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Kumjian–Pask algebras of finitely aligned higher-rank graphs ☆

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A B S T R A C T

We extend the definition of Kumjian–Pask algebras to include algebras associated to finitely aligned higher-rank graphs. We show that these Kumjian–Pask algebras are universally defined and have a graded uniqueness theorem. We also prove the Cuntz–Krieger uniqueness theorem; to do this, we use a groupoid approach. As a consequence of the graded uniqueness theorem, we show that every Kumjian–Pask algebra is isomorphic to the Steinberg algebra associated to its boundary path groupoid. We then use Steinberg algebra results to prove the Cuntz–Krieger uniqueness theorem and also to characterize simplicity and basic simplicity.

1. Introduction

In the 1990s, C^*-algebras of row-finite directed graphs were introduced in [7,16,17]. Since their first appearance, these C^*-algebras have been intensively studied (for example, see [24]). Some of the earliest results about these algebras include the existence

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of a universal family, the gauge-invariant uniqueness theorem, and the Cuntz–Krieger uniqueness theorem.

Higher-rank graph C^*-algebras were introduced by Kumjian and Pask in [15] as a generalisation of the C^*-algebras of directed graphs. In [15], Kumjian and Pask limit their focus to row-finite higher-rank graphs with no sources. Later, Raeburn, Sims and Yeend extended the coverage by introducing C^*-algebras of locally convex, row-finite higher-rank graphs in [21] and then finitely aligned higher-rank graphs in [22]. It is in the finitely aligned setting where graphs that fail to be row-finite are considered. Once again Raeburn, Sims and Yeend establish the existence of a universal family, the gauge-invariant uniqueness theorem, and the Cuntz–Krieger uniqueness theorem.

On the other hand, Leavitt path algebras were developed independently by Ara, Moreno, and Pardo in [4] and Abrams and Aranda Pino in [2]. A complex Leavitt path algebra is a purely algebraic structure constructed from a directed graph that sits densely inside the graph C^*-algebra. Tomforde showed in [30] that one can generalise further and define Leavitt path R-algebras where R is any commutative ring with identity. Tomforde proved the existence of a universal family, the graded uniqueness theorem (which is the algebraic analogue of the gauge-invariant uniqueness theorem), and the Cuntz–Krieger uniqueness theorem for Leavitt path R-algebras. Tomforde's proofs in [30] use techniques that are similar to those employed by Raeburn for Leavitt path $\mathbb{C}$-algebras in [6] and in Tomforde's earlier paper [29] for Leavitt path K-algebras where K is an arbitrary field.

Moving to higher-rank graphs, Kumjian–Pask R-algebras were introduced in [5] and include the class of Leavitt path algebras. Kumjian–Pask algebras are the algebraic analogue of the higher-rank graph C^*-algebras of [15]. As in [15], the authors of [5] consider row-finite higher-rank graphs with no sources. Later, Clark, Flynn and an Huef developed Kumjian–Pask algebras for locally convex, row-finite higher-rank graphs in [11]. To complete the final algebraic piece, in this paper we introduce Kumjian–Pask algebras for finitely aligned higher-rank graphs. We will establish the existence of a universal family, the graded uniqueness theorem, and the Cuntz–Krieger uniqueness theorem.

Our motivation to consider this class of higher-rank graphs comes from our desire to establish an algebraic version of [19, Theorem 4.1]: there Pangalela shows that the Toeplitz C^*-algebra associated to a row-finite graph Λ can be realized as the graph C^*-algebra associated to a higher-rank graph constructed from Λ, called $T\Lambda$. In this setting $T\Lambda$ has sources and is not locally convex.

Let Λ be a finitely aligned k-graph and let R be a commutative ring with identity. We define a Kumjian–Pask Λ-family (Definition 3.1) and show the existence of a universal Kumjian–Pask algebra $\mathrm{KP}_R(\Lambda)$ that is a $\mathbb{Z}^k$-graded R-algebra in Proposition 3.7. We then prove the graded uniqueness theorem in Theorem 4.1. Up to this point, our techniques mirror the C^*-algebraic techniques of [22]. However, the proof of the Cuntz–Krieger uniqueness theorem of [22] is highly analytic so we must use an alternate approach. We have chosen a groupoid approach.

In Section 5, we introduce groupoids and *Steinberg algebras*. Then, given a finitely aligned higher-rank graph Λ, we build the associated boundary-path groupoid $\mathcal{G}_\Lambda$ as in [32]. We then use the graded uniqueness theorem (Theorem 4.1) to show that the Kumjian–Pask algebra $\mathrm{KP}_R(\Lambda)$ is isomorphic to the Steinberg algebra $A_R(G_\Lambda)$ in Proposition 5.4. With this isomorphism in place, we aim to use results about Steinberg algebras to establish results about Kumjian–Pask algebras.

First we establish how certain properties of Λ translate to properties of $\mathcal{G}_\Lambda$; we do this in Section 6 and Section 7. Of interest in its own right, we show that a higher-rank graph Λ is *aperiodic* if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is *effective* in Proposition 6.3. We also show in Proposition 7.1, that a higher-rank graph Λ is *cofinal* if and only if $\mathcal{G}_\Lambda$ is *minimal*.

Now in Section 8, we prove the Cuntz–Krieger uniqueness theorem. This theorem only applies to Kumjian–Pask algebras associated to aperiodic graphs. The proof is simply an application of the Cuntz–Krieger uniqueness theorem for Steinberg algebras [9, Theorem 3.2] which applies to effective groupoids. Note that our technique gives an alternate proof of the Cuntz–Krieger uniqueness theorem in the special cases of Leavitt path algebras in [30] and the row-finite Kumjian–Pask algebras of [5,11].

Finally, in Section 9, we give necessary and sufficient conditions for $\mathrm{KP}_R(\Lambda)$ to be basically simple in Theorem 9.3 and simple in Theorem 9.4. These two results are a consequence of the characterisation of basic simplicity and simplicity of the Steinberg algebra $A_R(\mathcal{G}_\Lambda)$ (see Theorem 4.1 and Corollary 4.6 of [9]).

2. Background

Let $\mathbb{N}$ be the set of non-negative integers and let k be a positive integer. We write $n \in \mathbb{N}^k$ as $(n_1, \ldots, n_k)$ and for $m, n \in \mathbb{N}^k$, we write $m \leq n$ to denote $m_i \leq n_i$ for $1 \leq i \leq k$. We also write $m \vee n$ for their coordinate-wise maximum and $m \wedge n$ for their coordinate-wise minimum. We denote the usual basis in $\mathbb{N}^k$ by $\{e_i\}$.

A *directed graph* or 1-*graph* $E = (E^0, E^1, r, s)$ consists of countable sets of vertices E^0, edges E^1 and functions $r, s : E^1 \to E^0$, which denote range and source maps, respectively. We follow the conventions of [23] and write $\lambda\mu$ to denote the composition of paths λ and μ with $s(\lambda) = r(\mu)$. Thus a path of length $n \in \mathbb{N}$ is a sequence $\lambda = \lambda_1 \cdots \lambda_n$ of edges λ_i with $s(\lambda_i) = r(\lambda_{i+1})$ for $1 \leq i \leq n-1$. We also have $s(\lambda) := s(\lambda_n)$ and $r(\lambda) := r(\lambda_1)$.

Remark 2.1. We use this convention of paths because we view the collection of paths as a category.

2.1. Higher-rank graphs

For a positive integer k, we regard the additive semigroup $\mathbb{N}^k$ as a category with one object. A *higher-rank graph* or k-*graph* $\Lambda = (\Lambda^0, \Lambda, r, s)$ is a countable small category Λ with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the *factorisation property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$

such that $\lambda = \mu v$ and $d(\mu) = m$, $d(\nu) = n$. We then write $\lambda(0, m)$ for μ and $\lambda(m, m+n)$ for ν.

We write Λ^0 to denote the set of objects in Λ and we identify each object $v \in \Lambda^0$ with the identity morphism at the object, which, by the factorisation property, is the only morphism with range and source v. We then regard elements of Λ^0 as *vertices*. For $n \in \mathbb{N}^k$, we define

$$\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\}$$

and call the elements λ of Λ^n *paths of degree* n. For each $\lambda \in \Lambda$ we say λ has *source* $s(\lambda)$ and *range* $r(\lambda)$. For $v \in \Lambda^0$, $\lambda \in \Lambda$ and $E \subseteq \Lambda$, we define

$$\begin{aligned} vE &:= \{\mu \in E : r(\mu) = v\}, \\ \lambda E &:= \{\lambda\mu \in \Lambda : \mu \in E, r(\mu) = s(\lambda)\}, \\ E\lambda &:= \{\mu\lambda \in \Lambda : \mu \in E, s(\mu) = r(\lambda)\}. \end{aligned}$$

Remark 2.2. In older references, for example [15,21], $v\Lambda$ is denoted by $\Lambda(v)$.

Example 2.3 *([21, Example 2.2.(ii)])*. Let $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$. We define

$$\Omega_{k,m} := \left\{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\right\}.$$

This is a category with objects $\left\{p \in \mathbb{N}^k : p \leq m\right\}$, range map $r(p, q) = p$, source map $s(p, q) = q$, and degree map $d(p, q) = q - p$. Then $(\Omega_{k,m}, d)$ is a k-graph.

