Journal of Algebra 434 (2015) 46–64

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A decomposition rule for certain tensor product representations of the symmetric groups

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ARTICLE INFO

Article history:
Received 24 March 2009
Available online 14 April 2015
Communicated by Masaki Kashiwara

MSC:
05E10
20C30

Keywords:
Representations of symmetric groups
Tensor product representations
Kronecker coefficient

ABSTRACT

In this paper, we give a combinatorial rule to calculate the decomposition of the tensor product (Kronecker product) of two irreducible complex representations of the symmetric group $\mathfrak{S}_n$, when one of the representations corresponds to a hook $(n-m, 1^m)$.

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1. Introduction

Let G be a group such that each of its finite-dimensional complex representations is completely reducible. One of the basic problems of the representation theory of G is to give a decomposition rule

$$L(\lambda) \otimes L(\nu) \cong \bigoplus_{\mu \in P} m_{\lambda,\nu}^{\mu} L(\mu)$$

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http://dx.doi.org/10.1016/j.jalgebra.2015.03.016
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of the tensor product of two irreducible representations of G, where $\{L(\lambda) \mid \lambda \in P\}$ denotes a set of complete representatives of irreducible finite-dimensional representations of G. When G is the general linear group $GL(n, \mathbb{C})$, the multiplicity $m^{\mu}_{\lambda,\nu}$ is known as the famous Littlewood–Richardson coefficient $LR^{\mu}_{\lambda,\nu}$, which equals the number of certain combinatorial objects (see e.g. [4]). When G is an arbitrary complex simple Lie group (or rather, an arbitrary symmetrizable Kac–Moody algebra), inspired by Kashiwara's theory of crystals and works of Lakshmibai and Seshadri, Littelmann [9] gave combinatorial objects whose number equals $m^{\mu}_{\lambda,\nu}$.

Let G be the symmetric group $\mathfrak{S}_n$ of n letters. As usual, we use the set $\mathcal{P}_n$ of partitions of n as the set P of labels of irreducible representations of G. In despite of the long research history (see e.g. Murnaghan [10] for an early work), much less is known about $m^{\mu}_{\lambda,\nu}$ for $\mathfrak{S}_n$-representations, comparing with the Lie theoretic case. Lascoux [7], Garsia and Remmel [5], Remmel [11], Remmel and Whitehead [12] and Rosas [14] gave descriptions of $m^{\mu}_{\lambda,\nu}$, when λ and ν are either two-row partitions or hook partitions. Recently, Ballantine and Orellana [1] gave a combinatorial rule for $m^{\mu}_{\lambda,\nu}$ in the case where λ is a two-row partition $(n-p, p)$ and ν is not a partition inside the $2(p-1) \times 2(p-1)$ square.

In this paper, we give a combinatorial rule to calculate the number $m^{\mu}_{\lambda,\nu}$ for $\mathfrak{S}_n$-representations, when the partition ν is a hook $(n-m, 1^m)$. More precisely, we construct a set $\mathrm{PH}_m(\lambda, \mu)$ in a combinatorial manner, which satisfies

$$L(\lambda) \otimes L(n-m, 1^m) \cong \bigoplus_{\mu \in \mathcal{P}_n} |\mathrm{PH}_m(\lambda, \mu)| \cdot L(\mu)$$

for each $\lambda, \mu \in \mathcal{P}_n$ and $0 \leq m < n$.

Instead of dealing with the hook representation $L(n-m, 1^m)$ directly, we consider a slightly bigger representation $\Lambda_m(\mathbb{C}^n)$, the m-th exterior power of the defining representation of $\mathfrak{S}_n$. By considering a certain permutation representation $\mathbb{C}\Omega_m$, we show that the multiplicity of $L(\mu)$ in $L(\lambda) \otimes \Lambda_m(\mathbb{C}^n)$ is equal to $w_m := \sum_{\zeta \in \mathcal{P}_{n-m}} \sum_{\xi \in \mathcal{P}_m} LR^{\lambda}_{\zeta,\xi} \cdot LR^{\mu}_{\zeta,\xi^{\mathsf{t}}}$, where ξ^{t} denotes the transpose of ξ. Although we could not find the result in the literature, we suspect that it was known for experts, since the techniques to be used to prove are rather standard.

To give a decomposition of $L(\lambda) \otimes L(n-m, 1^m)$, we use a set $\mathrm{PW}_m(\lambda, \mu)$ such that $|\mathrm{PW}_m(\lambda, \mu)| = w_m$ and that each of its elements is a Zelevinsky's *picture* [16]. A picture is a bijective map between skew Young diagrams, which satisfies certain order-theoretic conditions, and it is identified with a tableau on a skew Young diagram satisfying some conditions. Using a variant of Zelevinsky's insertion algorithm for pictures [16], we construct the set $\mathrm{PH}_m(\lambda, \mu)$ as a subset of $\mathrm{PW}_m(\lambda, \mu)$.

2. Preliminaries

2.1. Partitions and diagrams

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a sequence of integers. We say that λ is a *partition* of n if $\lambda_1 \geq \ldots \geq \lambda_l > 0$ and $|\lambda| := \sum_i \lambda_i = n$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, we define

its *length* $l(\lambda)$ by $l(\lambda) = l$. Also, we define its *diagram* $D(\lambda)$ and its *i-th row* $D_i(\lambda)$ $(1 \leq i \leq l)$ by $D(\lambda) = \amalg_i D_i(\lambda)$ and $D_i(\lambda) = \{(i,1),(i,2),\dots,(i,\lambda_i)\}$, respectively. We denote by $\mathcal{P}_n$ the set of partitions of n. For a partition λ and $i \geq 1$, we define $\lambda_i \in \mathbb{Z}_{\geq 0}$ by $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)})$ and $\lambda_i = 0$ $(i > l(\lambda))$.

For two partitions $\lambda \in \mathcal{P}_n$ and $\zeta \in \mathcal{P}_m$, we write $\zeta \subseteq \lambda$ if $D(\zeta) \subseteq D(\lambda)$. When $\zeta \subseteq \lambda$, we denote the pair (λ, ζ) by λ/ζ and call it a *skew partition* of (n,m). For a skew partition λ/ζ, we define its *size*, its *length*, its *diagram* and its *i-th row* by $|\lambda/\zeta| := |\lambda| - |\zeta|$, $l(\lambda/\zeta) := l(\lambda)$, $D(\lambda/\zeta) := D(\lambda) \setminus D(\zeta)$ and $D_i(\lambda/\zeta) := D_i(\lambda) \setminus D_i(\zeta)$, respectively.

For a partition λ, we define its *conjugate* λ^{t} to be a unique partition satisfying $D(\lambda^{\mathsf{t}}) = D(\lambda)^{\mathsf{t}} := \{\mathbf{u}^{\mathsf{t}} \mid \mathbf{u} \in D(\lambda)\}$, where $(i,j)^{\mathsf{t}} := (j,i)$. For a skew partition λ/ζ, we set $(\lambda/\zeta)^{\mathsf{t}} := \lambda^{\mathsf{t}}/\zeta^{\mathsf{t}}$.

2.2. Representations

Let G be a finite group. Let M be a finite-dimensional $\mathbb{C}G$-module and let $L(\lambda)$ be a simple $\mathbb{C}G$-module labeled by λ. We denote by $[M]$, $[M : L(\lambda)]$ and M_λ the character of M, the multiplicity of $L(\lambda)$ in M and the homogeneous component of M corresponding to $L(\lambda)$, respectively.

Lemma 2.1. *Let G be a finite group and let H be its subgroup. For each $\mathbb{C}G$-module M and $\mathbb{C}H$-module N, we have the isomorphisms $M \otimes \mathrm{Ind}_H^G N \cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G M \otimes N)$; $x \otimes (\sigma \otimes_{\mathbb{C}H} y) \mapsto \sigma \otimes_{\mathbb{C}H} (\sigma^{-1}x \otimes y)$ $(x \in M,\ \sigma \in G,\ y \in N)$ of $\mathbb{C}G$-modules. Also, we have the isomorphism $\mathrm{Hom}_{\mathbb{C}H}(\mathrm{Res}_H^G M, N) \cong \mathrm{Hom}_{\mathbb{C}G}(M, \mathrm{Ind}_H^G N)$; $g \mapsto f$; $f(x) = \frac{1}{|H|}\sum_{\sigma \in G} \sigma \otimes_{\mathbb{C}H} g(\sigma^{-1}x)$ $(x \in M,\ g \in \mathrm{Hom}_{\mathbb{C}H}(\mathrm{Res}_H^G M, N))$ of vector spaces. Here Ind_H^G and Res_H^G denote the induction functor and the restriction functor, respectively.*

See e.g. [2, Corollary 10.20 and Proposition 10.21] for a proof.

Let $\mathfrak{S}_n$ be the symmetric group of n letters. For each $\lambda \in \mathcal{P}_n$, we denote the corresponding simple $\mathbb{C}\mathfrak{S}_n$-module by $L(\lambda)$ (see e.g. [4] or [6]). It is well-known that

$$L(\lambda)^* \cong L(\lambda), \quad \mathbb{C}_{sgn} \otimes L(\lambda) \cong L(\lambda^{\mathsf{t}}), \tag{2.1}$$

where $L(\lambda)^*$ denotes the dual module of $L(\lambda)$ and $\mathbb{C}_{sgn}$ denotes the module corresponding to the sign representation. For each $\sigma \in \mathfrak{S}_n$, we denote by $\sigma_\lambda \in GL(L(\lambda))$ the image of σ via the representation corresponding to $L(\lambda)$.

