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Hall algebras of cyclic quivers and q-deformed Fock spaces ☆

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ABSTRACT

Based on the work of Ringel and Green, one can define the (Drinfeld) double Ringel–Hall algebra $\boldsymbol{\mathcal{D}}(Q)$ of a quiver Q as well as its highest weight modules. The main purpose of the present paper is to show that the basic representation $L(\Lambda_0)$ of $\boldsymbol{\mathcal{D}}(\Delta_n)$ of the cyclic quiver Δ_n provides a realization of the q-deformed Fock space $\bigwedge^\infty$ defined by Hayashi. This is worked out by extending a construction of Varagnolo and Vasserot. By analysing the structure of nilpotent representations of Δ_n, we obtain a decomposition of the basic representation $L(\Lambda_0)$ which induces the Kashiwara–Miwa–Stern decomposition of $\bigwedge^\infty$ and a construction of the canonical basis of $\bigwedge^\infty$ defined by Leclerc and Thibon in terms of certain monomial basis elements in $\boldsymbol{\mathcal{D}}(\Delta_n)$.

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1. Introduction

In [36], Ringel introduced the Hall algebra $\boldsymbol{\mathcal{H}}(\Delta_n)$ of the cyclic quiver Δ_n with n vertices and showed that its subalgebra generated by simple representations, called the composition algebra, is isomorphic to the positive part $\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n)$ of the quantized enveloping algebra $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$. Schiffmann [37] further showed that $\boldsymbol{\mathcal{H}}(\Delta_n)$ is the tensor product of $\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n)$ with a central subalgebra which is the polynomial ring in infinitely many indeterminates. Following the approach in [42], the double Ringel–Hall algebra $\boldsymbol{\mathcal{D}}(\Delta_n)$ was defined in [6]. Based on [12,19] and an explicit description of central elements of $\boldsymbol{\mathcal{H}}(\Delta_n)$ in [18], it was shown in [6, Th. 2.3.3] that $\boldsymbol{\mathcal{D}}(\Delta_n)$ is isomorphic to the quantum affine algebra $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$ defined by Drinfeld's new presentation [10].

The q-deformed Fock space representation $\bigwedge^\infty$ of the quantized enveloping algebra $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ has been constructed by Hayashi [16], and its crystal basis was described by Misra and Miwa [32]. Further, by work of Kashiwara, Miwa, and Stern [24], the action of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ on $\bigwedge^\infty$ is centralized by a Heisenberg algebra which arises from affine Hecke algebras. This yields a bimodule isomorphism from $\bigwedge^\infty$ to the tensor product of the basic representation of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ and the Fock space representation of the Heisenberg algebra.

By defining a natural semilinear involution on $\bigwedge^\infty$, Leclerc and Thibon [26] obtained in an elementary way a canonical basis of $\bigwedge^\infty$. It was conjectured in [25,26] that for $q = 1$, the coefficients of the transition matrix of the canonical basis on the natural basis of $\bigwedge^\infty$ are equal to the decomposition numbers for Hecke algebras and quantum Schur algebras at roots of unity. These conjectures have been proved, respectively, by Ariki [1] and Varagnolo and Vasserot [43]. For the categorification of the Fock space, see, for example, [39,17,41].

In [43], Varagnolo and Vasserot extended the $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$-action on the Fock space $\bigwedge^\infty$ to that of the extended Ringel–Hall algebra $\boldsymbol{\mathcal{D}}(\Delta_n)^{\leqslant 0}$ of the cyclic quiver Δ_n. They also showed that the canonical basis of the Ringel–Hall algebra $\boldsymbol{\mathcal{H}}(\Delta_n)$ in the sense of Lusztig induces a basis of $\bigwedge^\infty$ which conjecturally coincides with the canonical basis constructed by Leclerc and Thibon [26]. This conjecture was proved by Schiffmann [37] by identifying the central subalgebra of $\boldsymbol{\mathcal{H}}(\Delta_n)$ with the ring of symmetric functions.

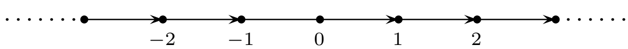
The main purpose of the present paper is to extend Varagnolo–Vasserot's construction to obtain a $\boldsymbol{\mathcal{D}}(\Delta_n)$-module structure on the Fock space $\bigwedge^\infty$ which is shown to be isomorphic to the basic representation $L(\Lambda_0)$ of $\boldsymbol{\mathcal{D}}(\Delta_n)$. Moreover, the central elements in the positive and negative parts of $\boldsymbol{\mathcal{D}}(\Delta_n)$ constructed by Hubery [18] give rise naturally to the operators introduced in [24] which generate the Heisenberg algebra. Furthermore, the structure of $\boldsymbol{\mathcal{D}}(\Delta_n)$ yields a decomposition of $L(\Lambda_0)$ which induces the Kashiwara–Miwa–Stern decomposition of $\bigwedge^\infty$. This also provides a way to construct the canonical basis of $\bigwedge^\infty$ in [26] in terms of certain monomial basis elements of $\boldsymbol{\mathcal{D}}(\Delta_n)$.

The paper is organized as follows. In Section 2 we review the classification of (nilpotent) representations of both infinite linear quiver Δ_∞ and the cyclic quiver Δ_n with n vertices and discuss their generic extensions. Section 3 recalls the definition of Ringel–Hall algebras $\boldsymbol{\mathcal{H}}(\Delta_\infty)$ and $\boldsymbol{\mathcal{H}}(\Delta_n)$ of Δ_∞ and Δ_n as well as the maps from the homogeneous spaces of $\boldsymbol{\mathcal{H}}(\Delta_n)$ to those of $\boldsymbol{\mathcal{H}}(\Delta_\infty)$ introduced in [43]. The images of basis elements of $\boldsymbol{\mathcal{H}}(\Delta_n)$ under these maps are described. In Section 4 we first follow the approach in [42] to present the construction of double Ringel–Hall algebras of both Δ_∞ and Δ_n and then study the irreducible highest weight $\boldsymbol{\mathcal{D}}(\Delta_n)$-modules based on the results in [21]. Section 5 recalls from [16,32,43] the Fock space representation $\bigwedge^\infty$ over $\mathbf{U}_v(\widehat{\mathfrak{sl}}_\infty)$ ($\cong \boldsymbol{\mathcal{D}}(\Delta_\infty)$) as well as over $\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n)$. In Section 6 we define the $\boldsymbol{\mathcal{D}}(\Delta_n)$-module structure on $\bigwedge^\infty$ based on [24,43]. It is shown in Section 7 that $\bigwedge^\infty$ is isomorphic to the basic representation of $\boldsymbol{\mathcal{D}}(\Delta_n)$. In the final section, we present a way to construct the canonical basis of $\bigwedge^\infty$ and interpret the "ladder method" construction of certain basis elements in $\bigwedge^\infty$ in terms of generic extensions of nilpotent representations of Δ_n.

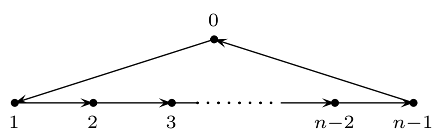
2. Nilpotent representations and generic extensions

In this section we consider nilpotent representations of both a cyclic quiver $\Delta = \Delta_n$ with n vertices ($n \geqslant 2$) and the infinite quiver $\Delta = \Delta_\infty$ of type A_∞^∞ and study their generic extensions. We show that the degeneration order of nilpotent representations of Δ_n induces the dominant order of partitions.

Let Δ_∞ denote the infinite quiver of type A_∞^∞



with vertex set $I = I_\infty = \mathbb{Z}$, and for $n \geqslant 2$, let Δ_n denote the cyclic quiver



with vertex set $I = I_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$. For each $i \in I_\infty = \mathbb{Z}$, let $\bar{i}$ denote its residue class in $I_n = \mathbb{Z}/n\mathbb{Z}$. We also simply write $\bar{i} \pm 1$ to denote the residue class of $i \pm 1$ in $\mathbb{Z}/n\mathbb{Z}$.

Given a field k, we denote by $\operatorname{Rep}^0 \Delta$ the category of finite dimensional nilpotent representations of Δ ($= \Delta_\infty$ or Δ_n) over k. (Note that each finite dimensional representation of Δ_∞ is automatically nilpotent.) Given a representation $V = (V_i, V_\rho) \in \operatorname{Rep}^0 \Delta$, the vector $\mathbf{dim} V = (\dim_k V_i)_{i \in I}$ is called the *dimension vector* of V. The Grothendieck group of $\operatorname{Rep}^0 \Delta$ is identified with the free abelian group $\mathbb{Z}I$ with basis I. Let $\{\varepsilon_i \mid i \in I\}$ denote the standard basis of $\mathbb{Z}I$. Thus, elements in $\mathbb{Z}I$ will be written as $\mathbf{d} = (d_i)_{i \in I}$ or $\mathbf{d} = \sum_{i \in I} d_i \varepsilon_i$. In case $I = \mathbb{Z}/n\mathbb{Z}$, we sometimes write $\mathbb{Z}^n$ for $\mathbb{Z}I$.

The Euler form $\langle -, - \rangle : \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ is defined by

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle = \dim_k \operatorname{Hom}_{k\Delta}(M, N) - \dim_k \operatorname{Ext}^1_{k\Delta}(M, N).$$

Its symmetrization

$$(\mathbf{dim}\, M, \mathbf{dim}\, N) = \langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle + \langle \mathbf{dim}\, N, \mathbf{dim}\, M\rangle$$

is called the symmetric Euler form.

It is well known that the isoclasses (isomorphism classes) of representations in $\mathrm{Rep}^0\,\Delta$ are parametrized by the set $\mathfrak{M}$ consisting of all multisegments

$$\mathfrak{m} = \sum_{i\in I,\, l\geqslant 1} m_{i,l}[i,l),$$

where all $m_{i,l} \in \mathbb{N}$ but finitely many are zero. More precisely, the representation $M(\mathfrak{m}) = M_k(\mathfrak{m})$ associated with $\mathfrak{m}$ is defined by

$$M(\mathfrak{m}) = \bigoplus_{i\in I, l\geqslant 1} m_{i,l} S_i[l],$$

where $S_i[l]$ denotes the indecomposable representation of Δ with the simple top S_i and length l. For each $\mathbf{d} \in \mathbb{N}I$, put

$$\mathfrak{M}^{\mathbf{d}} = \{\mathfrak{m} \in \mathfrak{M} \mid \mathbf{dim} M(\mathfrak{m}) = \mathbf{d}\}.$$

Furthermore, we will write $\mathfrak{M} = \mathfrak{M}_\infty$ (resp., $\mathfrak{M} = \mathfrak{M}_n$) if $I = \mathbb{Z}$ (resp., $I = \mathbb{Z}/n\mathbb{Z}$).

It is also known that there exist Auslander–Reiten sequences in $\mathrm{Rep}^0\,\Delta$, that is, for each $M \in \mathrm{Rep}^0\,\Delta$, there is an Auslander–Reiten sequence

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0,$$

where τM denotes the Auslander–Reiten translation of M. It is clear that τ induces an isomorphism $\tau : \mathbb{Z}I \to \mathbb{Z}I$ such that $\tau(\mathbf{dim}\, M) = \mathbf{dim}\, \tau M$. In particular, $\tau(\varepsilon_i) = \varepsilon_{i+1}$, $\forall\, i \in I$. If $\Delta = \Delta_n$, then $\tau^{sn} = \mathrm{id}$ for all $s \in \mathbb{Z}$. For $\mathfrak{m} \in \mathfrak{M}$, let $\tau\mathfrak{m}$ be defined by $M(\tau\mathfrak{m}) \cong \tau M(\mathfrak{m})$.

Given $\mathbf{d} \in \mathbb{N}I$, let $V = \oplus_{i\in I} V_i$ be an I-graded vector space with dimension vector $\mathbf{d}$. Consider

$$E_V = \{(x_i) \in \bigoplus_{i\in I} \mathrm{Hom}_k(V_i, V_{i+1}) \mid x_{n-1}\cdots x_0 \text{ is nilpotent if } \Delta = \Delta_n.\}.$$

Then each element $x \in E_V$ defines a representation (V, x) of dimension vector $\mathbf{d}$ in $\mathrm{Rep}^0\,\Delta$. Moreover, the group

$$G_V = \prod_{i\in I} \mathrm{GL}(V_i)$$

acts on E_V by conjugation, and there is a bijection between the G_V-orbits and the isoclasses of representations in $\operatorname{Rep}^0 \Delta$ of dimension vector $\mathbf{d}$. For each $x \in E_V$, by $\mathcal{O}_x$ we denote the G_V-orbit of x. In case k is algebraically closed, we have the equalities

$$\dim \mathcal{O}_x = \dim G_V - \dim \operatorname{End}_{k\Delta}(V, x) = \sum_{i \in I} d_i^2 - \dim \operatorname{End}_{k\Delta}(V, x). \tag{2.0.1}$$

By abuse of notation, for each $M \in \operatorname{Rep}^0 \Delta$, we denote by $\mathcal{O}_M$ the orbit of M.

Following [3,33,5], given two representations M, N in $\operatorname{Rep}^0 \Delta$, there exists a unique (up to isomorphism) extension G of M by N such that $\dim \operatorname{End}_{k\Delta}(G)$ is minimal. The extension G is called the *generic extension* of M by N, denoted by $M * N$. Moreover, generic extensions satisfy the associativity, i.e., for $L, M, N \in \operatorname{Rep}^0 \Delta$,

$$L * (M * N) \cong (L * M) * N.$$

Let $\mathcal{M}(\Delta)$ denote the set of isoclasses of representations in $\operatorname{Rep}^0 \Delta$. Define a multiplication on $\mathcal{M}(\Delta)$ by setting

$$[M] * [N] = [M * N].$$

Then $\mathcal{M}(\Delta)$ is a monoid with identity $[0]$, the isoclass of zero representation of Δ.

By [33,5], the generic extension $M*N$ can be also characterized as the unique maximal element among all the extensions of M by N with respect to the degeneration order $\leqslant_{\rm deg}$ which is defined by setting $M \leqslant_{\rm deg} N$ if $\mathbf{dim}\, M = \mathbf{dim}\, N$ and

$$\dim_k \operatorname{Hom}_{k\Delta}(M, X) \geqslant \dim_k \operatorname{Hom}_{k\Delta}(N, X), \quad \text{for all } X \in \operatorname{Rep}^0 \Delta. \tag{2.0.2}$$

If k is algebraically closed, then $M \leqslant_{\rm deg} N$ if and only if $\overline{\mathcal{O}}_M \subseteq \mathcal{O}_N$, where $\overline{\mathcal{O}}_M$ is the closure of $\mathcal{O}_M$. This defines a partial order relation on the set $\mathcal{M}(\Delta)$ of isoclasses of representations in $\operatorname{Rep}^0 \Delta$; see [44, Th. 2] or [5, Lem. 3.2]. By [33, 2.4], for $M, N, M', N' \in \operatorname{Rep}^0 \Delta$,

$$M' \leqslant_{\rm deg} M, N' \leqslant_{\rm deg} N \Longrightarrow M' * N' \leqslant_{\rm deg} M * N.$$

For $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{M}_n$ (resp., $\mathfrak{M}_\infty$), we write $\mathfrak{m} \leqslant_{\rm deg} \mathfrak{m}'$ (resp., $\mathfrak{m} \leqslant_{\rm deg}^{\infty} \mathfrak{m}'$) if $M(\mathfrak{m}) \leqslant_{\rm deg} M(\mathfrak{m}')$ in $\operatorname{Rep}^0 \Delta_n$ (resp., $\operatorname{Rep} \Delta_\infty$).

By [4,13], there is a covering functor

$$\mathscr{F} : \operatorname{Rep} \Delta_\infty \longrightarrow \operatorname{Rep}^0 \Delta_n$$

sending $S_i[l]$ to $S_{\bar{i}}[l]$ for $i \in \mathbb{Z}$ and $l \geqslant 1$. Moreover, $\mathscr{F}$ is dense and exact, and the Galois group of $\mathscr{F}$ is the infinite cyclic group G generated by τ^n, i.e., $\tau^n(S_i[l]) = S_{i+n}[l]$. For $\mathfrak{m} \in \mathfrak{M}_\infty$, let $\mathscr{F}(\mathfrak{m}) \in \mathfrak{M}_n$ be such that $M(\mathscr{F}(\mathfrak{m})) \cong \mathscr{F}(M(\mathfrak{m})) \in \operatorname{Rep}^0 \Delta_n$. From (2.0.2) we easily deduce that for $M, N \in \operatorname{Rep} \Delta_\infty$,

$$M \leqslant_{\deg} N \Longrightarrow \mathscr{F}(M) \leqslant_{\deg} \mathscr{F}(N). \tag{2.0.3}$$

The following two classes of representations will play an important role later on. For each $\mathbf{d} = (d_i) \in \mathbb{N}I$, we set

$$S_{\mathbf{d}} = \bigoplus_{i \in I} d_i S_i \in \operatorname{Rep}^0 \Delta.$$

In other words, $S_{\mathbf{d}}$ is the unique semisimple representation of dimension vector $\mathbf{d}$.

Let Π be the set of all partitions $\lambda = (\lambda_1, \ldots, \lambda_t)$ (i.e., $\lambda_1 \geqslant \cdots \geqslant \lambda_t \geqslant 1$). For each $\lambda \in \Pi$, define

$$\mathfrak{m}_\lambda = \sum_{s=1}^{t} [1 - s, \lambda_s) \in \mathfrak{M}.$$

Then

$$M(\mathfrak{m}_\lambda) = S_0[\lambda_1] \oplus S_{-1}[\lambda_2] \oplus \cdots \oplus S_{1-t}[\lambda_t] \in \operatorname{Rep}^0 \Delta.$$

If $\Delta = \Delta_\infty$, then we sometimes write $\mathfrak{m}_\lambda = \mathfrak{m}_\lambda^\infty \in \mathfrak{M}_\infty$ to make a distinction. It follows from the definition that $\mathscr{F}(\mathfrak{m}_\lambda^\infty) = \mathfrak{m}_\lambda$ for all $\lambda \in \Pi$.

Proposition 2.1. *Let $\lambda, \mu \in \Pi$.*

(1) *If $\Delta = \Delta_\infty$, then*

$$\mathbf{dim}\, M(\mathfrak{m}_\mu^\infty) = \mathbf{dim}\, M(\mathfrak{m}_\lambda^\infty) \Longleftrightarrow \mu = \lambda.$$

In particular, for each $\mathfrak{m} \in \mathfrak{M}_\infty$, there exists at most one $\nu \in \Pi$ such that $\mathfrak{m} = \mathfrak{m}_\nu^\infty$.

(2) *If $\Delta = \Delta_n$, then*

$$M(\mathfrak{m}_\mu) \leqslant_{\deg} M(\mathfrak{m}_\lambda) \Longrightarrow \mu \trianglelefteq \lambda,$$

where $\trianglelefteq$ is the dominance order on Π, i.e., $\mu \trianglelefteq \lambda \Longleftrightarrow \sum_{j=1}^{i} \mu_j \leqslant \sum_{j=1}^{i} \lambda_j$, $\forall\, i \geqslant 1$.

Proof. (1) By definition, both the socles of $M(\mathfrak{m}_\lambda^\infty)$ and $M(\mathfrak{m}_\mu^\infty)$ are multiplicity-free. Thus, comparing the socles of $S_0[\lambda_1]$ and $S_0[\mu_1]$ gives $\lambda_1 = \mu_1$. The lemma then follows from an inductive argument.

(2) Suppose $M(\mathfrak{m}_\mu) \leqslant_{\deg} M(\mathfrak{m}_\lambda)$. By viewing $\mathfrak{m}_\lambda$ and $\mathfrak{m}_\mu$ as multipartitions in $\mathfrak{M}_n$, we obtain by [7, Prop. 2.7] that for each $l \geqslant 1$,

$$\sum_{s=1}^{l} \widetilde{\mu}_s \geqslant \sum_{s=1}^{l} \widetilde{\lambda}_s,$$

where $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots)$ and $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$ are the dual partition of λ and μ, respectively, that is, $\widetilde{\mu} \trianglerighteq \widetilde{\lambda}$. By [31, 1.1], $\mu \trianglelefteq \lambda$. $\square$

3. Ringel–Hall algebra of the quiver $\boldsymbol{\Delta}$

In this section we introduce the Ringel–Hall algebra $\mathcal{H}(\Delta)$ of Δ ($= \Delta_n$ or Δ_∞) and the maps from homogeneous subspaces of $\mathcal{H}(\Delta_n)$ to those of $\mathcal{H}(\Delta_\infty)$ defined in [43, 6.1]. We also describe the images of basis elements of $\mathcal{H}(\Delta_n)$ under these maps.