One way to visualise k-graphs is to use coloured graphs. By choosing k different colours $c_1, \ldots, c_k$, we can view paths in Λ^{e_i} as edges of colour c_i. For a k-graph Λ, we call its corresponding coloured graph the *skeleton* of Λ. For further discussion about k-graphs and their skeletons, see [14].

Let Λ be a k-graph. For $\lambda, \mu \in \Lambda$, we say that τ is a *minimal common extension* of λ and μ if

$$d(\tau) = d(\lambda) \vee d(\mu), \ \tau(0, d(\lambda)) = \lambda \text{ and } \tau(0, d(\mu)) = \mu.$$

Let $\mathrm{MCE}(\lambda, \mu)$ denote the collection of all minimal common extensions of λ and μ. Then we write

$$\Lambda^{\min}(\lambda, \mu) := \{(\rho, \tau) \in \Lambda \times \Lambda : \lambda\rho = \mu\tau \in \mathrm{MCE}(\lambda, \mu)\}.$$

Meanwhile, for $E \subseteq \Lambda$ and $\lambda \in \Lambda$, we write

$$\mathrm{Ext}(\lambda; E) := \bigcup_{\mu \in E} \left\{\rho : (\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)\right\}.$$

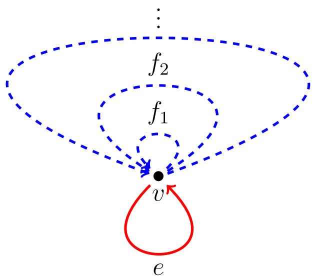
A set $E \subseteq v\Lambda$ is *exhaustive* if for every $\lambda \in v\Lambda$, there exists $\mu \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. We define

$$\mathrm{FE}(\Lambda) := \bigcup_{v \in \Lambda^0} \{E \subseteq v\Lambda \backslash \{v\} : E \text{ is finite and exhaustive}\}.$$

For $E \in \mathrm{FE}(\Lambda)$, we write $r(E)$ for the vertex v which satisfies $E \subseteq v\Lambda$.

We say that Λ is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$. We see that every 1-graph is finitely aligned. As in [15, Definition 1.4], we say that a k-graph Λ is *row-finite* if $v\Lambda^n$ is finite for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Note that for all $\lambda, \mu \in \Lambda$, we have $\left|\Lambda^{\min}(\lambda, \mu)\right| = |\mathrm{MCE}(\lambda, \mu)| \leq \left|r(\lambda)\Lambda^{d(\lambda)\vee d(\mu)}\right|$. Hence, every row-finite k-graph Λ is finitely aligned. On the other hand, a finitely aligned k-graph Λ is not necessarily row-finite.

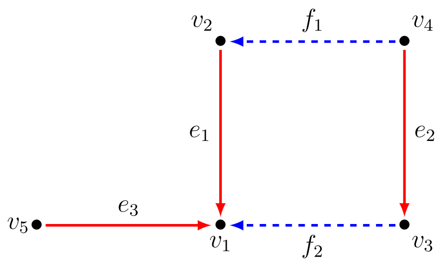
For example, consider the 2-graph Λ_1 which has skeleton



where $ef_i = f_ie$ for all positive integers i, the solid edge has degree $(1, 0)$ and dashed edges have degree $(0, 1)$. It is clearly not row-finite because $|v\Lambda_1^{(0,1)}| = \infty$. On the other hand, for $\lambda, \mu \in \Lambda$, $\left|\Lambda_1^{\min}(\lambda, \mu)\right|$ is either 0 or 1, and then Λ_1 is finitely aligned.

Following [15, Definition 1.4], a k-graph Λ has *no sources* if $v\Lambda^n$ is non-empty for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Meanwhile, recall from [21, Definition 3.9] that a k-graph Λ is *locally convex* if for all $v\Lambda^0$, $1 \leq i, j \leq k$ with $i \neq j$, $\lambda \in v\Lambda^{e_i}$ and $\mu \in v\Lambda^{e_j}$, the sets $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are non-empty.

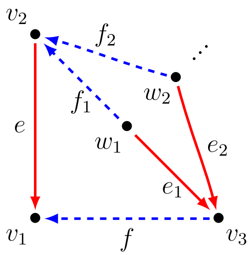
Consider the 2-graph Λ_2 with skeleton



where $e_1f_1 = f_2e_2$, solid edges have degree $(1, 0)$ and dashed edges have degree $(0, 1)$. Since v_5 does not receive edges with degree $(0, 1)$, then v_5 is a source of Λ_2. Furthermore,

Λ_2 fails to be locally-convex since $e_3 \in v_1\Lambda_2^{(1,0)}$, $f_2 \in v_1\Lambda_2^{(0,1)}$ but $s(e_3)\Lambda_2^{(0,1)} = \emptyset$. On the other hand, Λ_2 is row-finite thus Λ_2 is finitely aligned.

Next consider the 2-graph Λ_3 with skeleton



where $ef_i = fe_i$ for all positive integers i, solid edges have degree $(1,0)$ and dashed edges have degree $(0,1)$. Since $\left|\Lambda_3^{\min}(e,f)\right| = \infty$, then Λ_3 is not finitely aligned. Hence, not every k-graph is finitely aligned.

To summarise, finitely aligned k-graphs generalise both row-finite k-graphs with no sources and locally convex row-finite k-graphs. However, this class of k-graphs does not cover all k-graphs. In this paper, we focus on finitely aligned k-graphs. For other examples and further discussion, see [15,19,21,22,31].

2.2. Paths and boundary paths

Suppose that Λ is a finitely aligned k-graph. Recall from [21, Definition 3.1] that for $n \in \mathbb{N}^k$, we define

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n, \text{ and } d(\lambda)_i < n_i \text{ implies } s(\lambda)\Lambda^{e_i} = \emptyset\}.$$

Note that $v\Lambda^{\leq n} \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. This is because v is contained in $v\Lambda^{\leq n}$ whenever $v\Lambda^{\leq n}$ has no non-trivial paths of degree less than or equal to q. For further discussion, see [21, Remark 3.2].

Following [13, Definition 5.10], we say that a degree-preserving functor $x : \Omega_{k,m} \to \Lambda$ is a *boundary path* of Λ if for every $n \in \mathbb{N}^k$ with $n \leq m$ and for $E \in x(n,n)\mathrm{FE}(\Lambda)$, there exists $\lambda \in E$ such that $x(n, n + d(\lambda)) = \lambda$. We write $\partial\Lambda$ for the set of all boundary paths. Note that for $v \in \Lambda^0$, $v\partial\Lambda$ is non-empty [13, Lemma 5.15].

Remark 2.4. In the locally convex setting, the set $\Lambda^{\leq\infty}$ (as defined in [21, Definition 3.14]) is referred to as the "boundary path space". Indeed, if Λ is row-finite and locally convex, then $\Lambda^{\leq\infty} = \partial\Lambda$ [31, Proposition 2.12]. However, more generally, $\Lambda^{\leq\infty} \subseteq \partial\Lambda$ and the two can be different (see [31, Example 2.11]).

Let $x \in \partial\Lambda$. If $n \in \mathbb{N}^k$ and $n \leq d(x)$, we define $\sigma^n x$ by $\sigma^n x(0,m) = x(n, n+m)$ for all $m \leq d(x) - n$, and by [13, Lemma 5.13.(1)], $\sigma^n x$ also belongs to $\partial\Lambda$. We also write $x(n)$ for the vertex $x(n,n)$. Then the range of boundary path x is the vertex $r(x) := x(0)$. For $\lambda \in \Lambda x(0)$, we also have $\lambda x \in \partial\Lambda$ [13, Lemma 5.13.(2)].

2.3. Graded rings

Suppose that G is an additive abelian group. A ring A is G-*graded* if there are additive subgroups $\{A_g : g \in G\}$ satisfying:

$$A = \bigoplus{}_{g\in G} A_g \text{ and for } g,h \in G,\ A_g A_h \subseteq A_{g+h}.$$

If A and B are G-graded rings, a homomorphism $\pi : A \to B$ is G-*graded* if $\pi(A_g) \subseteq B_g$ for $g \in G$.

Let A be a G-graded ring. We say an ideal I of A is a G-*graded ideal* if $\{I \cap A_g : g \in G\}$ is a grading of I.

3. Kumjian–Pask Λ-families

Suppose that Λ is a finitely aligned k-graph and R is a commutative ring with identity 1. For $\lambda \in \Lambda$, we call λ^* a *ghost path* (λ^* is a formal symbol) and we define

$$G(\Lambda) := \{\lambda^* : \lambda \in \Lambda\}.$$

For $v \in \Lambda^0$, we define $v^* := v$. We also extend r and s to be defined on $G(\Lambda)$ by

$$r(\lambda^*) = s(\lambda) \text{ and } s(\lambda^*) = r(\lambda).$$

We then define composition on $G(\Lambda)$ by setting $\lambda^*\mu^* = (\mu\lambda)^*$ for $\lambda,\mu \in \Lambda$; and write $G\left(\Lambda^{\neq 0}\right)$ the set of ghost paths that are not vertices. Note that the factorisation property of Λ induces a similar factorisation property on $G(\Lambda)$.