Let $\Omega = \Omega_{1,n}$ be the set $\{1,2,\dots,n\}$ equipped with the natural action of $\mathfrak{S}_n$. Then the *defining module* $\mathbb{C}\Omega \cong \mathbb{C}^n$ has the following decomposition:

$$\mathbb{C}\Omega \cong L(n) \oplus L(n-1,1).$$

Moreover, we have

$$\Lambda_m(L(n-1,1)) \cong L(n-m, 1^m)$$

for each $0 \leq m \leq n-1$ (see e.g. [6, p. 391]), where Λ_m stands for the m-th exterior power and $(n-m, 1^m)$ stands for the *m-th hook*

$$(n-m, 1^m) = (n-m, \overbrace{1, 1, \ldots, 1}^{m \text{ times}}).$$

Since $L(n)$ is isomorphic to the unit module (trivial module) $\mathbb{C}_{unit}$, we also have

$$\Lambda_m(\mathbb{C}\Omega) \cong \begin{cases} L(n) & (m=0) \\ L(n-m+1, 1^{m-1}) \oplus L(n-m, 1^m) & (0<m<n) \\ L(1^n) & (m=n). \end{cases} \tag{2.2}$$

Let $\#_m \colon \mathfrak{S}_m \to \mathfrak{S}_n$ be the group homomorphism which sends each transposition $(i, i+1)$ to $(n-m+i, n-m+i+1)$. As usual, we identify the subgroup $\langle \rho \cdot \#_m(\tau) \,|\, \rho \in \mathfrak{S}_{n-m}, \tau \in \mathfrak{S}_m \rangle$ of $\mathfrak{S}_n$ with $\mathfrak{S}_{n-m} \times \mathfrak{S}_m$. The *Littlewood–Richardson coefficient* $LR^{\lambda}_{\zeta,\xi}$ is defined by the following decomposition rule:

$$[\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-m} \times \mathfrak{S}_m} L(\lambda)] = \sum_{\zeta \in \mathcal{P}_{n-m}} \sum_{\xi \in \mathcal{P}_m} LR^{\lambda}_{\zeta,\xi} [L(\zeta) \boxtimes L(\xi)], \tag{2.3}$$

where $\boxtimes$ stands for the outer tensor product.

Let M and N be $\mathbb{C}\mathfrak{S}_n$-modules and let $g \colon M \to N$ be a $\mathbb{C}\mathfrak{S}_n$-module map. For each $\lambda, \mu \in \mathcal{P}_n$, we define a vector space $\Upsilon_{\lambda\mu}(M)$ and a linear map $\Upsilon_{\lambda\mu}(g) : \Upsilon_{\lambda\mu}(M) \to \Upsilon_{\lambda\mu}(N)$ by

$$\Upsilon_{\lambda\mu}(M) = \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(L(\mu), L(\lambda) \otimes M)$$

and $\Upsilon_{\lambda\mu}(g)(f) = (\mathrm{id}_{L(\lambda)} \otimes g) \circ f$ $(f \in \Upsilon_{\lambda\mu}(M))$. Then $\Upsilon_{\lambda\mu}$ gives an exact functor from the category of finite-dimensional $\mathbb{C}\mathfrak{S}_n$-modules to the category of finite-dimensional $\mathbb{C}$-vector spaces. We note that

$$[L(\lambda) \otimes M] = \sum_{\mu \in \mathcal{P}_n} \dim(\Upsilon_{\lambda\mu}(M)) \cdot [L(\mu)]. \tag{2.4}$$

3. The module $\mathbb{C}\Omega_m$

For each $0 < m \leq n$, we consider the subset $\Omega_m = \Omega_{m,n}$ of the m-fold Cartesian product Ω^m defined by

$$\Omega_m := \left\{ (i_1, \ldots, i_m) \in \Omega^m \;\middle|\; i_1, \ldots, i_m \text{ are pairwise distinct} \right\}.$$

We define an action $(\sigma, I) \mapsto \sigma_V I$ $(\sigma \in \mathfrak{S}_n, I \in \Omega_m)$ of $\mathfrak{S}_n$ on Ω_m by

$$\sigma_V(i_1, \ldots, i_m) = (\sigma(i_1), \ldots, \sigma(i_m)) \tag{3.1}$$

and call it the *vertical action*. Also, we define the *horizontal action* $(\tau, I) \mapsto \tau_H I$ ($\tau \in \mathfrak{S}_m$, $I \in \Omega_m$) by

$$\tau_H(i_1, \dots, i_m) = (i_{\tau^{-1}(1)}, \dots, i_{\tau^{-1}(m)}).$$

Since these actions commute with each other, the linear span $\mathbb{C}\Omega_m$ becomes an $\mathfrak{S}_n \times \mathfrak{S}_m$-module.

It is easy to see that the vertical action of $\mathfrak{S}_n$ on Ω_m is transitive and that the stabilizer of $I_m := (n-m+1, n-m+2, \dots, n)$ is identified with $\mathfrak{S}_{n-m}$. Hence $\mathbb{C}\Omega_m$ is isomorphic to the induced module $\mathrm{Ind}_{\mathfrak{S}_{n-m}}^{\mathfrak{S}_n} L(n-m)$.

For each $f \in \Upsilon_{\lambda\mu}(\mathbb{C}\Omega_m)$ and $I \in \Omega_m$, we define $f_I \colon L(\mu) \to L(\lambda)$ by

$$f(v) = \sum_I f_I(v) \otimes I.$$

The following proposition follows immediately from Lemma 2.1.

Proposition 3.1. *For each $\lambda, \mu \in \mathcal{P}_n$, there exists a linear isomorphism*

$$\heartsuit_m \colon \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{n-m}}(L(\mu), L(\lambda)) \cong \Upsilon_{\lambda\mu}(\mathbb{C}\Omega_m) \tag{3.2}$$

defined by

$$\heartsuit_m(h)(v) = \frac{1}{(n-m)!} \sum_{\sigma \in \mathfrak{S}_n} \sigma h(\sigma^{-1} v) \otimes \sigma_V I_m \qquad (h \in \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{n-m}}(L(\mu), L(\lambda)), v \in L(\mu)). \tag{3.3}$$

The inverse of $\heartsuit_m$ is given by

$$\heartsuit_m^{-1}(f) = f_{I_m} \quad (f \in \Upsilon_{\lambda\mu}(\mathbb{C}\Omega_m)).$$

Since $\Upsilon_{\lambda\mu}$ is a functor, $\Upsilon_{\lambda\mu}(\mathbb{C}\Omega_m)$ becomes an $\mathfrak{S}_m$-module via $\tau \mapsto \Upsilon_{\lambda\mu}(\tau_H)$. On the other hand, the left-hand side of (3.2) also becomes an $\mathfrak{S}_m$-module via $\tau_{\lambda\mu}(h) := \#_m(\tau)_\lambda \circ h \circ \#_m(\tau)_\mu^{-1}$ ($h \in \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{n-m}}(L(\mu), L(\lambda))$, $\tau \in \mathfrak{S}_m$). In fact, we have the following:

Lemma 3.2. *The map $\heartsuit_m$ commutes with the actions of $\mathfrak{S}_m$. In particular, we have the following linear isomorphism of homogeneous components:*

$$\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{n-m}}(L(\mu), L(\lambda))_{(1^m)} \cong \Upsilon_{\lambda\mu}(\mathbb{C}\Omega_m)_{(1^m)}. \tag{3.4}$$

Proof. To prove the assertion, it suffices to show

$$\heartsuit_m^{-1}(\Upsilon_{\lambda\mu}(\tau_H)(f)) = \tau_{\lambda,\mu}(\heartsuit_m^{-1}(f)) \tag{3.5}$$

for each transposition $\tau = (p, p+1) \in \mathfrak{S}_m$ and $f \in \Upsilon_{\lambda\mu}(\mathbb{C}\Omega_m)$. For each $I = (i_1, i_2, \ldots, i_m) \in \Omega_m$, we have

$$\tau_H^{-1} I = (i_1, \ldots, i_{p-1}, i_{p+1}, i_p, i_{p+2}, \ldots, i_m) = (i_p, i_{p+1})_V I. \tag{3.6}$$

On the other hand, comparing

$$\sigma f(v) = \sum_J \sigma_\lambda f_J(v) \otimes \sigma_V J = \sum_I \sigma_\lambda f_{\sigma_V^{-1} I}(v) \otimes I$$

with

$$\sigma f(v) = f(\sigma_\mu v) = \sum_I f_I(\sigma_\mu v) \otimes I,$$

for each $\sigma \in \mathfrak{S}_n$ and $v \in L(\mu)$, we get

$$f_{\sigma_V^{-1} I}(v) = \sigma_\lambda^{-1} f_I(\sigma_\mu v). \tag{3.7}$$

In particular,

$$f_{\tau_H^{-1} I}(v) = (i_p, i_{p+1})_\lambda f_I\left((i_p, i_{p+1})_\mu^{-1} v\right)$$

by (3.6). Hence, we have

$$\begin{aligned}(\Upsilon_{\lambda\mu}(\tau_H)(f))(v) &= \sum_I f_{\tau_H^{-1} I}(v) \otimes I \\ &= \sum_{I=(i_1,\ldots,i_m)} (i_p, i_{p+1})_\lambda f_I\left((i_p, i_{p+1})_\mu^{-1} v\right) \otimes I.\end{aligned}$$

Since $(i_p, i_{p+1}) = \#_m(p, p+1)$ for $I = I_m$, we get (3.5). □

Let $(\mathbb{C}\Omega_m)_{-\boxtimes(1^m)}$ be the homogeneous component of $\mathbb{C}\Omega_m$ with respect to the horizontal action of $\mathfrak{S}_m$, which corresponds to $L(1^m)$. It is easy to see that the correspondence $(i_1) \wedge \cdots \wedge (i_m) \mapsto \sum_\tau \operatorname{sgn}(\tau)\ (i_{\tau^{-1}(1)}, \ldots, i_{\tau^{-1}(m)})$ gives an isomorphism

$$\Lambda_m(\mathbb{C}\Omega) \cong (\mathbb{C}\Omega_m)_{-\boxtimes(1^m)} \tag{3.8}$$

of $\mathbb{C}\mathfrak{S}_n$-modules.

Lemma 3.3. *We have the following linear isomorphisms:*

$$\Upsilon_{\lambda\mu}(\mathbb{C}\Omega_m)_{(1^m)} \cong \Upsilon_{\lambda\mu}((\mathbb{C}\Omega_m)_{-\boxtimes(1^m)}), \tag{3.9}$$

$$(L(\mu)^* \otimes L(\lambda))_{(n-m)\boxtimes(1^m)} \cong \operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{n-m}}(L(\mu), L(\lambda))_{(1^m)}, \tag{3.10}$$

where the left-hand side of the second isomorphism denotes the homogeneous component of $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-m}\times\mathfrak{S}_m}(L(\mu)^*\otimes L(\lambda))$*, which corresponds to* $L(n-m)\boxtimes L(1^m)$.

