R_1$, and $R_2$ be large reverse tableaux such that $R_1 \overset{\beta}{\sim} R_2$.*

1. *$\tilde{f}_i R$ is never zero.*
2. *$\tilde{e}_i R$ is either zero or large.*
3. *$\tilde{f}_i R_1 \overset{\beta}{\sim} \tilde{f}_i R_2$.*
4. *Either $\tilde{e}_i R_1$ and $\tilde{e}_i R_2$ are both zero, or $\tilde{e}_i R_1 \overset{\beta}{\sim} \tilde{e}_i R_2$.*

**Proof.** (1) The largeness of $R$ guarantees the existence at least one basic $i$-column, and the bottom row of any basic $i$-column is occupied by an $i$-box. The $i$-signature to be written under this $i$-box in the tensor product form of $R$ is 0, and the semi-standard condition on $R$ implies that the signature 0 will not be canceled out by signatures from boxes contained in any of the columns sitting to its left. This is sufficient to guarantee $\tilde{f}_i R$ to be nonzero.

(2) Let there be at least one 1 remaining in the $i$-signature for the tensor product form of $R$, after all $(0,1)$ pair cancelations, so that $\tilde{e}_i R$ is nonzero. Remark 4.2 implies that an $(i+1)$-box can only appear within some of the basic columns, $(i,m)$-collections, and $(i+1,m)$-collections. Since no basic column or $(i+1,m)$-collection can contribute a 1 to the $i$-signature of $R$, the $\tilde{e}_i$ operator must act on an $(i+1)$-box appearing in an $(i,m)$-collection of $R$, for some $m$. Such an action will not affect any basic column, so that $\tilde{e}_i R$ will remain large.

(3) Note that $\tilde{f}_i$ must act on either the rightmost basic $i$-column or on an $(i,m)$-collection, for some $m$, and that any $(i,m)$-collection is located to the right of all basic $i$-columns. Focusing on the $i$-signatures for $R_1$ and $R_2$ coming from just the two parts, one can see that the equivalence of the two reverse tableaux implies that $\tilde{f}_i$ will acts on two corresponding $i$-boxes, and that the results of the $\tilde{f}_i$ actions will be equivalent.

(4) The discussion already given for items (2) and (3) implies that $\tilde{e}_i$ will act on an $(i,m)$-collection that is located to the right of all basic $i$-columns. As before, equivalence of $R_1$ and $R_2$ forces actions of $\tilde{e}_i$ on the two reverse tableaux to be both zero or to materialize on corresponding $(i+1)$-boxes, so that the results are equivalent. $\square$

Let us consider two given large reverse tableaux $R_1$ and $R_2$. There are $\lambda_1, \lambda_2 \in P^+$, such that $R_1 \in R(\lambda_1)$ and $R_2 \in R(\lambda_2)$. If we assume $R_1 \stackrel{\alpha}{\sim} R_2$, we can write $R_1 = \tilde{f}_{i_t} \cdots \tilde{f}_{i_1} R_{\lambda_1}$ and $R_2 = \tilde{f}_{i_t} \cdots \tilde{f}_{i_1} R_{\lambda_2}$, for some set of indices $i_1, \dots, i_t \in I$. Here, $R_{\lambda_1}$ and $R_{\lambda_2}$ are the highest weight elements. Iterative applications of Lemma 4.5(2) to $R_1$ and $R_2$ imply that both $\tilde{f}_{i_k} \cdots \tilde{f}_{i_1} R_{\lambda_1}$ and $\tilde{f}_{i_k} \cdots \tilde{f}_{i_1} R_{\lambda_2}$ are large, for every $0 \leqslant k \leqslant t$. Then, since we know $R_{\lambda_1} \stackrel{\beta}{\sim} R_{\lambda_2}$, iterative applications of Lemma 4.5(3) imply $R_1 \stackrel{\beta}{\sim} R_2$.

Let us next assume $R_1 \stackrel{\beta}{\sim} R_2$ to argue in the converse direction. For some sequence of indices $i_1, \dots, i_t$, we have $\tilde{e}_{i_t} \cdots \tilde{e}_{i_1} R_1 = R_{\lambda_1}$. Iterative applications of items (2) and (4) of Lemma 4.5 imply that $\tilde{e}_{i_t} \cdots \tilde{e}_{i_1} R_2 \stackrel{\beta}{\sim} \tilde{e}_{i_t} \cdots \tilde{e}_{i_1} R_1$. Since $\tilde{e}_i R_{\lambda_1} = 0$ for all $i \in I$, Lemma 4.5(4) implies that $\tilde{e}_{i_t} \cdots \tilde{e}_{i_1} R_2$ must also be maximal, and we may write $\tilde{e}_{i_t} \cdots \tilde{e}_{i_1} R_2 = R_{\lambda_2}$. This shows that $R_1 = \tilde{f}_{i_1} \cdots \tilde{f}_{i_t} R_{\lambda_1}$ and $R_2 = \tilde{f}_{i_1} \cdots \tilde{f}_{i_t} R_{\lambda_2}$, so that $R_1 \stackrel{\alpha}{\sim} R_2$.

We have thus argued that the two equivalence relations $\stackrel{\alpha}{\sim}$ and $\stackrel{\beta}{\sim}$ given to $R^L$ are identical.

**Proposition 4.6.** *For large reverse tableaux $R_1$ and $R_2$, $R_1 \stackrel{\alpha}{\sim} R_2$ if and only if $R_1 \stackrel{\beta}{\sim} R_2$.*

Now, if we can locate at least one representative in $R^L = \bigcup_{\lambda \in P^+} R(\lambda)^L$, for every equivalence class of $B(\cup) = \bigcup_{\lambda \in P^+} B(\lambda)/\stackrel{\alpha}{\sim}$, then $R(\cup) = R^L/\stackrel{\beta}{\sim}$ could be taken as another realization for $B(\infty)$. The first item of the following lemma provides the representatives we need.

**Lemma 4.7.**

1. *Given any $b \in B(\infty)$, there exists a $\lambda \in P^+$ such that $\bar{\pi}_\lambda(b)$ is a large reverse tableau. Here, the map $\bar{\pi}_\lambda$ is interpreted as sending elements of $B(\infty)$ to elements of $R(\lambda) \cong B(\lambda)$.*
2. *Given any element of $R(\cup)$, it is always possible to choose its representative $R \in R^L$ in such a way that $\tilde{f}_i R$ is large.*

**Proof.** (1) The proof of this claim is identical to that of Lemma 3.2 appearing in [1], which our claim is very similar to. Our requirement for large reverse tableau replaces the original claim's requirement for a certain semi-standard tableau.

