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Frattini argument for Hall subgroups [☆]



Danila Olegovitch Revin ^{a,b}, Evgeny Petrovitch Vdovin ^{a,b,*}

^a Sobolev Institute of Mathematics, pr. Acad. Koptyg, 4, Novosibirsk, 630090, Russia

^b Novosibirsk State University, ul. Pirogova, 2, Novosibirsk, 630090, Russia

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ABSTRACT

In the paper, it is proved that if a finite group G possesses a π -Hall subgroup for a set π of primes, then every normal subgroup A of G possesses a π -Hall subgroup H such that $G = AN_G(H)$.

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Introduction

One of the most frequently used statements in the finite group theory is the following evident corollary to the Sylow theorem on the conjugacy of Sylow p -subgroups [1], [2, (6.2)].

Proposition (*Frattini argument*). *Let A be a normal subgroup of a finite group G , and let S be a Sylow p -subgroup of A for a prime p . Then $G = AN_G(S)$.*

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* Corresponding author.

E-mail addresses: revin@math.nsc.ru (D.O. Revin), vdovin@math.nsc.ru (E.P. Vdovin).

Notice that if A is normal in G , then the action of G by conjugation on the set of subgroups of A induces the action of G on the set

$$\{H^A \mid H \leq A\}$$

of conjugacy classes of subgroups of A (here and thereafter, for a subgroup H of G , we set $H^G = \{H^g \mid g \in G\}$). Moreover, the stabilizer of the class H^A consisting of subgroups conjugate with H in A coincides with $AN_G(H)$. Thus, the equality $G = AN_G(H)$ is equivalent for H^A to be invariant under the action of G , or in other words, is equivalent to the equality $H^A = H^G$ (cf. [2, (6.3)] for example).

Analogs of the Frattini argument are valid not only for Sylow subgroups, but also for some other classes of subgroups. For example, it is proved in [3] that every normal subgroup A of a finite group G possesses a maximal solvable subgroup S such that $G = AN_G(S)$, and, as a consequence, in every finite group, there is a subgroup that is a solvable injector and a solvable projector simultaneously.

The notion of a π -Hall subgroup is a natural generalization of the notion of Sylow subgroup. Let π be a set of primes. A subgroup H of a finite group G is called a π -Hall subgroup, if all prime divisors of the order of H lie in π , while the index of H is not divisible by the elements of π . We denote by $\text{Hall}_\pi(G)$ the set of π -Hall subgroups of a finite group G .

The Hall theorem states that, in a solvable group, π -Hall subgroups exist and are conjugate for every set π of primes. As a corollary, in every finite solvable group, for every set π of primes, a complete analogue of the Frattini argument for π -Hall subgroups holds. However, given a set π of primes, in a nonsolvable finite group π -Hall subgroups may fail to exist, and, even if they exist, they can be nonconjugate.

According to [4], we say that G satisfies E_π (or, briefly $G \in E_\pi$) if $\text{Hall}_\pi(G) \neq \emptyset$. If $G \in E_\pi$ and all π -Hall subgroups are conjugate then we say that G satisfies C_π ($G \in C_\pi$). A group satisfying E_π or C_π is called also an E_π - or a C_π -group, respectively.

It is easy to show that if A is a normal subgroup of a finite group G and $A \in C_\pi$ for some π then $G = AN_G(H)$ for every π -Hall subgroup H of A . This notice is a key idea in the proof of the famous Chunikhin theorem [5, (3.12)], claiming that an extension of a C_π -group by a C_π -group is a C_π -group as well. In general, distinct analogues of the Frattini argument for Hall subgroup are of extraordinary importance in the study of Hall subgroups and generalizations of the Sylow theorem (see [4,6–26]).

The following statement is the main result of the paper.