Proof. Define an action of $\mathfrak{S}_n\times\mathfrak{S}_m$ on $H:=\mathrm{Hom}_{\mathbb{C}}(L(\mu),L(\lambda)\otimes\mathbb{C}\Omega_m)$ by $(\sigma,\tau)f=(\sigma_\lambda\otimes(\sigma_V\tau_H))\circ f\circ\sigma_\mu^{-1}$. Then the left-hand side of (3.9) is identified with the homogeneous component $H_{(n)\boxtimes(1^m)}$, which corresponds to $L(n)\boxtimes L(1^m)$. Let $\iota:(\mathbb{C}\Omega_m)_{-\boxtimes(1^m)}\to\mathbb{C}\Omega_m$ be the natural injection. To prove (3.9), it is enough to show that the subspace $\mathrm{Im}(\Upsilon_{\lambda\mu}(\iota))$ of H coincides with $H_{(n)\boxtimes(1^m)}$. Let f be an element of $H_{(n)\boxtimes(1^m)}$. Since $f(v)\in(L(\lambda)\otimes\mathbb{C}\Omega_m)_{-\boxtimes(1^m)}=L(\lambda)\otimes(\mathbb{C}\Omega_m)_{-\boxtimes(1^m)}$ for each $v\in L(\mu)$, f gives a linear map $f':L(\mu)\to L(\lambda)\otimes(\mathbb{C}\Omega_m)_{-\boxtimes(1^m)}$. Since both ι and f commute with the actions of $\mathfrak{S}_n$, f' also commutes with the actions of $\mathfrak{S}_n$. This proves $\mathrm{Im}(\Upsilon_{\lambda\mu}(\iota))\supseteq H_{(n)\boxtimes(1^m)}$. Since the other inclusion "$\subseteq$" is obvious, we have completed the proof of (3.9). The isomorphism (3.10) is rather obvious, since both of the spaces in (3.10) are naturally identified with $\mathrm{Hom}_{\mathbb{C}}(L(\mu),L(\lambda))_{(n-m)\boxtimes(1^m)}$. □

Lemma 3.4. *We have*

$$\dim\left(L(\mu)^*\otimes L(\lambda)\right)_{(n-m)\boxtimes(1^m)}=\sum_{\zeta\in\mathcal{P}_{n-m}}\sum_{\xi\in\mathcal{P}_m}LR^{\lambda}_{\zeta,\xi}LR^{\mu}_{\zeta,\xi^{\mathrm{t}}}. \tag{3.11}$$

Proof. By (2.1) and (2.3), we have

$$\begin{aligned}&[\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-m}\times\mathfrak{S}_m}\left(L(\mu)^*\otimes L(\lambda)\right)\ :\ L(n-m)\boxtimes L(1^m)]\\&\quad=\sum_{\zeta,\zeta'\in\mathcal{P}_{n-m}}\sum_{\xi,\xi'\in\mathcal{P}_m}LR^{\lambda}_{\zeta,\xi}LR^{\mu}_{\zeta',\xi'}\\&\quad\times[L(\zeta')\otimes L(\zeta)\ :\ L(n-m)][L(\xi')\otimes L(\xi)\ :\ L(1^m)].\end{aligned} \tag{3.12}$$

On the other hand, by (2.1), we have

$$[L(\zeta')\otimes L(\zeta)\ :\ L(n-m)]=\dim\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{n-m}}(L(\zeta')^*,L(\zeta))=\delta_{\zeta\zeta'}$$

and

$$[L(\xi')\otimes L(\xi)\ :\ L(1^m)]=\dim\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_m}(L(1^m)\otimes L(\xi')^*,L(\xi))=\delta_{\xi^{\mathrm{t}}\xi'}.$$

Hence the left-hand side of (3.12) agrees with the right-hand side of (3.11). Since $L(n-m)\boxtimes L(1^m)$ is one-dimensional, this proves (3.11). □

Theorem 3.5.

(1) *The multiplicity* w_m *of* $L(\mu)$ *in the tensor product module* $L(\lambda)\otimes\Lambda_m(\mathbb{C}\Omega)$ *is equal to*

$$\sum_{\zeta\in\mathcal{P}_{n-m}}\sum_{\xi\in\mathcal{P}_m}LR^{\lambda}_{\zeta,\xi}LR^{\mu}_{\zeta,\xi^{\mathrm{t}}}. \tag{3.13}$$

(2) *The multiplicity* h_m *of* $L(\mu)$ *in* $L(\lambda)\otimes L(n-m,1^m)$ *is equal to* $\sum_{i=0}^{m}(-1)^{m-i}w_i$.

Proof. The number w_m is equal to $\dim \Upsilon_{\lambda\mu}(\mathbb{C}\Omega_m)$ by (2.4). Then, applying (3.8), (3.9), (3.4), (3.10) and (3.11) in this order, Part (1) follows. By (2.4), h_m is equal to $\dim \Upsilon_{\lambda\mu}(L(n-m,1^m))$. Hence, we have $w_0 = h_0$, $w_m = h_{m-1} + h_m$ $(0 < m < n)$ and $w_n = h_{n-1}$ by (2.2). Solving these equations, we get Part (2). □

4. A crystallized exact sequence

Let λ/ζ and ν/η be skew partitions of $(n, n-m)$. In [16], Zelevinsky has constructed a finite set $\mathrm{Pic}(\nu/\eta, \lambda/\zeta)$ of *pictures*, which satisfies

$$|\mathrm{Pic}(\nu/\eta, \lambda/\zeta)| = \sum_{\xi \in \mathcal{P}_m} LR^{\lambda}_{\zeta,\xi} \cdot LR^{\nu}_{\eta,\xi}.$$

By Theorem 3.5 (1) and the well-known symmetry $LR^{\mu^{\mathrm{t}}}_{\zeta^{\mathrm{t}},\omega^{\mathrm{t}}} = LR^{\mu}_{\zeta,\omega}$, the multiplicity of $L(\mu)$ in $L(\lambda) \otimes \Lambda_m(\mathbb{C}\Omega)$ is equal to the number of elements of the set

$$\mathrm{PW}_m(\lambda, \mu) := \amalg_{\zeta \in \mathcal{P}_{n-m}} \mathrm{PW}(\lambda, \mu; \zeta), \tag{4.1}$$

where

$$\mathrm{PW}(\lambda, \mu; \zeta) := \mathrm{Pic}((\mu/\zeta)^{\mathrm{t}}, \lambda/\zeta) \tag{4.2}$$

if $\zeta \subseteq \lambda, \mu$, and $\mathrm{PW}(\lambda, \mu; \zeta) = \emptyset$ if otherwise. In this section, we will construct a subset $\mathrm{PH}_m(\lambda, \mu) \subseteq \mathrm{PW}_m(\lambda, \mu)$ whose number of elements is equal to the multiplicity of $L(\mu)$ in $L(\lambda) \otimes L(n-m, 1^m)$. Also, we will give a bijection $E : \mathrm{PH}_m(\lambda, \mu) \cong \mathrm{PW}_{m+1}(\lambda, \mu) \setminus \mathrm{PH}_{m+1}(\lambda, \mu)$. For this purpose, we will use a backward analogue of Zelevinsky's insertion–deletion algorithm for pictures. As a first step, we introduce a backward analogue of Sagan–Stanley's insertion–deletion algorithm for tableaux on skew partitions (cf. [15]).

Remark. By (2.2), there exists an exact sequence

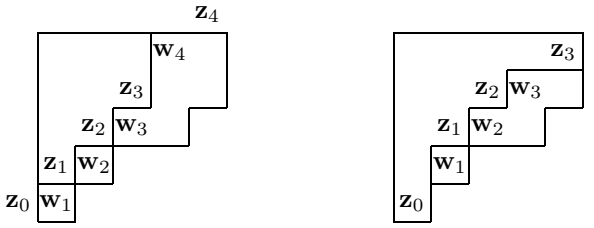
$$\mathbb{E} : 0 \to \Lambda_0(\mathbb{C}\Omega) \to \Lambda_1(\mathbb{C}\Omega) \to \cdots \to \Lambda_n(\mathbb{C}\Omega) \to 0$$

of $\mathbb{C}\mathfrak{S}_n$-modules. Since $\Upsilon_{\lambda\mu}$ is exact, there also exists an exact sequence $\Upsilon_{\lambda\mu}(\mathbb{E})$ of vector spaces. The above-mentioned construction may be viewed as a combinatorial counterpart of $\Upsilon_{\lambda\mu}(\mathbb{E})$.

4.1. Tableaux

For a partition $\lambda \in \mathcal{P}_n$, we say that a point $\mathbf{u} \in \mathbb{Z}^2_{>0}$ is a *corner* of λ if $\mathbf{u} \in D(\lambda)$ and $D(\lambda) \setminus \{\mathbf{u}\} = D(\xi)$ for some $\xi =: \lambda \setminus \mathbf{u} \in \mathcal{P}_{n-1}$. Also, we say that a point $\mathbf{v} \in \mathbb{Z}^2_{>0}$ is a *cocorner* of λ if $\mathbf{v} \notin D(\lambda)$ and $D(\lambda) \amalg \{\mathbf{v}\} = D(\omega)$ for some $\omega =: \lambda \amalg \mathbf{v} \in \mathcal{P}_{n+1}$. For a

skew partition λ/ζ, we say that a point $\mathbf{w} \in D(\lambda/\zeta)$ is an *inner corner* of λ/ζ if $\mathbf{w}$ is a cocorner of ζ. We denote by $\mathrm{IC}(\lambda/\zeta)$ the set of inner corners of λ/ζ. We say that a point $\mathbf{z} \in \mathbb{Z}^2_{>0}$ is an *inner cocorner* of λ/ζ if $\mathbf{z}$ is a corner of ζ. We say that $\mathbf{z} \in \mathbb{Z}^2$ is an *extreme cocorner* of λ/ζ if either $\mathbf{z} = (l(\lambda/\zeta), 0)$ and $(l(\lambda/\zeta), 1) \in D(\lambda/\zeta)$, or $\mathbf{z} = (0, \lambda_1)$ and $(1, \lambda_1) \in D(\lambda/\zeta)$. Let $\mathrm{ICC}(\lambda/\zeta)$ be the set of inner cocorners of λ/ζ and let $\overline{\mathrm{ICC}}(\lambda/\zeta)$ be the set of inner cocorners and extreme cocorners of λ/ζ. For example, when $\lambda/\zeta = (5,5,4,2,1)/(3,3,2,1)$, we have $\mathrm{IC}(\lambda/\zeta) = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$, $\mathrm{ICC}(\lambda/\zeta) = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ and $\overline{\mathrm{ICC}}(\lambda/\zeta) = \{\mathbf{z}_i \mid 0 \le i \le 4\}$, where $\mathbf{w}_1 = (5,1)$, $\mathbf{w}_2, \ldots, \mathbf{z}_4$ are as in the first figure of (4.3) below. When $\lambda/\zeta = (5,5,4,2,1)/(5,3,2,1,1)$, we have $\mathrm{IC}(\lambda/\zeta) = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and $\mathrm{ICC}(\lambda/\zeta) = \{\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\} = \overline{\mathrm{ICC}}(\lambda/\zeta)$, where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{z}_3$ are as in the second figure of (4.3).