The cyclic quiver Δ_n gives the $n \times n$ Cartan matrix $C_n = (a_{ij})_{i,j \in I}$ of type $\widehat{A}_{n-1}$, while Δ_∞ defines the infinite Cartan matrix $C_\infty = (a_{ij})_{i,j \in \mathbb{Z}}$. Thus, we have the associated quantum enveloping algebras $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ and $\mathbf{U}_v(\mathfrak{sl}_\infty)$ which are $\mathbb{Q}(v)$-algebras with generators $K_i^{\pm 1}, E_i, F_i, D^{\pm 1}$ $(i \in I = \mathbb{Z}/n\mathbb{Z})$ and $K_i^{\pm 1}, E_i, F_i$ $(i \in \mathbb{Z})$, respectively, and the quantum Serre relations. In particular, the relations involving the generator $D^{\pm 1}$ in $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ are

$$DD^{-1} = 1 = D^{-1}D,\ K_iD = DK_i,\ DE_i = v^{\delta_{0,i}}E_iD,\ DF_i = v^{-\delta_{0,i}}F_iD, \quad \forall\, i \in I;$$

see [2, Def. 3.16]. The subalgebra of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ generated by $K_i^{\pm 1}, E_i, F_i$ $(i \in I = \mathbb{Z}/n\mathbb{Z})$ is denoted by $\mathbf{U}_v'(\widehat{\mathfrak{sl}}_n)$; see [24, 1.1].

By [34,36,15], for $\mathfrak{p}, \mathfrak{m}_1, \ldots, \mathfrak{m}_t \in \mathfrak{M}$, there is a polynomial $\varphi^{\mathfrak{p}}_{\mathfrak{m}_1,\ldots,\mathfrak{m}_t}(q) \in \mathbb{Z}[q]$ (called Hall polynomial) such that for each finite field k,

$$\varphi^{\mathfrak{p}}_{\mathfrak{m}_1,\ldots,\mathfrak{m}_t}(|k|) = F^{M_k(\mathfrak{p})}_{M_k(\mathfrak{m}_1),\ldots,M_k(\mathfrak{m}_t)},$$

which is by definition the number of the filtrations

$$M_k(\mathfrak{p}) = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{t-1} \supseteq M_t = 0$$

such that $M_{s-1}/M_s \cong M_k(\mathfrak{m}_s)$ for all $1 \le s \le t$. By [35, Sect. 2], for each $\mathfrak{m} \in \mathfrak{M}$, there is a polynomial $a_{\mathfrak{m}}(q) \in \mathbb{Z}[q]$ such that for each finite field k,

$$a_{\mathfrak{m}}(|k|) = |\operatorname{Aut}_{k\Delta}(M_k(\mathfrak{m}))|.$$

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ be the Laurent polynomial ring over $\mathbb{Z}$ in indeterminate v. By definition, the (twisted generic) *Ringel–Hall algebra* $\mathcal{H}(\Delta)$ of Δ is the free $\mathcal{Z}$-module with basis $\{u_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}\}$ and multiplication given by

$$u_{\mathfrak{m}}u_{\mathfrak{m}'} = v^{\langle \mathbf{dim}\, M(\mathfrak{m}), \mathbf{dim} M(\mathfrak{m}')\rangle} \sum_{\mathfrak{p} \in \mathfrak{M}} \varphi^{\mathfrak{p}}_{\mathfrak{m},\mathfrak{m}'}(v^2) u_{\mathfrak{p}}. \tag{3.0.1}$$

In practice, we also write $u_{\mathfrak{m}} = u_{[M(\mathfrak{m})]}$ in order to make certain calculations in terms of modules. Furthermore, for each $\mathbf{d} \in \mathbb{N}I$, we simply write $u_{\mathbf{d}} = u_{[S_{\mathbf{d}}]}$.

For each $i \in I$, set $u_i = u_{[S_i]}$. We then denote by $\mathcal{C}(\Delta)$ the subalgebra of $\mathcal{H}(\Delta)$ generated by the divided power $u_i^{(t)} = u_i^t/[t]^!$, $i \in I$ and $t \geqslant 1$, called the *composition algebra* of Δ, where

$$[t]^! = [t][t-1]\cdots[1] \text{ with } [m] = (v^m - v^{-m})/(v - v^{-1}). \tag{3.0.2}$$

Moreover, both $\mathcal{H}(\Delta)$ and $\mathcal{C}(\Delta)$ are $\mathbb{N}I$-graded:

$$\mathcal{H}(\Delta) = \bigoplus_{\mathbf{d}\in\mathbb{N}I} \mathcal{H}(\Delta)_{\mathbf{d}} \text{ and } \mathcal{C}(\Delta) = \bigoplus_{\mathbf{d}\in\mathbb{N}I} \mathcal{C}(\Delta)_{\mathbf{d}}, \tag{3.0.3}$$

where $\mathcal{H}(\Delta)_{\mathbf{d}}$ is spanned by all $u_{\mathfrak{m}}$ with $\mathfrak{m} \in \mathfrak{M}^{\mathbf{d}}$ and $\mathcal{C}(\Delta)_{\mathbf{d}} = \mathcal{C}(\Delta) \cap \mathcal{H}(\Delta)_{\mathbf{d}}$. Since the Auslander–Reiten translate $\tau : \mathrm{Rep}^0\,\Delta \to \mathrm{Rep}^0\,\Delta$ is an auto-equivalence, it induces an automorphism $\tau : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta), u_{\mathfrak{m}} \mapsto u_{\tau\mathfrak{m}}$. We also consider the $\mathbb{Q}(v)$-algebras

$$\boldsymbol{\mathcal{H}}(\Delta) = \mathcal{H}(\Delta) \otimes_{\mathcal{Z}} \mathbb{Q}(v) \text{ and } \boldsymbol{\mathcal{C}}(\Delta_n) = \mathcal{C}(\Delta_n) \otimes_{\mathcal{Z}} \mathbb{Q}(v).$$

Remark 3.1. We remark that the Hall algebra of Δ defined in [43] is the opposite algebra of $\mathcal{H}(\Delta)$ given here with v being replaced by v^{-1}. Thus, v and v^{-1} should be swaped when comparing with the formulas in [43].

Following [34], $\mathcal{C}(\Delta_\infty) = \mathcal{H}(\Delta_\infty)$, and there is an isomorphism $\mathbf{U}_v^+(\mathfrak{sl}_\infty) \cong \boldsymbol{\mathcal{H}}(\Delta_\infty)$ taking $E_i \mapsto u_i$, $\forall\, i \in I_\infty = \mathbb{Z}$. But, for $n \geqslant 2$, $\mathcal{C}(\Delta_n)$ is a proper subalgebra of $\mathcal{H}(\Delta_n)$. By [36],

$$\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n) \cong \boldsymbol{\mathcal{C}}(\Delta_n),\ E_i \longmapsto u_i,\ \forall\, i \in I_n.$$

By [37, Th. 2.2], $\boldsymbol{\mathcal{H}}(\Delta_n)$ is decomposed into the tensor product of $\boldsymbol{\mathcal{C}}(\Delta_n)$ and a polynomial ring in infinitely many indeterminates which are central elements in $\boldsymbol{\mathcal{H}}(\Delta_n)$. Such central elements have been explicitly constructed in [18]. More precisely, for each $t \geqslant 1$, let

$$\boldsymbol{c}_t = (-1)^t v^{-2nt} \sum_{\mathfrak{m}} (-1)^{\dim \mathrm{End}(M(\mathfrak{m}))} a_{\mathfrak{m}}(v^2) u_{\mathfrak{m}} \in \mathcal{H}(\Delta_n), \tag{3.1.1}$$

where the sum is taken over all $\mathfrak{m} \in \mathfrak{M}_n$ such that $\mathbf{dim}\, M(\mathfrak{m}) = t\delta$ with $\delta = (1, \ldots, 1) \in \mathbb{N}I_n$, and $\mathrm{soc}\, M(\mathfrak{m})$ is square-free, i.e., $\mathbf{dim}\,\mathrm{soc}\, M(\mathfrak{m}) \leqslant \delta$. The following result is proved in [18].

Theorem 3.2. *The elements c_m are central in $\mathcal{H}(\Delta_n)$. Moreover, there is a decomposition*

$$\boldsymbol{\mathcal{H}}(\Delta_n) = \boldsymbol{\mathcal{C}}(\Delta_n) \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v)[\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots],$$

where $\mathbb{Q}(v)[\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots]$ is the polynomial algebra in $\boldsymbol{c}_t$ for $t \geqslant 1$. In particular, $\boldsymbol{\mathcal{H}}(\Delta_n)$ is generated by u_i and $\boldsymbol{c}_t$ for $i \in I_n$ and $t \geqslant 1$.

For each $\mathfrak{m} \in \mathfrak{M}$, set $d(\mathfrak{m}) = \dim M(\mathfrak{m})$, $\mathbf{d}(\mathfrak{m}) = \mathbf{dim}\, M(\mathfrak{m})$ and define

$$\widetilde{u}_{\mathfrak{m}} = v^{\dim \mathrm{End}_{k\Delta}(M(\mathfrak{m}))-d(\mathfrak{m})} u_{\mathfrak{m}}. \tag{3.2.1}$$

Then $\{\widetilde{u}_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}\}$ is also a $\mathcal{Z}$-basis of $\mathcal{H}(\Delta)$ which plays a role in the construction of the canonical basis. In particular,

$$\widetilde{u}_i = u_i \text{ for each } i \in I \text{ and } \widetilde{u}_{\mathbf{d}} = v^{\sum_i (d_i^2 - d_i)} u_{\mathbf{d}} \text{ for each } \mathbf{d} \in \mathbb{N}I.$$

Consider the map $\pi : \mathbb{Z}I_\infty \to \mathbb{Z}I_n, \mathbf{d} \mapsto \bar{\mathbf{d}}$, where $\pi(\mathbf{d}) = \bar{\mathbf{d}} = (d_{\bar{i}})$ is defined by

$$d_{\bar{i}} = \sum_{j \in \bar{i}} d_j, \ \forall \bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}.$$

In particular, for each representation $M \in \mathrm{Rep}\,\Delta_\infty$, $\mathbf{dim}\, \mathscr{F}(M) = \pi(\mathbf{dim}\, M)$.

In the following we briefly recall from [43, 6.1] the $\mathcal{Z}$-linear map

$$\gamma_{\mathbf{d}} : \mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}} \longrightarrow \mathcal{H}(\Delta_\infty)_{\mathbf{d}} \tag{3.2.2}$$

for each $\mathbf{d} \in \mathbb{N}I_\infty$. These maps play a crucial role in defining an action of $\mathcal{H}(\Delta_n)$ on the Fock space later on.

Let $k = \mathbb{F}_q$ be a finite filed with q elements and let $V = \oplus_{i \in I} V_i$ be an I-graded $\mathbb{F}_q$-vector space with dimension vector $\mathbf{d}$. Then we define $\mathbb{C}_{G_V}(E_V)$ to be the set of G_V-invariant functions $E_V \to \mathbb{C}$, which is a vector space over $\mathbb{C}$. Then $\mathcal{H}(\Delta)_{\mathbf{d}} \otimes_{\mathcal{Z}} \mathbb{C}$ (at $v = \sqrt{q}$) can be identified with $\mathbb{C}_{G_V}(E_V)$ via taking $u_{[(V,x)]}$ to the characteristic function of the G_V-orbit of x in E_V.

Now take $\mathbf{d} \in \mathbb{N}I_\infty$ and let $V = \oplus_{i \in \mathbb{Z}} V_i$ be an I_∞-graded $\mathbb{F}_q$-vector space of dimension vector $\mathbf{d}$. This gives an I_n-graded space $\overline{V} = \oplus_{\bar{i} \in I_n} V_{\bar{i}}$ of dimension vector $\bar{\mathbf{d}}$ with $\overline{V}_{\bar{i}} = \oplus_{j \in \bar{i}} V_j$, $\forall \bar{i} \in I_n$. Moreover, $\overline{V}$ admits a filtration by the subspaces

$$\overline{V}_{\geqslant i} = \bigoplus_{j \geqslant i} V_j, \ \forall i \in \mathbb{Z}.$$

Then the associated graded space $\oplus_{i \in \mathbb{Z}} \overline{V}_{\geqslant i} / \overline{V}_{\geqslant i-1}$ is naturally identified with the $\mathbb{Z}$-graded space V. Set

$$E_{\overline{V},V} = \{x \in E_{\overline{V}} \mid x(\overline{V}_{\geqslant i}) \subseteq \overline{V}_{\geqslant i+1}\} \subset E_{\overline{V}}.$$

This gives a map $p : E_{\overline{V},V} \to E_V$, which takes a representation of Δ_n in $E_{\overline{V}}$ to the induced representation of Δ_∞ in E_V, and the embedding $\iota : E_{\overline{V},V} \to E_{\overline{V}}$. By specializing v to $\sqrt{q}$, the map $\gamma_{\mathbf{d}}$ is then given by

$$(\gamma_{\mathbf{d}} \otimes_{\mathcal{Z}} \mathbb{C}) \mid_{v=\sqrt{q}} : \mathbb{C}_{G_{\overline{V}}}(E_{\overline{V}}) \longrightarrow \mathbb{C}_{G_V}(E_V), \ f \longmapsto \sqrt{q}^{h(\mathbf{d})} p_! \iota^*(f),$$

where $h(\mathbf{d}) = \sum_{i<j, \bar{i}=\bar{j}} d_i(d_{j+1} - d_j)$. Here we identify $\mathcal{H}(\Delta_n)_{\mathbf{d}} \otimes_{\mathcal{Z}} \mathbb{C}$ with $\mathbb{C}_{G_{\overline{V}}}(E_{\overline{V}})$ and $\mathcal{H}(\Delta_\infty)_{\mathbf{d}} \otimes_{\mathcal{Z}} \mathbb{C}$ with $\mathbb{C}_{G_V}(E_V)$.

The first two statements in the following lemma are taken from [43, Sect. 6.1], and the third one follows from the isomorphism $\tau : \mathcal{H}(\Delta_\infty) \to \mathcal{H}(\Delta_\infty)$.

Lemma 3.3. (1) *For each* $\mathbf{d} \in \mathbb{N}I_\infty$, $\gamma_{\mathbf{d}}(\widetilde{u}_{\bar{\mathbf{d}}}) = v^{-h(\mathbf{d})}\widetilde{u}_{\mathbf{d}}$.

(2) *Fix* $\alpha, \beta \in \mathbb{N}I_n$ *with* $\bar{\mathbf{d}} = \alpha + \beta$. *Then for* $x \in \mathcal{H}(\Delta_n)_\alpha$ *and* $y \in \mathcal{H}(\Delta_n)_\beta$,

$$\sum_{\mathbf{a},\mathbf{b}} v^{\kappa(\mathbf{a},\mathbf{b})}\gamma_{\mathbf{a}}(x)\gamma_{\mathbf{b}}(y) = \gamma_{\mathbf{d}}(xy), \tag{3.3.1}$$

where the sum is taken over all pairs $\mathbf{a}, \mathbf{b} \in \mathbb{N}I_\infty$ *satisfying* $\mathbf{a} + \mathbf{b} = \mathbf{d}$, $\bar{\mathbf{a}} = \alpha$, *and* $\bar{\mathbf{b}} = \beta$, *and* $\kappa(\mathbf{a}, \mathbf{b}) = \sum_{i>j, \bar{i}=\bar{j}} a_i(2b_j - b_{j-1} - b_{j+1})$.

(3) *For each* $\mathbf{d} \in \mathbb{N}I_\infty$ *and* $\mathfrak{m} \in \mathfrak{M}_n^{\bar{\mathbf{d}}}$, $\gamma_{\tau^n(\mathbf{d})}(\widetilde{u}_{\mathfrak{m}}) = \tau^n(\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}}))$.

We now describe the images of the basis elements $\widetilde{u}_{\mathfrak{m}}$ of $\mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}}$ under $\gamma_{\mathbf{d}}$.

Proposition 3.4. *Let* $\mathbf{d} \in \mathbb{N}I_\infty$ *and* $\mathfrak{m} \in \mathfrak{M}_n$ *be such that* $\alpha := \mathbf{dim}\, M(\mathfrak{m}) = \bar{\mathbf{d}}$. *Then*

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}}) \in \sum_{\mathfrak{z} \in \mathfrak{M}_\infty,\, \mathscr{F}(\mathfrak{z}) \leqslant_{\deg} \mathfrak{m}} \mathcal{Z}\widetilde{u}_{\mathfrak{z}}.$$

Proof. Consider the radical filtration of $M = M(\mathfrak{m})$

$$M = \mathrm{rad}^0 M \supseteq \mathrm{rad}^1 M (= \mathrm{rad}\, M) \supseteq \cdots \supseteq \mathrm{rad}^{\ell-1} M \supseteq \mathrm{rad}^\ell M = 0$$

with $\mathrm{rad}^{s-1} M / \mathrm{rad}^s M \cong S_{\alpha_s}$, where ℓ is the Loewy length of M and $\alpha_s \in \mathbb{N}I_n$ for $1 \leqslant s \leqslant \ell$. Then $M = S_{\alpha_1} * \cdots * S_{\alpha_\ell}$. Moreover, by [8, Sect. 9],

$$\widetilde{u}_{\alpha_1} \cdots \widetilde{u}_{\alpha_\ell} = \widetilde{u}_{\mathfrak{m}} + \sum_{\mathfrak{p} <_{\deg} \mathfrak{m}} f_{\mathfrak{m},\mathfrak{p}} \widetilde{u}_{\mathfrak{p}}, \quad \text{where } f_{\mathfrak{m},\mathfrak{p}} \in \mathcal{Z}.$$

On the one hand, by induction with respect to the order $\leqslant_{\deg}$, we may assume that for each $\mathfrak{p} \in \mathfrak{M}_n^{\mathbf{d}}$ with $\mathfrak{p} <_{\deg} \mathfrak{m}$, $\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{p}})$ is a $\mathcal{Z}$-linear combination of $\widetilde{u}_{\mathfrak{y}}$ with $\mathfrak{y} \in \mathfrak{M}_\infty$ satisfying $\mathscr{F}(\mathfrak{y}) \leqslant_{\deg} \mathfrak{p}$. Therefore,

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}}) = \gamma_{\mathbf{d}}(\widetilde{u}_{\alpha_1} \cdots \widetilde{u}_{\alpha_\ell}) + x, \tag{3.4.1}$$

where $x = -\sum_{\mathfrak{p} <_{\deg} \mathfrak{m}} f_{\mathfrak{m},\mathfrak{p}} \gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{p}})$ is a $\mathcal{Z}$-linear combination of $\widetilde{u}_{\mathfrak{z}}$ with $\mathscr{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}$.