Definition 3.1. A *Kumjian–Pask* Λ*-family* $\{S_\lambda, S_{\mu^*} : \lambda,\mu \in \Lambda\}$ in an R-algebra A consists of $S : \Lambda \cup G\left(\Lambda^{\neq 0}\right) \to A$ such that:

(KP1) $\left\{S_v : v \in \Lambda^0\right\}$ is a collection of mutually orthogonal idempotents;
(KP2) for $\lambda,\mu \in \Lambda$ with $s(\lambda) = r(\mu)$, we have $S_\lambda S_\mu = S_{\lambda\mu}$ and $S_{\mu^*}S_{\lambda^*} = S_{(\lambda\mu)^*}$;
(KP3) $S_{\lambda^*}S_\mu = \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} S_\rho S_{\tau^*}$ for all $\lambda,\mu \in \Lambda$; and
(KP4) $\prod_{\lambda\in E}\left(S_{r(E)} - S_\lambda S_{\lambda^*}\right) = 0$ for all $E \in \mathrm{FE}(\Lambda)$.

Remark 3.2. A number of aspects of these relations are worth commenting on:

(i) In previous references about Leavitt path algebras and Kumjian–Pask algebras, people usually distinguish the vertex idempotents as “P_v” (for example, see [1–5,11, 29,30]). We do not follow this convention because we do not want to make additional unnecessary cases in each proof.

(ii) (KP2) in [5,11] has more relations to check. However, using our notational convention, those relations can be simplified and are equivalent to our (KP2).
(iii) The restriction to finitely aligned k-graphs is necessary for the sum in (KP3) to be make sense (see [20]).
(iv) In (KP3), we interpret the empty sum as 0, so $S_{\lambda^*} S_\mu = 0$ whenever $\Lambda^{\min}(\lambda, \mu) = \emptyset$. We also have $S_{\lambda^*} S_\lambda = S_{s(\lambda)}$.
(v) (KP3–4) have been changed from those in [5, Definition 3.1] and [11, Definition 3.1]. We do this because we need to adjust the relations to deal with situation where k-graph is not locally convex. For further discussion, see Appendix A of [22].

The following lemma establishes some useful properties of a family satisfying (KP1–3).

Proposition 3.3. *Let Λ be a finitely aligned k-graph, R be a commutative ring with 1, and $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ be a family satisfying (KP1–3) in an R-algebra A. Then*

(a) $S_\lambda S_{\lambda^*} S_\mu S_{\mu^*} = \sum_{\lambda\rho \in \mathrm{MCE}(\lambda,\mu)} S_{\lambda\rho} S_{(\lambda\rho)^*}$ *for $\lambda, \mu \in \Lambda$; and $\{S_\lambda S_{\lambda^*} : \lambda \in \Lambda\}$ is a commuting family.*
(b) *The subalgebra generated by $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is*

$$\operatorname{span}_R\{S_\lambda S_{\mu^*} : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}.$$

(c) *For $n \in \mathbb{N}^k$ and $\lambda, \mu \in \Lambda^{\leq n}$, we have $S_{\lambda^*} S_\mu = \delta_{\lambda,\mu} S_{s(\lambda)}$.*
(d) *Suppose that $rS_v \neq 0$ for all $r \in R\setminus\{0\}$, $v \in \Lambda^0$ and that $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$. If $r \in R\setminus\{0\}$ and $G \subseteq s(\lambda)\Lambda$ is finite non-exhaustive, then*

$$rS_\lambda \neq 0 \text{ and } rS_\lambda \Big(\prod_{\nu \in G} \big(S_{s(\lambda)} - S_\nu S_{\nu^*}\big) \Big) S_{\mu^*} \neq 0.$$

Proof. To show (a), we take $\lambda, \mu \in \Lambda$ and then

$$\begin{aligned} S_\lambda S_{\lambda^*} S_\mu S_{\mu^*} &= S_\lambda \Big(\sum_{(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)} S_\rho S_{\tau^*} \Big) S_{\mu^*} = \sum_{(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)} S_{\lambda\rho} S_{(\mu\tau)^*} \\ &= \sum_{(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)} S_{\lambda\rho} S_{(\lambda\rho)^*} = \sum_{\lambda\rho \in \mathrm{MCE}(\lambda,\mu)} S_{\lambda\rho} S_{(\lambda\rho)^*}. \end{aligned}$$

Furthermore,

$$S_\lambda S_{\lambda^*} S_\mu S_{\mu^*} = \sum_{\lambda\rho \in \mathrm{MCE}(\lambda,\mu)} S_{\lambda\rho} S_{(\lambda\rho)^*} = \sum_{\mu\tau \in \mathrm{MCE}(\lambda,\mu)} S_{\mu\tau} S_{(\mu\tau)^*} = S_\mu S_{\mu^*} S_\lambda S_{\lambda^*},$$

as required.

Next we show (b). For $\lambda, \mu \in \Lambda$, we have $S_\lambda S_{\mu^*} = S_\lambda S_{s(\lambda)} S_{s(\mu)} S_{\mu^*}$ by (KP2). Then by (KP1), $S_\lambda S_{\mu^*} \neq 0$ implies $s(\lambda) = s(\mu)$. Therefore, the result follows from part (a), (KP2) and (KP3).

To show (c), we take $\lambda, \mu \in \Lambda^{\leq n}$. Suppose that $S_{\lambda^*} S_\mu \neq 0$. By (KP3), there exists $(\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)$ such that $\lambda\rho = \mu\tau$ and $d(\lambda\rho) \leq n$. Since $\lambda, \mu \in \Lambda^{\leq n}$, then $\rho = s(\lambda) = \tau$ and hence $\lambda = \mu$.

Finally, we show (d). Take $r \in R \setminus \{0\}$ and $\lambda \in \Lambda$. Suppose for contradiction that $rS_\lambda = 0$. Then

$$0 = S_{\lambda^*}(rS_\lambda) = rS_{\lambda^*} S_\lambda = rS_{s(\lambda)},$$

which contradicts with $rS_v \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$. Hence, $rS_\lambda \neq 0$.

Now take $r \in R \setminus \{0\}$, $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ and finite non-exhaustive $G \subseteq s(\lambda)\Lambda$. Suppose for contradiction that

$$rS_\lambda \Big(\prod_{\nu \in G} \big(S_{s(\lambda)} - S_\nu S_{\nu^*} \big) \Big) S_{\mu^*} = 0.$$

Since G is non-exhaustive, then there exists $\gamma \in s(\lambda)\Lambda$ such that $\mathrm{Ext}(\gamma; G) = \emptyset$. Hence $\Lambda^{\min}(\nu, \gamma) = \emptyset$ for every $\nu \in G$, and then by (KP3), $S_{\nu^*} S_\gamma = 0$ for $\nu \in G$. Therefore,

$$\begin{aligned} 0 &= \Big(rS_\lambda \Big(\prod_{\nu \in G} \big(S_{s(\lambda)} - S_\nu S_{\nu^*} \big) \Big) S_{\mu^*} \Big) S_{\mu\gamma} \\ &= rS_\lambda \Big(\prod_{\nu \in G} \big(S_{s(\lambda)} - S_\nu S_{\nu^*} \big) \Big) S_\gamma \\ &= rS_\lambda S_\gamma = rS_{\lambda\gamma}, \end{aligned}$$

which contradicts with $rS_{\lambda\gamma} \neq 0$. Hence, $rS_\lambda \big(\prod_{\nu \in G} \big(S_{s(\lambda)} - S_\nu S_{\nu^*} \big) \big) S_{\mu^*} \neq 0$. □

Remark 3.4. For $n \in \mathbb{N}^k$, we have $\Lambda^n \subseteq \Lambda^{\leq n}$. Hence, Proposition 3.3.(c) also implies that for $n \in \mathbb{N}^k$ and $\lambda, \mu \in \Lambda^n$, we have $S_{\lambda^*} S_\mu = \delta_{\lambda,\mu} S_{s(\lambda)}$.

Remark 3.5. Suppose that $rS_v \neq 0$ for all $r \in R \setminus \{0\}$, $v \in \Lambda^0$ and that $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$. Then the contrapositive of Proposition 3.3.(d) says: if $r \in R$ and $G \subseteq s(\lambda)\Lambda$ is finite such that $rS_\lambda \big(\prod_{\nu \in G} \big(S_{s(\lambda)} - S_\nu S_{\nu^*} \big) \big) S_{\mu^*} = 0$, then we have either $r = 0$ or G is exhaustive.

Now we give an example of a Kumjian–Pask Λ-family in a particular algebra of endomorphisms.