Theorem 1 (Frattini argument for Hall subgroups). *Let $G \in E_\pi$ for some set π of primes and $A \trianglelefteq G$. Then there exists a π -Hall subgroup H of A such that $G = AN_G(H)$. Moreover $N_G(H) \in E_\pi$ and every π -Hall subgroup of $N_G(H)$ is a π -Hall subgroup of G .*

Remark 1. It is well known that, if $A \trianglelefteq G$ and H is a π -Hall subgroup of G , then $H \cap A$ is a π -Hall subgroup of A (see Lemma 1). Thus, the condition $G \in E_\pi$ in Theorem 1 implies

$A \in E_\pi$. However we cannot replace the condition $G \in E_\pi$ with a weaker condition $A \in E_\pi$. Indeed, let $\pi = \{2, 3\}$ and $A = \text{GL}_3(2) = \text{SL}_3(2)$. Then A possesses exactly two classes of conjugate π -Hall subgroups with the representatives

$$H_1 = \begin{pmatrix} \boxed{\text{GL}_2(2)} & * \\ 0 & \boxed{1} \end{pmatrix}, \quad H_2 = \begin{pmatrix} \boxed{1} & * \\ 0 & \boxed{\text{GL}_2(2)} \end{pmatrix}.$$

The class H_1^A consists of the line stabilizers in the natural module for G , while H_2^A consists of the plane stabilizers. The map $\iota : x \in A \mapsto (x^t)^{-1}$ is an automorphism of order 2 of A (here and below x^t is the transpose of x). This automorphism interchanges the classes H_1^A and H_2^A . Consider the natural split extension $G = A : \langle \iota \rangle$. Then $N_G(H) \leq A$ for every π -Hall subgroup H of A and, in particular, $AN_G(H) = A < G$. Indeed, since $|G : A| = 2$, the assertion $N_G(H) \not\leq A$ would imply $G = AN_G(H)$ and $H^G = H^A$, in particular $H^\iota \in H^A$. But in this case the automorphism ι would leave the class H^A invariant, a contradiction. Now $G \notin E_\pi$, since otherwise, in view of the identity $|G : A| = 2$, the equality $G = AK$ would hold for every π -Hall subgroup K of G and, in particular, $K \not\leq A$. Since $K \cap A$ is a π -Hall subgroup of A and $K \leq N_G(K \cap A)$ it follows that $G = AN_G(K \cap A)$, a contradiction.

Remark 2. If one compares the conclusion of [Theorem 1](#) with the original Frattini argument, it will be natural to ask, whether the identity $G = AN_G(H)$ holds for every π -Hall subgroup H of A , or at least for every subgroup of the form $H = K \cap A$, where K is a π -Hall subgroup of G , if the condition of [Theorem 1](#) is satisfied? This question has a negative answer. Indeed, let $\pi = \{2, 3\}$ and $S = \text{GL}_3(2)$. Suppose that subgroups H_1 and H_2 are defined in the same way as in [Remark 1](#). Consider the direct product

$$A = \underbrace{S \times \cdots \times S}_{5 \text{ times}}.$$

Now A admits an automorphism $\tau : (x_1, x_2, \dots, x_5) \mapsto (x_5, x_1, \dots, x_4)$ and we denote by G the natural semidirect product of A and $\langle \tau \rangle$. Since $|G : A| = 5$ is a prime, $5 \notin \pi$, and $A \trianglelefteq G$, the sets $\text{Hall}_\pi(G)$ and $\text{Hall}_\pi(A)$ coincide. Consider π -Hall subgroups

$$K_1 = \underbrace{H_1 \times \cdots \times H_1}_{4 \text{ times}} \times H_2,$$

$$K_2 = H_2 \times \underbrace{H_1 \times \cdots \times H_1}_{4 \text{ times}}$$

of A (hence also of G). Clearly, $K_1 = K_2^\tau$, but K_1 and K_2 are not conjugate in A . Thus, the conjugacy classes of K_1 and K_2 in A are distinct, while they are fused in G . Whence $G \neq AN_G(K_i)$, $i = 1, 2$.

As a corollary to [Theorem 1](#) we obtain the main result of [\[19\]](#), thus we give a simpler proof to the following statement.

Corollary 1. *If $G \in C_\pi$, $A \trianglelefteq G$ and H is a π -Hall subgroup of G , then $HA \in C_\pi$.*

Corollary 2. *Let A be a normal subgroup of G . Then $G \in E_\pi$ if and only if $A \in E_\pi$, $G/A \in E_\pi$, and there exists $H \in \text{Hall}_\pi(A)$ such that $H^A = H^G$.*

Corollary 3. *Let $G \in E_\pi$, $A \leq \text{Aut}(G)$ and $(|G|, |A|) = 1$. Then there exists an A -invariant π -Hall subgroup H of G .*

1. Preliminary results

We always denote by π a set of primes and by π' the complement to π in the set of all primes. An integer n is called a π -number if every prime divisor of n belongs to π . A group G is called a π -group if $|G|$ is a π -number.