(4.3)

Let $\leq_{\nwarrow}$ and $\leq_{\swarrow}$ be partial orders on $\mathbb{Z}^2$ defined by

$$(i,j) \leq_{\nwarrow} (k,l) \Longleftrightarrow i \le k \text{ and } j \le l, \tag{4.4}$$

$$(i,j) \leq_{\swarrow} (k,l) \Longleftrightarrow k \le i \text{ and } j \le l. \tag{4.5}$$

We note that

$$\mathbf{x} \leq_{\swarrow} \mathbf{y} \Longleftrightarrow \mathbf{y}^{\mathrm{t}} \leq_{\swarrow} \mathbf{x}^{\mathrm{t}} \tag{4.6}$$

for each $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$. Also, we note that $(\mathrm{IC}(\lambda/\zeta) \amalg \overline{\mathrm{ICC}}(\lambda/\zeta), \leq_{\swarrow})$ is a totally ordered set whose order is expressed as

$$\mathbf{z}_0 <_{\swarrow} \mathbf{w}_1 <_{\swarrow} \mathbf{z}_1 <_{\swarrow} \mathbf{w}_2 <_{\swarrow} \cdots <_{\swarrow} \mathbf{w}_k <_{\swarrow} \mathbf{z}_k, \tag{4.7}$$

where $\mathrm{IC}(\lambda/\zeta) = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and $\overline{\mathrm{ICC}}(\lambda/\zeta) = \{\mathbf{z}_0, \ldots, \mathbf{z}_k\}$.

Let ν/η be a skew partition and let $T : D(\nu/\eta) \to \mathbb{Z}_{>0}$ be an injective map. We say that T is a *partial tableau* on ν/η if it is a map of ordered sets from $(D(\nu/\eta), \leq_{\nwarrow})$ to $(\mathbb{Z}_{>0}, \leq)$. Let T be a partial tableau on a skew partition ν/η of length $l = l(\nu/\eta)$. Let $a > 0$ be an integer such that $a \notin T(D(\nu/\eta))$. In order to construct the backward row insertion of a into T, we define the *bumping route* $\mathbf{x}_l, \mathbf{x}_{l-1}, \ldots, \mathbf{x}_r$ of (T, a) by the following lemma:

Lemma 4.1. *There exist a unique integer $l \ge r \ge 0$ and a unique sequence $\mathbf{x}_l, \mathbf{x}_{l-1}, \ldots, \mathbf{x}_r \in \mathbb{Z}^2$ satisfying the following three conditions:*

(1) $\mathbf{x}_i$ $(i > r)$ *is the right-most point of* $D_i(\nu/\eta)$ *which satisfies* $T(\mathbf{x}_i) < a_i$, *where* $a_j = T(\mathbf{x}_{j+1})$ *for* $r \leq j < l$ *and* $a_l = a$.
(2) *If* $r > 0$, *then* $a_r \leq T(\mathbf{x})$ *for every* $\mathbf{x} \in D_r(\nu/\eta)$.
(3) *Either* $r > 0$ *and* $\mathbf{x}_r = (r, \eta_r)$, *or* $r = 0$ *and* $\mathbf{x}_0 = (0, \nu_1)$.

The lemma is proved by determining $\mathbf{x}_l, \mathbf{x}_{l-1}, \ldots, \mathbf{x}_r$ inductively. We call $\mathbf{u} := \mathbf{x}_r \in \overline{\mathrm{ICC}}(\nu/\eta)$ the *bumping destination* of (T, a) and write $\mathbf{u} \overset{T}{\text{———}} a$. We say that a is an *addable integer* of T if $\mathbf{u} \in \mathrm{ICC}(\nu/\eta)$. In this case, we define the *backward row insertion* $E_a T$ to be the partial tableau on $\nu/(\eta \setminus \mathbf{u})$ determined by $(E_a T)(\mathbf{x}_i) = a_i$ $(l \geq i \geq r)$ and $(E_a T)(\mathbf{y}) = T(\mathbf{y})$ $(\mathbf{y} \neq \mathbf{x}_l, \mathbf{x}_{l-1}, \ldots, \mathbf{x}_r)$.

As well as the usual row insertion, the backward row insertion has an inverse operation. Let $\mathbf{v}$ be an inner corner of ν/η. Then there exists at most one pair (T', b) such that $T = E_b T'$ and that $\mathbf{v}$ is the bumping destination of (T', b). When there exists such a pair, we say that $\mathbf{v}$ is a *removable corner* of T and we write $T' = F_{\mathbf{v}} T$ and $\mathbf{v} \overset{T}{\text{———}} b$. More explicitly, $F_{\mathbf{v}} T$ is given as follows:

Lemma 4.2. *An inner corner* $\mathbf{v} = (s, j)$ *of* ν/η *is a removable corner of* T *if and only if there exists a (necessarily unique) sequence* $\mathbf{y}_s = \mathbf{v}, \mathbf{y}_{s+1}, \ldots, \mathbf{y}_l$ *of points of* $D(\nu/\eta)$ *such that* $\mathbf{y}_i$ *is the left-most point in* $D_i(\nu/\eta)$ *satisfying* $T(\mathbf{y}_{i-1}) < T(\mathbf{y}_i)$ *for each* $s < i \leq l$. *In this case, we have* $(F_{\mathbf{v}} T)(\mathbf{y}_i) = T(\mathbf{y}_{i-1})$ $(s < i \leq l)$, $(F_{\mathbf{v}} T)(\mathbf{x}) = T(\mathbf{x})$ $(\mathbf{x} \neq \mathbf{y}_{s+1}, \ldots, \mathbf{y}_l)$ *and* $\mathbf{v} \overset{T}{\text{———}} T(\mathbf{y}_l)$.

Example 4.3. Define a partial tableau T on $(5, 5, 4, 3)/(4, 3, 2)$ as follows:

$$T = \begin{array}{|c|c|c|c|c|} \hline & & & & 6 \\ \hline & & & 8 & 9 \\ \hline & & 3 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 5 & 10 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array} . \tag{4.8}$$

Then, the bumping route of $(T, 7)$ is $(4, 2)$, $(3, 3)$, $(2, 3)$. Hence $(2, 3) \overset{T}{\text{———}} 7$, and 7 is an addable integer of T. The tableau $E_7 T$ is given by the first equality of (4.9) below.

$$E_7 T = \begin{array}{|c|c|c|c|c|} \hline & & & & 6 \\ \hline & & 3 & 8 & 9 \\ \hline & & 5 & 11 & \multicolumn{1}{c}{} \\ \cline{1-4} 1 & 7 & 10 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array} , \qquad F_{(3,3)} T = \begin{array}{|c|c|c|c|c|} \hline & & & & 6 \\ \hline & & & 8 & 9 \\ \hline & & & 11 & \multicolumn{1}{c}{} \\ \cline{1-4} 1 & 3 & 10 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array} \tag{4.9}$$

On the other hand, $\mathbf{v} = (3, 3)$ is a removable corner of T and $F_{(3,3)} T$ is given by the second equality of (4.9). Moreover, we have $(3, 3) \overset{T}{\text{———}} 5$.

The following "bumping lemmas" are quite essential for our discussions.

Lemma 4.4. *For each partial tableau T on ν/η, we have the following:*

(1) *Let $a \leq a'$ be positive integers such that $a, a' \notin T(D(\nu/\eta))$. Define $\mathbf{u}, \mathbf{u}' \in \overline{\mathrm{ICC}}(\nu/\eta)$ by $\mathbf{u} \xrightarrow[T]{} a$ and $\mathbf{u}' \xrightarrow[T]{} a'$. Then, we have $\mathbf{u} \leq_{\swarrow} \mathbf{u}'$.*

(2) *Let $\mathbf{v}$ and $\mathbf{v}'$ be removable corners of T such that $\mathbf{v} \leq_{\swarrow} \mathbf{v}'$. Define integers b and b' by $\mathbf{v} \xrightarrow[T]{} b$ and $\mathbf{v}' \xrightarrow[T]{} b'$. Then, we have $b \leq b'$.*

(3) *Let a be an addable integer of T and let a' be another positive integer such that $a' \notin T(D(\nu/\eta)) \amalg \{a\}$. Define $\mathbf{u} \in \mathrm{ICC}(\nu/\eta)$ and $\mathbf{u}' \in \overline{\mathrm{ICC}}(\nu/(\eta \setminus \mathbf{u}))$ by $\mathbf{u} \xrightarrow[T]{} a$ and $\mathbf{u}' \xrightarrow[E_a T]{} a'$, respectively. Then we have $a < a' \Rightarrow \mathbf{u} <_{\swarrow} \mathbf{u}'$ and $a' < a \Rightarrow \mathbf{u}' <_{\swarrow} \mathbf{u}$.*

(4) *Let $\mathbf{v}$ be a removable corner of T and let $\mathbf{v}'$ be a removable corner of $F_{\mathbf{v}} T$. Define integers b and b' by $\mathbf{v} \xrightarrow[T]{} b$ and $\mathbf{v}' \xrightarrow[F_{\mathbf{v}} T]{} b'$, respectively. Then, we have $\mathbf{v} <_{\swarrow} \mathbf{v}' \Rightarrow b < b'$ and $\mathbf{v}' <_{\swarrow} \mathbf{v} \Rightarrow b' < b$.*

(5) *Let $a \notin T(D(\nu/\eta))$ be a positive integer and define $\mathbf{u} \in \overline{\mathrm{ICC}}(\nu/\eta)$ by $\mathbf{u} \xrightarrow[T]{} a$. Let $\mathbf{v}$ be a removable corner of T such that $\mathbf{u} <_{\swarrow} \mathbf{v}$. Then, we have $a < b$, where $\mathbf{v} \xrightarrow[T]{} b$.*