Proposition 3.6. *Let Λ be a finitely aligned k-graph and R be a commutative ring with 1. Let $\mathbb{F}_R(\partial\Lambda)$ be the free module with basis the boundary path space. Then for every $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda \setminus \Lambda^0$, there exist endomorphisms $S_v, S_\lambda, S_{\mu^*} : \mathbb{F}_R(\partial\Lambda) \to \mathbb{F}_R(\partial\Lambda)$ such that for $x \in \partial\Lambda$,*

$$S_v(x) = \begin{cases} x & \text{if } r(x) = v; \\ 0 & \text{otherwise,} \end{cases}$$

$$S_\lambda(x) = \begin{cases} \lambda x & \textit{if } s(\lambda) = r(x); \\ 0 & \textit{otherwise}, \end{cases}$$

$$S_{\mu^*}(x) = \begin{cases} \sigma^{d(\mu)}x & \textit{if } x(0, d(\mu)) = \mu; \\ 0 & \textit{otherwise}. \end{cases}$$

Furthermore, $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ *is a Kumjian–Pask* Λ*-family in the* R*-algebra* $\operatorname{End}(\mathbb{F}_R(\partial\Lambda))$ *with* $rS_v \neq 0$ *for all* $r \in R\setminus\{0\}$ *and* $v \in \Lambda^0$*.*

Proof. Take $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda\setminus\Lambda^0$. First note that for $x \in \partial\Lambda$ and $m \leq d(x)$, we have $\sigma^m x \in \partial\Lambda$. Now define functions f_v, f_λ, and $f_{\mu^*} : \partial\Lambda \to \mathbb{F}_R(\partial\Lambda)$ by

$$f_v(x) = \begin{cases} x & \text{if } r(x) = v; \\ 0 & \text{otherwise}, \end{cases}$$

$$f_\lambda(x) = \begin{cases} \lambda x & \text{if } s(\lambda) = r(x); \\ 0 & \text{otherwise}, \end{cases}$$

$$f_{\mu^*}(x) = \begin{cases} \sigma^{d(\mu)}x & \text{if } x(0, d(\mu)) = \mu; \\ 0 & \text{otherwise}. \end{cases}$$

The universal property of free modules gives nonzero endomorphisms

$$S_v, S_\lambda, S_{\mu^*} : \mathbb{F}_R(\partial\Lambda) \to \mathbb{F}_R(\partial\Lambda)$$

extending f_v, f_λ, and f_{μ^*}, as needed.

Now we claim that $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Kumjian–Pask Λ-family. To see (KP1), take $v \in \Lambda^0$ and $x \in \partial\Lambda$. Then we have $S_v^2(x) = x = S_v(x)$ if $r(x) = v$, and $S_v^2(x) = 0 = S_v(x)$ otherwise. Hence $S_v^2 = S_v$. Now take $v, w \in \Lambda^0$ with $v \neq w$ and $x \in \partial\Lambda$. Since $x \in w\partial\Lambda$ implies $x \notin v\partial\Lambda$, we have $S_vS_w(x) = 0$ for $x \in \partial\Lambda$ and $S_vS_w = 0$.

Next we show (KP2). Take $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$. Then for $x \in s(\mu)\partial\Lambda$, we have $\mu x \in s(\lambda)\partial\Lambda$. Then $S_\lambda S_\mu(x) = \lambda\mu x = S_{\lambda\mu}(x)$ if $x \in s(\mu)\partial\Lambda$, and $S_\lambda S_\mu(x) = 0 = S_{\lambda\mu}(x)$ otherwise. Hence $S_\lambda S_\mu = S_{\lambda\mu}$. Meanwhile, for $x \in r(\lambda)\partial\Lambda$ with $x(0, d(\lambda\mu)) = \lambda\mu$, we have $d(\lambda\mu) \leq d(x)$ and $\sigma^{d(\lambda\mu)}x \in s(\mu)\partial\Lambda$. Furthermore, $x(0, d(\lambda\mu)) = \lambda\mu$, implies $x(0, d(\lambda)) = \lambda$ and then we have $d(\lambda) \leq d(x)$ and $\sigma^{d(\lambda)}x \in s(\lambda)\partial\Lambda$. Hence,

$$S_{\mu^*}S_{\lambda^*}(x) = S_{\mu^*}\sigma^{d(\lambda)}x = \sigma^{d(\lambda)+d(\mu)}x = \sigma^{d(\lambda\mu)}x = S_{(\lambda\mu)^*}(x)$$

if $x(0, d(\lambda\mu)) = \lambda\mu$, and $S_{\mu^*}S_{\lambda^*}(x) = 0 = S_{(\lambda\mu)^*}(x)$ otherwise. Therefore, $S_{\mu^*}S_{\lambda^*} = S_{(\lambda\mu)^*}$.

Now we show (KP3). Take $\lambda, \mu \in \Lambda$. If $r(\lambda) \neq r(\mu)$, then $S_{\lambda^*}S_\mu = 0$ and $\Lambda^{\min}(\lambda, \mu) = \emptyset$, as required. Suppose $r(\lambda) = r(\mu)$. We have

$$S_{\lambda^*}S_{\mu}(x) = \begin{cases} (\mu x)(d(\lambda), d(\mu x)) & \text{if } x \in s(\mu)\partial\Lambda \text{ and } (\mu x)(0, d(\lambda)) = \lambda; \\ 0 & \text{otherwise.} \end{cases}$$

Take $x \in s(\mu)\partial\Lambda$. Note that $s(\mu) = r(\tau)$ for $(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)$. First suppose $(\mu x)(0, d(\lambda)) \neq \lambda$. Then for $(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)$,

$$(\mu x)(0, d(\lambda\rho)) \neq \lambda\rho \text{ and } (\mu x)(0, d(\mu\tau)) \neq \mu\tau.$$

Hence $x(0, d(\tau)) \neq \tau$ and $S_\rho S_{\tau^*}(x) = S_\rho(0) = 0$. Therefore

$$\sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} S_\rho S_{\tau^*}(x) = 0.$$

Next suppose $(\mu x)(0, d(\lambda)) = \lambda$. Since $(\mu x)(0, d(\lambda)) = \lambda$ and $(\mu x)(0, d(\mu)) = \mu$, there is $\tau \in s(\mu)\Lambda$ such that $(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)$ and $(\mu x)(0, d(\mu\tau)) = \mu\tau$. Therefore $x(0, d(\tau)) = \tau$. Note that this τ is unique by the factorisation property. Hence for $(\rho',\tau') \in \Lambda^{\min}(\lambda,\mu)$ such that $(\rho',\tau') \neq (\rho,\tau)$, we have $S_{\rho'}S_{\tau'^*}(x) = 0$. Also $x(0, d(\tau)) = \tau$, thus

$$\begin{aligned} S_\rho S_{\tau^*}(x) &= S_\rho(x(d(\tau), d(x))) = \rho[x(d(\tau), d(x))] \\ &= \rho[(\mu x)(d(\mu\tau), d(\mu x))] \\ &= \rho[(\mu x)(d(\lambda\rho), d(\mu x))] \text{ (since } \mu\tau = \lambda\rho) \\ &= (\mu x)(d(\lambda), d(\mu x)) \end{aligned}$$

and

$$\sum_{(\rho',\tau')\in\Lambda^{\min}(\lambda,\mu)} S_\rho S_{\tau^*}(x) = S_\rho S_{\tau^*}(x) = (\mu x)(d(\lambda), d(\mu x)) = S_{\lambda^*}S_\mu(x),$$

as required.

Finally, we show (KP4). Take $E \in \mathrm{FE}(\Lambda)$. Take $x \in r(E)\partial\Lambda$. Since $E \in x(0)\mathrm{FE}(\Lambda)$ and x is a boundary path, then there exists $\lambda \in E$ such that $x(0, d(\lambda)) = \lambda$. This implies

$$\begin{aligned} \left(S_{r(E)} - S_\lambda S_{\lambda^*}\right)(x) &= S_{r(E)}(x) - S_\lambda S_{\lambda^*}(x) \\ &= x - S_\lambda(x(d(\lambda), d(x))) \\ &= x - x = 0. \end{aligned}$$

Hence

$$\Big(\prod_{\lambda\in E}\left(S_{r(E)} - S_\lambda S_{\lambda^*}\right)\Big)(x) = 0$$

for $x \in r(E)\partial\Lambda$, and $\prod_{\lambda\in E}\left(S_{r(E)} - S_\lambda S_{\lambda^*}\right) = 0$.

Thus $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Kumjian–Pask Λ-family, as claimed. Now note that for $v \in \Lambda^0$, $v\partial\Lambda$ is non-empty. This implies that for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$, $rS_v \neq 0$. $\square$

Using an alternate construction of a Kumjian–Pask Λ-family, we next show that there is an R-algebra which is universal for Kumjian–Pask Λ-families.

Theorem 3.7. *Let Λ be a finitely aligned k-graph and R be a commutative ring with 1.*

(a) *There is a universal R-algebra $\mathrm{KP}_R(\Lambda)$ generated by a Kumjian–Pask Λ-family $\{s_\lambda, s_{\mu^*} : \lambda, \mu \in \Lambda\}$ such that whenever $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Kumjian–Pask Λ-family in an R-algebra A, then there exists a unique R-algebra homomorphism $\pi_S : \mathrm{KP}_R(\Lambda) \to A$ such that $\pi_S(s_\lambda) = S_\lambda$ and $\pi_S(s_{\mu^*}) = S_{\mu^*}$ for $\lambda, \mu \in \Lambda$.*
(b) *We have $rs_v \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$.*
(c) *The subsets*

$$\mathrm{KP}_R(\Lambda)_n := \mathrm{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda, d(\lambda) - d(\mu) = n\}$$

forms a $\mathbb{Z}^k$-grading of $\mathrm{KP}_R(\Lambda)$.