Lemma 1. *Let A be a normal subgroup of G . If H is a π -Hall subgroup of G , then $H \cap A$ is a π -Hall subgroup of A , and HA/A is a π -Hall subgroup of G/A .*

Proof. See [5, Chapter IV, (5.11)]. \square

Recall that a finite group G is said to be π -separable, if there is a normal series of G with all factors being either π - or π' -groups.

Lemma 2. *Every π -separable group satisfies C_π .*

Proof. See [5, Chapter V, Theorem 3.7]. \square

Lemma 3. *Let A be a normal subgroup of G such that G/A is a π -group, let U be a π -Hall subgroup of A . Then a π -Hall subgroup H of G with $H \cap A = U$ exists if and only if $U^G = U^A$.*

Proof. See [14, Lemma 2.1(e)]. \square

Lemma 4. *Let A be a normal subgroup of an E_π -group G . Then for every $K/A \in \text{Hall}_\pi(G/A)$ there exists $H \in \text{Hall}_\pi(G)$ such that $K = HA$.*

Proof. See [15, Corollary 9]. \square

If S is a subnormal subgroup of G , then by $\text{Hall}_\pi^G(S)$ we denote the set of subgroups of type $H \cap S$, where $H \in \text{Hall}_\pi(G)$. Clearly $\text{Hall}_\pi^G(S)$ is a union of several classes of conjugate π -Hall subgroups of S , and let $k_\pi^G(S)$ be the number of these classes.

Recall that a subgroup generated by all minimal normal subgroups is called the *socle*. A group is called *almost simple*, if its socle is a finite simple nonabelian group.

Lemma 5. *Let π be a set of primes and G be an almost simple E_π -group with (nonabelian simple) socle S . Then the following hold.*

- (1) *If $2 \notin \pi$, then $k_\pi^G(S) = 1$.*
- (2) *If $3 \notin \pi$, then $k_\pi^G(S) \in \{1, 2\}$.*
- (3) *If $2, 3 \in \pi$, then $k_\pi^G(S) \in \{1, 2, 3, 4, 9\}$.*

In particular, $k_\pi^G(S)$ is a π -number.

Proof. See [14, Theorem 1.1]. \square

Lemma 6. *Let π be a set of primes, G be an almost simple E_π -group with socle S and $T \leq G$. Consider the action of T on*

$$\Omega = \{(H \cap S)^S \mid H \in \text{Hall}_\pi(G)\}$$

by conjugation. Then the following hold:

- (1) *if $k_\pi^G(S) \neq 9$, then the length of every orbit is a π -number;*
- (2) *if $k_\pi^G(S) = 9$, then there exists an orbit such that its length is a π -number.*

Proof. By definition, $k_\pi^G(S) = |\Omega|$. Consider an orbit Δ of T on Ω . Since $|\Delta| \leq |\Omega|$, Lemma 5 implies that either $|\Delta|$ is a π -number, or $2, 3 \in \pi$, $k_\pi^G(S) = 9$, and $|\Delta| \in \{5, 7\}$. In the last case, the action of T on Ω is intransitive, every orbit under this action with the exception of Δ has the length at most $|\Omega| - |\Delta| \leq 4$, and so is a π -number. \square

Lemma 7. *Let G be an almost simple E_π -group, S be the socle of, and $H \in \text{Hall}_\pi(G)$. Then, for some $H \in \text{Hall}_\pi(G)$, the equality $(H \cap S)^G = (H \cap S)^S$ holds.*

Proof. Consider

$$\Omega = \{(H \cap S)^S \mid H \in \text{Hall}_\pi(G)\}.$$

Let T/S be a π' -Hall subgroup of a (solvable) group G/S . Consider the action of T on Ω induced by the action of T on the set of subgroups of S via conjugation. By Lemma 6 it follows that T possesses an orbit Δ of length being a π -number. On the other hand, since S is included in the kernel of the action of T on Ω and since T/S is a π' -group we obtain that $|\Delta|$ is a π' -number. Hence $|\Delta| = 1$.