(2) Given any representative $R' \in R^L$ of $b \in R(\cup)$, let us create a *larger* representative by inserting one copy of the basic $i$-column into the correct position within $R'$. It is clear that this new large

reverse tableau $R$ is a representative for $b$. Since at most one basic $i$-column may be affected during an $\tilde{f}_i$ action, the largeness of $\tilde{f}_i R$ will be ensured by the inserted basic $i$-column. $\square$

We can now rightfully state that $R(\cup)$ is equal to $B(\cup)$ as sets, under the identification $R(\lambda) \cong B(\lambda)$. Our next goal is to provide a crystal structure on $R(\cup)$ and compare it with that on $B(\cup)$.

Lemma 4.7(2) and Lemma 4.5(3) provide a natural definition for the Kashiwara operator $\tilde{f}_i$ action on $R(\cup)$. An analogous support for the $\tilde{e}_i$ operator is provided by Lemmas 4.5(2) and 4.5(4). Let us quickly discuss the remaining maps that are needed to define a crystal structure. It is clear that, given $b \in R(\cup)$, we may choose any representative $R \in R(\lambda)^L$ and define its weight to be $\mathrm{wt}(b) = \mathrm{wt}(R) - \lambda$. Lemma 4.5(4) indicates that we may define $\varepsilon_i(b) = \varepsilon_i(R)$ and $\varphi_i(b) = \varepsilon_i(b) + \mathrm{wt}(b)(h_i)$. We have introduced a crystal structure on $R(\cup)$ induced from the crystal structures of each $R(\lambda)$.

Since the crystal structures on $R(\cup)$ and $B(\cup)$ were both derived from those on $R(\lambda)$ and $B(\lambda)$, the two crystals $R(\cup)$ and $B(\cup)$ are isomorphic under the identification $R(\lambda) \cong B(\lambda)$ of crystals.

**Theorem 4.8.** *The set of equivalence classes $R(\cup) = \bigcup_{\lambda \in P^+} R(\lambda)^L / \overset{\beta}{\sim}$ can be given a crystal structure induced from those on each $R(\lambda)$. This crystal $R(\cup)$ is isomorphic to $B(\infty)$ as a crystal.*

## 4.2. Reverse tableau description of $B(\infty)$

To achieve our goal of giving an explicit description of $B(\infty)$ in terms of reverse tableaux, it suffices to provide an explicit set of representatives for $R(\cup)$ and translate the various maps on $R(\cup)$ to those on the representative set. Let us first introduce a new definition.

**Definition 4.9.** A large reverse tableau is *marginally large*, if it contains exactly one basic $i$-column, for each $i \in I$. We denote by $R(\infty)$, the set of all marginally large reverse tableaux.

**Example 4.10.** In the $A_3$-type case, any marginally large reverse tableau takes the following form:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
 & & & & 1 & 1 \cdots 1\ 2 \cdots 2 \\
\hline
 & & 1 & 1 \cdots 1\ 2 \cdots 2 & 2 & 2 \cdots 2\ 3 \cdots 3 \\
\hline
1 & 2 \cdots 2 & 2 & 3 \cdots 3 & 3 & 4 \cdots 4 \\
\hline
\end{array}$$

This has three basic columns. The parts without any shading are the basic columns, and exactly one must exist for each $i \in I$. The shaded parts are optional and may be of arbitrary widths.

The simplest $A_3$-type marginally large reverse tableau is the highest weight element

$$\begin{array}{|c|c|c|}
\hline
 & & 1 \\
\hline
 & 1 & 2 \\
\hline
1 & 2 & 3 \\
\hline
\end{array}$$

that consists of just the basic columns. This will serve as our representative for the highest weight element $b_\infty \in B(\infty)$, and we will use $R_\infty$ to denote this highest weight element, viewed as an element of $R(\infty)$.

Given any large reverse tableau, we can arrive at a marginally large reverse tableau by successively removing a suitable number of its basic columns. The definition of equivalence between reverse tableaux implies that we will arrive at the same marginally large reverse tableau even if we started from another equivalent large reverse tableau. This shows that every equivalence class of large reverse tableaux can be represented by a unique marginally large reverse tableaux. We can now state the following result.

**Theorem 4.11.** *The set $R(\infty)$ of marginally large reverse tableaux forms a set of representatives for $R(\cup)$. It can be given a crystal structure with which it becomes crystal isomorphic to $B(\infty)$.*

Let us explain how the crystal structure on $R(\cup)$ may be carries over to that on $R(\infty)$. Given a marginally large reverse tableau $R$, according to Lemma 4.7(2), one can choose an equivalent large reverse tableau $R'$ such that $\tilde{f}_i R'$ is large. Then basic columns can be removed from the resulting $\tilde{f}_i R'$, until we arrive at a marginally large reverse tableau. This process is the $\tilde{f}_i$ action on the set of marginally large reverse tableaux, but let us provide a more simplified version of this process.

Note that any given marginally large reverse tableau $R$ is contained in a single $R(\lambda)$ for some $\lambda \in P^+$. The following steps are taken to compute the Kashiwara operator $\tilde{f}_i$ on $R \in R(\infty)$.

1. If $\tilde{f}_i R$, computed as an element of the crystal $R(\lambda)$, is marginally large, we are done.
2. If otherwise, then $\tilde{f}_i$ must have acted on the $i$-box situated within the unique basic $i$-column. Insert a single basic $i$-column at the appropriate position to obtain a marginally large reverse tableau.

To apply $\tilde{e}_i$ to $R \in R(\infty)$, the following procedure is taken.

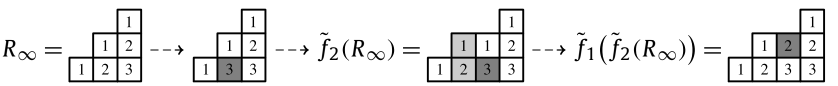
1. If $\tilde{e}_i R$, computed as an element of $R(\lambda)$, is either zero or marginally large, we are done.
2. If otherwise, then $\tilde{e}_i$ must have acted on the $(i+1)$-box in the column sitting to the right of the unique basic $i$-column. Remove the column containing the changed box, which will be a basic $i$-column, to arrive at a marginally large reverse tableau.