Proof. We use an argument similar to [5, Theorem 3.4] and [11, Theorem 3.7]. To show (a), first we define $X := \Lambda \cup G(\Lambda^{\neq 0})$ and $\mathbb{F}_R(w(X))$ be the free algebra on the set $w(X)$ of words on X. Let I be the ideal of $\mathbb{F}_R(w(X))$ generated by elements of the following sets:

(i) $\{vw - \delta_{v,w}v : v, w \in \Lambda^0\}$,
(ii) $\{\lambda - \mu\nu, \lambda^* - \nu^*\mu^* : \lambda, \mu, \nu \in \Lambda \text{ and } \lambda = \mu\nu\}$,
(iii) $\{\lambda^*\mu - \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} \rho\tau^* : \lambda, \mu \in \Lambda\}$, and
(iv) $\{\prod_{\lambda\in E}(r(E) - \lambda\lambda^*) : E \in \mathrm{FE}(\Lambda)\}$.

Now define $\mathrm{KP}_R(\Lambda) := \mathbb{F}_R(w(X))/I$ and $q : \mathbb{F}_R(w(X)) \to \mathbb{F}_R(w(X))/I$ be the quotient map. Define $s_\lambda := q(\lambda)$ for $\lambda \in \Lambda$, and $s_{\mu^*} := q(\mu^*)$ for $\mu^* \in G(\Lambda^{\neq 0})$. Then $\{s_\lambda, s_{\mu^*} : \lambda \in \Lambda, \mu^* \in G(\Lambda^{\neq 0})\}$ is a Kumjian–Pask Λ-family in $\mathrm{KP}_R(\Lambda)$.

Now let $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ be a Kumjian–Pask Λ-family in an R-algebra A. Define $f : X \to A$ by $f(\lambda) := S_\lambda$ for $\lambda \in \Lambda$, and $f(\mu^*) := S_{\mu^*}$ for $\mu^* \in G(\Lambda^{\neq 0})$. The universal property of $\mathbb{F}_R(w(X))$ gives a unique R-algebra homomorphism $\phi : \mathbb{F}_R(w(X)) \to A$ such that $\phi|_X = f$. Since $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Kumjian–Pask Λ-family, then $I \subseteq \ker(\phi)$. Thus there exists an R-algebra homomorphism $\pi_S : \mathrm{KP}_R(\Lambda) \to A$ such that $\pi_S \circ q = \phi$. The homomorphism π_S is unique since the elements in X generate $\mathbb{F}_R(w(X))$ as an algebra. Furthermore, we have $\pi_S(s_\lambda) = S_\lambda$ for $\lambda \in \Lambda$ and $\pi_S(s_{\mu^*}) = S_{\mu^*}$ for $\mu^* \in G(\Lambda^{\neq 0})$, as required.

To show (b), let $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ be the Kumjian–Pask Λ-family as in Proposition 3.6. Then $rS_v \neq 0$ for $v \in \Lambda^0$. Since $\pi_S(rs_v) = rS_v \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$, we have $rs_v \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$.

Next we show (c). We first extend the degree map to $w(X)$ by $d(w) := \sum_{i=1}^{|w|} d((w_i))$ for $w \in w(X)$. By [5, Proposition 2.7], $\mathbb{F}_R(w(X))$ is $\mathbb{Z}^k$-graded by the subgroups

$$\mathbb{F}_R(w(X))_n := \left\{ \sum_{w \in w(X)} r_w w : r_w \neq 0 \text{ implies } d(w) = n \right\}.$$

Now we claim that the ideal I defined in (a) is a graded ideal. It suffices to show that I is generated by elements in $\mathbb{F}_R(w(X))_n$ for some $n \in \mathbb{Z}^k$. Since $d(v) = 0$ for $v \in \Lambda^0$, then the generators in (i) belong to $\mathbb{F}_R(w(X))_0$. If $\lambda = \mu\nu$ in Λ, then $\lambda - \mu\nu$ belongs to $\mathbb{F}_R(w(X))_{d(\lambda)}$ and $\lambda^* - \nu^*\mu^*$ belongs to $\mathbb{F}_R(w(X))_{-d(\lambda)}$. For $\lambda, \mu \in \Lambda$ and $(\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)$, we have

$$d(\rho) - d(\tau) = (d(\lambda) \vee d(\mu) - d(\lambda)) - (d(\lambda) \vee d(\mu) - d(\mu)) = -d(\lambda) + d(\mu)$$

and then the generators in (iii) belong to $\mathbb{F}_R(w(X))_{-d(\lambda)+d(\mu)}$. Finally, a word $\lambda\lambda^*$ has degree 0 and then the generators in (iv) belong to $\mathbb{F}_R(w(X))_0$. Thus I is a graded ideal.

Since I is graded, then $\mathrm{KP}_R(\Lambda) = \mathbb{F}_R(w(X))/I$ is graded by the subgroups

$$(\mathbb{F}_R(w(X))/I)_n := \operatorname{span}_R \{q(w) : w \in w(X), d(w) = n\}.$$

By Proposition 3.3.(b), we have $\mathrm{KP}_R(\Lambda) = \operatorname{span}_R \{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}$. We have to show that

$$\mathrm{KP}_R(\Lambda)_n := \operatorname{span}_R \{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda, d(\lambda) - d(\mu) = n\} = (\mathbb{F}_R(w(X))/I)_n.$$

Take $\lambda, \mu \in \Lambda$ with $d(\lambda) - d(\mu) = n$. Then $s_\lambda s_{\mu^*} = q(\lambda) q(\mu^*) = q(\lambda\mu^*)$ and $d(\lambda\mu^*) = d(\lambda) - d(\mu) = n$. Hence $s_\lambda s_{\mu^*} \in (\mathbb{F}_R(w(X))/I)_n$ and $\mathrm{KP}_R(\Lambda)_n \subseteq (\mathbb{F}_R(w(X))/I)_n$.

To prove $(\mathbb{F}_R(w(X))/I)_n \subseteq \mathrm{KP}_R(\Lambda)_n$, we first establish the following claim:

Claim 3.8. *Let $X := \Lambda \cup G(\Lambda^{\neq 0})$ and $q : \mathbb{F}_R(w(X)) \to \mathrm{KP}_R(\Lambda)$ be the quotient map. Then for $w \in w(X)$, we have $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$.*

Proof of Claim 3.8. We are modifying the proof of [5, Lemma 3.5] and [11, Lemma 3.8] using our version of (KP3). We prove the claim by induction on $|w|$. For $|w| = 0$, we have $w = v$ for some $v \in \Lambda^0$. Then $q(w) = s_v = s_v s_{v^*}$ and $d(v) - d(v) = 0$. So $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$.

For $|w| = 1$, we have two possibilities. First suppose $w = \lambda$ for $\lambda \in \Lambda$. Then $q(w) = s_\lambda = s_\lambda s_{s(\lambda)^*}$ and $d(\lambda) - d(s(\lambda)) = d(\lambda)$. So $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$. Next suppose $w = \lambda^*$ for $\lambda \in \Lambda$. Then $q(w) = s_{\lambda^*} = s_{s(\lambda)} s_{\lambda^*}$ and $d(s(\lambda)) - d(\lambda) = -d(\lambda) = d(\lambda^*)$. So $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$.

For $|w| = 2$, we have four possibilities: $w = \lambda\mu^*$, $w = \lambda\mu$, $w = \mu^*\lambda^*$, or $w = \lambda^*\mu$. For the first three cases, we have

$$q(\lambda\mu^*) = s_\lambda s_{\mu^*} \text{ and } d(\lambda) - d(\mu) = d(\lambda\mu^*),$$
$$q(\lambda\mu) = s_{\lambda\mu} s_{s(\mu)^*} \text{ and } d(\lambda\mu) - d(s(\mu)) = d(\lambda\mu),$$
$$q(\mu^*\lambda^*) = s_{s(\mu)} s_{(\lambda\mu)^*} \text{ and } d(s(\mu)) - d\left((\lambda\mu)^*\right) = d(\mu^*\lambda^*),$$

as required. Suppose $w = \lambda^*\mu$. By (KP3), we have

$$q(\lambda^*\mu) = s_{\lambda^*} s_\mu = \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} s_\rho s_{\tau^*}.$$

For $(\rho,\tau) \in \Lambda^{\min}(\lambda,\mu)$, we have $\lambda\rho = \mu\tau$ and then $d(w) = d(\mu) - d(\lambda) = d(\rho) - d(\rho)$. So $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$.

Now suppose that $n \geq 2$ and $q(y) \in \mathrm{KP}_R(\Lambda)_{d(y)}$ for every word y with $|y| \leq n$. Let w be a word with $|w| = n+1$ and $q(w) \neq 0$. If w contains a subword $w_i w_{i+1} = \lambda\mu$, then λ and μ are composable in Λ since otherwise $q(\lambda\mu) = 0$. Now let w' be the word obtained from w by replacing $w_i w_{i+1}$ with the single path $\lambda\mu$, and then

$$q(w) = s_{w_1} \cdots s_{w_{i-1}} s_\lambda s_\mu s_{w_{i+2}} s_{w_{n+1}} = s_{w_1} \cdots s_{w_{i-1}} s_{\lambda\mu} s_{w_{i+2}} s_{w_{n+1}} = q(w').$$

Since $|w'| = n$ and $d(w') = d(w)$, the inductive hypothesis implies $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$. A similar argument shows $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$ whenever w contains a subword $w_i w_{i+1} = \mu^*\lambda^*$.