(6) *Let a and $\mathbf{u}$ be as in Part (5). If a cocorner $\mathbf{v} = (s, j)$ of η satisfies $\mathbf{v} <_{\swarrow} \mathbf{u}$, then $\mathbf{v}$ is a removable corner of T. Moreover, we have $b < a$, where $\mathbf{v} \xrightarrow[T]{} b$.*

Proof. All of these can be proved in a standard manner (cf. [4, p. 9]). Here we will prove Part (6) using Lemma 4.2. Let $\mathbf{x}_l, \mathbf{x}_{l-1}, \ldots, \mathbf{x}_r = \mathbf{u}$ be the bumping route of (T, a). Since $s > r$, we have $D_s(\nu/\eta) \supseteq \{\mathbf{x}_s\} \neq \emptyset$. Hence $\mathbf{v}$ is an inner corner of ν/η. Suppose there exist an integer $s \leq t < l$ and a sequence $\mathbf{y}_s := \mathbf{v}, \mathbf{y}_{s+1}, \ldots, \mathbf{y}_t$, such that each $\mathbf{y}_i$ $(s < i \leq t)$ satisfies the condition in Lemma 4.2 and that $\mathbf{y}_t$ is left of or equal to $\mathbf{x}_t$. Since $T(\mathbf{y}_t) \leq T(\mathbf{x}_t) < T(\mathbf{x}_{t+1})$, we have $S := \{\mathbf{y} \in D_{t+1}(\nu/\eta) \mid T(\mathbf{y}_t) < T(\mathbf{y})\} \neq \emptyset$. Let $\mathbf{y}_{t+1}$ be the left-most element of S. Then, it is obvious that $\mathbf{y}_{t+1}$ satisfies the condition in Lemma 4.2 for $i = t+1$ and that $\mathbf{y}_{t+1}$ is left of or equal to $\mathbf{x}_{t+1}$. By induction, we see that there exists a sequence $\mathbf{y}_s, \mathbf{y}_{s+1}, \ldots, \mathbf{y}_l$ satisfying the condition of Lemma 4.2 and that $\mathbf{y}_l$ is left of or equal to $\mathbf{x}_l$. This proves the first assertion as well as the second assertion, since $b = T(\mathbf{y}_l) \leq T(\mathbf{x}_l) < a$. □

4.2. Pictures

Next, we recall the definition of the picture of Zelevinsky, following Fomin and Greene [3] (or rather, Leeuwen [8]).

Let ν/η and λ/ζ be skew partitions of $(n, n-m)$. We say that a bijection $\Pi\colon D(\nu/\eta) \to D(\lambda/\zeta)$ is a *picture* from ν/η onto λ/ζ if both $\Pi : (D(\nu/\eta), \leq_{\nwarrow}) \to (D(\lambda/\zeta), \leq_{\swarrow})$ and $\Pi^{-1} : (D(\lambda/\zeta), \leq_{\nwarrow}) \to (D(\nu/\eta), \leq_{\swarrow})$ are maps of ordered sets. We denote by $\mathrm{Pic}(\nu/\eta, \lambda/\zeta)$ the set of pictures from ν/η onto λ/ζ.

Let $R\colon (D(\lambda/\zeta), \leq_{\swarrow}) \to \mathbb{Z}_{>0}$ be an injective map of ordered sets. A typical example $R_{\mathrm{row}} = R_{\mathrm{row},\lambda/\zeta}$ of such maps (called the *row reading* of λ/ζ) is given by

$$R_{\mathrm{row}}(i,j) = |\{(i',j') \in D(\lambda/\zeta) \mid i < i' \text{ or } i = i',\ j' \le j\}|.$$

For example, if $\lambda/\zeta = (5,5,4,2,1)/(3,2,1)$, then R_{row} is expressed as

$$R_{\mathrm{row}} = \begin{array}{|c|c|c|c|c|}\hline \multicolumn{2}{c}{} & \multicolumn{1}{c}{} & 10 & 11 \\ \cline{3-5} \multicolumn{2}{c}{} & 7 & 8 & 9 \\ \cline{3-5} \multicolumn{1}{c}{} & 4 & 5 & 6 & \multicolumn{1}{c}{} \\ \cline{2-4} 2 & 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 1 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}. \tag{4.10}$$

We say that a partial tableau T on ν/η is a *Remmel–Whitney tableau* of type R [13] if it satisfies the following three conditions:

(1) The image of T coincides with that of R.
(2) For each $(i,j),(i,j+1) \in D(\lambda/\zeta)$, we have $T^{-1}(a) \le_{\swarrow} T^{-1}(b)$, where $a = R(i,j)$ and $b = R(i,j+1)$.
(3) For each $(i,j),(i+1,j) \in D(\lambda/\zeta)$, we have $T^{-1}(a) \le_{\swarrow} T^{-1}(b)$, where $a = R(i,j)$ and $b = R(i+1,j)$.

The correspondence $\Pi \mapsto R \circ \Pi$ gives a bijection from $\mathrm{Pic}(\nu/\eta, \lambda/\zeta)$ onto the set $\mathrm{RW}(\nu/\eta; R)$ of Remmel–Whitney tableau on ν/η of type R (cf. [3]).

Example 4.5. Let λ, μ and ζ be $(5,5,4,2,1)$, $(4,4,4,3,2)$ and $(3,3,2,1)$, respectively. Let $R\colon D(\lambda/\zeta) \to \mathbb{Z}_{>0}$ be the restriction of (4.10) and let T be as in (4.8). By checking six order relations including

$$T^{-1}(R(1,4)) = T^{-1}(10) = (4,3) \le_{\swarrow} (3,4) = T^{-1}(11) = T^{-1}(R(1,5)),$$

we see that T is an element of $\mathrm{RW}((\mu/\zeta)^{\mathsf{t}}; R)$. Hence, there exists a unique $\Lambda \in \mathrm{Pic}((\mu/\zeta)^{\mathsf{t}}, \lambda/\zeta)$ which satisfies $T = R \circ \Lambda$. Explicitly, Λ is given by $\Lambda(a) = A$, $\Lambda(b) = B, \dots, \Lambda(h) = H$, where $a = (4,1), b, \dots, H \in \mathbb{Z}^2_{>0}$ are as in (4.11) below, since $\Lambda(a) = R^{-1}(T(a)) = R^{-1}(1) = A$, for example.

$$\begin{array}{|c|c|c|c|}\hline \multicolumn{3}{c}{} & d \\ \cline{4-4} \multicolumn{2}{c}{} & e & f \\ \cline{3-4} \multicolumn{1}{c}{} & b & h & \multicolumn{1}{c}{} \\ \cline{2-3} a & c & g & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array} \quad \xrightarrow{\ \Lambda\ } \quad \begin{array}{|c|c|c|c|}\hline \multicolumn{2}{c}{} & G & H \\ \cline{3-4} \multicolumn{2}{c}{} & E & F \\ \cline{3-4} \multicolumn{1}{c}{} & C & D & \multicolumn{1}{c}{} \\ \cline{2-3} B & \multicolumn{3}{c}{} \\ \cline{1-1} A & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}. \tag{4.11}$$

Let $\Pi \in \mathrm{Pic}(\nu/\eta, \lambda/\zeta)$ be a picture and let $\mathbf{z}$ be an element of $\overline{\mathrm{ICC}}(\lambda/\zeta)$. Let $R : (D(\lambda/\zeta) \amalg \{\mathbf{z}\}, \le_{\swarrow}) \to \mathbb{Z}_{>0}$ be an injective map of ordered sets. Then the bumping route $\mathbf{x}_l, \mathbf{x}_{l-1}, \dots, \mathbf{x}_r$ of $(R \circ \Pi, R(\mathbf{z}))$ does not depend on the choice of R. In fact, the route is characterized by the following three conditions (cf. [8, p. 338]):

(1) $\Pi(\mathbf{x}_i)$ $(i > r)$ is the predecessor of $\mathbf{y}_i$ in the totally ordered set $(\Pi(D_i(\nu/\eta)) \amalg \{\mathbf{y}_i\}, \leq_{\swarrow})$, where $\mathbf{y}_j = \Pi(\mathbf{x}_{j+1})$ for $r \leq j < l$ and $\mathbf{y}_l = \mathbf{z}$.
(2) If $r > 0$, then $\mathbf{y}_r$ is the minimal element of $(\Pi(D_r(\nu/\eta)) \amalg \{\mathbf{y}_r\}, \leq_{\swarrow})$.
(3) Either $r > 0$ and $\mathbf{x}_r = (r, \eta_r)$, or $r = 0$ and $\mathbf{x}_0 = (0, \nu_1)$.

We call $\mathbf{x}_l, \mathbf{x}_{l-1}, \ldots, \mathbf{x}_r$ and $\mathbf{u} := \mathbf{x}_r \in \overline{\mathrm{ICC}}(\nu/\eta)$ the *bumping route* and the *bumping destination* of $(\Pi, \mathbf{z})$, respectively. Also, we write $\mathbf{u} \overset{\Pi}{\text{———}} \mathbf{z}$. When $\mathbf{z} \in \mathrm{ICC}(\lambda/\zeta)$ and $\mathbf{u} \in \mathrm{ICC}(\nu/\eta)$, we say that $\mathbf{z}$ is an *addable cocorner* of Π. In this case, we define the *row insertion* $E_{\mathbf{z}}\Pi$ to be the unique picture from $\nu/(\eta \setminus \mathbf{u})$ onto $\lambda/(\zeta \setminus \mathbf{z})$ satisfying the condition

$$E_{R(\mathbf{z})}(R \circ \Pi) = R \circ (E_{\mathbf{z}}\Pi) \tag{4.12}$$

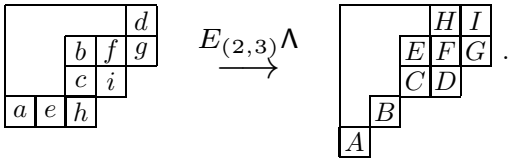
(cf. [16,8]). More explicitly, the picture $E_{\mathbf{z}}\Pi$ is given by $(E_{\mathbf{z}}\Pi)(\mathbf{x}_l) = \mathbf{z}$, $(E_{\mathbf{z}}\Pi)(\mathbf{x}_i) = \Pi(\mathbf{x}_{i+1})$ $(l > i \geq r)$ and $(E_{\mathbf{z}}\Pi)(\mathbf{y}) = \Pi(\mathbf{y})$ $(\mathbf{y} \neq \mathbf{x}_l, \mathbf{x}_{l-1}, \ldots, \mathbf{x}_r)$.