**Remark 4.12.** Whenever Kashiwara operator acts on a marginally large reverse tableau, the $\tilde{f}_i$ acts on either an $i$-box in an $(i,m)$-collection for some $m$ or the $i$-box in the basic $i$-column, and the $\tilde{e}_i$ acts on an $(i+1)$-box in an $(i,m)$-collection for some $m$.

Note that the disappearance of a basic $i$-column during an $\tilde{f}_i$ action or the appearance of a new basic $i$-column during an $\tilde{e}_i$ computation implies that the operator has acted on the first row.

As for the remaining maps wt, $\varepsilon_i$, and $\varphi_i$, it suffices to adopt the corresponding maps defined in the previous subsection on $R(\cup)$.

**Example 4.13.** We illustrate the $\tilde{f}_i$ actions on a marginally large reverse tableau, for the $A_3$ type. In order to apply $\tilde{f}_2$ to $R_\infty$, we first compute the action according to the crystal structure of $R(\lambda)$, where $\lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$. Since the result does not contain any basic 2-column, the reverse tableau is not large. Thus, a basic 2-column is inserted during the $\tilde{f}_2$ action.




The dark shaded boxes are the ones changed by direct application of the $\tilde{f}_i$ operator, and the light shadings indicate the columns inserted to preserve largeness.

In the final section, we will present a simple description of $B(\lambda)$, using the elements of $R(\infty)$.

### 4.3. *Review of the tableau description for $B(\infty)$*

In this section, we briefly recall the tableau description of $B(\infty)$ presented by [3]. This is given as a subset of the union of tableau descriptions $T(\lambda)$ for $B(\lambda)$ introduced in [11].

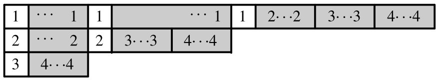
In the remainder of this work, the top row of a (usual, non-reverse) tableau will be referred to as its first row.

**Definition 4.14.** For $i \in I$, a *basic i-column* is a single column of $i$-many boxes, with the box at its $k$-th row containing the entry $k$, for each $1 \leqslant k \leqslant i$.

A semi-standard tableau with entries from $\{1, \dots, n+1\}$ is *marginally large*, if it contains exactly one basic $i$-column, for each $i \in I$. The set of all marginally large tableaux is denoted by $T(\infty)$.

It may be inferred from the definition that, for each $i \in I$, the number of $i$-boxes appearing in the $i$-th row of any marginally large tableaux is exactly one larger than the total number of boxes appearing its $(i+1)$-th row.

**Example 4.15.** The set $T(\infty)$, in the $A_3$-type case, consists of all tableaux of the following form.




The non-shaded parts, which are basic columns, must exist, whereas the shaded parts are optional and can be of arbitrary widths.

**Theorem 4.16.** *The set $T(\infty)$ of all marginally large tableaux forms a crystal and is isomorphic to the crystal $B(\infty)$.*

Let us describe the crystal structure on this set. We start with the description of the Kashiwara operator actions on $T(\infty)$. To apply $\tilde{f}_i$ to a marginally large tableau, we go through the following procedure.

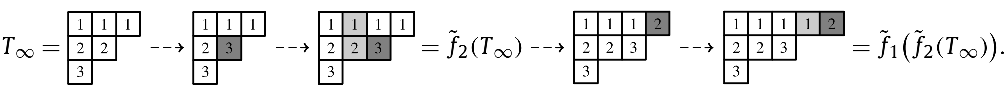
1. Apply $\tilde{f}_i$ to the tableau as usual. That is, write it in tensor product form, apply tensor product rule, and assemble back into original tableau form.
2. If the result is a large tableau, it is automatically marginally large, and we are done.
3. If the result is not large, then $\tilde{f}_i$ must have been applied to the $i$-box in the basic $i$-column. Insert one basic $i$-column to the left of the box $\tilde{f}_i$ acted on.

Analogous process for the $\tilde{e}_i$ operator is as follows.

1. Apply $\tilde{e}_i$ to the tableau as usual.
2. If the result is zero or a marginally large tableau, we are done.
3. Otherwise, the result is large, but not marginally large. The $\tilde{e}_i$ operator must have acted on the box sitting to the right of the $i$-box in the basic $i$-column. Remove the column containing the changed box, which must be a new basic $i$-column.

Any given marginally large tableau $T$ is contained in exactly one $T(\lambda)$ for some $\lambda \in P^+$. It is clear that we can set the weight $\mathrm{wt}(T)$ to be $\lambda$ less than the weight of this tableau $T$ in the crystal $T(\lambda)$. Furthermore, $\varepsilon_i(T)$ is set to the corresponding value computed for $T$ as an element of $T(\lambda)$, and $\varphi_i(T) = \varepsilon_i(T) + \mathrm{wt}(T)(h_i)$.

**Example 4.17.** The simplest marginally large tableau is $T_\infty$, the first diagram given below, which consists of just the basic columns. It corresponds to the highest weight element $b_\infty \in B(\infty)$. We illustrate the $\tilde{f}_2$ action on $T_\infty$ and the $\tilde{f}_1$ action on $\tilde{f}_2(T_\infty)$.




The dark shaded boxes are the ones $\tilde{f}_i$ has acted on, and the light shadings show columns inserted to preserve largeness.

We remark that, even though both $R(\infty)$ and $T(\infty)$ are tableau descriptions of $B(\infty)$, it is easy to verify that neither the *Schützenberger sliding algorithm* [18] nor the *Schensted bumping algorithm* [13,17] can be used to make the isomorphism explicit.

## 5. Monomial descriptions of $B(\lambda)$

There are multiple existing descriptions of the crystal $B(\lambda)$, for the special linear Lie algebra types, that are based on the Nakajima monomial theory. In this section, we present two more such descriptions of $B(\lambda)$. The two are the connected components from the crystal $\mathcal{M}$ that contain the highest weight elements $\prod_{i\in I} Y_i(1)^{\lambda(h_i)}$ and $\prod_{i\in I} Y_i(-i)^{\lambda(h_i)}$, and both of these have already been described before [5,14]. However, we will describe the two sets using different expressions that are more suitable for our needs.