So suppose w contains no subword of the form $\lambda\mu$ or $\mu^*\lambda^*$. Since $|w| \geq 3$, either $w_1 w_2$ or $w_2 w_3$ has the form $\lambda^*\mu$. By (KP3), we write $q(w)$ as a sum of terms $q(y^i)$ with $|y^i| = n+1$ and $d(y^i) = d(w)$. Since $|w| \geq 3$, each nonzero summand $q(y^i)$ contains a factor of the form $s_\gamma s_\rho$ or one of the form $s_{\tau^*} s_{\gamma^*}$. Then the previous argument shows that every $q(y^i) \in \mathrm{KP}_R(\Lambda)_{d(w)}$ and $q(w) \in \mathrm{KP}_R(\Lambda)_{d(w)}$, as required. □ Claim 3.8

Every element in $(\mathbb{F}_R(w(X))/I)_n$ is in the form $q(w)$ with $w \in w(X)$ and $d(w) = n$, which, by Claim 3.8, belongs to $\mathrm{KP}_R(\Lambda)_n$. Then $(\mathbb{F}_R(w(X))/I)_n \subseteq \mathrm{KP}_R(\Lambda)_n$, as required. □

Definition 3.9. Suppose that $\{S_\lambda, S_{\mu^*} : \lambda,\mu \in \Lambda\}$ is the Kumjian–Pask Λ-family in the R-algebra $\mathrm{End}(\mathbb{F}_R(\partial\Lambda))$ as in Proposition 3.6. We call the R-algebra homomorphism $\pi_S : \mathrm{KP}_R(\Lambda) \to \mathrm{End}(\mathbb{F}_R(\partial\Lambda))$ obtained from Theorem 3.7.(a), the *boundary path representation of* $\mathrm{KP}_R(\Lambda)$.

4. The graded uniqueness theorem

Throughout this section, Λ is a finitely aligned k-graph and R is a commutative ring with identity 1.

Theorem 4.1 *(The graded uniqueness theorem). Let Λ be a finitely aligned k-graph, R be a commutative ring with* 1*, and A be a $\mathbb{Z}^k$-graded R-algebra. Suppose that $\pi : \mathrm{KP}_R(\Lambda) \to A$*

is a $\mathbb{Z}^k$-graded ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$. Then π is injective.

We start the proof of Theorem 4.1 by adapting some C^*-algebra results used to prove the gauge-invariant uniqueness theorem [22, Theorem 4.2] to Kumjian–Pask algebras. Although the argument is rather technical, the point is that most of the argument in the C^*-algebra setting also works in our situation.

First we recall from [22, Definition 2.5] that a *Cuntz–Krieger Λ-family* is a collection $\{T_\lambda : \lambda \in \Lambda\}$ of partial isometries (in other words, it satisfies $T_\lambda = T_\lambda T_\lambda^* T_\lambda$ for $\lambda \in \Lambda$, see [23, Appendix A]) in a C^*-algebra B satisfying:

(TCK1) $\left\{T_v : v \in \Lambda^0\right\}$ is a collection of mutually orthogonal projections;
(TCK2) $T_\lambda T_\mu = T_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(TCK3) $T_\lambda^* T_\mu = \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} T_\rho T_\tau^*$ for all $\lambda, \mu \in \Lambda$; and
(CK) $\prod_{\lambda\in E}\left(T_{r(E)} - T_\lambda T_\lambda^*\right) = 0$ for all $E \in \mathrm{FE}(\Lambda)$.

For a finitely aligned k-graph Λ, there exists a universal C^*-algebra $C^*(\Lambda)$ generated by the universal Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$. Now suppose that $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Kumjian–Pask Λ-family in an R-algebra A and we define $T_\lambda := S_\lambda$ for $\lambda \in \Lambda$ and $T_\mu^* := S_{\mu^*}$ for $\mu \in G\left(\Lambda^{\neq 0}\right)$. Then $\{T_\lambda : \lambda \in \Lambda\}$ is a collection satisfying $T_\lambda = T_\lambda T_\lambda^* T_\lambda$ for $\lambda \in \Lambda$, (TCK1–3) and (CK). (Note that we do not say that $\{T_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family, since we need a C^*-algebra containing T_λ, T_μ^*.) Similarly, a Cuntz–Krieger Λ-family in a C^*-algebra gives a Kumjian–Pask Λ-family. Thus one can translate proofs about Cuntz–Krieger Λ-families to proofs about Kumjian–Pask Λ-families.

The key ingredient to proof of Theorem 4.1 is proving that the uniqueness theorem holds on the core $\mathrm{KP}_R(\Lambda)_0 := \mathrm{span}_R\{s_\lambda s_{\mu^*} : d(\lambda) = d(\mu)\}$ (Theorem 4.4). First we establish some preliminary results and notation.

Following [22, Lemma 3.2], for every finite set $E \subseteq \Lambda$, there exists a finite set $F \subseteq \Lambda$ which contains E and satisfies

$$\begin{gathered}\lambda, \mu, \rho, \tau \in F,\ d(\lambda) = d(\mu),\ d(\rho) = d(\tau),\ s(\lambda) = s(\mu), \text{ and } s(\rho) = s(\tau) \\ \text{imply } \left\{\lambda\alpha, \tau\beta : (\alpha, \beta) \in \Lambda^{\min}(\mu, \rho)\right\} \subseteq F.\end{gathered} \tag{4.1}$$

We then write

$$\Pi E := \bigcap\{F \subseteq \Lambda : E \subseteq F \text{ and } F \text{ satisfies (4.1)}\}$$

and $\Pi E \times_{d,s} \Pi E$ for the set $\{(\lambda, \mu) \in \Pi E \times \Pi E : d(\lambda) = d(\mu), s(\lambda) = s(\mu)\}$. Note that ΠE is finite. Now recall from Notation 3.12 of [22] that for $\lambda \in \Pi E$, we write

$$T(\lambda) := \{\nu \in s(\lambda)\Lambda : d(\nu) \neq 0, \lambda\nu \in \Pi E\}.$$

Since $\lambda T(\lambda) \subseteq \Pi E$ and ΠE is finite, then $T(\lambda)$ is also finite.

Now suppose that $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Kumjian–Pask Λ-family in an R-algebra A. The argument of Lemma 3.2 of [22] shows that the set

$$M^S_{\Pi E} := \operatorname{span}_R \{S_\lambda S_{\mu^*} : (\lambda, \mu) \in \Pi E \times_{d,s} \Pi E\}$$

is closed under multiplication. For $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, define

$$\Theta(S)^{\Pi E}_{\lambda,\mu} := S_\lambda \Big(\prod_{\nu \in T(\lambda)} \big(S_{s(\lambda)} - S_{\lambda\nu} S_{(\lambda\nu)^*}\big) \Big) S_{\mu^*}.$$

Applying the argument of Proposition 3.9 and Proposition 3.11 of [22] gives the following.

Lemma 4.2. *Let $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ be a Kumjian–Pask Λ-family in an R-algebra A and $E \subseteq \Lambda$ be finite. For $(\lambda, \mu), (\rho, \tau) \in \Pi E \times_{d,s} \Pi E$, we have*

$$\Theta(S)^{\Pi E}_{\lambda,\mu} \Theta(S)^{\Pi E}_{\rho,\tau} = \delta_{\mu,\rho} \Theta(S)^{\Pi E}_{\lambda,\tau}, \quad S_\lambda S_{\mu^*} = \sum_{\lambda\nu \in \Pi E} \Theta(S)^{\Pi E}_{\lambda\nu,\mu\nu}$$

and $M^S_{\Pi E}$ is spanned by the set $\{\Theta(S)^{\Pi E}_{\lambda,\mu} : (\lambda, \mu) \in \Pi E \times_{d,s} \Pi E\}$.

Lemma 4.3. *Let Λ be a finitely aligned k-graph, R be a commutative ring with 1 and $E \subseteq \Lambda$ be finite. Suppose that $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$. Let $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$. Then the following conditions are equivalent:*

(a) $\pi\big(\Theta(s)^{\Pi E}_{\lambda,\mu}\big) = 0$.
(b) $\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$.
(c) $T(\lambda)$ *is exhaustive.*

Furthermore, for $r \in R \setminus \{0\}$ we have

$$\pi\big(r\Theta(s)^{\Pi E}_{\lambda,\mu}\big) = 0 \textit{ if and only if } r\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$$

and π is injective on $M^s_{\Pi E}$.

Proof. By following the argument of Proposition 3.13 and Corollary 3.17 of [22], we have the three equivalent conditions. Now take $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$ and $r \in R \setminus \{0\}$. If $r\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$, we trivially have $\pi\big(r\Theta(s)^{\Pi E}_{\lambda,\mu}\big) = 0$. So suppose $\pi\big(r\Theta(s)^{\Pi E}_{\lambda,\mu}\big) = 0$. Since $\pi(rs_v) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$, then by Remark 3.5, $\pi\big(r\Theta(s)^{\Pi E}_{\lambda,\mu}\big) = 0$ implies that $T(\lambda)$ is exhaustive (since $r \neq 0$) and by (c)⇒(b), $\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$. So $r\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$, as required.