On the other hand, by applying (3.3.1) inductively, we obtain

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\alpha_1} \cdots \widetilde{u}_{\alpha_\ell}) = \sum_{\mathbf{a}_1, \ldots, \mathbf{a}_\ell} v^{\sum_{s<t} \kappa(\mathbf{a}_s, \mathbf{a}_t) - \sum_s h(\mathbf{a}_s)} \widetilde{u}_{\mathbf{a}_1} \cdots \widetilde{u}_{\mathbf{a}_\ell}, \tag{3.4.2}$$

where the sum is taken over all sequences $\mathbf{a}_1, \ldots, \mathbf{a}_\ell \in \mathbb{N}I_\infty$ satisfying

$$\mathbf{a}_1 + \cdots + \mathbf{a}_\ell = \mathbf{d} \text{ and } \overline{\mathbf{a}_s} = \alpha_s, \ \forall\, 1 \leqslant s \leqslant \ell.$$

By the definition, each term $\widetilde{u}_{\mathbf{a}_1} \cdots \widetilde{u}_{\mathbf{a}_\ell}$ is a $\mathcal{Z}$-linear combination of $\widetilde{u}_{\mathfrak{y}}$ such that $M(\mathfrak{y})$ admits a filtration

$$M(\mathfrak{y}) = X_0 \supset X_1 \supset \cdots \supset X_{\ell-1} \supset X_\ell = 0$$

satisfying $X_{s-1}/X_s \cong S_{\mathbf{a}_s}$ for all $1 \leqslant s \leqslant \ell$. Applying the exact functor $\mathscr{F}$ gives a filtration of $\mathscr{F}(M(\mathfrak{y}))$

$$\mathscr{F}(M(\mathfrak{y})) = \mathscr{F}(X_0) \supset \mathscr{F}(X_1) \supset \cdots \supset \mathscr{F}(X_{\ell-1}) \supset \mathscr{F}(X_\ell) = 0$$

such that

$$\mathscr{F}(X_{s-1})/\mathscr{F}(X_s) \cong \mathscr{F}(X_{s-1}/X_s) \cong S_{\alpha_s}, \ \forall\, 1 \leqslant s \leqslant \ell.$$

Therefore,

$$\mathscr{F}(M(\mathfrak{y})) = M(\mathscr{F}(\pi)) \leqslant_{\deg} S_{\alpha_1} * \cdots * S_{\alpha_\ell} = M(\mathfrak{m}),$$

that is, $\mathscr{F}(\mathfrak{y}) \leqslant_{\deg} \mathfrak{m}$.

In conclusion, we obtain that

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}}) \in \sum_{\mathfrak{z} \in \mathfrak{M}_\infty,\, \mathscr{F}(\mathfrak{z}) \leqslant_{\deg} \mathfrak{m}} \mathcal{Z}\widetilde{u}_{\mathfrak{z}}. \qquad \square$$

Fix $\lambda \in \Pi$ and write

$$\mathbf{d}(\lambda) = \mathbf{dim}\, M(\mathfrak{m}_\lambda^\infty) \in \mathbb{N}I_\infty \text{ and } \alpha(\lambda) = \mathbf{dim} M(\mathfrak{m}_\lambda) \in \mathbb{N}I_n.$$

By the definition of $M(\mathfrak{m}_\lambda^\infty)$ and $M(\mathfrak{m}_\lambda)$, the radical filtration of $\widetilde{M} = M(\mathfrak{m}_\lambda^\infty)$

$$\widetilde{M} = \operatorname{rad}^0 \widetilde{M} \supseteq \operatorname{rad} \widetilde{M} \supseteq \cdots \supseteq \operatorname{rad}^{\ell-1} \widetilde{M} \supseteq \operatorname{rad}^\ell \widetilde{M} = 0$$

gives rise to the radical filtration of $M(\mathfrak{m}_\lambda) = \mathscr{F}(\widetilde{M})$

$$M(\mathfrak{m}_\lambda) = \mathscr{F}(\operatorname{rad}^0 \widetilde{M}) \supseteq \mathscr{F}(\operatorname{rad} \widetilde{M}) \supseteq \cdots \supseteq \mathscr{F}(\operatorname{rad}^{\ell-1} \widetilde{M}) \supseteq \mathscr{F}(\operatorname{rad}^\ell \widetilde{M}) = 0,$$

that is, $\mathscr{F}(\operatorname{rad}^s \widetilde{M}) = \operatorname{rad}^s(M(\mathfrak{m}_\lambda))$ for $1 \leqslant s \leqslant \ell$. Let $\mathbf{d}(\lambda)_s \in \mathbb{N}I_\infty$ and $\alpha(\lambda)_s \in \mathbb{N}I_n$, $1 \leqslant s \leqslant \ell$, be such that

$$\operatorname{rad}^{s-1} \widetilde{M}/\operatorname{rad}^s \widetilde{M} \cong S_{\mathbf{d}(\lambda)_s} \text{ and } \operatorname{rad}^{s-1} M(\mathfrak{m}_\lambda)/\operatorname{rad}^s M(\mathfrak{m}_\lambda) \cong S_{\alpha(\lambda)_s}.$$

Then $\overline{\mathbf{d}(\lambda)_s} = \alpha(\lambda)_s$ for $1 \leqslant s \leqslant \ell$. Applying (3.4.1) and (3.4.2) to $\mathfrak{m}_\lambda$ gives the following result.

Corollary 3.5. (1) *Let $\lambda \in \Pi$ and keep the notation above. Then*

$$\gamma_{\mathbf{d}(\lambda)}(\widetilde{u}_{\mathfrak{m}_\lambda}) \in v^{\theta(\lambda)}\widetilde{u}_{\mathfrak{m}_\lambda^\infty} + \sum_{\mathfrak{z}\in\mathfrak{M}_\infty,\, \mathscr{F}(\mathfrak{z})<_{\deg}\mathfrak{m}_\lambda} \mathcal{Z}\widetilde{u}_{\mathfrak{z}},$$

where $\theta(\lambda) = \sum_{s<t}\kappa(\mathbf{d}(\lambda)_s, \mathbf{d}(\lambda)_t) - \sum_{s=1}^{\ell} h(\mathbf{d}(\lambda)_s)$.

(2) *Let $\mathbf{d} \in \mathbb{N}I_\infty$ with $\overline{\mathbf{d}} = \alpha(\lambda)$. If $\mathbf{d} = \tau^{rm}(\mathbf{d}(\lambda))$ for some $r \in \mathbb{Z}$, then*

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}_\lambda}) \in v^{\theta(\lambda)}\widetilde{u}_{\tau^{rm}(\mathfrak{m}_\lambda^\infty)} + \sum_{\mathfrak{z}\in\mathfrak{M}_\infty,\, \mathscr{F}(\mathfrak{z})<_{\deg}\mathfrak{m}_\lambda} \mathcal{Z}\widetilde{u}_{\mathfrak{z}}.$$

Otherwise,

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathfrak{m}_\lambda}) \in \sum_{\mathfrak{z}\in\mathfrak{M}_\infty^{\mathbf{d}},\, \mathscr{F}(\mathfrak{z})<_{\deg}\mathfrak{m}_\lambda} \mathcal{Z}\widetilde{u}_{\mathfrak{z}}.$$

In the following we briefly recall the canonical basis of $\mathcal{H}(\Delta)$ for $\Delta = \Delta_n$ or Δ_∞. By [27] and [43, Prop. 7.5], there is a semilinear ring involution $\iota : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta)$ taking $v \mapsto v^{-1}$ and $\widetilde{u}_{\mathbf{d}} \mapsto \widetilde{u}_{\mathbf{d}}$ for all $\mathbf{d} \in \mathbb{Z}I$. It is often called the bar-involution, usually written as $\bar{x} = \iota(x)$. The canonical basis (or the global crystal basis in the sense of Kashiwara) $\mathbf{B} := \{b_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}\}$ for $\mathcal{H}(\Delta)$ (at $v = \infty$) can be characterized as follows:

$$\overline{b}_{\mathfrak{m}} = b_{\mathfrak{m}}, \quad b_{\mathfrak{m}} \in \widetilde{u}_{\mathfrak{m}} + \sum_{\mathfrak{p}<_{\deg}\mathfrak{m}} v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}_{\mathfrak{p}}; \tag{3.5.1}$$

see [27]. The canonical basis elements $b_{\mathfrak{m}}$ also admit a geometric characterization given in [28,43]. Let $H^i_{\mathcal{O}_{\mathfrak{p}}}(IC_{\mathcal{O}_{\mathfrak{m}}})$ be the stalk at a point of $\mathcal{O}_{\mathfrak{p}}$ of the i-th intersection cohomology sheaf of the closure $\overline{\mathcal{O}_{\mathfrak{m}}}$ of $\mathcal{O}_{\mathfrak{m}}$. Then

$$b_{\mathfrak{m}} = \sum_{\substack{i\in\mathbb{N}\\ \mathfrak{p}\leqslant_{\deg}\mathfrak{m}}} v^{i-\dim\mathcal{O}_{\mathfrak{m}}+\dim\mathcal{O}_{\mathfrak{p}}} \dim H^i_{\mathcal{O}_{\mathfrak{p}}}(IC_{\mathcal{O}_{\mathfrak{m}}})\widetilde{u}_{\mathfrak{p}}.$$

For the cyclic quiver case, by [29], the subset of $\mathbf{B}$

$$\mathbf{B}^{\mathrm{ap}} := \{b_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{M}_n^{\mathrm{ap}}\}$$

is the canonical basis of $\mathcal{C}(\Delta_n)$, where $\mathfrak{M}_n^{\mathrm{ap}}$ denotes the set of aperiodic multisegments, that is, those multisegments $\mathfrak{m} = \sum_{i\in I_n,\, l\geqslant 1} m_{i,l}[i,l)$ satisfying that for each $l \geqslant 1$, there is some $i \in I_n$ such that $m_{i,l} = 0$. In other words, $\mathbf{B}^{\mathrm{ap}}$ is the canonical basis of $\mathbf{U}_v^{\pm}(\widehat{\mathfrak{sl}}_n)$. Note that for each $\lambda = (\lambda_1, \dots, \lambda_m) \in \Pi$, the corresponding multisegment $\mathfrak{m}_\lambda$ is aperiodic if and only if λ is n-regular which, by definition, satisfies $\lambda_s > \lambda_{s+n-1}$ for $1 \leqslant s \leqslant s+n-1 \leqslant m$.

4. Double Ringel–Hall algebras and highest weight modules

In this section we follow [42,6] to define the double Ringel–Hall algebra $\boldsymbol{\mathcal{D}}(\Delta)$ of the quiver $\Delta = \Delta_n$ or Δ_∞ and study the irreducible highest weight modules of $\boldsymbol{\mathcal{D}}(\Delta_n)$ associated with integral dominant weights in terms of a quantized generalized Kac–Moody algebra.

The Ringel–Hall algebra $\boldsymbol{\mathcal{H}}(\Delta)$ of Δ can be extended to a Hopf algebra $\boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0}$ which is a $\mathbb{Q}(v)$-vector space with a basis $\{u^+_{\mathfrak{m}} K_\alpha \mid \alpha \in \mathbb{Z}I, \mathfrak{m} \in \mathfrak{M}\}$; see [34,14,42] or [6, Prop. 1.5.3]. Its algebra structure is given by

$$\begin{aligned} &K_\alpha K_\beta = K_{\alpha+\beta}, \quad K_\alpha u^+_{\mathfrak{m}} = v^{(\mathbf{d}(\mathfrak{m}),\alpha)} u^+_{\mathfrak{m}} K_\alpha, \\ &u^+_{\mathfrak{m}} u^+_{\mathfrak{m}'} = \sum_{\mathfrak{p}\in\mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}),\mathbf{d}(\mathfrak{m}')\rangle} \varphi^{\mathfrak{p}}_{\mathfrak{m},\mathfrak{m}'}(v^2) u^+_{\mathfrak{p}}, \end{aligned} \tag{4.0.2}$$

where $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{M}$ and $\alpha, \beta \in \mathbb{Z}I$, and its coalgebra structure is given by

$$\begin{aligned} &\Delta(u^+_{\mathfrak{m}}) = \sum_{\mathfrak{m}',\mathfrak{m}''\in\mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}'),\mathbf{d}(\mathfrak{m}'')\rangle} \frac{\mathfrak{a}_{\mathfrak{m}'}(v^2)\mathfrak{a}_{\mathfrak{m}''}(v^2)}{\mathfrak{a}_{\mathfrak{m}}(v^2)} \varphi^{\mathfrak{m}}_{\mathfrak{m}',\mathfrak{m}''}(v^2) u^+_{\mathfrak{m}''} \otimes u^+_{\mathfrak{m}'} K_{\mathbf{d}(\mathfrak{m}'')}, \\ &\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \varepsilon(u^+_{\mathfrak{m}}) = 0 \ (\mathfrak{m} \neq 0), \quad \varepsilon(K_\alpha) = 1, \end{aligned} \tag{4.0.3}$$

where $\mathfrak{m} \in \mathfrak{M}$ and $\alpha \in \mathbb{Z}I$. We refer to [42] or [6] for the definition of the antipode.

Dually, there is a Hopf algebra $\boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0}$ with basis $\{K_\alpha u^-_{\mathfrak{m}} \mid \alpha \in \mathbb{Z}I, \mathfrak{m} \in \mathfrak{M}\}$. In particular, the multiplication is given by

$$\begin{aligned} &K_\alpha K_\beta = K_{\alpha+\beta}, \quad K_\alpha u^-_{\mathfrak{m}} = v^{-(\mathbf{d}(\mathfrak{m}),\alpha)} u^-_{\mathfrak{m}} K_\alpha, \\ &u^-_{\mathfrak{m}} u^-_{\mathfrak{m}'} = \sum_{\mathfrak{p}\in\mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}'),\mathbf{d}(\mathfrak{m})\rangle} \varphi^{\mathfrak{p}}_{\mathfrak{m}',\mathfrak{m}}(v^2) u^-_{\mathfrak{p}}, \end{aligned} \tag{4.0.4}$$

where $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{M}$ and $\alpha, \beta \in \mathbb{Z}I$. The comultiplication and the counit are given by

$$\begin{aligned} &\Delta(u^-_{\mathfrak{m}}) = \sum_{\mathfrak{m}',\mathfrak{m}''\in\mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}'),\mathbf{d}(\mathfrak{m}'')\rangle} \frac{\mathfrak{a}_{\mathfrak{m}'}\mathfrak{a}_{\mathfrak{m}''}}{\mathfrak{a}_{\mathfrak{m}}} \varphi^{\mathfrak{m}}_{\mathfrak{m}',\mathfrak{m}''}(v^2) u^-_{\mathfrak{m}''} K_{-\mathbf{d}(\mathfrak{m}')} \otimes u^-_{\mathfrak{m}'}, \\ &\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \varepsilon(u^-_{\mathfrak{m}}) = 0 \ (\mathfrak{m} \neq 0), \quad \varepsilon(K_\alpha) = 1, \end{aligned} \tag{4.0.5}$$

where $\alpha \in \mathbb{Z}I$ and $\mathfrak{m} \in \mathfrak{M}$.

It is routine to check that the bilinear form $\psi : \boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0} \times \boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0} \to \mathbb{Q}(v)$ defined by

$$\psi(K_\alpha u^+_{\mathfrak{m}}, K_\beta u^-_{\mathfrak{m}'}) = v^{(\alpha,\beta)-\langle \mathbf{d}(\mathfrak{m}),\mathbf{d}(\mathfrak{m})\rangle + 2d(\mathfrak{m})} \frac{\delta_{\mathfrak{m},\mathfrak{m}'}}{\mathfrak{a}_{\mathfrak{m}}(v^2)} \tag{4.0.6}$$

is a skew-Hopf pairing in the sense of [22]; see, for example, [6, Prop. 2.1.3].

Following [42] or [6, §2.1], with the triple $(\boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0}, \boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0}, \psi)$ we obtain the associated reduced *double Ringel–Hall algebra* $\boldsymbol{\mathcal{D}}(\Delta)$ which inherits a Hopf algebra structure from those of $\boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0}$ and $\boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0}$. In particular, for all elements $x \in \boldsymbol{\mathcal{D}}(\Delta)^{\geqslant 0}$ and $y \in \boldsymbol{\mathcal{D}}(\Delta)^{\leqslant 0}$, we have in $\boldsymbol{\mathcal{D}}(\Delta)$ the following relations

$$\sum \psi(x_1, y_1) y_2 x_2 = \sum \psi(x_2, y_2) x_1 y_1, \tag{4.0.7}$$

where $\Delta(x) = \sum x_1 \otimes x_2$ and $\Delta(y) = \sum y_1 \otimes y_2$ (here we use the Sweedler notation). Moreover, $\boldsymbol{\mathcal{D}}(\Delta)$ admits a triangular decomposition

$$\boldsymbol{\mathcal{D}}(\Delta) = \boldsymbol{\mathcal{D}}(\Delta)^{+} \otimes \boldsymbol{\mathcal{D}}(\Delta)^{0} \otimes \boldsymbol{\mathcal{D}}(\Delta)^{-}, \tag{4.0.8}$$

where $\boldsymbol{\mathcal{D}}(\Delta)^{\pm}$ are subalgebras generated by $u_{\mathfrak{m}}^{\pm}$ ($\mathfrak{m} \in \mathfrak{M}$), and $\boldsymbol{\mathcal{D}}(\Delta)^{0}$ is generated by K_α ($\alpha \in \mathbb{Z}I$). Thus, $\boldsymbol{\mathcal{D}}(\Delta)^{0}$ is identified with the Laurent polynomial ring $\mathbb{Q}(v)[K_i^{\pm 1} : i \in I]$,

$$\begin{aligned} \boldsymbol{\mathcal{H}}(\Delta) = \mathcal{H}(\Delta) \otimes_{\mathcal{Z}} \mathbb{Q}(v) &\xrightarrow{\sim} \boldsymbol{\mathcal{D}}(\Delta)^{+}, \;\; u_{\mathfrak{m}} \longmapsto u_{\mathfrak{m}}^{+}, \\ \boldsymbol{\mathcal{H}}(\Delta)^{\mathrm{op}} = \mathcal{H}(\Delta)^{\mathrm{op}} \otimes_{\mathcal{Z}} \mathbb{Q}(v) &\xrightarrow{\sim} \boldsymbol{\mathcal{D}}(\Delta)^{-}, \;\; u_{\mathfrak{m}} \longmapsto u_{\mathfrak{m}}^{-}. \end{aligned}$$

For $i \in I$, $\alpha \in \mathbb{N}I$ and $\mathfrak{m} \in \mathfrak{M}$, we write

$$u_i^{\pm} = u_{[S_i]}^{\pm}, \;\; u_\alpha^{\pm} = u_{[S_\alpha]}^{\pm}, \quad \text{and} \quad \widetilde{u}_{\mathfrak{m}}^{\pm} = v^{\dim \mathrm{End}_\Delta(M(\mathfrak{m})) - \dim M(\mathfrak{m})} u_{\mathfrak{m}}^{\pm}.$$

The canonical basis of $\mathcal{H}(\Delta)$ in (3.5.1) gives the canonical bases $\mathbf{B}^{\pm} := \{b_{\mathfrak{m}}^{\pm} \mid \mathfrak{m} \in \mathfrak{M}\}$ of $\boldsymbol{\mathcal{D}}(\Delta)^{\pm}$ satisfying

$$b_{\mathfrak{m}}^{\pm} \in \widetilde{u}_{\mathfrak{m}}^{\pm} + \sum_{\mathfrak{p} <_{\deg} \mathfrak{m}} v^{-1} \mathbb{Z}[v^{-1}] \widetilde{u}_{\mathfrak{p}}^{\pm}. \tag{4.0.9}$$

It is known that $\boldsymbol{\mathcal{D}}(\Delta_\infty)$ is generated by $u_i^{\pm}, K_i^{\pm 1}$ ($i \in \mathbb{Z}$) and is isomorphic to $\mathbf{U}_v(\mathfrak{sl}_\infty)$. By [36], the $\mathbb{Q}(v)$-subalgebra of $\boldsymbol{\mathcal{D}}(\Delta_n)$ generated by $u_i^{\pm}, K_i^{\pm 1}$ ($i \in I_n = \mathbb{Z}/n\mathbb{Z}$) is isomorphic to $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$, while $\boldsymbol{\mathcal{D}}(\Delta_n)$ gives a realization of $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$; see [38,19,6]. From now on, we write for notational simplicity,

$$\boldsymbol{\mathcal{D}}(\infty) = \boldsymbol{\mathcal{D}}(\Delta_\infty) \;\text{ and }\; \boldsymbol{\mathcal{D}}(n) = \boldsymbol{\mathcal{D}}(\Delta_n).$$

Remarks 4.1. (1) The construction of $\boldsymbol{\mathcal{D}}(n)$ is slightly different from that in [6, §2.1]. In particular, the K_i here play a role as $\widetilde{K}_i = K_i K_{i+1}^{-1}$ there. In particular, they do not satisfy the equality $K_0 K_1 \cdots K_{n-1} = 1$.