One characteristic of our new description is that the expression for each element displays all the Kashiwara operator actions that can be used to reach the element from the highest weight element. This characteristic has helped us in formulating the descriptions of $B(\lambda)$ in terms of elements from $R(\infty)$ and $T(\infty)$, given in the final section.

### 5.1. Connected component of $\prod_{i\in I} Y_i(1)^{\lambda(h_i)}$

Recall from Theorem 3.1 that, for a highest weight element $M \in \mathcal{M}$, the connected component of the crystal $\mathcal{M}$ containing $M$ is isomorphic to $B(\mathrm{wt}(M))$. In this section, we find the connected component of $\mathcal{M}$, containing the highest weight element $M_\lambda = \prod_{i\in I} Y_i(1)^{\lambda(h_i)}$, which is of weight $\lambda = \sum_{i\in I} \lambda(h_i)\Lambda_i$.

Let $M(\lambda)$ be the set of all monomials of the form

$$\prod_{i\in I}\left(Y_i(1)^{\lambda(h_i)} \cdot \prod_{1\leqslant m\leqslant n+1-i} U_i(m)^{-u_{i,m}}\right), \tag{5.1}$$

with the exponents satisfying

$$\begin{aligned} &0 \leqslant u_{i,1} \leqslant u_{i-1,1} + \lambda(h_i), \\ &0 \leqslant u_{i,m} \leqslant \min\{u_{i-1,m} + \lambda(h_{i-1+m}), u_{i+1,m-1}\}, \quad \text{for } 2 \leqslant m \leqslant n+1-i, \end{aligned} \tag{5.2}$$

for each $i \in I$, where we take $u_{0,m} = 0$, for every $m$. The vector $M_\lambda$ is contained in the set $M(\lambda)$, as can be seen by taking $u_{i,m} = 0$, for every $i$ and $m$.

**Proposition 5.3.** *The set $M(\lambda)$ is the connected component of crystal $\mathcal{M}$, containing the vector $M_\lambda = \prod_{i\in I} Y_i(1)^{\lambda(h_i)}$ of weight $\lambda$, and $M(\lambda) \cong B(\lambda)$.*

**Proof.** Since the final claim follows from Theorem 3.1, it suffices to show that the actions of the Kashiwara operators on $M(\lambda)$ satisfy the properties

$$\tilde{f}_i M(\lambda) \subset M(\lambda) \cup \{0\}, \qquad \tilde{e}_i M(\lambda) \subset M(\lambda) \cup \{0\},$$

for all $i \in I$, and that every element of $M(\lambda)$ is connected to the element $M_\lambda$, by Kashiwara operator actions.

Fix any $M \in M(\lambda)$ and $i \in I$, and let us suppose that $\tilde{f}_i M \notin M(\lambda) \cup \{0\}$. Then, since $M \in M(\lambda)$ and $\tilde{f}_i M \neq 0$, we must have $\varphi_i(M) > 0$ and

$$\tilde{f}_i M = U_i(m)^{-1} M,$$

for some $1 \leqslant m \leqslant n+1-i$.

Let us first assume $m = 1$ was used. Then, the fact $m_f(M, i) = m = 1$ implies that the exponent $y_i(1) = -u_{i,1} + u_{i-1,1} + \lambda(h_i)$ of $Y_i(1)$ appearing in the monomial $M$ is positive, so that we have

$$u_{i-1,1} + \lambda(h_i) > u_{i,1}.$$

On the other hand, since the assumption $\tilde{f}_i M \notin M(\lambda)$ can only be associated with the violation of the first line of (5.2), we can state

$$u_{i,1} + 1 > u_{i-1,1} + \lambda(h_i).$$

Since these two inequalities cannot be true simultaneously, we have a contradiction.

Let us next treat the $m > 1$ case. As before, the process of obtaining $m = m_f(M, i)$ implies that $y_i(m) > 0$, and we know

$$u_{i+1,m-1} - u_{i,m-1} + u_{i-1,m} > u_{i,m}.$$

On the other hand, assumption $\tilde{f}_i M \notin M(\lambda)$ implies

$$u_{i,m} + 1 > \min\{u_{i-1,m} + \lambda(h_{i-1+m}), u_{i+1,m-1}\}.$$

A combination of the two inequalities implies

$$u_{i+1,m-1} - u_{i,m-1} + u_{i-1,m} > \min\{u_{i-1,m} + \lambda(h_{i-1+m}), u_{i+1,m-1}\},$$

so that at least one of

$$\begin{aligned} u_{i+1,m-1} &> u_{i,m-1} + \lambda(h_{i-1+m}), \\ u_{i-1,m} &> u_{i,m-1}, \end{aligned}$$

must be true. However, both of these contradict the assumption $M \in M(\lambda)$. Verification of $\tilde{e}_i M(\lambda) \subset M(\lambda) \cup \{0\}$ can be done similarly.

To show the connectedness of $M(\lambda)$, it suffices to show that the only maximal element in $M(\lambda)$ is $M_\lambda$. Suppose $M \in M(\lambda)$ is such that $\tilde{e}_i(M) = 0$, for all $i \in I$, and suppose $M \neq M_\lambda$. The latter assumption is that there is at least one positive $u_{i,m}$. We first locate the largest $m$ for which there is a positive $u_{i,m}$ and then choose the largest $i$ for which $u_{i,m}$ is positive, with the $m$ already fixed. For such $i$ and $m$, we know

$$y_i(m+1) = u_{i+1,m} - u_{i,m} - u_{i,m+1} + u_{i-1,m+1} = -u_{i,m} < 0$$

and that $y_i(k) = 0$ for all $k > m + 1$. This implies $-\sum_{k>m} y_i(k) > 0$, so that

$$\varepsilon_i(M) = \max\Big\{-\sum_{k>j} y_i(k) \;\Big|\; j \in \mathbf{Z}\Big\} > 0,$$

and this contradicts the assumption of $M$ being a maximal element. $\square$

**Example 5.4.** For the $A_3$-type case, the crystal $M(\lambda)$ consists of all monomials of the form

$$\begin{aligned} Y_1(1)^{\lambda(h_1)} Y_2(1)^{\lambda(h_2)} Y_3(1)^{\lambda(h_3)} \cdot {} & U_1(1)^{-u_{1,1}} U_2(1)^{-u_{2,1}} U_3(1)^{-u_{3,1}} \\ & U_1(2)^{-u_{1,2}} U_2(2)^{-u_{2,2}} \\ & U_1(3)^{-u_{1,3}} \end{aligned} \tag{5.5}$$

satisfying the following conditions:

$$\begin{aligned} & 0 \leqslant u_{1,1} \leqslant \lambda(h_1), \qquad 0 \leqslant u_{2,1} \leqslant u_{1,1} + \lambda(h_2), \qquad 0 \leqslant u_{3,1} \leqslant u_{2,1} + \lambda(h_3), \\ & 0 \leqslant u_{1,2} \leqslant \min\{\lambda(h_2), u_{2,1}\}, \qquad 0 \leqslant u_{2,2} \leqslant \min\{u_{1,2} + \lambda(h_3), u_{3,1}\}, \\ & 0 \leqslant u_{1,3} \leqslant \min\{\lambda(h_3), u_{2,2}\}. \end{aligned}$$