Thus, $(H \cap S)^T = (H \cap S)^S$ for some $H \in \text{Hall}_\pi(G)$. Since $G = HT$ we also have

$$(H \cap S)^G = (H \cap S)^{HT} = (H \cap S)^T = (H \cap S)^S,$$

and the lemma follows. \square

Let A, B, H be subgroups of G such that $B \trianglelefteq A$. By $N_H(A/B)$ we denote the intersection $N_H(A) \cap N_H(B)$. Then each element $x \in N_H(A/B)$ induces an automorphism on A/B acting by $Ba \mapsto Bx^{-1}ax$. Thus there exists a homomorphism $N_H(A/B) \rightarrow \text{Aut}(A/B)$. Denote by $\text{Aut}_H(A/B)$ the image of the homomorphism and call it a *group of H -induced automorphisms on A/B* . The kernel of the homomorphism is denoted by $C_H(A/B)$. If $B = 1$, then $\text{Aut}_H(A/B)$ is denoted by $\text{Aut}_H(A)$.

Lemma 8. *Let S be a simple subnormal subgroup of an E_π -group G . Then $\text{Aut}_G(S) \in E_\pi$.*

Proof. There exists a composition series

$$1 = G_0 < G_1 < \dots < G_n = G$$

of G such that $S = G_1$. The claim of the lemma follows from [15, Theorem 4] and the equality $\text{Aut}_G(G_1/G_0) = \text{Aut}_G(S)$. \square

Lemma 9. *Let S be a simple subnormal E_π -subgroup of G and $U \in \text{Hall}_\pi(S)$. Choose a right transversal g_1, \dots, g_n for $N_G(S)$ in G and let*

$$V = \langle U^{g_i} \mid i = 1, \dots, n \rangle.$$

Then $V \in \text{Hall}_\pi(\langle S^G \rangle)$.

Proof. One may assume that S is nonabelian. Let $A = \langle S^G \rangle$. Since S is nonabelian simple, we derive that A is a direct product of S^{g_1}, \dots, S^{g_n} and, as a consequence, V is a direct product of U^{g_1}, \dots, U^{g_n} . In particular, V is a π -group and $|A : V| = |S : U|^n$ is a π' -number, i.e. $V \in \text{Hall}_\pi(A)$. \square

Lemma 10. *Let S be a simple subnormal subgroup of G and a subgroup U of S be chosen so that the equality $U^{\text{Aut}_G(S)} = U^S$ holds. Take a right transversal g_1, \dots, g_n for $N_G(S)$ in G . Let*

$$A = \langle S^{g_i} \mid i = 1, \dots, n \rangle \quad \text{and} \quad V = \langle U^{g_i} \mid i = 1, \dots, n \rangle.$$

Then $A = \langle S^G \rangle \trianglelefteq G$ and $V^G = V^A$. Moreover, if $U \in \text{Hall}_\pi(S)$, then $V \in \text{Hall}_\pi(A)$.

Proof. In view of the choice of elements g_1, \dots, g_n , for every $g \in G$ we have $S^g \in \{S^{g_i} \mid i = 1, \dots, n\}$ and so $A = \langle S^G \rangle \trianglelefteq G$. Notice also that since S is simple and subnormal, we have $[S^{g_i}, S^{g_j}] = 1$ for $i \neq j$.

Let $g \in G$. There exist a permutation $\sigma \in \text{Sym}_n$ and $x_1, \dots, x_n \in N_G(S)$ such that $g_i g = x_i g_{i\sigma}$. Now consider $\gamma_i \in \text{Aut}_G(S)$, where the map $\gamma_i : S \rightarrow S$ is given by $s \mapsto s^{x_i}$. By condition, $U^{x_i} = U^{\gamma_i} = U^{s_i}$ for some $s_i \in S$. Set $a_i = s_{i\sigma^{-1}}^{g_i}$ and $a = a_1 \cdots a_n$. Clearly $a \in A$. We remain to show that $V^g = V^a$, and therefore the equality $V^G = V^A$ holds.