(2) We can extend $\boldsymbol{\mathcal{D}}(n)$ to the $\mathbb{Q}(v)$-algebra $\widehat{\boldsymbol{\mathcal{D}}}(n)$ by adding new generators $D^{\pm 1}$ with relations

$$\begin{aligned} DD^{-1} = 1 = D^{-1}D, \, K_i D = DK_i, \, DE_i = v^{\delta_{0,i}} E_i D, \, DF_i \\ = v^{-\delta_{0,i}} F_i D, \, Du_{\mathfrak{m}}^{\pm} = v^{\pm a_0} u_{\mathfrak{m}}^{\pm} D \end{aligned}$$

for all $i \in I_n$ and $\mathfrak{m} \in \mathfrak{M}$, where $\mathbf{d}(\mathfrak{m}) = (a_i)_{i \in I_n}$. Then $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ clearly becomes a subalgebra of $\widehat{\boldsymbol{\mathcal{D}}}(n)$.

As in (3.1.1), define for each $t \geqslant 1$,

$$\boldsymbol{c}_t^{\pm} = (-1)^t v^{-2tn} \sum_{\mathfrak{m}} (-1)^{\dim \operatorname{End}(M(\mathfrak{m}))} \mathfrak{a}_{\mathfrak{m}}(v^2) u_{\mathfrak{m}}^{\pm} \in \boldsymbol{\mathcal{D}}(n)^{\pm}.$$

By Theorem 3.2, the elements $\boldsymbol{c}_t^+$ and $\boldsymbol{c}_t^-$ are central in $\boldsymbol{\mathcal{D}}(n)^+$ and $\boldsymbol{\mathcal{D}}(n)^-$, respectively. Following [19, Sect. 4], define recursively for $t \geqslant 1$,

$$\boldsymbol{x}_t^{\pm} = t\boldsymbol{c}_t^{\pm} - \sum_{s=1}^{t-1} \boldsymbol{x}_s^{\pm} \boldsymbol{c}_{t-s}^{\pm} \in \boldsymbol{\mathcal{D}}(n)^{\pm}.$$

Clearly, $\boldsymbol{x}_t^+$ and $\boldsymbol{x}_t^-$ are again central elements in $\boldsymbol{\mathcal{D}}(n)^+$ and $\boldsymbol{\mathcal{D}}(n)^-$, respectively. By applying [18, Cor. 10 & 12], the $\boldsymbol{x}_t^{\pm}$ are primitive, i.e.,

$$\Delta(\boldsymbol{x}_t^+) = \boldsymbol{x}_t^+ \otimes K_{t\delta} + 1 \otimes \boldsymbol{x}_t^+ \quad \text{and} \quad \Delta(\boldsymbol{x}_t^-) = \boldsymbol{x}_t^- \otimes 1 + K_{-t\delta} \otimes \boldsymbol{x}_t^-,$$

and they satisfy

$$\psi(\boldsymbol{x}_t^+, \boldsymbol{x}_s^-) = v^{2tn}\{\boldsymbol{x}_t, \boldsymbol{x}_s\} = \delta_{t,s} t v^{2tn} v^{-2tn}(1 - v^{-2tn}) = \delta_{t,s} t(1 - v^{-2tn}).$$

Finally, as in [6, § 2.2], we scale the elements $\boldsymbol{x}_t^{\pm}$ by setting

$$\boldsymbol{z}_t^{\pm} = \frac{v^{tn}}{v^t - v^{-t}} \boldsymbol{x}_t^{\pm} \in \boldsymbol{\mathcal{D}}(n)^{\pm} \quad \text{for } t \geqslant 1.$$

Then

$$\Delta(\boldsymbol{z}_t^+) = \boldsymbol{z}_t^+ \otimes K_{t\delta} + 1 \otimes \boldsymbol{z}_t^+, \ \Delta(\boldsymbol{z}_t^-) = \boldsymbol{z}_t^- \otimes 1 + K_{-t\delta} \otimes \boldsymbol{z}_t^-, \tag{4.1.1}$$

and

$$\psi(\boldsymbol{z}_t^+, \boldsymbol{z}_s^-) = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2}.$$

Lemma 4.2. (1) *For each $i \in I_n$,*

$$[u_i^+, u_i^-] = \frac{K_i - K_i^{-1}}{v - v^{-1}}.$$

(2) *For $\alpha \in \mathbb{N}I_n$ and $t, s \geqslant 1$, $K_\alpha \boldsymbol{z}_t^{\pm} = \boldsymbol{z}_t^{\pm} K_\alpha$ and*

$$[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2} (K_{t\delta} - K_{-t\delta}). \tag{4.2.1}$$

Moreover, for each $i \in I_n$ and $t \geqslant 1$,

$$[u_i^+, \boldsymbol{z}_t^-] = 0 = [u_i^-, \boldsymbol{z}_t^+].$$

Proof. We only prove the formula (4.2.1). The remaining ones are easy calculations. Since $\Delta(\boldsymbol{z}_t^+) = \boldsymbol{z}_t^+ \otimes K_{t\delta} + 1 \otimes \boldsymbol{z}_t^+$ and $\Delta(\boldsymbol{z}_s^-) = \boldsymbol{z}_s^- \otimes 1 + K_{-s\delta} \otimes \boldsymbol{z}_s^-$, we have by (4.0.7) that

$$\begin{aligned} & K_{t\delta}\psi(\boldsymbol{z}_t^+, \boldsymbol{z}_s^-) + \boldsymbol{z}_t^+\psi(1, \boldsymbol{z}_s^-) + \boldsymbol{z}_s^- K_{t\delta}\psi(\boldsymbol{z}_t^+, K_{-s\delta}) + \boldsymbol{z}_s^-\boldsymbol{z}_t^+\psi(1, K_{-s\delta}) \\ =& \boldsymbol{z}_t^+\boldsymbol{z}_s^-\psi(K_{t\delta}, 1) + \boldsymbol{z}_s^-\psi(\boldsymbol{z}_t^+, 1) + \boldsymbol{z}_t^+ K_{-s\delta}\psi(K_{t\delta}, \boldsymbol{z}_s^-) + K_{-s\delta}\psi(\boldsymbol{z}_t^+, \boldsymbol{z}_s^-). \end{aligned}$$

This implies that

$$[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] = \psi(\boldsymbol{z}_t^+, \boldsymbol{z}_s^-)(K_{t\delta} - K_{-s\delta}) = \delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t - v^{-t})^2}(K_{t\delta} - K_{-t\delta})$$

since $\psi(1, \boldsymbol{z}_s^-) = \psi(\boldsymbol{z}_t^+, K_{s\delta}) = \psi(\boldsymbol{z}_t^+, 1) = \psi(K_{t\delta}, \boldsymbol{z}_s^-) = 0$ and $\psi(1, K_{s\delta}) = \psi(K_{-t\delta}, 1) = 1$. $\square$

Using arguments similar to those in the proof of [6, Th. 2.3.1], we obtain a presentation of $\boldsymbol{\mathcal{D}}(n)$. More precisely, $\boldsymbol{\mathcal{D}}(n)$ is the $\mathbb{Q}(v)$-algebra generated by $K_i^{\pm 1}$, $u_i^+ = E_i$, $u_i^- = F_i$, and $\boldsymbol{z}_t^{\pm}$ for $i \in I_n$ and $t \geqslant 1$ with defining relations:

(DH1) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1 = K_i^{-1}K_i$;
(DH2) $K_iE_j = v^{a_{ij}}E_jK_i,\ K_iF_j = v^{-a_{ij}}F_jK_i,\ K_i\boldsymbol{z}_t^{\pm} = \boldsymbol{z}_t^{\pm}K_i$;
(DH3) $[E_i, F_j] = \delta_{i,j}\frac{K_i - K_i^{-1}}{v - v^{-1}}$, $[E_i, \boldsymbol{z}_t^-] = 0$,
$[\boldsymbol{z}_t^+, F_i] = 0$, $[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] = \delta_{t,s}\frac{t(v^{2tn}-1)}{(v^t - v^{-t})^2}(K_{t\delta} - K_{-t\delta})$;
(DH4) $\displaystyle\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1-c_{i,j} \\ a \end{bmatrix} E_i^aE_jE_i^b = 0$ for $i \neq j$,
$\boldsymbol{z}_t^+\boldsymbol{z}_s^+ = \boldsymbol{z}_s^+\boldsymbol{z}_t^+$, $E_i\boldsymbol{z}_t^+ = \boldsymbol{z}_t^+E_i$;
(DH5) $\displaystyle\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1-c_{i,j} \\ a \end{bmatrix} F_i^aF_jF_i^b = 0$ for $i \neq j$,
$\boldsymbol{z}_t^-\boldsymbol{z}_s^- = \boldsymbol{z}_s^-\boldsymbol{z}_t^-$, $F_i\boldsymbol{z}_t^- = \boldsymbol{z}_t^-F_i$,

where $i, j \in I_n$ and $t, s \geqslant 1$.

In the following we simply identify $I_n = \mathbb{Z}/n\mathbb{Z}$ with the subset $\{0, 1, \dots, n-1\}$ of $\mathbb{Z}$. Let $P^\vee = (\oplus_{i \in I_n}\mathbb{Z}h_i) \oplus \mathbb{Z}d$ be the free abelian group with basis $\{h_i \mid i \in I_n\} \cup \{d\}$. Set $\mathfrak{h} = P^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ and define

$$P = \{\Lambda \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q}) \mid \Lambda(P^\vee) \subset \mathbb{Z}\}.$$

Then $P = (\oplus_{i \in I_n}\mathbb{Z}\Lambda_i) \oplus \mathbb{Z}\omega$, where $\{\Lambda_i \mid i \in I_n\} \cup \{\omega\}$ is the dual basis of $\{h_i \mid i \in I_n\} \cup \{d\}$. This gives rise to the Cartan datum $(P^\vee, P, \Pi^\vee, \Pi)$ associated with the

Cartan matrix $C_n = (a_{ij})$, where $\Pi^\vee = \{h_i \mid i \in I_n\}$ is the set of simple coroots and $\Pi = \{\alpha_i \mid i \in I_n\}$ is the set of simple roots defined by

$$\alpha_i(h_j) = a_{ji}, \ \alpha_i(d) = \delta_{0,i} \ \text{ for all } i, j \in I_n.$$

Finally, let

$$P^+ = \{\Lambda \in P \mid \Lambda(h_i) \geqslant 0, \ \forall\, i \in I_n\} = \Big(\bigoplus_{i\in I_n} \mathbb{N}\Lambda_i\Big) \oplus \mathbb{Z}\omega$$

denote the set of dominant weights.

For each $\Lambda \in P$, consider the left ideal J_Λ of $\boldsymbol{\mathcal{D}}(n)$ defined by

$$\begin{aligned} J_\Lambda &= \sum_{\mathfrak{m}\in\mathfrak{M}_n\setminus\{0\}} \boldsymbol{\mathcal{D}}(n) u_{\mathfrak{m}}^+ + \sum_{\alpha\in\mathbb{Z}I_n} \boldsymbol{\mathcal{D}}(n)(K_\alpha - v^{\Lambda(\alpha)}) \\ &= \sum_{\mathfrak{m}\in\mathfrak{M}_n\setminus\{0\}} \boldsymbol{\mathcal{D}}(n) u_{\mathfrak{m}}^+ + \sum_{i\in I_n} \boldsymbol{\mathcal{D}}(n)(K_i - v^{\Lambda(h_i)}), \end{aligned}$$

where $\Lambda(\alpha) = \sum_{i\in I_n} a_i \Lambda(h_i)$ if $\alpha = \sum_{i\in I_n} a_i \varepsilon_i \in \mathbb{Z}I_n$. The quotient module

$$M(\Lambda) := \boldsymbol{\mathcal{D}}(n)/J_\Lambda$$

is called the Verma module which is a highest weight module with highest vector $\eta_\Lambda := 1 + J_\Lambda$. Applying the triangular decomposition (4.0.8) shows that

$$\boldsymbol{\mathcal{D}}(n)^- \longrightarrow M(\Lambda), \ x^- \longmapsto x^- + J_\Lambda$$

is an isomorphism of $\mathbb{Q}(v)$-vector spaces. Via this isomorphism, $\boldsymbol{\mathcal{D}}(n)^-$ becomes a $\boldsymbol{\mathcal{D}}(n)$-module. It is clear that $M(\Lambda)$ contains a unique maximal submodule M' which gives rise to an irreducible $\boldsymbol{\mathcal{D}}(n)$-module $L(\Lambda) = M(\Lambda)/M'$.

Remark 4.3. By the construction, if $\Lambda, \Lambda' \in P^+$ satisfy $\Lambda - \Lambda' \in \mathbb{Z}\omega$, then $L(\Lambda) = L(\Lambda')$. Therefore, it might be more appropriate to work with the algebra $\widehat{\boldsymbol{\mathcal{D}}}(n)$ defined in Remark 4.1(2).

Theorem 4.4. *Let $\Lambda = \sum_{i\in I_n} a_i \Lambda_i + b\omega \in P^+$ be a dominant weight with $\sum_{i\in I_n} a_i > 0$. Then*

$$L(\Lambda) \cong \boldsymbol{\mathcal{D}}(n)^-/\Big(\sum_{i\in I_n} \boldsymbol{\mathcal{D}}(n)^-(u_i^-)^{a_i+1}\Big).$$

Proof. As in [9, Sect. 3], we extend the Cartan matrix $C = (a_{ij})_{i,j\in I_n}$ to a Borcherds–Cartan matrix $\widetilde{C} = (\widetilde{a}_{ij})_{i,j\in\mathbb{N}}$ by setting $\widetilde{a}_{ij} = a_{ij}$ for $0 \leqslant i, j < n$ and $\widetilde{a}_{ij} = 0$ otherwise. Consider the free abelian group $\widetilde{P}^\vee = (\oplus_{i\in\mathbb{N}}\mathbb{Z}h_i) \oplus (\oplus_{i\in\mathbb{N}}\mathbb{Z}d_i)$ and define

$$\widetilde{P} = \{\theta \in (\widetilde{P}^\vee \otimes \mathbb{Q})^* \mid \theta(\widetilde{P}^\vee) \subset \mathbb{Z}\}.$$

We then obtain a Cartan datum of type $\widetilde{C}$

$$(\widetilde{P}^\vee, \widetilde{P}, \widetilde{\Pi}^\vee = \{h_i \mid i \in \mathbb{N}\}, \widetilde{\Pi} = \{\widetilde{\alpha}_i \mid i \in \mathbb{N}\})$$

where the $\widetilde{\alpha}_i$ are defined by

$$\widetilde{\alpha}_i(h_j) = \widetilde{a}_{ji} \text{ and } \widetilde{\alpha}_i(d_j) = \delta_{i,j}, \ \forall\, i, j \in \mathbb{N}.$$

Following [23, Def. 2.1] or [21, Def. 1.3], with the above Cartan datum we have the associated quantum generalized Kac–Moody algebra $\mathbf{U}_v(\widetilde{C})$ which is by definition a $\mathbb{Q}(v)$-algebra generated by $K_i^{\pm 1}, D_i^{\pm 1}, E_i, F_i$ for $i \in \mathbb{N}$ with relations; see [21, (1.4)] for the details. Clearly, the subalgebra of $\mathbf{U}_v(\widetilde{C})$ generated by $K_i^{\pm 1}, D_0^{\pm 1}, E_i, F_i$ for $0 \leqslant i < n$ is isomorphic to $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$.

In order to make a comparison with $\boldsymbol{\mathcal{D}}(n)$, we consider the subalgebra $\widetilde{\mathbf{U}}$ of $\mathbf{U}_v(\widetilde{C})$ generated by $K_i^{\pm 1}, E_i, F_i$ for $i \in \mathbb{N}$. Then $\widetilde{\mathbf{U}}$ admits a triangular decomposition

$$\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}^- \otimes \widetilde{\mathbf{U}}^0 \otimes \widetilde{\mathbf{U}}^+,$$

where $\widetilde{\mathbf{U}}^-$, $\widetilde{\mathbf{U}}^+$, and $\widetilde{\mathbf{U}}^0$ are subalgebras generated by F_i, E_i, and $K_i^{\pm 1}$ for $i \in \mathbb{N}$, respectively. In particular, $\widetilde{\mathbf{U}}^0 = \mathbb{Q}(v)[K_i^{\pm 1} : i \in \mathbb{N}]$. It follows from the definition that there is a surjective algebra homomorphism $\Psi : \widetilde{\mathbf{U}} \to \boldsymbol{\mathcal{D}}(n)$ given by

$$\Psi(E_i) = \begin{cases} u_i^+, & \text{if } 0 \leqslant i < n; \\ y_{i-n+1} z_{i-n+1}^+, & \text{if } i \geqslant n, \end{cases} \quad \Psi(F_i) = \begin{cases} u_i^-, & \text{if } 0 \leqslant i < n; \\ z_{i-n+1}^-, & \text{if } i \geqslant n \end{cases}, \quad \text{and}$$

$$\Psi(K_i^{\pm 1}) = \begin{cases} K_i^{\pm 1}, & \text{if } 0 \leqslant i < n; \\ K_{(i-n+1)\delta}^{\pm 1}, & \text{if } i \geqslant n, \end{cases}$$

where $y_t = t(v^{2tn} - 1)(v - v^{-1})/(v^t - v^{-t})^2$ for $t \geqslant 1$; see (4.2.1). Hence, each $\boldsymbol{\mathcal{D}}(n)$-module can be viewed as a $\widetilde{\mathbf{U}}$-module via the homomorphism Ψ. By the definition, Ψ induces isomorphisms $\widetilde{\mathbf{U}}^\pm \cong \boldsymbol{\mathcal{D}}(n)^\pm$. Thus, in what follows, we will identify $\widetilde{\mathbf{U}}^\pm$ with $\boldsymbol{\mathcal{D}}(n)^\pm$ via Ψ.

As defined in [21, Sect. 2.1], for each $\theta \in \widetilde{P}$, there is an associated irreducible $\widetilde{\mathbf{U}}$-module $L(\theta)$. By [21, Prop. 3.3], $L(\theta)$ is integrable if and only if θ is dominant, that is,

$$\theta \in \widetilde{P}^+ = \{\rho \in (\widetilde{P}^\vee \otimes \mathbb{Q})^* \mid \rho(\widetilde{P}^\vee) \subset \mathbb{N}\}.$$

Moreover, by [23, Cor. 4.7], for $\theta \in \widetilde{P}^+$,

$$L(\theta) \cong \widetilde{\mathbf{U}}^- / \big(\sum_{i \in I_n} \widetilde{\mathbf{U}}^- F_i^{\theta(h_i)+1} + \sum_{i \geqslant n, \theta(h_i) = 0} \widetilde{\mathbf{U}}^- F_i \big).$$

Viewing the irreducible $\boldsymbol{\mathcal{D}}(n)$-module $L(\Lambda)$ as a $\widetilde{\mathbf{U}}$-module, it is then isomorphic to $L(\widetilde{\Lambda})$, where $\widetilde{\Lambda} \in \widetilde{P}$ is defined by

$$\widetilde{\Lambda}(h_i) = \begin{cases} \Lambda(h_i) = a_i, & \text{if } 0 \leqslant i < n; \\ (i-n+1)\sum_{0\leqslant j<n} a_j, & \text{if } i \geqslant n \end{cases} \quad \text{and} \quad \widetilde{\Lambda}(d_i) = \delta_{i,0}b.$$

From the assumption $\sum_{i\in I} a_i > 0$ it follows that $\widetilde{\Lambda}(h_i) > 0$ for all $i \geqslant n$. Consequently,

$$L(\Lambda) \cong L(\widetilde{\Lambda}) \cong \widetilde{\mathbf{U}}^-/\big(\sum_{i\in I_n} \widetilde{\mathbf{U}}^- F_i^{a_i+1}\big) = \boldsymbol{\mathcal{D}}(n)^-/\big(\sum_{i\in I_n} \boldsymbol{\mathcal{D}}(n)^-(u_i^-)^{a_i+1}\big). \qquad \square$$

For each $\Lambda \in P$, let $L_0(\Lambda)$ denote the irreducible $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module of highest weight Λ. Applying Theorem 3.2 gives the following result.