The highest weight element of $M(\lambda)$ is $M_\lambda = Y_1(1)^{\lambda(h_1)} Y_2(1)^{\lambda(h_2)} Y_3(1)^{\lambda(h_3)}$, and the element given by (5.5) can be reached from $M_\lambda$ through applications of $(u_{1,1} + u_{1,2} + u_{1,3})$-many $\tilde{f}_1$, $(u_{2,1} + u_{2,2})$-many $\tilde{f}_2$, and $u_{3,1}$-many $\tilde{f}_3$, in some order.

### 5.2. Connected component of $\prod_{i \in I} Y_i(-i)^{\lambda(h_i)}$

In this section, we present the connected component of the crystal $\mathcal{M}$, containing the highest weight element $N_\lambda = \prod_{i \in I} Y_i(-i)^{\lambda(h_i)}$ of weight $\lambda$.

Let $N(\lambda)$ be the set of all monomials of the form

$$\prod_{i \in I} \left( Y_i(-i)^{\lambda(h_i)} \cdot \prod_{1 \leqslant m \leqslant i} U_i(-m)^{-u_{i,-m}} \right), \tag{5.6}$$

with the exponents satisfying

$$\begin{aligned} & 0 \leqslant u_{i,-m} \leqslant \min\{u_{i+1,-(m+1)} + \lambda(h_m), u_{i-1,-m}\}, \quad \text{for } 1 \leqslant m \leqslant i-1, \\ & 0 \leqslant u_{i,-i} \leqslant u_{i+1,-(i+1)} + \lambda(h_i), \end{aligned} \tag{5.7}$$

for each $i \in I$, where we take $u_{n+1,-m} = 0$, for every $m$.

The proof of the following claim is similar to that of Proposition 5.3.

**Proposition 5.8.** *The set $N(\lambda)$ is the connected component of crystal $\mathcal{M}$, containing the vector $N_\lambda = \prod_{i \in I} Y_i(-i)^{\lambda(h_i)}$ of weight $\lambda$, and $N(\lambda) \cong B(\lambda)$.*

**Example 5.9.** In the $A_3$-type case, the crystal $N(\lambda)$ consists of all monomials of the form

$$\begin{aligned} Y_1(-1)^{\lambda(h_1)} Y_2(-2)^{\lambda(h_2)} Y_3(-3)^{\lambda(h_3)} \cdot {} & U_1(-1)^{-u_{1,-1}} U_2(-2)^{-u_{2,-2}} U_3(-3)^{-u_{3,-3}} \\ & U_2(-1)^{-u_{2,-1}} U_3(-2)^{-u_{3,-2}} \\ & U_3(-1)^{-u_{3,-1}} \end{aligned} \tag{5.10}$$

satisfying the following conditions:

$$0 \leqslant u_{1,-1} \leqslant u_{2,-2} + \lambda(h_1), \qquad 0 \leqslant u_{2,-2} \leqslant u_{3,-3} + \lambda(h_2), \qquad 0 \leqslant u_{3,-3} \leqslant \lambda(h_3),$$
$$0 \leqslant u_{2,-1} \leqslant \min\{u_{3,-2} + \lambda(h_1), u_{1,-1}\}, \qquad 0 \leqslant u_{3,-2} \leqslant \min\{\lambda(h_2), u_{2,-2}\},$$
$$0 \leqslant u_{3,-1} \leqslant \min\{\lambda(h_1), u_{2,-1}\}.$$

The weight of the element given by (5.10) is

$$\lambda - (u_{1,-1})\alpha_1 - (u_{2,-1} + u_{2,-2})\alpha_2 - (u_{3,-1} + u_{3,-2} + u_{3,-3})\alpha_3.$$

## 6. Crystal $B(\lambda)$ as a subset of $B(\infty)$

Two descriptions of the crystal $B(\infty)$ were discussed in Section 4. These were the new reverse tableau description $R(\infty)$ and the existing tableau description $T(\infty)$ from [3]. In this section, we describe the crystal $B(\lambda)$ for the special linear Lie algebra types using elements from these two tableau descriptions of $B(\infty)$.

We will base our realization on the fact [16] that the connected component in the crystal $B(\infty) \otimes \mathsf{T}_\lambda$ containing the element $b_\infty \otimes \mathsf{t}_\lambda$ is isomorphic to $B(\lambda)$. Here, $\lambda \in P^+$, and the crystal $\mathsf{T}_\lambda$ is the single-element set $\{\mathsf{t}_\lambda\}$ with the following crystal structure:

$$\mathrm{wt}(\mathsf{t}_\lambda) = \lambda, \qquad \varepsilon_i(\mathsf{t}_\lambda) = -\lambda(h_i), \qquad \varphi_i(\mathsf{t}_\lambda) = 0, \qquad \tilde{e}_i(\mathsf{t}_\lambda) = 0, \qquad \tilde{f}_i(\mathsf{t}_\lambda) = 0.$$

Our goal will be achieved by finding the connected components containing $R_\infty \otimes \mathsf{t}_\lambda$ and $T_\infty \otimes \mathsf{t}_\lambda$ in $R(\infty) \otimes \mathsf{T}_\lambda$ and $T(\infty) \otimes \mathsf{T}_\lambda$, respectively, where $R_\infty$ and $T_\infty$ denote the highest weight elements of $R(\infty)$ and $T(\infty)$. After presenting the two sets that are isomorphic to $B(\lambda)$, we will discuss how each of them are in natural correspondence with one of the two monomial descriptions found in the previous section.