By the definition, it follows that $a_i \in S^{g_i}$ and $U^{g_i a} = U^{g_i a_i}$. We have

$$\begin{aligned} V^g &= \langle U^{g_i g} \mid i = 1, \dots, n \rangle = \langle U^{x_i g_{i\sigma}} \mid i = 1, \dots, n \rangle \\ &= \langle U^{s_i g_{i\sigma}} \mid i = 1, \dots, n \rangle = \langle U^{s_{i\sigma^{-1} g_i}} \mid i = 1, \dots, n \rangle \\ &= \langle U^{g_i s_{i\sigma^{-1}}^{g_i}} \mid i = 1, \dots, n \rangle = \langle U^{g_i a_i} \mid i = 1, \dots, n \rangle \\ &= \langle U^{g_i a} \mid i = 1, \dots, n \rangle = V^a, \end{aligned}$$

and the lemma follows. \square

2. Proof of the main results

Proof of Theorem 1. Let $G \in E_\pi$ and $A \trianglelefteq G$. We show by induction by the order of G , that A possesses a π -Hall subgroup H such that $G = AN_G(H)$.

If G is simple, we have nothing to prove.

Firstly, consider the case, where A is a minimal normal subgroup of G . Let S be a minimal subnormal subgroup of A . By Lemma 8 we have $\text{Aut}_G(S) \in E_\pi$. Lemma 7 implies that S possesses a π -Hall subgroup U such that $U^{\text{Aut}_G(S)} = U^S$. Choose a right transversal g_1, \dots, g_n of G by $N_G(S)$, and let

$$H = \langle U_i^g \mid i = 1, \dots, n \rangle.$$

Since A is a minimal normal subgroup of G we obtain $A = \langle S^G \rangle$. By Lemma 9, H is a π -Hall subgroup of A . By Lemma 10 it follows that $H^G = H^A$ or equivalently $G = AN_G(H)$.

Thus we may assume that there exists a minimal normal subgroup M of G such that $M < A$. As we proved above, the identity $G = MN_G(V)$ holds for a π -Hall subgroup V of M . Set $K = N_G(V)$. Notice that by Lemma 3 the identity $V = M \cap T$ holds for some π -Hall subgroup T of G . Since $V = M \cap T \trianglelefteq T$, it follows that $T \leq K$, whence $K \in E_\pi$.

Suppose, $K < G$. By induction, $K = (K \cap A)N_K(H)$ for some $H \in \text{Hall}_\pi(K \cap A)$. Notice that $|A : (K \cap A)|$ divides $|A : (T \cap A)|$, since $T \leq K$. Now $(T \cap A) \in \text{Hall}_\pi(A)$, hence $|A : (K \cap A)|$ is a π' -number and $\text{Hall}_\pi(K \cap A) \subseteq \text{Hall}_\pi(A)$. In particular, $H \in \text{Hall}_\pi(A)$. We have

$$G = MK = M(K \cap A)N_K(H) \leq AN_G(H).$$

Now assume that $K = G$, i.e. $V \trianglelefteq G$. By induction

$$G/M = (A/M)N_{G/M}(X/M) = AN_G(X)/M$$

for some $X/M \in \text{Hall}_\pi(A/M)$ and thus $G = AN_G(X)$.

Suppose that $V \neq 1$, i.e. $M = V$. Then $X \in \text{Hall}_\pi(A)$, and the conclusion of the theorem holds.

Consider the remaining case $V = 1$. In other words, consider the case, where M is a π' -group. By the Schur–Zassenhaus theorem [2, Theorem 18.1], X possesses a π -Hall subgroup H , moreover $H \in \text{Hall}_\pi(A)$ and $X = HM$. Let $g \in G$. In view of the identity $G = AN_G(X)$ there exists $a \in A$ such that $X^g = X^a$. Now H^g and H^a are π -Hall subgroups of X^g and, by the Schur–Zassenhaus theorem, H^g and H^a are conjugate in $X^g \leq A$. Hence, H and H^g are conjugate in A , the class H^A is G -invariant and $G = AN_G(H)$.

In order to complete the proof, it remains to show that if $G \in E_\pi$, $A \trianglelefteq G$ and $H \in \text{Hall}_\pi(A)$ is such that $G = AN_G(H)$, then $N_G(H) \in E_\pi$ and $\text{Hall}_\pi(N_G(H)) \subseteq \text{Hall}_\pi(G)$.