Corollary 4.5. *Let $\Lambda = \sum_{i\in I_n} a_i\Lambda_i + b\omega \in P^+$ with $\sum_{i\in I_n} a_i > 0$. Then $L_0(\Lambda)$ is the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-submodule of $L(\Lambda)$ generated by the highest weight vector η_Λ and there is a vector space decomposition*

$$L(\Lambda) = L_0(\Lambda) \otimes \mathbb{Q}(v)[\boldsymbol{z}_1^-, \boldsymbol{z}_2^-, \ldots].$$

In particular, if $L(\Lambda)|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)}$ denotes the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module via restriction, then

$$L(\Lambda)|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m\geqslant 0} L_0(\Lambda - m\delta^*)^{\oplus p(m)}, \tag{4.5.1}$$

where $\delta^* = \sum_{i\in I_n} \alpha_i$ and $p(m)$ is the number of partitions of m.

Proof. By Theorem 3.2,

$$\boldsymbol{\mathcal{D}}(n)^- = \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) \otimes \mathbb{Q}(v)[\boldsymbol{z}_1^-, \boldsymbol{z}_2^-, \ldots].$$

This implies that

$$\begin{aligned} L(\Lambda) &\cong \boldsymbol{\mathcal{D}}(n)^-/\big(\sum_{i\in I_n} \boldsymbol{\mathcal{D}}(n)^-(u_i^-)^{a_i+1}\big) \\ &\cong \big(\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)/\big(\sum_{i\in I_n} \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)F_i^{a_i+1}\big)\big) \otimes \mathbb{Q}(v)[\boldsymbol{z}_1^-, \boldsymbol{z}_2^-, \ldots]. \end{aligned}$$

By [30, Cor. 6.2.3], $L_0(\Lambda) \cong \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)/\big(\sum_{i\in I_n} \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)F_i^{a_i+1}\big)$. Hence, $L_0(\Lambda)$ is the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-submodule of $L(\Lambda)$ generated by η_Λ and the desired decomposition is obtained.

For each family of nonnegative integers $\{m_t \mid t \geqslant 1\}$ satisfying all but finitely many m_t are zero, $L_0(\Lambda) \otimes \prod_{t\geqslant 1}(\boldsymbol{z}_t^-)^{m_t}$ is a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-submodule of $L(\Lambda)$ since $[u_i^\pm, \boldsymbol{z}_t^-] = 0$ for all $i \in I_n$ and $t \geqslant 1$. It is easy to see that

$$L_0(\Lambda) \otimes \prod_{t \geqslant 1} (\boldsymbol{z}_t^-)^{m_t} \cong L_0(\Lambda - (\sum_{t \geqslant 1} m_t)\delta^*).$$

We conclude that

$$L(\Lambda)|_{\mathbf{U}_v'(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m \geqslant 0} L_0(\Lambda - m\delta^*)^{\oplus p(m)}. \qquad \square$$

By [30, Th. 14.4.11], for each $\Lambda \in P^+$, the canonical basis $\{b_{\mathfrak{m}}^- \mid \mathfrak{m} \in \mathfrak{M}_n^{\mathrm{ap}}\}$ of $\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)$ gives rise to the canonical basis

$$\{b_{\mathfrak{m}}^- \eta_\Lambda \neq 0 \mid \mathfrak{m} \in \mathfrak{M}_n^{\mathrm{ap}}\}$$

of $L_0(\Lambda)$. On the other hand, the crystal basis theory for the quantum generalized Kac–Moody algebra $\mathbf{U}(\widetilde{C})$ has been developed in [21]. Since all the F_i for $i \geqslant n$ correspond to imaginary simple roots and are central in $\widetilde{\mathbf{U}}^- = \boldsymbol{\mathcal{D}}(n)^-$, applying the construction in [21, Sect. 6] shows that the set

$$\mathbf{B}' := \big\{ \big(\prod_{i \geqslant n} F_i^{m_i} \big) b_{\mathfrak{m}}^- \mid \mathfrak{m} \in \mathfrak{M}_n^{\mathrm{ap}} \text{ and all } m_i \in \mathbb{N} \text{ but finitely many are zero} \big\}$$

forms the global crystal basis of $\widetilde{\mathbf{U}}^- = \boldsymbol{\mathcal{D}}(n)^-$. We remark that $\mathbf{B}'$ does not coincide with the canonical basis $\mathbf{B}^-$ of $\boldsymbol{\mathcal{D}}(n)^-$ in (4.0.9).

5. The q-deformed Fock space I: $\boldsymbol{\mathcal{D}}(\infty)$-module

In this section we introduce the q-deformed Fock space Λ^∞ from [16] and review its module structure over $\boldsymbol{\mathcal{D}}(\infty) = \mathbf{U}_v(\mathfrak{sl}_\infty)$ defined in [32,43], as well as its $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$-module structure. We also provide a proof of [43, Prop. 5.1] by using the properties of representations of Δ_∞. Throughout this section, we identify $\boldsymbol{\mathcal{D}}(\infty)$ with $\mathbf{U}_v(\widehat{\mathfrak{sl}}_\infty)$ via taking $u_i^+ \mapsto E_i$, $u_i^- \mapsto F_i$ for all $i \in I_\infty = \mathbb{Z}$.

For each partition $\lambda \in \Pi$, let $T(\lambda)$ denote the tableau of shape λ whose box in the intersection of the i-th row and the j-th column is labelled with $j - i$ (the box is then said to be with color $j - i$). For example, if $\lambda = (4, 2, 2, 1)$, then $T(\lambda)$ has the form

$$\begin{array}{|c|c|c|c|}
\cline{1-1}
-3 \\
\cline{1-2}
-2 & -1 \\
\cline{1-2}
-1 & 0 \\
\hline
0 & 1 & 2 & 3 \\
\hline
\end{array}$$

For given $i \in \mathbb{Z}$, a removable i-box of $T(\lambda)$ is by definition a box with color i which can be removed in such a way that the new tableau has the form $T(\mu)$ for some $\mu \in \Pi$. On the contrary, an indent i-box of $T(\lambda)$ is a box with color i which can be added to $T(\lambda)$. For $i \in \mathbb{Z}$ and $\lambda \in \Pi$, define

$$n_i(\lambda) = |\{\text{indent } i\text{-boxes of } T(\lambda)\}| - |\{\text{removable } i\text{-boxes of } T(\lambda)\}|.$$

Let $\bigwedge^\infty$ be the $\mathbb{Q}(v)$-vector space with basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$. Following [43, 4.2], there is a left $\mathbf{U}_v(\mathfrak{sl}_\infty)$-module structure on $\bigwedge^\infty$ defined by

$$K_i \cdot |\lambda\rangle = v^{n_i(\lambda)}|\lambda\rangle,\ \ E_i \cdot |\lambda\rangle = |\nu\rangle,\ \ F_i \cdot |\lambda\rangle = |\mu\rangle,\ \forall\, i \in \mathbb{Z}, \lambda \in \Pi, \tag{5.0.2}$$

where $\mu, \nu \in \Pi$ are such that $T(\mu) - T(\lambda)$ and $T(\lambda) - T(\nu)$ are a box with color i. As remarked in [32, Sect. 2] and [43, 4.2], $\bigwedge^\infty$ is isomorphic to the basic representation of $\mathbf{U}_v(\mathfrak{sl}_\infty)$ with the canonical basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$.

Lemma 5.1. (1) *For $i \in \mathbb{Z}$ and $\lambda, \mu \in \Pi$, if $u_i^- \cdot |\mu\rangle = |\lambda\rangle$, then there is an exact sequence*

$$0 \longrightarrow S_i \longrightarrow M(\mathfrak{m}_\lambda) \longrightarrow M(\mathfrak{m}_\mu) \longrightarrow 0.$$

(2) *Let $\mathfrak{m} = [i, l)$ for some $i \in \mathbb{Z}$ and $l \geqslant 1$. Then $\widetilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\lambda\rangle$ if $i \leqslant 0$ and $i + l - 1 \geqslant 0$ and 0 otherwise, where $\lambda = (i + l, 1^{(-i)})$. In particular, if $i = 0$, then $\widetilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle = |\lambda\rangle$.*

Proof. (1) This follows directly from the definition.

(2) We proceed induction on l. The statement is trivial if $l = 1$. Suppose now $l > 1$. By the definition, $M(\mathfrak{m}) = S_i[l]$ with $\mathbf{dim}\, M(\mathfrak{m}) = \sum_{j=i}^{i+l-1} \varepsilon_j$. Then

$$u_{i+l-1}^- \cdots u_{i+1}^- u_i^- = v^{1-l} u_{\mathfrak{m}}^- + \sum_{\mathfrak{z} <_{\deg}^\infty \mathfrak{m}} v^{1-l} u_{\mathfrak{z}}^-.$$

For each $\mathfrak{z}$ with $\mathfrak{z} <_{\deg}^\infty \mathfrak{m}$, $M(\mathfrak{z})$ is decomposable. Thus, we may write

$$M(\mathfrak{z}) = M(\mathfrak{y}) \oplus M(\mathfrak{z}_1),$$

where $\mathfrak{y} \in \mathfrak{M}_\infty$ and $\mathfrak{z}_1 = [j, i + l - j)$ for some $i < j \leqslant i + l - 1$. This implies that

$$u_{\mathfrak{y}}^- u_{\mathfrak{z}_1}^- = u_{\mathfrak{z}}^-.$$

By the induction hypothesis,

$$u_{\mathfrak{z}_1}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle \ \text{ if } j \leqslant 0 \text{ and } i + l - 1 \geqslant 0,$$

and 0 otherwise, where $\mu = (i + l, 1^{(-j)})$. Let now $j \leqslant 0$ and $i + l - 1 \geqslant 0$ and let $k_1, \ldots, k_{j-i}$ be a permutation of $i, i + 1, \ldots, j - 1$. Then

$$(u_{k_1}^- u_{k_2}^- \cdots u_{k_{j-i}}^-) \cdot |\mu\rangle = 0$$

unless $k_1 = i, k_2 = i+1 \ldots, k_{j-i} = j-1$, and moreover

$$(u_i^- u_{i+1}^- \cdots u_{j-1}^-) \cdot |\mu\rangle = |\lambda\rangle.$$

Since $u_{\mathfrak{y}}^-$ is a $\mathcal{Z}$-linear combination of the monomials $u_{k_1}^- u_{k_2}^- \cdots u_{k_{j-i}}^-$, we have $\widetilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\lambda\rangle$.

Now let $i = 0$. Then $u_{\mathfrak{z}_1}^- \cdot |\emptyset\rangle = 0$ for each $\mathfrak{z}_1 = [j, i+l-j)$ with $0 < j \leqslant i+l-1$. Hence,

$$\widetilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle = v^{1-l} u_{\mathfrak{m}}^- \cdot |\emptyset\rangle = (u_{l-1}^- \cdots u_1^- u_0^-) \cdot |\emptyset\rangle + \sum_{\mathfrak{z} <_{\deg}^{\infty} \mathfrak{m}} u_{\mathfrak{z}}^- \cdot |\emptyset\rangle = |\lambda\rangle. \qquad \square$$

Lemma 5.2. *Let* $\mathfrak{m} = \sum_{l \geqslant 1} m_{i,l}[i,l) \in \mathfrak{M}_\infty$ *and* $\lambda \in \Pi$.

(1) *If there is* $j \in \mathbb{Z}$ *such that* $\sum_{l \geqslant 1} m_{j,l} \geqslant 2$, *then* $\widetilde{u}_{\mathfrak{m}}^- \cdot |\lambda\rangle = 0$. *In particular, for each* $i \in \mathbb{Z}$ *and* $t \geqslant 2$, $(u_i^-)^{(t)} \cdot |\lambda\rangle = 0$, *where* $(u_i^-)^{(t)} = (u_i^-)^t/[t]!$; *see* (3.0.2).
(2) *The element* $\widetilde{u}_{\mathfrak{m}}^- \cdot |\lambda\rangle$ *is a* $\mathcal{Z}$*-linear combination of* $|\mu\rangle$ *with* $\mu \in \Pi$.

Proof. (1) For each $i \in \mathbb{Z}$, we put

$$m_i = \sum_{l \geqslant 1} m_{i,l} \quad \text{and} \quad M_i = \bigoplus_{l \geqslant 1} m_{i,l} S_i[l].$$

Then $M = M(\mathfrak{m}) = \oplus_{i \in \mathbb{Z}} M_i$, where all but finitely many M_i are zero and

$$u_{\mathfrak{m}}^- = v^{-\sum_{i>j} \langle \mathbf{dim}\, M_i, \mathbf{dim} M_j \rangle} (\cdots u_{[M_{-1}]}^- u_{[M_0]}^- u_{[M_1]}^- \cdots).$$

Suppose there is $j \in \mathbb{Z}$ with $m = m_j \geqslant 2$. Then M_j admits a decomposition

$$M_j = S_j[a_1] \oplus \cdots \oplus S_j[a_m] \text{ with } a_1 \geqslant \cdots \geqslant a_m \geqslant 1.$$

This implies that

$$u_{[S_j[a_m]]}^- \cdots u_{[S_j[a_1]]}^- = v^{b_j} u_{[M_j]}^-,$$

where $b_j = \sum_{1 \leqslant p < q \leqslant m} \langle \mathbf{dim} S_j[m_p], \mathbf{dim}\, S_j[m_q] \rangle$. Hence, it suffices to show that for each $\mu \in \Pi$,

$$u_{[M_j]}^- \cdot |\mu\rangle = v^{-b_j} (u_{[S_j[a_m]]}^- \cdots u_{[S_j[a_1]]}^-) \cdot |\mu\rangle = 0.$$

By the definition, $u_{[S_j[a_1]]}^- \cdot |\mu\rangle$ is a $\mathbb{Q}(v)$-linear combination of those ν which are obtained from μ by adding a $(j+r)$-box for each $0 \leqslant r < a_1$. Thus, each such ν does not admit an indent j-box. Then $u^-_{[S_j[a_1]]}\cdot|\nu\rangle = 0$ and, hence, $(u^-_{[S_j[a_m]]}\cdots u^-_{[S_j[a_1]]})\cdot|\mu\rangle = 0$. We conclude that $\widetilde{u}^-_{\mathfrak{m}}\cdot|\lambda\rangle = 0$.

(2) It is known that $\widetilde{u}^-_{\mathfrak{m}}$ is a $\mathcal{Z}$-linear combination of monomials of divided powers $(u_i^-)^{(t)}$ for $i\in\mathbb{Z}$ and $t\geqslant 1$. Since by (1), $(u_i^-)^{(t)}\cdot|\mu\rangle = 0$ for all $i\in\mathbb{Z}$, $\mu\in\Pi$ and $t\geqslant 2$, it follows that $\widetilde{u}^-_{\mathfrak{m}}\cdot|\lambda\rangle$ is a $\mathcal{Z}$-linear combination of $(u^-_{i_1}\cdots u^-_{i_m})\cdot|\lambda\rangle$, where $m=\dim M(\mathfrak{m})$ and $i_1,\ldots,i_m\in\mathbb{Z}$. By the definition, $(u^-_{i_1}\cdots u^-_{i_m})\cdot|\lambda\rangle$ is either zero or equal to $|\mu\rangle$ for some $\mu\in\Pi$. Therefore, $\widetilde{u}^-_{\mathfrak{m}}\cdot|\lambda\rangle$ is a $\mathcal{Z}$-linear combination of $|\mu\rangle$ with $\mu\in\Pi$. $\square$

Proposition 5.3. (1) *For each* $\mathfrak{m}\in\mathfrak{M}_\infty$,

$$\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle\in\mathcal{Z}|\lambda\rangle \ \textit{ for some } \lambda\in\Pi \textit{ with } \mathfrak{m}_\lambda\leqslant^\infty_{\deg}\mathfrak{m}.$$

(2) *For each* $\lambda\in\Pi$,

$$\widetilde{u}^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle = |\lambda\rangle \ \textit{ and } \ \widetilde{u}^-_{\mathfrak{p}}\cdot|\emptyset\rangle = 0 \ \textit{ for all } \mathfrak{p}\in\mathfrak{M} \textit{ with } \mathfrak{p}<^\infty_{\deg}\mathfrak{m}_\lambda.$$

In particular, $b^-_{\mathfrak{m}_\lambda}\cdot|\emptyset\rangle = |\lambda\rangle$.

Proof. (1) If $\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle = 0$, there is nothing to prove. Now suppose $\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle\neq 0$. By Lemma 5.2(2), we write

$$\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle = \sum_{\lambda\in\Pi} f_\lambda(v)|\lambda\rangle,$$

where all $f_\lambda(v)\in\mathcal{Z}$ but finitely many are zero. If $f_\lambda(v)\neq 0$, then $\mathbf{dim}\,M(\mathfrak{m}_\lambda) = \mathbf{dim}\,M(\mathfrak{m})$. By Lemma 2.1(1), such a $\lambda\in\Pi$ is unique. Hence, we may suppose $\widetilde{u}^-_{\mathfrak{m}}\cdot|\emptyset\rangle = f(v)|\lambda\rangle$ for some $0\neq f(v)\in\mathcal{Z}$ and $\lambda\in\Pi$. It remains to show that $\mathfrak{m}_\lambda\leqslant^\infty_{\deg}\mathfrak{m}$.

Applying Lemma 5.2(1) implies that

$$M = M(\mathfrak{m}) = S_{i_1}[a_1]\oplus\cdots\oplus S_{i_t}[a_t],$$

where $i_1<\cdots<i_t$ and $a_1,\ldots,a_t\geqslant 1$. Then

$$u^-_{[S_{i_1}[a_1]]}\cdots u^-_{[S_{i_t}[a_t]]} = v^a u^-_{\mathfrak{m}},$$

where $a=\sum_{1\leqslant p<q\leqslant t}\langle\mathbf{dim}S_{i_q}[a_q],\mathbf{dim}\,S_{i_p}[a_p]\rangle$.

We proceed induction on t to show that $M(\mathfrak{m}_\lambda)\leqslant^\infty_{\deg} M = M(\mathfrak{m})$. If $t=1$, this follows from Lemma 5.1(2). Let now $t>1$ and let $\mu\in\Pi$ be such that

$$(u^-_{[S_{i_2}[a_2]]}\cdots u^-_{[S_{i_t}[a_t]]})\cdot|\emptyset\rangle = g(v)|\mu\rangle \ \text{ for some } 0\neq g(v)\in\mathcal{Z}.$$

Then $u^-_{[S_{i_1}[a_1]]}\cdot|\mu\rangle = v^a f(v)g(v)^{-1}|\lambda\rangle$. By the induction hypothesis,

$$M(\mathfrak{m}_\mu)\leqslant^\infty_{\deg} S_{i_2}[a_2]\oplus\cdots\oplus S_{i_t}[a_t].$$

By writing $u^-_{[S_{i_1}[a_1]]}$ as a $\mathcal{Z}$-linear combination of monomials of u_i^-'s and applying Lemma 5.1(1), there exists $X \in \operatorname{Rep}\Delta_\infty$ satisfying $\mathbf{dim}\, X = \mathbf{dim}\, S_{i_1}[a_1]$ with an exact sequence

$$0 \longrightarrow X \longrightarrow M(\mathfrak{m}_\lambda) \longrightarrow M(\mathfrak{m}_\mu) \longrightarrow 0.$$

Since $S_{i_1}[a_1]$ is indecomposable, it follows that $X \leqslant^\infty_{\deg} S_{i_1}[a_1]$. Therefore,

$$\begin{aligned} M(\mathfrak{m}_\lambda) \leqslant^\infty_{\deg} M(\mathfrak{m}_\mu) * X \leqslant^\infty_{\deg} & (S_{i_2}[a_2] \oplus \cdots \oplus S_{i_t}[a_t]) * S_{i_1}[a_1] \\ = & (S_{i_2}[a_2] \oplus \cdots \oplus S_{i_t}[a_t]) \oplus S_{i_1}[a_1] = M(\mathfrak{m}), \end{aligned}$$

that is, $\mathfrak{m}_\lambda \leqslant^\infty_{\deg} \mathfrak{m}$.