### 6.1. *Connected component of $R_\infty \otimes \mathsf{t}_\lambda$*

Let us first work to find the connected component in $R(\infty) \otimes \mathsf{T}_\lambda$ containing $R_\infty \otimes \mathsf{t}_\lambda$. The reader is asked to recall the notion of $(i,m)$-collection and Remark 4.2 from Section 4 before continuing.

Let us write $r_{i,m}$ to denote the number of $(i+1)$-boxes that appear in the $(i,m)$-collection of a given marginally large reverse tableau. We will not make the dependence of $r_{i,m}$ on the reverse tableau explicit, as our use of this notation will always be in such a way that the reverse tableau under consideration is unambiguous. For each $\lambda \in P^+$, let $R(\infty)^\lambda$ be the set of all marginally large reverse tableaux, such that the $r_{i,m}$'s satisfy the condition

$$0 \leqslant r_{i,m} \leqslant r_{i-1,m} + \lambda(h_{i-1+m}), \tag{6.1}$$

for every $i \in I$ and $1 \leqslant m \leqslant n+1-i$, where we take $r_{0,m} = 0$, for all $m$. We also define the set $R(\infty)_\lambda = \{R \otimes \mathsf{t}_\lambda \mid R \in R(\infty)^\lambda\}$.

**Example 6.2.** In the $A_3$-type case, the set $R(\infty)^\lambda$ consists of all marginally large reverse tableaux, for which the $r_{i,m}$'s satisfy the following conditions:

$$\begin{array}{lll} & & 0 \leqslant r_{13} \leqslant \lambda(h_3), \\ & 0 \leqslant r_{12} \leqslant \lambda(h_2), & 0 \leqslant r_{22} \leqslant r_{12} + \lambda(h_3), \\ 0 \leqslant r_{11} \leqslant \lambda(h_1), & 0 \leqslant r_{21} \leqslant r_{11} + \lambda(h_2), & 0 \leqslant r_{31} \leqslant r_{21} + \lambda(h_3). \end{array}$$

An element from the set $R(\infty)^\lambda$ takes the following general form.

$$\begin{array}{|c|c|c|c|c|c|}
\cline{5-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 \cdots 1 \; 2 \cdots 2 \\
\cline{3-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 \cdots 1 \; 2 \cdots 2 & 2 & 2 \cdots 2 \; 3 \cdots 3 \\
\hline
1 & 2 \cdots 2 & 2 & 3 \cdots 3 & 3 & 4 \cdots 4 \\
\hline
\end{array}$$

We have shaded the $r_{i,m}$-many $(i+1)$-boxes appearing within each $(i,m)$-collection, for each $1 \leqslant i \leqslant 3$ and $1 \leqslant m \leqslant 3+1-i$ pair. For example, in the second row, the $r_{1,2}$-many 2-boxes contained in the $(1,2)$-collection and the $r_{2,2}$-many 3-boxes appearing in the $(2,2)$-collection have been shaded.

The highest weight element

$$R_\infty = \begin{array}{|c|c|c|}
\cline{3-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\
\cline{2-3}
\multicolumn{1}{c|}{} & 1 & 2 \\
\hline
1 & 2 & 3 \\
\hline
\end{array}$$

of $R(\infty)$ has $r_{i,m} = 0$, for all $i$ and $m$, and belongs to the set $R(\infty)^\lambda$, for any $\lambda \in P^+$.

The following claim is one of our main results.

**Theorem 6.3.** *The set $R(\infty)_\lambda$ is the connected component containing the element $R_\infty \otimes \mathsf{t}_\lambda$ in the crystal $R(\infty) \otimes \mathsf{T}_\lambda$.*

**Proof.** It suffices to prove that the actions of the Kashiwara operators on $R(\infty)_\lambda$ satisfy the properties

$$\tilde{f}_i R(\infty)_\lambda \subset R(\infty)_\lambda \cup \{0\}, \qquad \tilde{e}_i R(\infty)_\lambda \subset R(\infty)_\lambda \cup \{0\},$$

for all $i \in I$, and that every element of $R(\infty)_\lambda$ is connected to the maximal element $R_\infty \otimes \mathsf{t}_\lambda$, by Kashiwara operator actions.

Fix any $i \in I$ and suppose that $R \otimes \mathsf{t}_\lambda \in R(\infty)_\lambda$ is such that $\tilde{f}_i(R \otimes \mathsf{t}_\lambda)$ is nonzero and not contained in $R(\infty)_\lambda$. The assumption and tensor product rule imply that

$$\varphi_i(R) > \varepsilon_i(\mathsf{t}_\lambda) \quad \text{and} \quad \tilde{f}_i(R \otimes \mathsf{t}_\lambda) = \tilde{f}_i(R) \otimes \mathsf{t}_\lambda, \tag{6.4}$$

and that $\tilde{f}_i(R)$ is not contained in $R(\infty)^\lambda$. In other words, $\tilde{f}_i(R)$ is a marginally large reverse tableau which does not satisfy the condition (6.1).

Now, suppose that $\tilde{f}_i$ has acted on the first row of $R$. In this case, the $\tilde{f}_i$ action must materialize as action on an $i$-box that is contained in the unique basic $i$-column of $R$, and a single basic $i$-column must have been inserted into $R$ during the $\tilde{f}_i$ action. Since $\tilde{f}_i(R)$ does not satisfy the condition (6.1), we have

$$r_{i,1} + 1 > r_{i-1,1} + \lambda(h_i),$$

and this contradicts the fact

$$\varphi_i(R) - \varepsilon_i(\mathsf{t}_\lambda) = -r_{i,1} + r_{i-1,1} + \lambda(h_i) > 0,$$

which follows from (6.4).

Next, suppose that $\tilde{f}_i$ has acted on a row of $R$ other than the first. Then $\tilde{f}_i$ must have acted on an $i$-box located in an $(i,m)$-collection of $R$, where $2 \leqslant m \leqslant n+1-i$, so that

$$(r_{i+1,m-1} - r_{i,m}) - (r_{i,m-1} - r_{i-1,m}) > 0. \tag{6.5}$$

On the other hand, since $\tilde{f}_i(R)$ does not satisfy the condition (6.1), we have

$$r_{i,m} + 1 > r_{i-1,m} + \lambda(h_{i-1+m}).$$

Thus, we obtain

$$r_{i,m} = r_{i-1,m} + \lambda(h_{i-1+m}) \tag{6.6}$$

from the condition (6.1), for $R \in R(\infty)^\lambda$. By substituting (6.6) into the inequality (6.5), we obtain

$$(r_{i+1,m-1} - r_{i,m}) - (r_{i,m-1} - r_{i-1,m}) = r_{i+1,m-1} - r_{i,m-1} - \lambda(h_{i-1+m}) > 0.$$

This contradicts

$$r_{i+1,m-1} - r_{i,m-1} - \lambda(h_{i+m-1}) \leqslant 0,$$

which is the condition (6.1) for $R \in R(\infty)^\lambda$. The $\tilde{e}_i R(\infty)_\lambda \subset R(\infty)_\lambda \cup \{0\}$ claim may be verified similarly.