Notice that

$$N_G(H)/N_A(H) \cong AN_G(H)/A = G/A \in E_\pi$$

by Lemma 1. Lemma 3 implies that a complete preimage of a π -Hall subgroup of $N_G(H)/N_A(H)$ possesses a π -Hall subgroup Y such that $Y \cap N_A(H) = H$. Notice that $|A : (A \cap Y)|$ divides $|A : H|$ and so it is a π' -number. Since $|N_G(H) : YA|$ is a π' -number,

$$|N_G(H) : Y| = |N_G(H) : YA| |YA : Y| = |N_G(H) : YA| |A : (A \cap Y)|$$

is a π' -number as well. Therefore $Y \in \text{Hall}_\pi(N_G(H))$ and $N_G(H) \in E_\pi$. Now

$$|G : N_G(H)| = |AN_G(H) : N_G(H)| = |A : N_A(H)|$$

divides $|A : H|$ and so is a π' -number. Whence

$$\text{Hall}_\pi(N_G(H)) \subseteq \text{Hall}_\pi(G),$$

and the theorem follows. \square

Proof of Corollary 1. Let $G \in C_\pi$, $A \trianglelefteq G$, and $H \in \text{Hall}_\pi(G)$. We need to show that $HA \in C_\pi$.

Let

$$\Gamma = \{K \cap A \mid K \in \text{Hall}_\pi(G)\}.$$

In view of $G \in C_\pi$, G acts transitively on Γ by conjugations.

By Theorem 1, there exists $X \in \text{Hall}_\pi(A)$ such that $G = AN_G(X)$ (equivalently, $X^G = X^A$), $N_G(X) \in E_\pi$, and $\text{Hall}_\pi(N_X(K)) \subseteq \text{Hall}_\pi(G)$. Choose arbitrary $Y \in \text{Hall}_\pi(N_G(X))$. It is easy to see that $Y \cap A = X$ and so $X \in \Gamma$. By the transitivity of G on Γ we have $\Gamma = X^G = X^A$, and this identity implies the transitivity of A on Γ .

Now choose arbitrary $K \in \text{Hall}_\pi(HA) \subseteq \text{Hall}_\pi(G)$. Since $H \cap A \in \Gamma$ and $K \cap A \in \Gamma$, by the transitivity of A on Γ , there exists $a \in A \leq HA$ such that $K \cap A = H^a \cap A$. The group $N_{HA}(H^a \cap A)$ is π -separable, since all sections of the normal series

$$N_{HA}(H^a \cap A) \supseteq N_A(H^a \cap A) \supseteq H^a \cap A \supseteq 1$$

are either π - or π' -groups. Hence $N_{HA}(H^a \cap A) \in C_\pi$ by Lemma 2. Now $H^a, K \in \text{Hall}_\pi(N_{HA}(H^a \cap A))$, so there exists $x \in N_{HA}(H^a \cap A)$ such that $K = H^{ax}$. Thus we obtain that arbitrary $K \in \text{Hall}_\pi(HA)$ is conjugate with H in HA , whence $HA \in C_\pi$. \square

Proof of Corollary 2. Let $A \trianglelefteq G$.

Suppose, $A \in E_\pi$, $G/A \in E_\pi$, and $H^A = H^G$ for some $H \in \text{Hall}_\pi(A)$. Let M be the full preimage of a π -Hall subgroup of G/A . Then $M \in E_\pi$ in view of Lemma 3 and, moreover, $\text{Hall}_\pi(M) \subseteq \text{Hall}_\pi(G)$, since $|G : M| = |G/A : M/A|$ is a π -number. Thus, $G \in E_\pi$.

The converse statement is a straight consequence of Lemma 1 and Theorem 1. \square

Proof of Corollary 3. Let $G \in E_\pi$ and $A \leq \text{Aut}(G)$ with $(|G|, |A|) = 1$. Denote by G^* the natural semidirect product of G and A . Then $G^* \in E_\pi$ and, by Theorem 1, there exists $H \in \text{Hall}_\pi(G)$ such that $G^* = GN_{G^*}(H)$. Furthermore,

$$N_{G^*}(H)/N_G(H) \cong G^*/G \cong A.$$

Hence $(|N_{G^*}(H)/N_G(H)|, |N_G(H)|) = 1$. In view of the Schur–Zassenhaus theorem [2, Theorem 18.1], there is $B \leq N_{G^*}(H)$ isomorphic to A and $A = B^x$ for some $x \in G^*$. Now

$$A = B^x \leq N_{G^*}(H)^x = N_{G^*}(H^x)$$

and H^x is an A -invariant π -Hall subgroup of G . \square

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