(2) Write $\lambda = (\lambda_1, \ldots, \lambda_t)$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_t \geqslant 1$. Since

$$M(\mathfrak{m}_\lambda) = S_0[\lambda_1] \oplus S_{-1}[\lambda_2] \oplus \cdots \oplus S_{1-t}[\lambda_t],$$

we have that

$$u^-_{[S_{1-t}[\lambda_t]]} \cdots u^-_{[S_{-1}[\lambda_2]]} u^-_{[S_0[\lambda_1]]} = v^c u^-_{\mathfrak{m}_\lambda},$$

where

$$c = \sum_{1 \leqslant r < s \leqslant t} \langle \mathbf{dim} S_{1-r}[\lambda_r], \mathbf{dim}\, S_{1-s}[\lambda_s] \rangle = \sum_{1 \leqslant r < s \leqslant t} \dim \operatorname{Hom}_{\Delta_\infty}(S_{1-r}[\lambda_r], S_{1-s}[\lambda_s]).$$

By using an argument similar to that in the proof of Lemma 5.1(2), we obtain that

$$\begin{aligned} v^c u^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle &= (u^-_{[S_{1-t}[\lambda_t]]} \cdots u^-_{[S_{-1}[\lambda_2]]} u^-_{[S_0[\lambda_1]]}) \cdot |\emptyset\rangle \\ &= v^{\lambda_1 - 1} (u^-_{[S_{1-t}[\lambda_t]]} \cdots u^-_{[S_{-1}[\lambda_2]]}) \cdot |(\lambda_1)\rangle \\ &= c^{\lambda_1 + \lambda_2 - 2} (u^-_{[S_{1-t}[\lambda_t]]} \cdots u^-_{[S_{-2}[\lambda_3]]}) \cdot |(\lambda_1, \lambda_2)\rangle \\ &= v^{\lambda_1 + \cdots + \lambda_t - t} |(\lambda_1, \ldots, \lambda_t)\rangle = v^{\lambda_1 + \cdots + \lambda_t - t} |\lambda\rangle. \end{aligned}$$

Since

$$\dim \operatorname{End}_{\Delta_\infty}(M(\mathfrak{m}_\lambda)) = \sum_{1 \leqslant r \leqslant s \leqslant t} \dim \operatorname{Hom}_{\Delta_\infty}(S_{1-r}[\lambda_r], S_{1-s}[\lambda_s]) = c + t$$

and $\dim M(\mathfrak{m}_\lambda) = \lambda_1 + \cdots + \lambda_t$, it follows that

$$\widetilde{u}^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle = v^{c+t-(\lambda_1 + \cdots + \lambda_t)} u^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle = |\lambda\rangle.$$

Now let $\mathfrak{p} <^\infty_{\deg} \mathfrak{m}_\lambda$ and suppose $\widetilde{u}^-_{\mathfrak{p}} \cdot |\emptyset\rangle \neq 0$. By (1), there exists $\mu \in \Pi$ with $\mathfrak{m}_\mu \leqslant^\infty_{\deg} \mathfrak{p}$ such that $\widetilde{u}^-_{\mathfrak{p}} \cdot |\emptyset\rangle = f(v)|\mu\rangle$ for some $f(v) \in \mathcal{Z}$. Thus, $\mathfrak{m}_\mu <^\infty_{\deg} \mathfrak{m}_\lambda$. By Lemma 2.1(1), $\mu = \lambda$ since $\mathbf{dim} M(\mathfrak{m}_\mu) = \mathbf{dim}\, M(\mathfrak{m}_\lambda)$. This is a contradiction. Hence, $\widetilde{u}^-_{\mathfrak{p}} \cdot |\emptyset\rangle = 0$.

By (4.0.9),

$$b^-_{\mathfrak{m}_\lambda} \in \widetilde{u}^-_{\mathfrak{m}_\lambda} + \sum_{\mathfrak{p} <^\infty_{\deg} \mathfrak{m}_\lambda} v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}^-_{\mathfrak{p}}.$$

We conclude that $b^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle = \widetilde{u}^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle = |\lambda\rangle$. $\square$

As a consequence of the proposition above, we obtain [43, Prop. 5.1] as follows.

Corollary 5.4. *The subspace $\mathcal{I}$ of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$ spanned by $b^-_{\mathfrak{m}}$ with $\mathfrak{m} \in \mathfrak{M} - \{\mathfrak{m}_\lambda \mid \lambda \in \Pi\}$ is a left ideal of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$. Moreover, the map*

$$\mathbf{U}_v^-(\mathfrak{sl}_\infty)/\mathcal{I} \longrightarrow \textstyle\bigwedge^\infty, \ b^-_{\mathfrak{m}_\lambda} + \mathcal{I} \longmapsto |\lambda\rangle, \ \forall\, \lambda \in \Pi$$

is an isomorphism of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$-modules.

Proof. On the one hand, by [30, Th. 14.4.11], the set

$$\{b^-_{\mathfrak{m}} \cdot |\emptyset\rangle \neq 0 \mid \mathfrak{m} \in \mathfrak{M}_\infty\}$$

is a basis of $\bigwedge^\infty$. On the other hand, there is a $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$-module homomorphism

$$\phi : \mathbf{U}_v^-(\mathfrak{sl}_\infty) \longrightarrow \textstyle\bigwedge^\infty, \ x \longmapsto x \cdot |\emptyset\rangle.$$

It follows from Proposition 5.3(2) that $\mathcal{I} = \operatorname{Ker}\phi$ is a left ideal of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$ and that ϕ induces the desired isomorphism. $\square$

Finally, for $i \in \mathbb{Z}$ and $\lambda \in \Pi$, put

$$n_i^-(\lambda) = \sum_{j<i,\, j\in\bar{i}} n_j(\lambda), \ n_i^+(\lambda) = \sum_{j>i,\, j\in\bar{i}} n_j(\lambda), \quad \text{and} \ \ n_{\bar{i}}(\lambda) = \sum_{j\in\bar{i}} n_j(\lambda).$$

By [16,32], there is a $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$-module structure on $\bigwedge^\infty$ defined by

$$K_{\bar{i}} \cdot |\lambda\rangle = v^{n_{\bar{i}}(\lambda)}|\lambda\rangle, \ E_{\bar{i}} \cdot |\lambda\rangle = \sum_{j\in\bar{i}} v^{n_j^-(\lambda)} E_j \cdot |\lambda\rangle, \ F_{\bar{i}} \cdot |\lambda\rangle = \sum_{j\in\bar{i}} v^{-n_j^+(\lambda)} F_j \cdot |\lambda\rangle, \tag{5.4.1}$$

where $\bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}$.

6. The q-deformed Fock space II: $\mathcal{D}(n)$-module

In this section we first recall the left $\mathcal{D}(n)^{\leqslant 0}$-module structure on the Fock space $\bigwedge^\infty$ defined by Varagnolo and Vasserot in [43] and then extend their construction to obtain a $\mathcal{D}(n)$-module structure on $\bigwedge^\infty$.

For each $x = \sum_{\mathfrak{m}} x_{\mathfrak{m}} u_{\mathfrak{m}} \in \mathcal{H}(\Delta)$ with $\Delta = \Delta_n$ or Δ_∞, we write

$$x^\pm = \sum_{\mathfrak{m}} x_{\mathfrak{m}} u_{\mathfrak{m}}^\pm \in \boldsymbol{\mathcal{D}}(\Delta)^\pm.$$

Then for each $\mathbf{d} \in \mathbb{N}I_\infty$, the map $\gamma_{\mathbf{d}} : \mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}} \to \mathcal{H}(\Delta_\infty)_{\mathbf{d}}$ defined in Section 3 induces $\mathbb{Q}(v)$-linear maps

$$\gamma_{\mathbf{d}}^\pm : \boldsymbol{\mathcal{D}}(n)_{\bar{\mathbf{d}}}^\pm \longrightarrow \boldsymbol{\mathcal{D}}(\infty)_{\mathbf{d}}^\pm$$

such that $\gamma_{\mathbf{d}}^\pm(x^\pm) = (\gamma_{\mathbf{d}}(x))^\pm$ for each $x \in \mathcal{H}(\Delta_\infty)$.

Following [43, 6.2], for each $\bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}$, $\lambda \in \Pi$ and $x \in \boldsymbol{\mathcal{D}}(n)_\alpha^-$, define

$$K_{\bar{i}} \cdot |\lambda\rangle = v^{n_{\bar{i}}(\lambda)}|\lambda\rangle \quad \text{and} \quad x \cdot |\lambda\rangle = \sum_{\mathbf{d}} \left(\gamma_{\mathbf{d}}^-(x) K_{-\mathbf{d}'}\right) \cdot |\lambda\rangle, \tag{6.0.2}$$

where the sum is taken over all $\mathbf{d} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{d}} = \alpha$ and $\mathbf{d}' = \sum_{i>j,\, \bar{i}=\bar{j}} d_j \varepsilon_i$. By [43, Cor. 6.2], this defines a left $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module structure on $\bigwedge^\infty$ which extends the Hayashi action of $\mathbf{U}_v^{\leqslant 0}(\widehat{\mathfrak{sl}}_n)$ on $\bigwedge^\infty$ defined in (5.4.1).

Dually, for each $\lambda \in \Pi$ and $x \in \boldsymbol{\mathcal{D}}(n)_\alpha^+$, define

$$x \cdot |\lambda\rangle = \sum_{\mathbf{d}} \left(\gamma_{\mathbf{d}}^+(x) K_{\mathbf{d}''}\right) \cdot |\lambda\rangle, \tag{6.0.3}$$

where the sum is taken over all $\mathbf{d} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{d}} = \alpha$ and $\mathbf{d}'' = \sum_{i<j,\, \bar{i}=\bar{j}} d_j \varepsilon_i$. Then we have the following result whose proof is analogous to that of [43, Cor. 6.2].

Proposition 6.1. *The formula (6.0.3) defines a left $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module structure on $\bigwedge^\infty$ which extends the Hayashi action of $\mathbf{U}_v^{\geqslant 0}(\widehat{\mathfrak{sl}}_n)$ on $\bigwedge^\infty$.*

Proof. Let $x \in \boldsymbol{\mathcal{D}}(n)_\alpha^+$ and $y \in \boldsymbol{\mathcal{D}}(n)_\beta^+$, where $\alpha, \beta \in \mathbb{N}I_n$. By the definition, we have, on the one hand, that

$$(xy) \cdot |\lambda\rangle = \sum_{\mathbf{d}} \left(\gamma_{\mathbf{d}}^+(xy) K_{\mathbf{d}''}\right) \cdot |\lambda\rangle$$

and, on the other hand, that

$$x \cdot (y \cdot |\lambda\rangle) = \sum_{\mathbf{a},\mathbf{b}} \left(\gamma_{\mathbf{a}}^+(x) K_{\mathbf{a}''} \gamma_{\mathbf{b}}^+(y) K_{\mathbf{b}''}\right) \cdot |\lambda\rangle,$$

where the sum is taken over all $\mathbf{a}, \mathbf{b} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{a}} = \alpha$ and $\bar{\mathbf{b}} = \beta$.

Since $K_{\mathbf{a}''} \gamma_{\mathbf{b}}^+(y) = v^{(\mathbf{a}'',\mathbf{b})} \gamma_{\mathbf{b}}^+(y) K_{\mathbf{a}''}$, we obtain that

$$x \cdot (y \cdot |\lambda\rangle) = \sum_{\mathbf{d}} \sum_{\mathbf{a}+\mathbf{b}=\mathbf{d}} v^{(\mathbf{a}'',\mathbf{b})} \left(\gamma_{\mathbf{a}}^+(x) \gamma_{\mathbf{b}}^+(y) K_{\mathbf{d}''}\right) \cdot |\lambda\rangle.$$

By the definition,

$$(\mathbf{a}'', \mathbf{b}) = \Big(\sum_{i<j,\, \bar{i}=\bar{j}} a_j \varepsilon_i, \sum_i b_i \varepsilon_i \Big) = \sum_{i<j,\, \bar{i}=\bar{j}} b_i(2a_j - a_{j-1} - a_{j+1}) = \kappa(\mathbf{a}, \mathbf{b}).$$

Applying Lemma 3.3(2) gives that

$$(xy) \cdot |\lambda\rangle = x \cdot (y \cdot |\lambda\rangle).$$

Hence, $\bigwedge^\infty$ becomes a left $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module.

For each $\bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}$ and $\lambda \in \Pi$, we have

$$u_{\bar{i}}^+ \cdot |\lambda\rangle = \sum_{j \in \bar{i}} (u_j^+ K_{-\varepsilon_j''}) \cdot |\lambda\rangle.$$

Since $\varepsilon_j'' = \sum_{l<j,\, \bar{l}=\bar{j}} \varepsilon_l$ for each $j \in \bar{i}$, it follows that

$$K_{\varepsilon_j''} \cdot |\lambda\rangle = \prod_{l<j,\, \bar{l}=\bar{j}} K_{\varepsilon_l} \cdot |\lambda\rangle = v^{\sum_{l<j, \bar{l}=\bar{j}} n_l(\lambda)} |\lambda\rangle = v^{n_j^-(\lambda)} |\lambda\rangle.$$

This implies that

$$u_{\bar{i}}^+ \cdot |\lambda\rangle = \sum_{j \in \bar{i}} v^{n_j^-(\lambda)} u_j^+ \cdot |\lambda\rangle,$$

which coincides with the formula for $E_{\bar{i}} \cdot |\lambda\rangle$ in (5.4.1), as required. □

After the work of Varagnolo and Vasserot [43], one naturally expects to extend their construction to obtain a $\boldsymbol{\mathcal{D}}(n)$-module structure on $\bigwedge^\infty$. The formula for the actions $u_\alpha^+ \cdot |\lambda\rangle$ of the semisimple generators u_α^+ in $\boldsymbol{\mathcal{D}}(n)^+$ was given by Stroppel and Webster [41, Lem. 7.5]. The formula (6.0.3) is modified from (6.0.2) in terms of the Varagnolo–Vasserot's map in (3.2.2).

The rest of this section is devoted to verifying that formulas (6.0.2) and (6.0.3) indeed define a $\boldsymbol{\mathcal{D}}(n)$-module structure on $\bigwedge^\infty$. The strategy is to pass to the semi-infinite v-wedge spaces studied in [40,24] and then to compare the actions of the central elements $\boldsymbol{z}_m^\pm$ in $\boldsymbol{\mathcal{D}}(n)$ with those of the Heisenberg operators defined in [24].

Let Ω denote the $\mathbb{Q}(v)$-vector space with basis $\{\omega_i \mid i \in \mathbb{Z}\}$. By [6, Prop. 3.5], Ω admits a $\boldsymbol{\mathcal{D}}(n)$-module structure defined by

$$\begin{aligned} u_i^+ \cdot \omega_s &= \delta_{i+1,\bar{s}} \omega_{s-1}, \quad u_i^- \cdot \omega_s = \delta_{i,\bar{s}} \omega_{s+1} \\ K_i^{\pm 1} \cdot \omega_s &= v^{\pm \delta_{i,\bar{s}} \mp \delta_{i+1,\bar{s}}} \omega_s, \quad \boldsymbol{z}_m^\pm \cdot \omega_s = \omega_{s \mp mn} \end{aligned} \tag{6.1.1}$$

for all $i \in I_n$ and $s, m \in \mathbb{Z}$ with $m \geqslant 1$. In particular, $K_\delta^{\pm 1} \cdot \omega_s = \omega_s$ for each $s \in \mathbb{Z}$. This is an extension of the $\mathbf{U}_v'(\widehat{\mathfrak{sl}}_n)$-action on Ω defined in [24, 1.1] as well as an extension of the $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-action on Ω defined in [43, 8.1]; see [6, 3.5].

For a fixed positive integer r, consider the r-fold tensor product $\Omega^{\otimes r}$ which has a basis

$$\{\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \mid \mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r\}.$$

The Hopf algebra structure of $\boldsymbol{\mathcal{D}}(n)$ induces a $\boldsymbol{\mathcal{D}}(n)$-module structure on the r-fold tensor product $\Omega^{\otimes r}$. By (4.1.1), we have for each $t \geqslant 1$,

$$\begin{aligned} \Delta^{(r-1)}(\boldsymbol{z}_t^+) &= \sum_{s=0}^{r-1} \underbrace{1 \otimes \cdots \otimes 1}_{s} \otimes \boldsymbol{z}_t^+ \otimes \underbrace{K_{t\delta} \otimes \cdots \otimes K_{t\delta}}_{r-s-1} \quad \text{and} \\ \Delta^{(r-1)}(\boldsymbol{z}_t^-) &= \sum_{s=0}^{r-1} \underbrace{K_{-t\delta} \otimes \cdots \otimes K_{-t\delta}}_{s} \otimes \boldsymbol{z}_t^- \otimes \underbrace{1 \otimes \cdots \otimes 1}_{r-s-1}. \end{aligned} \tag{6.1.2}$$

This implies particularly that for each $t \geqslant 1$ and $\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \in \Omega^{\otimes r}$,

$$\boldsymbol{z}_t^{\pm} \cdot \omega_{\mathbf{i}} = \sum_{s=1}^{r} \omega_{i_1} \otimes \cdots \otimes \omega_{i_{s-1}} \otimes \omega_{i_s \mp tn} \otimes \omega_{i_{s+1}} \otimes \cdots \otimes \omega_{i_r}. \tag{6.1.3}$$

By (4.0.3) and (4.0.5), for each $\alpha \in \mathbb{N}I_n$, we have

$$\begin{aligned} \Delta^{(r-1)}(\widetilde{u}_\alpha^+) = & \sum_{\alpha=\alpha^{(1)}+\cdots+\alpha^{(r)}} v^{\sum_{s>t}\langle \alpha^{(s)}, \alpha^{(t)} \rangle} \times \\ & \widetilde{u}^+_{\alpha^{(1)}} \otimes \widetilde{u}^+_{\alpha^{(2)}} K_{\alpha^{(1)}} \otimes \cdots \otimes \widetilde{u}^+_{\alpha^{(r)}} K_{(\alpha^{(1)}+\alpha^{(2)}+\cdots+\alpha^{(r-1)})}, \\ \Delta^{(r-1)}(\widetilde{u}_\alpha^-) = & \sum_{\alpha=\alpha^{(1)}+\cdots+\alpha^{(r)}} v^{\sum_{s>t}\langle \alpha^{(s)}, \alpha^{(t)} \rangle} \times \\ & \widetilde{u}^-_{\alpha^{(1)}} K_{-(\alpha^{(2)}+\cdots+\alpha^{(r)})} \otimes \cdots \otimes u^-_{\alpha^{(r-1)}} K_{-\alpha^{(r)}} \otimes \widetilde{u}^-_{\alpha^{(r)}}. \end{aligned} \tag{6.1.4}$$

This gives the following lemma; see [43, Lem. 8.3] and [6, Cor. 3.5.8].

Lemma 6.2. *Let $\alpha \in \mathbb{N}I_n$ and $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r$. Then*

$$\widetilde{u}_\alpha^+ \cdot \omega_{\mathbf{i}} = \sum_{\mathbf{n}} v^{c^+(\mathbf{i}, \mathbf{i}-\mathbf{n})} \omega_{\mathbf{i}-\mathbf{n}}, \tag{6.2.1}$$

where the sum is taken over the sequences $\mathbf{n} = (n_1, \ldots, n_r) \in \{0,1\}^r$ satisfying $\alpha = \sum_{s=1}^r n_s \varepsilon_{\overline{i_s - 1}}$ and

$$c^+(\mathbf{i}, \mathbf{i}-\mathbf{n}) = \sum_{1 \leqslant s < t \leqslant r} n_s(n_t - 1)\langle \varepsilon_{\bar{i}_t}, \varepsilon_{\bar{i}_s} \rangle;$$

$$\widetilde{u}_\alpha^- \cdot \omega_{\mathbf{i}} = \sum_{\mathbf{n}} v^{c^-(\mathbf{i}, \mathbf{i}+\mathbf{n})} \omega_{\mathbf{i}+\mathbf{n}}, \tag{6.2.2}$$

where the sum is taken over the sequences $\mathbf{n} = (n_1, \ldots, n_r) \in \{0,1\}^r$ *satisfying* $\alpha = \sum_{s=1}^r n_s \varepsilon_{\bar{i}_s}$ *and*

$$c^-(\mathbf{i}, \mathbf{i}+\mathbf{n}) = \sum_{1 \leqslant s < t \leqslant r} n_t(n_s - 1)\langle \varepsilon_{\bar{i}_t}, \varepsilon_{\bar{i}_s} \rangle.$$

On the other hand, let $\widehat{\mathbf{H}}(r)$ be the Hecke algebra of affine symmetric group of type A which is by definition a $\mathbb{Q}(v)$-algebra with generators T_i and $X_j^{\pm 1}$ for $i = 1, \ldots, r-1$, $j = 1, \ldots, r$ and relations:

$$\begin{aligned}
&(T_i + 1)(T_i - v^2) = 0,\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \ \ T_i T_j = T_j T_i \ (|i-j| > 1),\\
&X_i X_i^{-1} = 1 = X_i^{-1} X_i, \ \ X_i X_j = X_j X_i,\\
&T_i X_i T_i = v^2 X_{i+1}, \ \ X_j T_i = T_i X_j \ (j \neq i, i+1).
\end{aligned}$$

This is the so-called *Bernstein presentation* of $\widehat{\mathbf{H}}(r)$.