Let $\Pi \in \mathrm{Pic}(\nu/\eta, \lambda/\zeta)$ be a picture and let $\mathbf{v}$ be an inner corner of ν/η. Then there exists at most one pair $(\Pi', \mathbf{w})$ such that $\Pi = E_{\mathbf{w}}\Pi'$ and that $\mathbf{v}$ is the bumping destination of $(\Pi', \mathbf{w})$. When there exists such a pair, we say that $\mathbf{v}$ is a *removable corner* of Π and we write $\Pi' = F_{\mathbf{v}}\Pi$ and $\mathbf{v} \overset{\Pi}{\text{———}} \mathbf{w}$. We note that $\mathbf{w}$ is an inner corner of λ/ζ. Let $R\colon (D(\lambda/\zeta), \leq_{\swarrow}) \to \mathbb{Z}_{>0}$ be an injective map of ordered sets. Then, $\mathbf{v}$ is a *removable corner* of Π if and only if it is a removable corner of $R \circ \Pi$. Moreover, we have

$$F_{\mathbf{v}}(R \circ \Pi) = R \circ (F_{\mathbf{v}}\Pi), \tag{4.13}$$

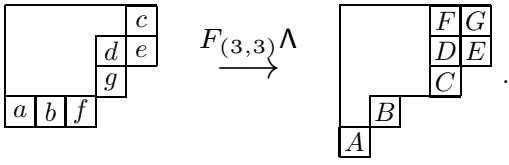
$$\mathbf{v} \overset{R \circ \Pi}{\text{———}} R(\mathbf{w}). \tag{4.14}$$

Example 4.6. Let λ, μ, ζ and Λ be as in Example 4.5. Let $R\colon D(\lambda/\zeta) \amalg \{(2,3)\} \to \mathbb{Z}_{>0}$ be the restriction of (4.10). Then $R(2,3) = 7$ and $T = R \circ \Lambda$ is given by (4.8). Hence by Example 4.3 and (4.12), we have $(2,3) \overset{\Lambda}{\text{———}} (2,3)$ and



(4.15)

Here the notation is the same as in (4.11), that is, $(E_{(2,3)}\Lambda)(a) = A$, for example. Similarly, by the last assertions of Example 4.3 and (4.13), we have $(3,3) \overset{\Lambda}{\text{———}} (3,3)$ and



(4.16)

Lemma 4.7.

(1) *Let* $\mathbf{z}$ *be an addable cocorner of* Π *and set* $\mathbf{u} \overset{\Pi}{\text{———}} \mathbf{z}$. *Then* $\mathbf{u}$ *is a removable corner of* $E_{\mathbf{z}}\Pi$, *which satisfies* $\mathbf{u} \overset{E_{\mathbf{z}}\Pi}{\text{———}} \mathbf{z}$. *Moreover, we have*

$$F_{\mathbf{u}}(E_{\mathbf{z}}\Pi) = \Pi. \tag{4.17}$$

(2) *Let* $\mathbf{v}$ *be a removable corner of* Π *and set* $\mathbf{v} \overset{}{\text{———}}_{\Pi} \mathbf{w}$. *Then* $\mathbf{w}$ *is an addable cocorner of* $F_{\mathbf{v}}\Pi$, *which satisfies* $\mathbf{v} \overset{F_{\mathbf{v}}\Pi}{\text{———}} \mathbf{w}$. *Moreover, we have*

$$E_{\mathbf{w}}(F_{\mathbf{v}}\Pi) = \Pi. \tag{4.18}$$

As an immediate consequence of Lemma 4.4, we have the following:

Lemma 4.8. *For each* $\Pi \in \mathrm{Pic}(\nu/\eta, \lambda/\zeta)$, *we have the following:*

(1) *Let* $\mathbf{z}$ *and* $\mathbf{z}'$ *be elements of* $\overline{\mathrm{ICC}}(\lambda/\zeta)$ *such that* $\mathbf{z} \leq_{\swarrow} \mathbf{z}'$. *Define* $\mathbf{u}, \mathbf{u}' \in \overline{\mathrm{ICC}}(\mu/\eta)$ *by* $\mathbf{u} \overset{\Pi}{\text{———}} \mathbf{z}$, $\mathbf{u}' \overset{\Pi}{\text{———}} \mathbf{z}'$. *Then we have* $\mathbf{u} \leq_{\swarrow} \mathbf{u}'$.

(2) *Let* $\mathbf{v}$ *and* $\mathbf{v}'$ *be removable corners of* Π *such that* $\mathbf{v} \leq_{\swarrow} \mathbf{v}'$. *Define* $\mathbf{w}, \mathbf{w}' \in \mathrm{IC}(\lambda/\zeta)$ *by* $\mathbf{v} \overset{}{\text{———}}_{\Pi} \mathbf{w}$, $\mathbf{v}' \overset{}{\text{———}}_{\Pi} \mathbf{w}'$. *Then, we have* $\mathbf{w} \leq_{\swarrow} \mathbf{w}'$.

(3) *Let* $\mathbf{z}$ *be an addable cocorner of* Π *and let* $\mathbf{z}'$ *be an inner cocorner of* $\lambda/(\zeta \setminus \mathbf{z})$. *Define points* $\mathbf{u}$ *and* $\mathbf{u}'$ *by* $\mathbf{u} \overset{\Pi}{\text{———}} \mathbf{z}$ *and* $\mathbf{u}' \overset{E_{\mathbf{z}}\Pi}{\text{———}} \mathbf{z}'$, *respectively. Then we have* $\mathbf{z} <_{\swarrow} \mathbf{z}' \Rightarrow \mathbf{u} <_{\swarrow} \mathbf{u}'$ *and* $\mathbf{z}' <_{\swarrow} \mathbf{z} \Rightarrow \mathbf{u}' <_{\swarrow} \mathbf{u}$.

(4) *Let* $\mathbf{v}$ *be a removable corner of* Π *and let* $\mathbf{v}'$ *be a removable corner of* $F_{\mathbf{v}}\Pi$. *Define points* $\mathbf{w}$ *and* $\mathbf{w}'$ *by* $\mathbf{v} \overset{}{\text{———}}_{\Pi} \mathbf{w}$ *and* $\mathbf{v}' \overset{}{\text{———}}_{F_{\mathbf{v}}\Pi} \mathbf{w}'$, *respectively. Then, we have* $\mathbf{v} <_{\swarrow} \mathbf{v}' \Rightarrow \mathbf{w} <_{\swarrow} \mathbf{w}'$ *and* $\mathbf{v}' <_{\swarrow} \mathbf{v} \Rightarrow \mathbf{w}' <_{\swarrow} \mathbf{w}$.

(5) *Let* $\mathbf{z}$ *be an element of* $\overline{\mathrm{ICC}}(\lambda/\zeta)$ *and define* $\mathbf{u} \in \overline{\mathrm{ICC}}(\nu/\eta)$ *by* $\mathbf{u} \overset{\Pi}{\text{———}} \mathbf{z}$. *Let* $\mathbf{v}$ *be a removable corner of* Π *such that* $\mathbf{u} <_{\swarrow} \mathbf{v}$. *Then, we have* $\mathbf{z} <_{\swarrow} \mathbf{w}$, *where* $\mathbf{v} \overset{}{\text{———}}_{\Pi} \mathbf{w}$.

(6) *Let* $\mathbf{z}$ *and* $\mathbf{u}$ *be as in Part (5). If a cocorner* $\mathbf{v}$ *of* η *satisfies* $\mathbf{v} <_{\swarrow} \mathbf{u}$, *then* $\mathbf{v}$ *is a removable corner of* Π. *Moreover, we have* $\mathbf{w} <_{\swarrow} \mathbf{z}$, *where* $\mathbf{v} \overset{}{\text{———}}_{\Pi} \mathbf{w}$.

4.3. Balanced corners

Let λ and μ be partitions of n and let ζ be a partition of $n - m$, which satisfies both $\zeta \subseteq \lambda$ and $\zeta \subseteq \mu$. We define a set $\mathrm{PW}(\lambda, \mu; \zeta)$ by (4.2) and call its element a *picture of type* $(\lambda, \mu; \zeta)$. Also, we define a set $\mathrm{PW}_m(\lambda, \mu)$ by (4.1) and call its element a *picture of type* (λ, μ) with size m. For convenience, we set $\mathrm{PW}_0(\lambda, \mu) = \emptyset$ if $\lambda \neq \mu$ and

$$\mathrm{PW}_0(\lambda, \lambda) = \mathrm{PW}(\lambda, \lambda; \lambda) = \{\mathrm{id}_\emptyset\}.$$

Let Λ be an element of $\mathrm{PW}_m(\lambda,\mu)$ and let $\mathbf{v}$ be its removable corner. We say that $\mathbf{v}$ is a *balanced corner* of Λ if

$$\mathbf{v} \overset{}{\underset{\Lambda}{---}} \mathbf{v}^{\mathsf{t}}.$$

Also, we say that an addable cocorner $\mathbf{z}$ of $\mathsf{V} \in \mathrm{PW}_m(\lambda,\mu)$ is a *balanced cocorner* if $\mathbf{z}^{\mathsf{t}} \overset{\mathsf{V}}{---} \mathbf{z}$.

Example 4.9. Let $\Lambda \in \mathrm{PW}((5,5,4,2,1),(4,4,4,3,2);(3,3,2,1))$ be as in (4.11). By Example 4.6, $(3,3)$ is a balanced corner of Λ, while $(2,3)$ is not a balanced cocorner of Λ.