Next we show that π is injective on $M^s_{\Pi E}$. Take $a \in M^s_{\Pi E}$ such that $\pi(a) = 0$. We have to show $a = 0$. Since $a \in M^s_{\Pi E}$ and $M^s_{\Pi E} = \operatorname{span}_R\{\Theta(s)^{\Pi E}_{\lambda,\mu} : (\lambda,\mu) \in \Pi E \times_{d,s} \Pi E\}$ (Lemma 4.2), we write $a = \sum_{(\lambda,\mu)\in F} r_{\lambda,\mu}\Theta(s)^{\Pi E}_{\lambda,\mu}$ where $F \subseteq \Pi E \times_{d,s} \Pi E$ is finite and for all $(\lambda,\mu) \in F$, we have $r_{\lambda,\mu} \in R$ and $\Theta(s)^{\Pi E}_{\lambda,\mu} \neq 0$. If $T(\lambda)$ is exhaustive for some $(\lambda,\mu) \in F$, then by (c)⇒(b), $\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$, which contradicts $\Theta(s)^{\Pi E}_{\lambda,\mu} \neq 0$. So $T(\lambda)$ is non-exhaustive for all $(\lambda,\mu) \in F$. Since $\pi(a) = 0$, then for $(\rho,\tau) \in F$, we have

$$\begin{aligned}
0 &= \pi\big(\Theta(s)^{\Pi E}_{\rho,\rho}\big)\pi(a)\pi\big(\Theta(s)^{\Pi E}_{\tau,\tau}\big) \\
&= \pi\big(\Theta(s)^{\Pi E}_{\rho,\rho}\big)\pi\Big(\sum_{(\lambda,\mu)\in F} r_{\lambda,\mu}\Theta(s)^{\Pi E}_{\lambda,\mu}\Big)\pi\big(\Theta(s)^{\Pi E}_{\tau,\tau}\big) \\
&= r_{\rho,\tau}\pi\big(\Theta(s)^{\Pi E}_{\rho,\tau}\big) = r_{\rho,\tau}\Theta(\pi(s))^{\Pi E}_{\rho,\tau} \quad \text{(by Lemma 4.2)}.
\end{aligned}$$

But now since $\pi(rs_v) \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$, then by Remark 3.5, $r_{\rho,\tau}\Theta(\pi(s))^{\Pi E}_{\rho,\tau} = 0$ implies that $r_{\rho,\tau} = 0$ (since $T(\rho)$ is non-exhaustive). Therefore, $a = 0$ and π is injective on $M^s_{\Pi E}$. □

A direct consequence of Lemma 4.3 is:

Theorem 4.4. *Let Λ be a finitely aligned k-graph and R be a commutative ring with 1. Suppose that $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$. Then π is injective on $\mathrm{KP}_R(\Lambda)_0$.*

Proof. Take $a \in \mathrm{KP}_R(\Lambda)_0$ such that $\pi(a) = 0$. We have to show $a = 0$. Write $a = \sum_{(\lambda,\mu)\in F} r_{\lambda,\mu}s_\lambda s_{\mu^*}$ with $d(\lambda) = d(\mu)$ for $(\lambda,\mu) \in F$. Define $E := \{\lambda,\mu : (\lambda,\mu) \in F\}$ and then $a \in M^s_{\Pi E}$. Since π is injective on $M^s_{\Pi E}$ (Lemma 4.3), $a = 0$. □

Now we establish the last stepping stone result before proving Theorem 4.1.

Lemma 4.5. *Let I be a graded ideal of $\mathrm{KP}_R(\Lambda)$. Then I is generated as an ideal by the set $I_0 := I \cap \mathrm{KP}_R(\Lambda)_0$.*

Proof. We generalise the argument of [30, Lemma 5.1]. Take $n \in \mathbb{Z}^k$ and write $n = n_1 - n_2$ such that $n_1, n_2 \in \mathbb{N}^k$ and $|n_1 + n_2|$ as minimum as possible. We show that $I_n := I \cap \mathrm{KP}_R(\Lambda)_n$ is contained in $\mathrm{KP}_R(\Lambda)_{n_1} I_0 \mathrm{KP}_R(\Lambda)_{n_2}$. Now take $a \in I_n$ and write $a = \sum_{(\lambda,\mu)\in F} r_{\lambda,\mu}s_\lambda s_{\mu^*}$. Note that $d(\lambda) - d(\mu) = n$ for $(\lambda,\mu) \in F$. Since $n = n_1 - n_2$ with $n_1, n_2 \in \mathbb{N}^k$ and $|n_1 + n_2|$ as minimum as possible, then for every $(\lambda,\mu) \in F$, $d(\lambda) \geq n_1$ and $d(\mu) \geq n_2$, so by the factorisation property, there exist $\lambda_1, \lambda_2, \mu_1, \mu_2$ such that

$$\lambda = \lambda_1\lambda_2,\ \mu = \mu_1\mu_2,\ d(\lambda_1) = n_1,\ d(\mu_1) = n_2, \text{ and } d(\lambda_2) = d(\mu_2).$$

Hence

$$a = \sum_{(\lambda,\mu)\in F} r_{\lambda_1\lambda_2,\mu_1\mu_2} s_{\lambda_1} \left(s_{\lambda_2} s_{\mu_2^*}\right) s_{\mu_1^*}.$$

Take $(\alpha,\beta) \in F$ and write $\alpha = \alpha_1\alpha_2$ and $\beta = \beta_1\beta_2$. Note that for $\nu,\gamma \in \Lambda$ with $d(\nu) = d(\gamma)$, Remark 3.4 gives $s_{\nu^*}s_\gamma = 0$ for $\nu \neq \gamma$. Then

$$\begin{aligned} s_{\alpha_1^*} a s_{\beta_1} &= \sum_{(\lambda,\mu)\in F} r_{\lambda_1\lambda_2,\mu_1\mu_2} \left(s_{\alpha_1^*} s_{\lambda_1}\right)\left(s_{\lambda_2} s_{\mu_2^*}\right)\left(s_{\mu_1^*} s_{\beta_1}\right) \\ &= \sum_{\{(\lambda,\mu)\in F:\lambda_1=\alpha_1,\mu_1=\beta_1\}} r_{\alpha_1\lambda_2,\beta_1\mu_2} s_{\lambda_2} s_{\mu_2^*} \end{aligned}$$

since $d(\alpha_1) = n_1 = d(\lambda_1)$ and $d(\beta_1) = n_2 = d(\mu_1)$ for $(\lambda,\mu) \in F$. Since $a \in I$, we have $s_{\alpha_1^*} a s_{\beta_1} \in I$. Since $d(\lambda_2) = d(\mu_2)$ for $(\alpha_1\lambda_2, \beta_1\mu_2) \in F$, we have $s_{\alpha_1^*} a s_{\beta_1} \in \mathrm{KP}_R(\Lambda)_0$. Hence

$$\sum_{\{(\lambda,\mu)\in F:\lambda_1=\alpha_1,\mu_1=\beta_1\}} r_{\alpha_1\lambda_2,\beta_1\mu_2} s_{\lambda_2} s_{\mu_2^*} = s_{\alpha_1^*} a s_{\beta_1} \in I_0$$

and

$$\begin{aligned} &\sum_{\{(\lambda,\mu)\in F:\lambda_1=\alpha_1,\mu_1=\beta_1\}} r_{\alpha_1\lambda_2,\beta_1\mu_2} s_{\alpha_1\lambda_2} s_{(\beta_1\mu_2)^*} \\ &\quad = s_{\alpha_1}\left(s_{\alpha_1^*} a s_{\beta_1}\right) s_{\beta_1^*} \in \mathrm{KP}_R(\Lambda)_{n_1} I_0 \mathrm{KP}_R(\Lambda)_{n_2}. \end{aligned}$$

Therefore

$$\begin{aligned} a &= \sum_{(\lambda,\mu)\in F} r_{\lambda_1\lambda_2,\mu_1\mu_2} s_{\lambda_1\lambda_2} s_{(\mu_1\mu_2)^*} \\ &= \sum_{\{(\alpha_1,\beta_1):(\alpha,\beta)\in F\}} \left(\sum_{\{(\lambda,\mu)\in F:\lambda_1=\alpha_1,\mu_1=\beta_1\}} r_{\alpha_1\lambda_2,\beta_1\mu_2} s_{\alpha_1\lambda_2} s_{(\beta_1\mu_2)^*} \right) \end{aligned}$$

also belongs to $\mathrm{KP}_R(\Lambda)_{n_1} I_0 \mathrm{KP}_R(\Lambda)_{n_2}$, and $I_n \subseteq \mathrm{KP}_R(\Lambda)_{n_1} I_0 \mathrm{KP}_R(\Lambda)_{n_2}$.