By [43, Sect. 8.2], there is a right $\widehat{\mathbf{H}}(r)$-module structure on $\Omega^{\otimes r}$ defined by

$$\begin{aligned}
&\omega_{\mathbf{i}} \cdot X_t = \omega_{i_1} \cdots \omega_{i_{t-1}} \omega_{i_t - n} \omega_{i_{t+1}} \cdots \omega_{i_r},\\
&\omega_{\mathbf{i}} \cdot T_k = \begin{cases} v^2 \omega_{\mathbf{i}}, & \text{if } i_k = i_{k+1};\\ v\omega_{\mathbf{i}s_k}, & \text{if } -n < i_k < i_{k+1} \leqslant 0;\\ v\omega_{\mathbf{i}s_k} + (v^2-1)\omega_{\mathbf{i}}, & \text{if } -n < i_{k+1} < i_k \leqslant 0, \end{cases}
\end{aligned} \tag{6.2.3}$$

where $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r$, $\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r}$ and

$$\omega_{\mathbf{i}s_k} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_{k+1}} \otimes \omega_{i_k} \otimes \cdots \otimes \omega_{i_r}.$$

Following [6, Prop. 3.5.5], the tensor space $\Omega^{\otimes r}$ is indeed a $\boldsymbol{\mathcal{D}}(n)$-$\widehat{\mathbf{H}}(r)$-bimodule. Set

$$\Xi^r = \sum_{i=1}^{r-1} \mathrm{Im}(1 + T_i) \subseteq \Omega^{\otimes r},$$

which is clearly a $\boldsymbol{\mathcal{D}}(n)$-submodule of $\Omega^{\otimes r}$. Thus, the quotient space $\Omega^{\otimes r}/\Xi^r$ becomes a $\boldsymbol{\mathcal{D}}(n)$-module. For each $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r$, write

$$\wedge \omega_{\mathbf{i}} = \omega_{i_1} \wedge \ldots \wedge \omega_{i_r} = \omega_{\mathbf{i}} + \Xi^r \in \Omega^{\otimes r}/\Xi^r.$$

By [24, Prop. 1.3], the set

$$\{\wedge \omega_{\mathbf{i}} \mid i_1 > \cdots > i_r\}$$

forms a basis of $\Omega^{\otimes r}/\Xi^r$.

For each $m \in \mathbb{Z}$, let $\mathscr{B}_m$ denote the set of sequences $\mathbf{i} = (i_1, i_2, \dots) \in \mathbb{Z}^\infty$ satisfying that $i_s = m - s + 1$ for $s \gg 0$, and set $\mathscr{B}_\infty = \cup_{m\in\mathbb{Z}} \mathscr{B}_m$. As in [43, Sect. 10.1], let Ω^∞ denote the space spanned by semi-infinite monomials

$$\omega_{\mathbf{i}} = \omega_{i_1} \otimes \omega_{i_2} \otimes \cdots, \quad \text{where } \mathbf{i} = (i_1, i_2, \dots) \in \mathscr{B}_\infty.$$

Then the affine Hecke algebra $\widehat{\mathbf{H}}(\infty)$ acts on Ω^∞ via the formulas in (6.2.3). Set

$$\Xi^\infty = \sum_{i=1}^{\infty} \mathrm{Im}(1 + T_i) \subseteq \Omega^\infty.$$

For each $\mathbf{i} = (i_1, i_2, \dots) \in \mathscr{B}_\infty$ as above, write

$$\wedge\omega_{\mathbf{i}} = \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots = \omega_{\mathbf{i}} + \Xi^\infty \in \Omega^\infty/\Xi^\infty.$$

For each $m \in \mathbb{Z}$, let $\mathcal{F}_{(m)}$ be the subspace of Ω^∞/Ξ^∞ spanned by $\wedge\omega_{\mathbf{i}}$ with $\mathbf{i} \in \mathscr{B}_m$. Then

$$\Omega^\infty/\Xi^\infty = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_{(m)}.$$

By [24, 1.4], the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module structure on $\Omega^{\otimes r}/\Xi^r$ induces a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module structure on $\mathcal{F}_{(m)}$ for each $m \in \mathbb{Z}$ and, hence, a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module structure on Ω^∞/Ξ^∞ as well. Moreover, by [24, Prop. 1.4], the injective map

$$\kappa : \bigwedge^\infty \longrightarrow \Omega^\infty/\Xi^\infty, \ |\lambda\rangle \longmapsto \wedge\omega_{\mathbf{i}_\lambda}$$

is a homomorphism of $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-modules which induces an isomorphism $\bigwedge^\infty \cong \mathcal{F}_{(0)}$, where $\mathbf{i}_\lambda = (i_1, i_2, \dots)$ with $i_s = \lambda_s + 1 - s$, $\forall\, s \geqslant 1$.

As in [24, (49)], for each $m \in \mathbb{Z}$, write

$$|m\rangle = \omega_m \wedge \omega_{m-1} \wedge \omega_{m-2} \wedge \cdots.$$

Clearly, for each $\mathbf{i} = (i_1, i_2, \dots) \in \mathscr{B}_m$, there exists a sufficiently large N such that

$$\wedge\omega_{\mathbf{i}} = (\omega_{i_1} \wedge \cdots \wedge \omega_{i_N}) \wedge |m - N\rangle.$$

By [24, Lem. 2.2] and (6.1.4), for given $\alpha \in \mathbb{N}I$ and $\mathbf{i} \in \mathscr{B}_m$, there is $t \gg 0$ such that

$$u_\alpha^- \cdot (\wedge\omega_{\mathbf{i}}) = \big(u_\alpha^- \cdot (\omega_{i_1} \wedge \cdots \wedge \omega_{i_t})\big) \wedge |m - t\rangle.$$

Hence, the $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module structure on $\Omega^{\otimes r}/\Xi^r$ induces a $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module structure on Ω^∞/Ξ^∞; see [43, Sect. 10.1]. Moreover, by [43, Lem. 10.1], the map $\kappa : \bigwedge^\infty \to \Omega^\infty/\Xi^\infty$ is a $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module homomorphism.

Dually, for each given $\mathbf{i} \in \mathscr{B}_m$, there is $t \gg 0$ such that

$$u_\alpha^+ \cdot (\wedge\omega_{\mathbf{i}}) = \big(u_\alpha^+ \cdot (\omega_{i_1} \wedge \cdots \wedge \omega_{i_t})\big) \wedge \big(K_\alpha \cdot |m-t\rangle\big).$$

Thus, Ω^∞/Ξ^∞ becomes a left $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module, too. We have the following result whose proof is similar to that of [43, Lem. 10.1].

Proposition 6.3. *The map κ is a $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module homomorphism.*

Proof. We need to show that for each $\lambda \in \Pi$ and $\alpha \in \mathbb{N}I_n$,

$$\kappa(\widetilde{u}_\alpha^+ \cdot |\lambda\rangle) = \widetilde{u}_\alpha^+(\kappa(|\lambda\rangle)).$$

For simplicity, write $\mathbf{i} := \mathbf{i}_\lambda = (i_1, i_2, \ldots)$. By (6.0.3),

$$\widetilde{u}_\alpha^+ \cdot |\lambda\rangle = \sum_{\mathbf{d}} (\gamma_{\mathbf{d}}^+(\widetilde{u}_\alpha^+) K_{\mathbf{d}''}) \cdot |\lambda\rangle = \sum_{\mathbf{d}} v^{-h(\mathbf{d})} (\widetilde{u}_{\mathbf{d}}^+ K_{\mathbf{d}''}) \cdot |\lambda\rangle,$$

where the sum is taken over all $\mathbf{d} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{d}} = \alpha$ and $h(\mathbf{d}) = \sum_{i<j, \bar{i}=\bar{j}} d_i(d_{j+1} - d_j)$.

For each fixed $\mathbf{d} = (d_i) \in \mathbb{N}I_\infty$ with $\bar{\mathbf{d}} = \alpha$, we have

$$\widetilde{u}_{\mathbf{d}}^+ = \cdots \widetilde{u}_{d_1\varepsilon_1}^+ \widetilde{u}_{d_0\varepsilon_0}^+ \widetilde{u}_{d_{-1}\varepsilon_{-1}}^+ \cdots = \prod_{i\in\mathbb{Z}} \widetilde{u}_{d_i\varepsilon_i}^+.$$

By the definition, $\widetilde{u}_{\mathbf{d}}^+ \cdot |\lambda\rangle \neq 0$ implies that

$$\mathbf{d} = \sum_{s\geqslant 1} n_s \varepsilon_{i_s - 1},$$

where $n_s \in \{0, 1\}$ for all $s \geqslant 1$. Moreover, if this is the case, then

$$\widetilde{u}_{\mathbf{d}}^+ \cdot |\lambda\rangle = |\mu_{\mathbf{n}}\rangle,$$

where $\mathbf{n} = (n_1, n_2, \ldots)$ and $\mu_{\mathbf{n}} = \mu \in \Pi$ is determined by $\mathbf{i}_\mu = \mathbf{i} - \mathbf{n}$. Therefore, for $\mathbf{d} \in \mathbb{N}I_\infty$ with $\mathbf{d} = \sum_{s\geqslant 1} n_s \varepsilon_{i_s-1}$,

$$K_{\mathbf{d}''} = \prod_{\bar{i}_s = \bar{i}_t,\, i_s > i_t} K_{i_t-1}^{n_s} \quad \text{and}$$

$$h(\mathbf{d}) = \sum_{i_s > i_t} -n_s n_t (\delta_{\bar{i}_s, \bar{i}_t} - \delta_{\bar{i}_s, \overline{i_t+1}}) = -\sum_{i_s > i_t} n_s n_t \langle \varepsilon_{\bar{i}_t}, \varepsilon_{\bar{i}_s} \rangle.$$

A calculation together with (6.2.1) implies that

$$\kappa(\widetilde{u}_\alpha^+ \cdot |\lambda\rangle) = \widetilde{u}_\alpha^+(\wedge\omega_{\mathbf{i}}) = \widetilde{u}_\alpha^+(\kappa(|\lambda\rangle)). \qquad \square$$

As a consequence of the results above, to prove that the formulas (6.0.2) and (6.0.3) define a $\boldsymbol{\mathcal{D}}(n)$-module structure on $\bigwedge^\infty$, it suffices to show that the $\boldsymbol{\mathcal{D}}(n)^{\leqslant 0}$-module and $\boldsymbol{\mathcal{D}}(n)^{\geqslant 0}$-module structures on Ω^∞/Ξ^∞ define a $\boldsymbol{\mathcal{D}}(n)$-module structure. In other words, we need to show that the actions of $K_i^{\pm 1}, u_i^+, u_i^-$ $(i \in I_n)$ and $\boldsymbol{z}_s^+, \boldsymbol{z}_s^-$ $(s \geqslant 1)$ on Ω^∞/Ξ^∞ satisfy the relations (DH1)–(DH5) in Section 4.

Since, as discussed above, Ω^∞/Ξ^∞ is a $\mathbf{U}_v'(\widehat{\mathfrak{sl}}_n)$-module, all the relations in (DH1)–(DH5) in which the $\boldsymbol{z}_s^\pm$ are not involved are satisfied. In the following we are going to check the relations

$$[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] = \delta_{t,s}\,\frac{t(v^{2tn}-1)}{(v^t - v^{-t})^2}(K_{t\delta} - K_{-t\delta}), \;\; \forall\, s,t \geqslant 1.$$

By [24, §2], for each $t \geqslant 1$, there are Heisenberg operators

$$B_t^\pm : \Omega^\infty/\Xi^\infty \to \Omega^\infty/\Xi^\infty, \;\; \wedge\omega_{\mathbf{i}} \longmapsto \sum_{s=1}^{\infty} \wedge\omega_{\mathbf{i}\mp tn\mathbf{e}_s},$$

where $\mathbf{i} \in \mathscr{B}_\infty$ and $\mathbf{e}_s = (\delta_{i,s})_{i\geqslant 1} \in \mathbb{Z}^\infty$. Note that for each $\mathbf{i} \in \mathscr{B}_\infty$, $\wedge\omega_{\mathbf{i}\mp tn\mathbf{e}_s} = 0$ for $s \gg 0$.

Proposition 6.4. *For each $t \geqslant 1$ and $\mathbf{i} \in \mathscr{B}_\infty$,*

$$B_t^+(\wedge\omega_{\mathbf{i}}) = v^t \boldsymbol{z}_t^+ \cdot (\wedge\omega_{\mathbf{i}}) \;\; \textit{and} \;\; B_t^-(\wedge\omega_{\mathbf{i}}) = \boldsymbol{z}_t^- \cdot (\wedge\omega_{\mathbf{i}}).$$

Proof. For each $m \in \mathbb{Z}$, recall the element

$$|m\rangle = \omega_m \wedge \omega_{m-1} \wedge \omega_{m-2} \wedge \cdots \in \Omega^\infty/\Xi^\infty.$$

Then $\boldsymbol{z}_t^+ \cdot |m\rangle = 0$ and $K_\delta \cdot |m\rangle = q|m\rangle$. Write

$$\wedge\omega_{\mathbf{i}} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_N} \wedge |N-m\rangle.$$

Applying (6.1.2) gives that

$$\begin{aligned}
&\boldsymbol{z}_t^+ \cdot (\wedge\omega_{\mathbf{i}}) \\
&= \textstyle\sum_{s=0}^N \underbrace{\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}}_{s} \wedge \boldsymbol{z}_t^+ \cdot \omega_{i_{s+1}} \wedge \underbrace{K_{t\delta}\cdot\omega_{i_{s+2}} \wedge \cdots \wedge K_{t\delta}\cdot\omega_{i_N}}_{N-s-1} \wedge (K_{t\delta}\cdot|N-m\rangle) \\
&= \textstyle\sum_{s=0}^N v^t\, \underbrace{\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}}_{s} \wedge \omega_{i_{s+1}+tn} \wedge \underbrace{\omega_{i_{s+2}} \wedge \cdots \wedge \omega_{i_N}}_{N-s-1} \wedge |N-m\rangle \\
&= v^t B_t^+(\wedge\omega_{\mathbf{i}}) \quad (\text{since } B_t^+(|N-m\rangle) = 0),
\end{aligned}$$

that is, $B_t^+(\wedge\omega_{\mathbf{i}}) = v^t \boldsymbol{z}_t^+ \cdot (\wedge\omega_{\mathbf{i}})$. The second equality can be proved similarly. $\square$

Corollary 6.5. *Let $t, s \geqslant 1$. Then for each $\mathbf{i} \in \mathscr{B}_\infty$,*

$$[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] \cdot (\wedge\omega_{\mathbf{i}}) = \delta_{t,s} \frac{t(v^{2tn}-1)}{(v^t - v^{-t})^2}(K_{t\delta} - K_{-t\delta}) \cdot (\wedge\omega_{\mathbf{i}}).$$

Proof. By [24, Prop. 2.2 & 2.6] (with $q = v$),

$$[B_t^+, B_s^-] = \delta_{t,s} \frac{t(1-v^{2tn})}{1-v^{2n}}.$$

This together with Proposition 6.4 implies that for each $\mathbf{i} \in \mathscr{B}_\infty$,

$$[\boldsymbol{z}_t^+, \boldsymbol{z}_s^-] \cdot (\wedge\omega_{\mathbf{i}}) = v^t [B_t^+, B_s^-]\delta_{t,s} \cdot (\wedge\omega_{\mathbf{i}}) = \delta_{t,s} \frac{tv^t(1-v^{2tn})}{1-v^{2n}}(\wedge\omega_{\mathbf{i}}).$$

On the other hand,

$$\begin{aligned}\delta_{t,s} \frac{t(v^{2tn}-1)}{(v^t - v^{-t})^2}(K_{t\delta} - K_{-t\delta}) \cdot (\wedge\omega_{\mathbf{i}}) &= \delta_{t,s} \frac{t(v^{2tn}-1)}{(v^t - v^{-t})^2}(v^t - v^{-t})(\wedge\omega_{\mathbf{i}}) \\ &= \delta_{t,s} \frac{tv^t(1-v^{2tn})}{1-v^{2n}}(\wedge\omega_{\mathbf{i}}).\end{aligned}$$

This gives the desired equality. $\square$

By [24, Prop. 2.1] (or direct calculations), the actions of $\boldsymbol{z}_t^\pm$ on Ω^∞/Ξ^∞ commutes with that of $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$. In conclusion, the actions of $K_i^{\pm 1}, u_i^+, u_i^-$ ($i \in I_n$) and $\boldsymbol{z}_s^+, \boldsymbol{z}_s^-$ ($s \geqslant 1$) on Ω^∞/Ξ^∞ satisfy the relations (DH1)–(DH5). Therefore, the formulas (6.0.2) and (6.0.3) define a $\boldsymbol{\mathcal{D}}(n)$-module structure on $\bigwedge^\infty$.

7. An isomorphism from $L(\Lambda_0)$ to $\bigwedge^\infty$

In this section we show that the Fock space $\bigwedge^\infty$ as a $\boldsymbol{\mathcal{D}}(n)$-module is isomorphic to the basic representation $L(\Lambda_0)$ defined in Section 4. As an application, the decomposition of $L(\Lambda_0)$ in Corollary 4.5 induces the Kashiwara–Miwa–Stern decomposition of $\bigwedge^\infty$ in [24].

Proposition 7.1. *For each $\mathfrak{m} \in \mathfrak{M}_n$, $\widetilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle$ is a $\mathcal{Z}$-linear combination of those $|\mu\rangle$ satisfying $\mathfrak{m}_\mu \leqslant_{\deg} \mathfrak{m}$.*

Proof. By (6.0.2),

$$\widetilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle = \sum_{\mathbf{d}} \big(\gamma_{\mathbf{d}}^-(\widetilde{u}_{\mathfrak{m}}^-)K_{-\mathbf{d}'}\big) \cdot |\emptyset\rangle, \quad \text{where } \mathbf{d}' = \sum_i \Big(\sum_{j<i,\, \bar{j}=\bar{i}} d_j\Big)\varepsilon_i.$$

Since $K_i \cdot |\emptyset\rangle = v^{\delta_{i,0}}|\emptyset\rangle$ for $i \in \mathbb{Z}$, it follows that $K_{-\mathbf{d}'} \cdot |\emptyset\rangle = v^{-\sum_{j<0,\, \bar{j}=\bar{0}} d_j}|\emptyset\rangle$. By Proposition 3.4,

$$\gamma^-_{\mathbf{d}}(\widetilde{u}^-_{\mathfrak{m}}) \in \sum_{\mathfrak{z}} \mathcal{Z}\widetilde{u}^-_{\mathfrak{z}},$$

where the sum is taken over $\mathfrak{z} \in \mathfrak{M}_\infty$ with $\mathscr{F}(\mathfrak{z}) \leqslant^\infty_{\deg} \mathfrak{m}$. Further, by Proposition 5.3(1),

$$\widetilde{u}^-_{\mathfrak{z}} \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle$$

for some $\mu \in \Pi$ with $\mathfrak{m}^\infty_\mu \leqslant^\infty_{\deg} \mathfrak{z}$. This implies that

$$\mathfrak{m}_\mu = \mathscr{F}(\mathfrak{m}^\infty_\mu) \leqslant_{\deg} \mathscr{F}(\mathfrak{z}) \leqslant_{\deg} \mathfrak{m}.$$

This finishes the proof. □

For each $\mathbf{d} = (d_i) \in \mathbb{N}I_\infty$, set

$$\sigma(\mathbf{d}) = -\sum_{i<0,\, \bar{i}=\bar{0}} d_i.$$

For $\lambda \in \Pi$, we write $\sigma(\lambda) = \sigma(\mathbf{dim} M(\mathfrak{m}^\infty_\lambda))$. The following result was proved in [43, 9.2 & 10.1]. We provide here a direct proof for completeness.