Lemma 4.10. *For each $\Lambda,\mathsf{V} \in \mathrm{PW}(\lambda,\mu;\zeta)$, we have the following:*

(1) *The picture Λ has at most one balanced corner. Also, it has at most one balanced cocorner.*
(2) *If V has a balanced cocorner $\mathbf{z}$, then $E_{\mathbf{z}}\mathsf{V}$ has no balanced cocorners. If Λ has a balanced corner $\mathbf{v}$, then $F_{\mathbf{v}}\Lambda$ has no balanced corners.*
(3) *If V has a balanced cocorner, then V has no balanced corners. If Λ has a balanced corner, then Λ has no balanced cocorners.*

Proof. Let $\mathbf{z}$ and $\mathbf{z}'$ be balanced cocorners of Λ. Since $(\mathrm{ICC}(\lambda/\zeta),\leq_{\swarrow})$ is totally ordered, we may assume $\mathbf{z} \leq_{\swarrow} \mathbf{z}'$. Then, by Lemma 4.8 (1), we have $\mathbf{z}^{\mathsf{t}} \leq_{\swarrow} (\mathbf{z}')^{\mathsf{t}}$. Hence, we have $\mathbf{z}' \leq_{\swarrow} \mathbf{z}$ by (4.6). This proves the second statement of Part (1). Similarly, the first statement of Part (1) follows from Lemma 4.8 (2), Part (2) follows from Lemma 4.8 (3), (4) and Part (3) follows from Lemma 4.8 (5), (6). □

Lemma 4.11. *Suppose that $(\mu/\zeta)^{\mathsf{t}}$ satisfies $D_l((\mu/\zeta)^{\mathsf{t}}) = \emptyset$, where $l = l((\mu/\zeta)^{\mathsf{t}})$. Then, for each picture $\mathsf{V} \in \mathrm{PW}(\lambda,\mu;\zeta)$, $((\zeta^{\mathsf{t}})_l, l)$ is a balanced cocorner of V.*

Proof. Since $(\zeta^{\mathsf{t}})_l = (\mu^{\mathsf{t}})_l > 0$, $\mathbf{u} := (l,(\zeta^{\mathsf{t}})_l)$ is a corner of ζ^{t}. On the other hand, by the definition of the backward row insertion, we have $\mathbf{u} \overset{\mathsf{V}}{---} \mathbf{z}$ for every $\mathbf{z} \in \mathrm{ICC}(\lambda/\zeta)$. Hence $\mathbf{u}^{\mathsf{t}}$ is a balanced cocorner of V. □

Lemma 4.12. *If Λ does not have a balanced cocorner, then Λ has a balanced corner.*

We will give a proof of the lemma above in Section 4.5.

4.4. Picture of hook shape

Let V be an element of $\mathrm{PW}(\lambda,\mu;\zeta)$. We say that V is a *picture of hook shape* of type $(\lambda,\mu;\zeta)$ if it has a (necessarily unique) balanced cocorner. Let $\mathrm{PH}(\lambda,\mu;\zeta)$ be the set of

pictures of hook shape of type $(\lambda, \mu; \zeta)$ and let $\mathrm{PH}_m(\lambda, \mu)$ be the set $\coprod_{\zeta \in \mathcal{P}_{n-m}} \mathrm{PH}(\lambda, \mu; \zeta)$. By Lemma 4.10 (3) and Lemma 4.12, we have the following decomposition:

$$\mathrm{PW}_m(\lambda, \mu) = \mathrm{PH}_m(\lambda, \mu) \amalg \mathrm{PH}^c_m(\lambda, \mu), \tag{4.19}$$

where $\mathrm{PH}^c_m(\lambda, \mu)$ denotes the set of pictures of type (λ, μ) with size m, which have balanced corners. Since $\mathrm{PH}^c_0(\lambda, \mu) = \mathrm{PH}_n(\lambda, \mu) = \emptyset$, we have in particular

$$\mathrm{PW}_0(\lambda, \mu) = \mathrm{PH}_0(\lambda, \mu), \quad \mathrm{PW}_n(\lambda, \mu) = \mathrm{PH}^c_n(\lambda, \mu). \tag{4.20}$$

Given $\mathsf{V} \in \mathrm{PH}_m(\lambda, \mu)$ and $\Lambda \in \mathrm{PH}^c_m(\lambda, \mu)$, we define pictures $E\mathsf{V}$ and $F\Lambda$ by $E\mathsf{V} = E_{\mathbf{z}}\mathsf{V}$ and $F\Lambda = F_{\mathbf{v}}\Lambda$ respectively, where $\mathbf{z}$ denotes the unique balanced cocorner of V and $\mathbf{v}$ denotes the unique balanced corner of Λ. By Lemma 4.10 (2) and (4.19), we have $E\mathsf{V} \in \mathrm{PH}^c_{m+1}(\lambda, \mu)$ and $F\Lambda \in \mathrm{PH}_{m-1}(\lambda, \mu)$. Hence, by Lemma 4.7, the operator E gives a bijection

$$\mathrm{PH}_m(\lambda, \mu) \cong \mathrm{PH}^c_{m+1}(\lambda, \mu) \tag{4.21}$$

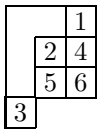
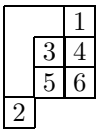
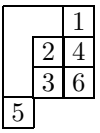
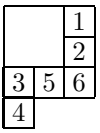
whose inverse is given by F.

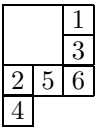
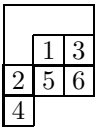
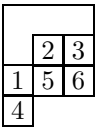
Theorem 4.13. *For each $\lambda \in \mathcal{P}_n$ and $0 \le m < n$, we have*

$$[L(\lambda) \otimes L(n-m, 1^m)] = \sum_{\mu \in \mathcal{P}_n} |\mathrm{PH}_m(\lambda, \mu)| \cdot [L(\mu)]. \tag{4.22}$$

Proof. By (4.19), (4.20) and (4.21), the integers $h'_m := |\mathrm{PH}_m(\lambda, \mu)|$ satisfy $w_0 = h'_0$, $w_m = h'_{m-1} + h'_m$ $(0 < m < n)$ and $w_n = h'_{n-1}$. Solving these equations, we get $h'_m = \sum_{i=0}^{m} (-1)^{m-i} w_i$. Therefore, the theorem follows from Part (2) of Theorem 3.5. □

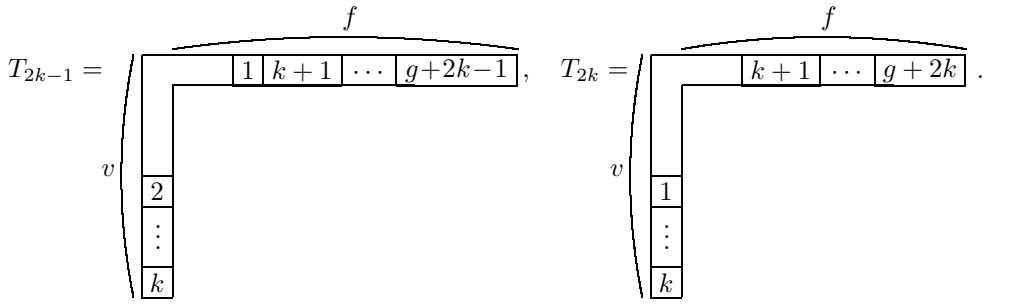
Example 4.14. For $\lambda = (5, 3, 1, 1)$ and $\mu = (4, 3, 3)$, the set $\mathrm{PW}_6(\lambda, \mu)$ consists of 7 elements Λ_i $(i = 1, \dots, 7)$, where the corresponding Remmel–Whitney tableaux $T_i = R_{row} \circ \Lambda_i$ are given by

$T_1 =$  , $T_2 =$  , $T_3 =$  , $T_4 =$  ,

$T_5 =$  , $T_6 =$  , $T_7 =$  . (4.23)

The point $(1, 3)$ is a balanced cocorner of both Λ_1 and Λ_2. While the points $(1, 4)$ and $(1, 3)$ are balanced corners of Λ_3 and Λ_i $(i \ge 4)$, respectively. Hence, the multiplicity of $L(4, 3, 3)$ in $L(5, 3, 1, 1) \otimes L(4, 1^6)$ is 2 and that of $L(4, 3, 3)$ in $L(5, 3, 1, 1) \otimes L(5, 1^5)$ is 5.

Example 4.15. (See [14, Theorem 3].) Let λ be $(n-e,1^e)$ and let μ be $(n-f,1^f)$, where $2e,2f \le n$ and $g := f-e \ge 0$. Then $\mathrm{PH}_{g+i}(\lambda,\mu) \neq \emptyset$ if and only if either $e+f<n$ and $0 \le i \le 2e$, or $e+f=n$ and $0 \le i \le n-2$. In this case, we have $\mathrm{PH}_{g+i}(\lambda,\mu) = \{\mathsf{V}_i\}$, where $T_i = R_{row} \circ \mathsf{V}_i$ are given by



(4.24)

4.5. The exactness

In this section, we will give a proof of Lemma 4.12. Throughout this section, we fix a picture Λ of type $(\lambda,\mu;\zeta)$, which has no balanced cocorners. For each $\mathbf{z} \in \overline{\mathrm{ICC}}(\lambda/\zeta)$, define $\mathbf{u}(\mathbf{z}) \in \overline{\mathrm{ICC}}((\mu/\zeta)^{\mathsf{t}})$ by $\mathbf{u}(\mathbf{z}) \xrightarrow{\Lambda} \mathbf{z}$. We also use the following notations:

$$
\begin{aligned}
l &= l((\mu/\zeta)^{\mathsf{t}}),\\
\mathbf{u}_{--} := \min\nolimits_{\le_{\swarrow}} \overline{\mathrm{ICC}}((\mu/\zeta)^{\mathsf{t}}), &\quad \mathbf{u}_{++} := \max\nolimits_{\le_{\swarrow}} \overline{\mathrm{ICC}}((\mu/\zeta)^{\mathsf{t}}),\\
\mathbf{z}_{--} := \min\nolimits_{\le_{\swarrow}} \overline{\mathrm{ICC}}(\lambda/\zeta), &\quad \mathbf{z}_{++} := \max\nolimits_{\le_{\swarrow}} \overline{\mathrm{ICC}}(\lambda/\zeta),\\
C_- &:= \{\mathbf{z} \in \overline{\mathrm{ICC}}(\lambda/\zeta)) \mid \mathbf{z} <_{\swarrow} \mathbf{u}(\mathbf{z})^{\mathsf{t}}\},\\
C_+ &:= \{\mathbf{z} \in \overline{\mathrm{ICC}}(\lambda/\zeta)) \mid \mathbf{u}(\mathbf{z})^{\mathsf{t}} <_{\swarrow} \mathbf{z}\}.
\end{aligned}
$$

Example 4.16. Let Λ be as in Example 4.9 and let $\mathbf{z}_i$ be as in the first figure of (4.3). Then we have $\mathbf{z}_0 <_{\swarrow} (0,4) = \mathbf{u}(\mathbf{z}_0)^{\mathsf{t}}$, $\mathbf{z}_1 <_{\swarrow} \mathbf{z}_2 <_{\swarrow} \mathbf{z}_3 = \mathbf{u}(\mathbf{z}_1)^{\mathsf{t}} = \mathbf{u}(\mathbf{z}_2)^{\mathsf{t}}$ and $\mathbf{u}(\mathbf{z}_3)^{\mathsf{t}} = \mathbf{u}(\mathbf{z}_4)^{\mathsf{t}} = \mathbf{z}_2 <_{\swarrow} \mathbf{z}_3 <_{\swarrow} \mathbf{z}_4$. Hence $C_- = \{\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2\}$ and $C_+ = \{\mathbf{z}_3, \mathbf{z}_4\}$.