It only remains to show that $R_\infty \otimes \mathrm{t}_\lambda$ is the only maximal element of $R(\infty)_\lambda$. Let us assume a fixed $i \in I$ and take any $R \otimes \mathrm{t}_\lambda \in R(\infty)_\lambda$. For $R \in R(\infty)^\lambda$, which is assumed to have its associated $r_{i,m}$ satisfy condition (6.1), a careful computation through the definition $\varphi_i(R) = \varepsilon_i(R) + \mathrm{wt}(R)(h_i)$ reveals that

$$\varphi_i(R) = \max\left\{ \sum_{1 \leqslant k \leqslant j} r_i(k) \,\middle|\, j \in \mathbf{Z}_{\leqslant n+1-i} \right\},$$

where

$$r_i(k) = \begin{cases} -r_{i,1} + r_{i-1,1} & \text{for } k = 1, \\ (r_{i+1,k-1} - r_{i,k}) - (r_{i,k-1} - r_{i-1,k}) & \text{for } 2 \leqslant k \leqslant n+1-i. \end{cases}$$

Hence, we can state

$$\varphi_i(R) \geqslant r_i(1) \geqslant -\lambda(h_i) = \varepsilon_i(\mathrm{t}_\lambda), \tag{6.7}$$

where the second inequality is implied by the condition (6.1). The tensor product rule now implies that $\tilde{e}_i(R \otimes \mathrm{t}_\lambda) = \tilde{e}_i(R) \otimes \mathrm{t}_\lambda$. Unless $R = R_\infty$, there will be an $i$ such that $\tilde{e}_i(R) \neq 0$, so that no element of $R(\infty)_\lambda$ other than $R_\infty \otimes \mathrm{t}_\lambda$ can be a maximal element. □

Given an element from our new description $R(\infty)_\lambda$ of $B(\lambda)$, one can directly recognize how many $\tilde{f}_i$ actions, for each $i \in I$, were required to arrive at the element from the highest weight element, and hence also obtain its weight.

Reviewing Examples 5.4 and 6.2, focusing on the many inequality conditions, one can notice a similarity between the two general elements from the $A_3$-type crystals $M(\lambda)$ and $R(\infty)_\lambda$. More generally, we wish to show that there is a natural correspondence between the crystals $M(\lambda)$ of Section 5.1 and the crystal $R(\infty)_\lambda$ of this section.

Recalling Remark 4.2, it is easy to see that a system of the $r_{i,m}$'s will uniquely identify an element of $R(\infty)$. That is, a reverse tableau that contains $r_{i,m}$-many $(i+1)$-boxes in its $(i,m)$-collection may not exist unless all meaningful inequalities $r_{i,m} \leqslant r_{i+1,m-1}$ are satisfied, but there will be at most one such marginally large reverse tableau. Given any $\lambda \in P^+$, we define the map $\phi : M(\lambda) \to R(\infty)_\lambda$ to send the general monomial $M$ of (5.1) satisfying (5.2) to $R \otimes \mathrm{t}_\lambda$, where $R$ is the unique marginally large reverse tableau such that $r_{i,m} = u_{i,m}$ for all $i \in I$ and $1 \leqslant m \leqslant n+1-i$. The condition (5.2) ensures that such a reverse tableau exists and that the condition (6.1) is satisfied.

**Proposition 6.8.** *The map $\phi : M(\lambda) \to R(\infty)_\lambda$ is a crystal isomorphism.*

**Proof.** It is easy to see that the map $\phi$ is a bijection. We will focus our effort in showing that the map $\phi$ commutes with the Kashiwara operators $\tilde{f}_i$. All other parts that need to be checked are either similar or easy.

Let us assume a fixed $i \in I$ throughout this proof. For a reverse tableau $R \in R(\infty)^\lambda$, which is assumed to have its associated $r_{i,m}$ satisfy condition (6.1), we know from the inequality (6.7) that

$$\varphi_i(R) - \varepsilon_i(\mathrm{t}_\lambda) = \max\left\{ \sum_{1 \leqslant k \leqslant j} r_i(k) \;\middle|\; j \in \mathbf{Z}_{\leqslant n+1-i} \right\} + \lambda(h_i) \geqslant r_i(1) + \lambda(h_i) \geqslant 0.$$

On the other hand, for the monomial $M \in M(\lambda)$ of (5.1), we can write

$$\varphi_i(M) = \max\left\{ \sum_{k \leqslant j} y_i(k) \;\middle|\; j \in \mathbf{Z} \right\} = \max\left\{ \sum_{k \leqslant j} y_i(k) \;\middle|\; j \in \mathbf{Z}_{\leqslant n+1-i} \right\},$$

where

$$y_i(k) = \begin{cases} 0 & \text{for } k \leqslant 0, \\ -u_{i,1} + u_{i-1,1} + \lambda(h_i) & \text{for } k = 1, \\ u_{i+1,k-1} - u_{i,k-1} - u_{i,k} + u_{i-1,k} & \text{for } 2 \leqslant k \leqslant n+1-i. \end{cases}$$

Now, let us take $\phi(M) = R \otimes \mathrm{t}_\lambda$, i.e., assume $u_{i,m} = r_{i,m}$, for all indices. Note that the discussion given so far implies

$$\varphi_i(M) = \varphi_i(R) - \varepsilon_i(\mathrm{t}_\lambda).$$

When $\tilde{f}_i M = 0$, we have $\varphi_i(M) = 0$ and $\varphi_i(R) - \varepsilon_i(\mathrm{t}_\lambda) = 0$, so that

$$\tilde{f}_i(R \otimes \mathrm{t}_\lambda) = R \otimes \tilde{f}_i(\mathrm{t}_\lambda) = 0,$$

by the tensor product rule.