Corollary 7.2. *For each $\lambda \in \Pi$,*

$$\widetilde{u}^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle \in |\lambda\rangle + \sum_{\mu \lhd \lambda} \mathcal{Z}|\mu\rangle.$$

In particular, the $\boldsymbol{\mathcal{D}}(n)$-module $\bigwedge^\infty$ is generated by $|\emptyset\rangle$ and the set

$$\{b^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle \mid \lambda \in \Pi\}$$

is a basis of $\bigwedge^\infty$.

Proof. Applying Corollary 3.5 gives that

$$\begin{aligned}
\widetilde{u}^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle &= \sum_{\mathbf{d}} \big(\gamma^-_{\mathbf{d}}(\widetilde{u}^-_{\mathfrak{m}_\lambda}) K_{-\mathbf{d}'}\big) \cdot |\emptyset\rangle = \sum_{\mathbf{d}} v^{\sigma(\mathbf{d})} \gamma^-_{\mathbf{d}}(\widetilde{u}^-_{\mathfrak{m}_\lambda}) \cdot |\emptyset\rangle \\
&= \sum_{r\in\mathbb{Z}} v^{\theta(\lambda)+\sigma(\lambda)} \widetilde{u}^-_{\tau^{rm}(\mathfrak{m}^\infty_\lambda)} \cdot |\emptyset\rangle + \sum_{\mathfrak{z}\in\mathfrak{M}_\infty,\, \mathscr{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}_\lambda} f_{\lambda,\mathfrak{z}}\, \widetilde{u}^-_{\mathfrak{z}} \cdot |\emptyset\rangle,
\end{aligned}$$

where $f_{\lambda,\mathfrak{z}} \in \mathcal{Z}$. By Proposition 5.3 and its proof,

$$\widetilde{u}^-_{\mathfrak{m}^\infty_\lambda} \cdot |\emptyset\rangle = |\lambda\rangle \ \text{ and } \ \widetilde{u}^-_{\tau^{rm}(\mathfrak{m}^\infty_\lambda)} \cdot |\emptyset\rangle = 0 \text{ for } r > 0.$$

Furthermore, for each $r < 0$, $\widetilde{u}^-_{\tau^{rm}(\mathfrak{m}^\infty_\lambda)} \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle$ such that $\mathfrak{m}^\infty_\mu \leqslant^\infty_{\deg} \tau^{rm}(\mathfrak{m}^\infty_\lambda)$. Then $\mathfrak{m}_\mu = \mathscr{F}(\mathfrak{m}^\infty_\mu) \leqslant_{\deg} \mathscr{F}(\tau^{rm}(\mathfrak{m}^\infty_\lambda)) = \mathfrak{m}_\lambda$, which implies that $\mu \unlhd \lambda$. Since $M(\tau^{rm}(\mathfrak{m}^\infty_\lambda))$

does not have a composition factor isomorphic to S_{λ_1-1}, μ does not contain a box with color λ_1-1. Thus, $\mu\neq\lambda$ and $\mu\lhd\lambda$.

Finally, by Proposition 7.1, for each $\mathfrak{z}\in\mathfrak{M}_\infty$ with $\mathscr{F}(\mathfrak{z})<_{\deg}\mathfrak{m}_\lambda$, $\widetilde{u}_{\mathfrak{z}}^-\cdot|\emptyset\rangle$ is a $\mathcal{Z}$-linear combination of $|\mu\rangle$ satisfying $\mathfrak{m}_\mu\leqslant_{\deg}\mathscr{F}(\mathfrak{z})$. Thus, $\mathfrak{m}_\mu\leqslant_{\deg}\mathscr{F}(\mathfrak{z})<_{\deg}\mathfrak{m}_\lambda$, which by Lemma 2.1 implies that $\mu\lhd\lambda$. Hence, each $\widetilde{u}_{\mathfrak{z}}^-\cdot|\emptyset\rangle$ is a $\mathcal{Z}$-linear combination of $|\mu\rangle$ with $\mu\lhd\lambda$. Consequently,

$$\widetilde{u}_{\mathfrak{m}_\lambda}^-\cdot|\emptyset\rangle\in v^{\theta(\lambda)+\sigma(\lambda)}|\lambda\rangle+\sum_{\mu\lhd\lambda}\mathcal{Z}|\mu\rangle.$$

Therefore, it remains to show that

$$\theta(\lambda)+\sigma(\lambda)=0.$$

Write $\lambda=(\lambda_1,\ldots,\lambda_m)$ with $\lambda_1\geqslant\cdots\geqslant\lambda_m\geqslant 1$ and set $|\lambda|=\sum_{s=1}^m\lambda_s$. We proceed induction on $|\lambda|$ to show that $\theta(\lambda)+\sigma(\lambda)=0$. By the definition,

$$\theta(\lambda)=\sum_{s<t}\kappa(\mathbf{d}_s,\mathbf{d}_t)-\sum_{s=1}^{\ell}h(\mathbf{d}_s),$$

where $\ell=\lambda_1$ is the Loewy length of $M=M(\mathfrak{m}_\lambda^\infty)$ and $S_{\mathbf{d}_s}\cong\operatorname{rad}^{s-1}M/\operatorname{rad}^sM$ for $1\leqslant s\leqslant\ell$. Let $1\leqslant t\leqslant m$ be such that $\lambda_1=\cdots=\lambda_t>\lambda_{t+1}$ and define

$$\lambda'=(\lambda_1,\cdots,\lambda_{t-1},\lambda_t-1,\lambda_{t+1},\lambda_m).$$

Then $|\lambda'|=|\lambda|-1$. By the induction hypothesis, we have $\theta(\lambda')+\sigma(\lambda')=0$.

For each $1\leqslant s\leqslant\ell$, let $\mathbf{d}'_s\in\mathbb{N}I_\infty$ be defined by setting $S_{\mathbf{d}'_i}\cong\operatorname{rad}^{s-1}M'/\operatorname{rad}^sM'$, where $M'=M(\mathfrak{m}_{\lambda'}^\infty)$. Then

$$\mathbf{d}'_\ell=\mathbf{d}_\ell-\varepsilon_{\ell-t}\ \text{ and }\ \mathbf{d}'_s=\mathbf{d}_s\ \text{ for } 1\leqslant s<\ell.$$

This implies that

$$\sum_{s=1}^{\ell}h(\mathbf{d}_s)-\sum_{s=1}^{\ell}h(\mathbf{d}'_s)=h(\mathbf{d}_\ell)-h(\mathbf{d}'_\ell)=-\delta_{\bar{t},\bar{1}}\ \text{ and}$$
$$\sum_{s<t}\kappa(\mathbf{d}_s,\mathbf{d}_t)-\sum_{s<t}\kappa(\mathbf{d}'_s,\mathbf{d}'_t)=\sum_{1\leqslant s<\ell}\kappa(\mathbf{d}_s,\varepsilon_{\ell-t}).$$

Hence,

$$\theta(\lambda)-\theta(\lambda')=\sum_{1\leqslant s<\ell}\kappa(\mathbf{d}_s,\varepsilon_{\ell-t})+\delta_{\bar{t},\bar{1}}.$$

On the other hand, $\sigma(\lambda) = \sigma(\lambda') - 1$ if $\ell - t < 0$ and $\bar{\ell} = \bar{t}$, and $\sigma(\lambda) = \sigma(\lambda')$ otherwise. A direct calculation shows that if $\ell - t \geqslant 0$, then

$$\sum_{1 \leqslant s < \ell} \kappa(\mathbf{d}_s, \varepsilon_{\ell - t}) = -\delta_{\bar{t}, \bar{1}},$$

and if $\ell - t < 0$, then

$$\sum_{1 \leqslant s < \ell} \kappa(\mathbf{d}_s, \varepsilon_{\ell - t}) = \begin{cases} \delta_{\bar{\ell}, \bar{t}} - 1, & \text{if } \bar{t} = \bar{1}; \\ \delta_{\bar{\ell}, \bar{t}}, & \text{if } \bar{t} \neq \bar{1}. \end{cases}$$

We conclude that in all cases,

$$\theta(\lambda) + \sigma(\lambda) = \theta(\lambda') + \sigma(\lambda') = 0. \qquad \square$$

By the definition, for each $i \in I_n = \mathbb{Z}/n\mathbb{Z}$,

$$K_i|\emptyset\rangle = v^{\delta_{i,0}}|\emptyset\rangle.$$

This together with the corollary above implies that $\bigwedge^\infty$ is a highest weight $\boldsymbol{\mathcal{D}}(n)$-module of highest weight Λ_0. Consequently, there is a unique surjective $\boldsymbol{\mathcal{D}}(n)$-module homomorphism

$$\varphi : \boldsymbol{\mathcal{D}}(n)^- = M(\Lambda_0) \longrightarrow \bigwedge^{\infty}, \ \eta_{\Lambda_0} \longmapsto |\emptyset\rangle.$$

Theorem 7.3. *The homomorphism φ induces an isomorphism of $\boldsymbol{\mathcal{D}}(n)$-modules*

$$\bar{\varphi} : L(\Lambda_0) \longrightarrow \bigwedge^{\infty}.$$

Proof. By definition, we have

$$F_i \cdot |\emptyset\rangle = 0 \ \text{ for } i \in I_n \backslash \{0\} \text{ and } \ F_0^2 \cdot |\emptyset\rangle = 0.$$

This together with Theorem 4.4 implies that φ induces a surjective homomorphism

$$\bar{\varphi} : L(\Lambda_0) = \boldsymbol{\mathcal{D}}(n)^- / \big(\sum_{i \in I_n} \boldsymbol{\mathcal{D}}(n)^- F_i^{\Lambda_0(h_i)+1} \big) \longrightarrow \bigwedge^{\infty}.$$

Since $L(\Lambda_0)$ is simple, we conclude that $\bar{\varphi}$ is an isomorphism. $\square$

Combining the theorem with Corollary 4.5 gives the decomposition of $\bigwedge^\infty$ obtained by Kashiwara, Miwa and Stern in [24, Prop. 2.3].

Corollary 7.4. *As a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$-module, $\bigwedge^\infty$ has a decomposition*

$$\bigwedge^\infty|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m\geqslant 0} L_0(\Lambda_0 - m\delta^*)^{\oplus p(m)}.$$

8. The canonical basis for $\bigwedge^\infty$

In this section we show that the canonical basis of $\bigwedge^\infty$ defined in [26] can be constructed by using the monomial basis of the Ringel–Hall algebra of Δ_n given in [8]. We also interpret the "ladder method" in [25] in terms of generic extensions defined in Section 2.

Recall that there is a bar-involution $a \mapsto \iota(a) = \overline{a}$ on $\boldsymbol{\mathcal{D}}(n)^-$ which takes $\overline{v} \mapsto v^{-1}$ and fixes all $\widetilde{u}_\alpha^-$ for $\alpha \in \mathbb{N}I_n$. Then it induces a semilinear involution on the basic representation $L(\Lambda_0)$ by setting

$$\overline{a\eta_{\Lambda_0}} = \overline{a}\eta_{\Lambda_0} \quad \text{for all } a \in \boldsymbol{\mathcal{D}}(n)^-.$$

On the other hand, by [26], there is a semilinear involution $x \mapsto \overline{x}$ on $\bigwedge^\infty$ which, by [43], satisfies

(i) $\overline{|\emptyset\rangle} = |\emptyset\rangle$,
(ii) $\overline{ax} = \overline{a}\,\overline{x}$ for all $a \in \boldsymbol{\mathcal{D}}(n)^-$ and $x \in \bigwedge^\infty$.

Therefore, the isomorphism $L(\Lambda_0) \to \bigwedge^\infty$ given in Theorem 7.3 is compatible with the bar-involutions.

It is proved in [26, Th. 3.3] that for each $\lambda \in \Pi$,

$$\overline{|\lambda\rangle} = |\lambda\rangle + \sum_{\mu\lhd\lambda} a_{\mu,\lambda}|\mu\rangle, \quad \text{where } a_{\mu,\lambda} \in \mathcal{Z}. \tag{8.0.1}$$

Then applying the standard linear algebra method to the basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$ in [27] (or see [11] for more details) gives rise to an "IC basis" $\{b_\lambda \mid \lambda \in \Pi\}$ which is characterized by

$$\overline{b_\lambda} = b_\lambda \quad \text{and} \quad b_\lambda \in |\lambda\rangle + \sum_{\mu\lhd\lambda} v^{-1}\mathbb{Z}[v^{-1}]|\mu\rangle.$$

The basis $\{b_\lambda \mid \lambda \in \Pi\}$ is called the *canonical basis* of $\bigwedge^\infty$. In other words, the basis elements b_λ are uniquely determined by the polynomials $a_{\mu,\lambda}$.

Remark 8.1. Varagnolo and Vasserot [43] have conjectured that

$$b^-_{\mathfrak{m}_\lambda} \cdot |\emptyset\rangle = b_\lambda \quad \text{for each } \lambda \in \Pi.$$

This conjecture was proved by Schiffmann [37].

In the following we provide a way to deduce (8.0.1) by using the monomial basis of the Ringel–Hall algebra of Δ_n given in [8]. As in [8, Sect. 3], set

$$I^e = I_n \cup \{\text{all sincere vectors in } \mathbb{N}I_n\}$$

and consider the set Σ of all words on the alphabet I^e. Recall that a vector $\mathbf{a} = (a_i) \in \mathbb{N}I_n$ is called sincere if $a_i \neq 0$ for all $i \in I_n$. Since $\boldsymbol{\mathcal{D}}(n)^-$ is isomorphic to the opposite Ringel–Hall algebra of Δ_n, we define

$$M *' N = N * M.$$

This gives the map

$$\wp^{\mathrm{op}} : \Sigma \longrightarrow \mathfrak{M},\ w = \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_t \longmapsto S_{\mathbf{a}_1} *' S_{\mathbf{a}_2} *' \cdots *' S_{\mathbf{a}_t}.$$

By [8, Sect. 9], for each $\mathfrak{m} \in \mathfrak{M}$, there is a distinguished word $w_{\mathfrak{m}} \in (\wp^{\mathrm{op}})^{-1}(\mathfrak{m})$ which defines a monomial $m^{(w_{\mathfrak{m}})}$ on $\widetilde{u}_{\mathbf{a}}^-$ with $\mathbf{a} \in \widetilde{I}$ such that

$$m^{(w_{\mathfrak{m}})} = \widetilde{u}_{\mathfrak{m}}^- + \sum_{\mathfrak{p} <_{\deg} \mathfrak{m}} \theta_{\mathfrak{p},\mathfrak{m}} \widetilde{u}_{\mathfrak{p}}^- \quad \text{for some } \theta_{\mathfrak{m},\mathfrak{p}} \in \mathcal{Z};$$

see [8, (9.1.1)]. If $\mathfrak{m} = \mathfrak{m}_\lambda$ for some $\lambda \in \Pi$, we simply write $w_{\mathfrak{m}_\lambda} = w_\lambda$. Thus,

$$m^{(w_\lambda)} = \widetilde{u}_{\mathfrak{m}_\lambda}^- + \sum_{\mathfrak{p} <_{\deg} \mathfrak{m}_\lambda} \theta_{\mathfrak{p},\mathfrak{m}_\lambda} \widetilde{u}_{\mathfrak{p}}^-. \tag{8.1.1}$$

This together with Proposition 7.1 and Corollary 7.2 implies that

$$m^{(w_\lambda)}|\emptyset\rangle = |\lambda\rangle + \sum_{\mu \lhd \lambda} \tau_{\mu,\lambda} |\mu\rangle, \tag{8.1.2}$$

where $\tau_{\mu,\lambda} \in \mathcal{Z}$. Since the monomials $m^{(w_\lambda)}$ are bar-invariant, we deduce that for each $\lambda \in \Pi$,

$$\overline{|\lambda\rangle} = |\lambda\rangle + \sum_{\mu \lhd \lambda} a'_{\mu,\lambda} |\mu\rangle \quad \text{for some } a'_{\mu,\lambda} \in \mathcal{Z}.$$

Comparing with (8.0.1) gives that

$$a_{\mu,\lambda} = a'_{\mu,\lambda} \quad \text{for all } \mu \lhd \lambda.$$

In case λ is n-regular, then $\mathfrak{m}_\lambda$ is aperiodic and the word w_λ can be chosen in Ω, the subset of all words on the alphabet $I_n = \mathbb{Z}/n\mathbb{Z}$; see [8, Sect. 4]. In other words, $m^{(w_\lambda)}$ is a monomial of the divided powers $(u_i^-)^{(t)} = F_i^{(t)}$ for $i \in I_n$ and $t \geqslant 1$. We now interpret the "ladder method" in [25, Sect. 6] in terms of the generic extension map. Let $\lambda = (\lambda_1, \dots, \lambda_t) \in \Pi$ be n-regular. Recall the corresponding nilpotent representation

$$M(\mathfrak{m}_\lambda) = \bigoplus_{a=1}^{t} S_{1-a}[\lambda_a],$$

where $1 - a$ is viewed as an element in I_n. Take $1 \leqslant s \leqslant t$ with $\lambda_1 = \cdots = \lambda_s > \lambda_{s+1}$ ($\lambda_{t+1} = 0$ by convention) and let $k \geqslant 0$ be maximal such that

$$\lambda_{s+l(n-1)+1} = \cdots = \lambda_{s+(l+1)(n-1)} \text{ and } \lambda_{s+l(n-1)} = \lambda_{s+l(n-1)+1} + 1 \text{ for } 0 \leqslant l \leqslant k-1.$$

Let $i_1 \in I$ be such that $\mathrm{soc}(S_{1-s}[\lambda_s]) = S_{i_1}$. Then for each $a = s + l(n-1)$ with $0 \leqslant l \leqslant k$,

$$\mathrm{soc}(S_{1-a}[\lambda_a]) = S_{i_1}.$$

Define $\mu = (\mu_1, \dots, \mu_t) \in \Pi$ by setting

$$\mu_a = \begin{cases} \lambda_a - 1, & \text{if } a = s + l(n-1) \text{ for some } 0 \leqslant l \leqslant k; \\ \lambda_a, & \text{otherwise.} \end{cases}$$

It is easy to see from the construction that μ is again n-regular. Moreover, by applying an argument similar to that in the proof of [5, Prop. 3.7],

$$(k+1)S_{i_1} *' M(\mathfrak{m}_\mu) = M(\mathfrak{m}_\mu) * (k+1)S_{i_1} = M(\mathfrak{m}_\lambda).$$

Repeating the above process, we finally obtain a sequence $i_1, \dots, i_d$ in I_n and positive integers $k_1 = k+1, \dots, k_d$ such that

$$(k_1 S_{i_1}) *' \cdots *' (k_d S_{i_d}) = M(\mathfrak{m}_\lambda).$$

In other word, the word $w_\lambda := i_1^{k_1} \cdots i_d^{k_d}$ lies in $(\wp^{\mathrm{op}})^{-1}(\mathfrak{m}_\lambda)$. It can be also checked that the word w_λ is distinguished. Thus, the corresponding monomial

$$m^{(w_\lambda)} = (u_{i_1}^-)^{(k_1)} \cdots (u_{i_d}^-)^{(k_d)} = F_{i_1}^{(k_1)} \cdots F_{i_d}^{(k_d)}$$

gives rise to the equality (8.1.2) for the element $m^{(w_\lambda)}|\emptyset\rangle$. We remark that $m^{(w_\lambda)}|\emptyset\rangle$ coincides with the element $A(\lambda)$ constructed in [25, (8)] by using the "ladder method" of James and Kerber [20].

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