Lemma 4.17. *We have* $\mathbf{u}(\mathbf{z}_{--}) = \mathbf{u}_{--} <_{\swarrow} \mathbf{u}(\mathbf{z}_{++})$.

Proof. Let $R\colon (D(\lambda/\zeta) \amalg \{\mathbf{z}_{--}, \mathbf{z}_{++}\}, \le_{\swarrow}) \to \mathbb{Z}_{>0}$ be an injective map of ordered sets. Then, we have $R(\mathbf{z}_{--}) < T(\mathbf{x}) < R(\mathbf{z}_{++})$ for every $\mathbf{x} \in D((\mu/\zeta)^{\mathsf{t}})$, where $T := R \circ \Lambda$. On the other hand, by Lemma 4.11, the last row $D_l((\mu/\zeta)^{\mathsf{t}})$ of $(\mu/\zeta)^{\mathsf{t}}$ is not empty. Hence the assertion follows immediately from the definition of the backward row insertion. □

Lemma 4.18. *We have* $\mathbf{z}_{--} \in C_-$ *and* $\mathbf{z}_{++} \in C_+$. *In particular,* $C_-, C_+ \neq \emptyset$.

Proof. Suppose that $\mathbf{u}(\mathbf{z}_{++})$ is not extreme. Then we have $\mathbf{u}(\mathbf{z}_{++})^{\mathsf{t}} \in \mathrm{ICC}(\lambda/\zeta)$. Hence $\mathbf{u}(\mathbf{z}_{++})^{\mathsf{t}} \le_{\swarrow} \mathbf{z}_{++}$. Since $\mathbf{z}_{++}$ is not a balanced cocorner even if $\mathbf{z}_{++} \in \mathrm{ICC}(\lambda/\zeta)$, this

proves $\mathbf{z}_{++} \in C_+$. Next, suppose that $\mathbf{u}(\mathbf{z}_{++})$ is extreme. By Lemma 4.17, we have $\mathbf{u}(\mathbf{z}_{++}) = \mathbf{u}_{++} = (0, (\mu^{\mathrm{t}})_1)$. Hence $\mathbf{z}_{++} \in C_+$. The proof of $\mathbf{z}_{--} \in C_-$ is similar and more easy, since $\mathbf{u}(\mathbf{z}_{--}) = \mathbf{u}_{--}$. □

In view of the lemma above, we may define $\mathbf{z}_- \in C_-$ and $\mathbf{z}_- \in C_+$ by

$$\mathbf{z}_- := \max_{\leq_{\swarrow}} C_-, \quad \mathbf{z}_+ := \min_{\leq_{\swarrow}} C_+.$$

Lemma 4.19.

(1) *If* $\mathbf{z} \in \overline{\mathrm{ICC}}(\lambda/\zeta)$ *satisfies* $\mathbf{z} \leq_{\swarrow} \mathbf{z}_-$, *then* $\mathbf{z} \in C_-$.
(2) *For each* $\mathbf{z} \in C_-$ *and* $\mathbf{z}' \in C_+$, *we have* $\mathbf{z} <_{\swarrow} \mathbf{z}'$. *In particular,* $\mathbf{z}_+$ *is the successor of* $\mathbf{z}_-$ *in* $(\overline{\mathrm{ICC}}(\lambda/\zeta), \leq_{\swarrow})$.

Proof. If $z \in \overline{\mathrm{ICC}}(\lambda/\zeta)$ satisfies $\mathbf{z} \leq_{\swarrow} \mathbf{z}_-$, then we have $\mathbf{u}(\mathbf{z}) \leq_{\swarrow} \mathbf{u}(\mathbf{z}_-)$ by Lemma 4.8 (1). Hence $\mathbf{z} \leq_{\swarrow} \mathbf{z}_- <_{\swarrow} \mathbf{u}(\mathbf{z}_-)^{\mathrm{t}} \leq_{\swarrow} \mathbf{u}(\mathbf{z})^{\mathrm{t}}$. This proves Part (1). Part (2) follows immediately from Part (1). □

By (4.7) and Part (2) of the lemma above, there exists a unique cocorner $\mathbf{w}_B$ of ζ such that

$$\mathbf{z}_- <_{\swarrow} \mathbf{w}_B <_{\swarrow} \mathbf{z}_+. \tag{4.25}$$

We will show that $\mathbf{v}_B := (\mathbf{w}_B)^{\mathrm{t}}$ is a balanced corner of Λ.

Lemma 4.20. *The point* $\mathbf{v}_B$ *is a removable corner of* Λ *and satisfies*

$$\mathbf{u}(\mathbf{z}_-) <_{\swarrow} \mathbf{v}_B <_{\swarrow} \mathbf{u}(\mathbf{z}_+).$$

Proof. We give a proof of the second inequality. The proof of the first inequality is similar and the first assertion follows from the second inequality and Lemma 4.8 (6). Suppose that $\mathbf{u}(\mathbf{z}_+)$ is not an extreme cocorner of $(\mu/\zeta)^{\mathrm{t}}$. Then, we have $\mathbf{u}(\mathbf{z}_+)^{\mathrm{t}} \in \mathrm{ICC}(\lambda/\zeta)$. Since $\mathbf{z}_-$ is the predecessor of $\mathbf{z}_+$ in $\overline{\mathrm{ICC}}(\lambda/\zeta)$, $\mathbf{z}_+ \in C_+$ implies $\mathbf{u}(\mathbf{z}_+)^{\mathrm{t}} \leq_{\swarrow} \mathbf{z}_-$. Hence we have $\mathbf{v}_B <_{\swarrow} \mathbf{u}(\mathbf{z}_+)$ by (4.25). Next, suppose that $\mathbf{u}(\mathbf{z}_+)$ is an extreme cocorner of $(\mu/\zeta)^{\mathrm{t}}$. Since the inequality is obvious if $\mathbf{u}(\mathbf{z}_+) = \mathbf{u}_{++}$, it suffices to show that $\mathbf{u}(\mathbf{z}_+) \neq \mathbf{u}_{--}$. Suppose that $\mathbf{u}(\mathbf{z}_+) = \mathbf{u}_{--}$ to the contrary. By Lemma 4.17, we have $\mathbf{z}_+ \neq \mathbf{z}_{++}$. Hence $\mathbf{z}_+ <_{\swarrow} (\mathbf{u}_{--})^{\mathrm{t}} = \mathbf{u}(\mathbf{z}_+)^{\mathrm{t}}$. This contradicts to the fact that $\mathbf{z}_+ \in C_+$. □

Define $\mathbf{w}$ by $\mathbf{v}_B \underset{\Lambda}{\text{---}} \mathbf{w}$. Then we have $\mathbf{z}_- <_{\swarrow} \mathbf{w} <_{\swarrow} \mathbf{z}_+$ by Lemma 4.8 (5), (6) and the lemma above. Since $\mathbf{w}_B \in \mathrm{IC}(\lambda/\zeta)$ is characterized by (4.25), this proves $\mathbf{w} = \mathbf{w}_B$. Thus, we get $\mathbf{v}_B \underset{\Lambda}{\text{---}} (\mathbf{v}_B)^{\mathrm{t}}$, which completes the proof of Lemma 4.12.

Note. After this paper was submitted, another description of $m^{\mu}_{\lambda,(n-m,1^m)}$ was obtained by J. Blasiak: arXiv:1209.2018.

References

[1] C.M. Ballantine, R.C. Orellana, A combinatorial interpretation for the coefficients in the Kronecker product $s_{(n-p,p)} * s_\lambda$, Sém. Lothar. Combin. 54A (2005/07), Art. B54Af, 29 pp. (electronic).
[2] C.W. Curtis, I. Reiner, Methods of Representation Theory, with Applications to Finite Groups and Orders, vol. I, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1981.
[3] S. Fomin, C. Greene, A Littlewood–Richardson miscellany, European J. Combin. 14 (1993) 191–212.
[4] W. Fulton, Young Tableaux, with Applications to Representation Theory and Geometry, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.
[5] A.M. Garsia, J. Remmel, Shuffles of permutations and the Kronecker product, Graphs Combin. 1 (1985) 217–263.
[6] R. Goodman, N.R. Wallach, Representations and Invariants of the Classical Groups, Cambridge University Press, Cambridge, 1998.
[7] A. Lascoux, Produit de Kronecker des representations du groupe symmetrique, in: Lecture Notes in Math., vol. 795, Springer, Berlin, 1980, pp. 319–329.
[8] M.A.A. van Leeuwen, Tableau algorithms defined naturally for pictures, Discrete Math. 157 (1996) 321–362.
[9] P. Littelmann, A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras, Invent. Math. 116 (1994) 329–346.
[10] F. Murnaghan, The analysis of the Kronecker product of irreducible representations of the symmetric group, Amer. J. Math. 60 (1938) 761–784.
[11] J.B. Remmel, Formulas for the expansion of the Kronecker products $S_{(m,n)} \otimes S_{(1^{p-r},r)}$ and $S_{(1^k,2^l)} \otimes S_{(1^{p-r},r)}$, Discrete Math. 99 (1992) 265–287.
[12] J.B. Remmel, T. Whitehead, On the Kronecker product of Schur functions of two row shapes, Bull. Belg. Math. Soc. Simon Stevin 1 (1994) 649–683.
[13] J.B. Remmel, R. Whitney, Multiplying Schur functions, J. Algorithms 5 (1984) 471–487.
[14] M.H. Rosas, The Kronecker product of Schur functions indexed by two-row shapes or hook shapes, J. Algebraic Combin. 14 (2001) 153–173.
[15] B.E. Sagan, R.P. Stanley, Robinson–Schensted algorithms for skew tableaux, J. Combin. Theory Ser. A 55 (1990) 161–193.
[16] A.V. Zelevinsky, A generalization of the Littlewood–Richardson rule and the Robinson–Schensted–Knuth correspondence, J. Algebra 69 (1981) 82–94.