When $\tilde{f}_i M \neq 0$, we can write $\tilde{f}_i M = U_i(m)^{-1} M$, for some $1 \leqslant m \leqslant n+1-i$. Since

$$\varphi_i(R) - \varepsilon_i(\mathrm{t}_\lambda) = \varphi_i(M) > 0,$$

the tensor product rule implies,

$$\tilde{f}_i(R \otimes \mathrm{t}_\lambda) = \tilde{f}_i(R) \otimes \mathrm{t}_\lambda.$$

In the $m = 1$ case, $\tilde{f}_i$ acts on the $i$-box in a basic $i$-column of $R$, and in the $2 \leqslant m \leqslant n+1-i$ case, $\tilde{f}_i$ acts on an $i$-box in an $(i,m)$-collection of $R$. In both cases, the only (effective) difference between $R$ and $\tilde{f}_i R$ is that their $(i,m)$-collections contain $r_{i,m}$-many and $(r_{i,m}+1)$-many $(i+1)$-boxes, respectively. Since we already know through Theorem 6.3 that $\tilde{f}_i R$ belongs to $R(\infty)^\lambda$, $\tilde{f}_i(R) \otimes \mathrm{t}_\lambda$ must be the image of $U_i(m)^{-1} M$ under $\phi$. We have shown that the map $\phi$ commutes with the Kashiwara operator $\tilde{f}_i$. $\square$

### 6.2. *Connected component of $T_\infty \otimes \mathrm{t}_\lambda$*

The connected component in $T(\infty) \otimes \mathrm{T}_\lambda$ containing $T_\infty \otimes \mathrm{t}_\lambda$ is discussed now. All results of this subsection may be proved as in the previous subsection, and we will not write down any of the proofs.

Given a marginally large tableau, for each pair of indices $1 \leqslant m \leqslant n$ and $m \leqslant i \leqslant n$, define $t_{i,-m}$ to be the combined number of all boxes labeled $i+1$ through $n+1$, appearing on the $m$-th row of the tableau, and also set $t_{n+1,-m} = 0$, for all $m$. Note that $t_{i,-m} - t_{i+1,-m}$ becomes the number of $(i+1)$-boxes appearing on the $m$-th row. For each $\lambda \in P^+$, let $T(\infty)^\lambda$ be the set of all marginally large tableaux, such that

$$0 \leqslant t_{i,-m} \leqslant t_{i+1,-(m+1)} + \lambda(h_m), \tag{6.9}$$

for all possible indices, and let $T(\infty)_\lambda = \{T \otimes \mathrm{t}_\lambda \mid T \in T(\infty)^\lambda\}$.

**Example 6.10.** In the $A_3$-type case, the set $T(\infty)^\lambda$ consists of all tableaux of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & \cdots\ 1 & 1 & \cdots\ 1 & 1 & 2\cdots 2 & 3\cdots 3 & 4\cdots 4 \\
\hline
2 & \cdots\ 2 & 2 & 3\cdots 3 & 4\cdots 4 \\
\cline{1-5}
3 & 4\cdots 4 \\
\cline{1-2}
\end{array}$$

The shaded part of the bottom row consists of $t_{3,-3}$-many 4-boxes. The shaded part of the second row consists of $t_{3,-2}$-many 4-boxes and $(t_{2,-2} - t_{3,-2})$-many 3-boxes. The $t_{i,-m}$'s satisfy the following conditions:

$$\begin{array}{lll}
0 \leqslant t_{1,-1} \leqslant \lambda(h_1) + t_{2,-2}, & 0 \leqslant t_{2,-1} \leqslant \lambda(h_1) + t_{3,-2}, & 0 \leqslant t_{3,-1} \leqslant \lambda(h_1), \\
0 \leqslant t_{2,-2} \leqslant \lambda(h_2) + t_{3,-3}, & 0 \leqslant t_{3,-2} \leqslant \lambda(h_2), & \\
0 \leqslant t_{3,-3} \leqslant \lambda(h_3). & &
\end{array}$$

The smallest tableau

$$T_\infty = \begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 & 2 \\
\cline{1-2}
3 \\
\cline{1-1}
\end{array}$$

from $T(\infty)$ belongs to the set $T(\infty)^\lambda$, for any $\lambda \in P^+$, with all its associated $t_{i,-m} = 0$.

**Theorem 6.11.** *The set $T(\infty)_\lambda$ is the connected component containing the element $T_\infty \otimes \mathrm{t}_\lambda$ in the crystal $T(\infty) \otimes \mathrm{T}_\lambda$.*

Analogous to the situation in the previous subsection, a system of the $t_{i,-m}$'s will uniquely identify an element of $T(\infty)$, as long as $t_{i,-m} \leqslant t_{i-1,-m}$ is satisfied by all applicable indices. Given any $\lambda \in P^+$, we define the map $\psi : N(\lambda) \to T(\infty)_\lambda$, to send the monomial of (5.6) satisfying (5.7) to $T \otimes \mathrm{t}_\lambda$, where $T$ is the unique marginally large tableau such that $t_{i,-m} = u_{i,-m}$, for all applicable indices. The condition (5.7) ensures that such a tableau exists and that the condition (6.9) is satisfied.

**Proposition 6.12.** *The map $\psi : N(\lambda) \to T(\infty)_\lambda$ is a crystal isomorphism.*

We remark that Kashiwara [8] and Nakashima [16] has shown the image of $B(\lambda) \hookrightarrow B(\infty) \otimes \mathrm{T}_\lambda$ to be

$$\left\{ b \otimes \mathrm{t}_\lambda \in B(\infty) \otimes \mathrm{T}_\lambda \;\middle|\; \varepsilon_i^*(b) \leqslant \langle h_i, \lambda \rangle \text{ for any } i \in I \right\},$$

for all symmetrizable Kac–Moody algebras and any $\lambda \in P^+$. (The crystal $\mathsf{T}_\lambda$ used in [8] differs from that commonly used by [16] and this paper, but this difference is not important.) Since our descriptions $R(\infty)_\lambda$ and $T(\infty)_\lambda$ for $B(\lambda)$ are explicit specifications, for $A_n$ type, of exactly the same image set, the conditions (6.1) and (6.9) should each be equivalent to appropriate translations of the condition $\varepsilon_i^*(b) \leqslant \langle h_i, \lambda\rangle$ to the $R(\infty)$ and $T(\infty)$ situations.

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