Now since I is a graded ideal and $I = \bigoplus_{n\in\mathbb{Z}^k} I_n$, we have that I is generated as an ideal by I_0. □

Proof of Theorem 4.1. Because π is graded, we have that $\ker\pi$ is a graded ideal. By Lemma 4.5, the ideal $\ker\pi$ is generated by the set $\ker\pi \cap \mathrm{KP}_R(\Lambda)_0$. Thus it suffices to show $\pi|_{\mathrm{KP}_R(\Lambda)_0} : \mathrm{KP}_R(\Lambda)_0 \to A$ is injective. However, the injectivity follows from Theorem 4.4. □

One immediate application of Theorem 4.1 is:

Proposition 4.6. *Let Λ be a finitely aligned k-graph. Let $\{s_\lambda, s_{\mu^*} : \lambda,\mu \in \Lambda\}$ be the universal Kumjian–Pask Λ-family for $R = \mathbb{C}$ and $\{t_\lambda : \lambda \in \Lambda\}$ be the universal Cuntz–Krieger*

Λ-family. Then there is an isomorphism $\pi_t : \mathrm{KP}_{\mathbb{C}}(\Lambda) \to \mathrm{span}_{\mathbb{C}}\left\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\right\}$ such that $\pi_t(s_\lambda) = t_\lambda$ and $\pi_t(s_{\mu^*}) = t_\mu^*$ for $\lambda, \mu \in \Lambda$. In particular, $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ is isomorphic to a dense subalgebra of $C^*(\Lambda)$.

Proof. Since $\{t_\lambda : \lambda \in \Lambda\}$ satisfies (TCK1–3) and (CK), then $\left\{t_\lambda, t_\mu^* : \lambda, \mu \in \Lambda\right\}$ also satisfies (KP1–4) and is a Kumjian–Pask Λ-family in $C^*(\Lambda)$. Thus the universal property of $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ gives a homomorphism π_t from $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ onto the dense subalgebra

$$A := \mathrm{span}_{\mathbb{C}}\left\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\right\}$$

of $C^*(\Lambda)$.

Next we show the injectivity of π_t. By Theorem 4.1, it suffices to show that π_t is a $\mathbb{Z}^k$-graded ring homomorphism. We claim that A is graded by

$$A_n := \mathrm{span}_{\mathbb{C}}\left\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda, d(\lambda) - d(\mu) = n\right\}.$$

Note that for $\lambda, \mu, \rho, \tau \in \Lambda$ with $d(\lambda) - d(\mu) = n$ and $d(\rho) - d(\tau) = m$, we have

$$\begin{aligned} t_\lambda t_\mu^* t_\rho t_\tau^* &= t_\lambda \Big(\sum_{(\mu', \rho') \in \Lambda^{\min}(\mu, \rho)} t_{\mu'} t_{\rho'}^* \Big) t_\tau^* \text{ (by (TCK3))} \\ &= \sum_{(\mu', \rho') \in \Lambda^{\min}(\mu, \rho)} t_{\lambda\mu'} t_{\tau\rho'}^* \end{aligned}$$

and for $(\mu', \rho') \in \Lambda^{\min}(\mu, \rho)$,

$$\begin{aligned} d(\lambda\mu') - d(\tau\rho') &= d(\lambda) + d(\mu') - d(\tau) - d(\rho') \\ &= d(\lambda) + (d(\mu) \vee d(\rho) - d(\mu)) \\ &\quad - d(\tau) - (d(\mu) \vee d(\rho) - d(\rho)) \\ &= (d(\lambda) - d(\mu)) - (d(\tau) - d(\rho)) \\ &= n + m. \end{aligned}$$

Hence $A_n A_m \subseteq A_{n+m}$. Since each spanning element $t_\lambda t_\mu^*$ belongs to $A_{d(\lambda)-d(\mu)}$, every element a of A can be written as a finite sum $\sum a_n$ with $a_n \in A_n$. For $a_n \in A_n$ such that a finite sum $\sum a_n = 0$, then we have each $a_n = 0$ by following the argument of [5, Lemma 7.4]. Thus $\left\{A_n : n \in \mathbb{Z}^k\right\}$ is a grading of A, as claimed. Then π_t is a $\mathbb{Z}^k$-grading and by Theorem 4.1, π_t is injective. □

5. Steinberg algebras

Steinberg algebras were introduced by Steinberg in [28] and are algebraic analogues of groupoid C^*-algebras. In [12], Clark and Sims show that for every 1-graph E, its Leavitt path algebra is isomorphic to a Steinberg algebra. In this section, we show that for every finitely aligned k-graph Λ, its Kumjian–Pask algebra is isomorphic to a Steinberg algebra (Proposition 5.4). We start out with an introduction to groupoids and Steinberg algebras in general.

A groupoid $\mathcal{G}$ is a small category in which every morphism has an inverse. For a groupoid $\mathcal{G}$, we write $r(a)$ and $s(a)$ to denote the *range* and *source* of $a \in \mathcal{G}$. Because $r(a) = s\left(a^{-1}\right)$ for $a \in \mathcal{G}$, then r and s have the common image. We call this common image the *unit space* of $\mathcal{G}$ and denote it $\mathcal{G}^{(0)}$. A pair $(a, b) \in \mathcal{G} \times \mathcal{G}$ is said *composable* if $s(a) = r(b)$. We then use notation $\mathcal{G}^{(2)}$ to denote the collection of composable pairs in $\mathcal{G}$. For $A, B \subseteq \mathcal{G}$, we write

$$AB := \left\{ab : a \in A, b \in B, (a, b) \in \mathcal{G}^{(2)}\right\}.$$

We say $\mathcal{G}$ is a *topological groupoid* if $\mathcal{G}$ is endowed with a topology such that composition and inversion on $\mathcal{G}$ are continuous. We also call an open set $U \subseteq \mathcal{G}$ an *open bisection* if s and r restricted to U are homeomorphisms into $\mathcal{G}^{(0)}$. Finally, we call $\mathcal{G}$ *ample* if $\mathcal{G}$ has a basis of compact open bisections.

Remark 5.1. Note that if $\mathcal{G}$ is ample, then $\mathcal{G}$ is locally compact and étale. In fact, $\mathcal{G}$ is Hausdorff ample if and only if $\mathcal{G}$ is locally compact, Hausdorff and étale with totally disconnected unit space.

Now suppose that $\mathcal{G}$ is a Hausdorff ample groupoid and R is a commutative ring with 1. As in [9, Section 2.2], the Steinberg algebra[1] associated to $\mathcal{G}$ is

$$A_R(\mathcal{G}) := \{f : \mathcal{G} \to R : f \text{ is locally constant and has compact support}\}$$

where addition and scalar multiplication are defined pointwise, and convolution is given by

$$(f \star g)(a) := \sum_{r(a)=r(b)} f(b)\, g\left(b^{-1}a\right).$$

Furthermore, for compact open bisections U and V, we have the characteristic function $1_U \in A_R(\mathcal{G})$ and

$$1_U \star 1_V = 1_{UV}$$

[28, Proposition 4.3]. Note that for $f \in A_R(\mathcal{G})$, $\operatorname{supp}(f)$ is clopen ([9, Remark 2.1]).

Example 5.2. To each finitely aligned k-graph Λ, we define the associated *boundary-path groupoid* $\mathcal{G}_\Lambda$ from [32, Definition 4.8] as follows. Write

[1] In [28], Steinberg writes $R\mathcal{G}$ to denote $A_R(\mathcal{G})$.

$$\Lambda *_s \Lambda := \{(\lambda, \mu) \in \Lambda \times \Lambda : s(\lambda) = s(\mu)\}.$$

The objects of $\mathcal{G}_\Lambda$ are

$$\mathrm{Obj}(\mathcal{G}_\Lambda) := \partial\Lambda.$$

The morphisms are

$$\begin{aligned}\mathrm{Mor}(\mathcal{G}_\Lambda) &:= \{(\lambda z, d(\lambda) - d(\mu), \mu z) \in \partial\Lambda \times \mathbb{Z}^k \times \partial\Lambda : \\ &\qquad (\lambda, \mu) \in \Lambda *_s \Lambda, z \in s(\lambda)\partial\Lambda\} \\ &= \{(x, m, y) \in \partial\Lambda \times \mathbb{Z}^k \times \partial\Lambda : \text{there exists } p, q \in \mathbb{N}^k \text{ such that} \\ &\qquad p \leq d(x), q \leq d(y), p - q = m \text{ and } \sigma^p x = \sigma^q y\}.\end{aligned}$$

The range and source maps are given by $r(x, m, y) := x$ and $s(x, m, y) := y$, and composition is defined such that

$$((x_1, m_1, y_1), (y_1, m_2, y_2)) \mapsto (x_1, m_1 + m_2, y_2).$$

Finally inversion is given by $(x, m, y) \mapsto (y, -m, x)$.

Next, we show how to realise $\mathcal{G}_\Lambda$ as a topological groupoid. For $(\lambda, \mu) \in \Lambda *_s \Lambda$ and finite non-exhaustive subset $G \subseteq s(\lambda)\Lambda$, we write

$$\begin{aligned}Z_\Lambda(\lambda) &:= \lambda\partial\Lambda, \\ Z_\Lambda(\lambda \setminus G) &:= Z_\Lambda(\lambda) \setminus \Big(\bigcup_{\nu \in G} Z_\Lambda(\lambda\nu)\Big), \\ Z_\Lambda(\lambda *_s \mu) &:= \{(x, d(\lambda) - d(\mu), y) \in \mathcal{G}_\Lambda : x \in Z_\Lambda(\lambda), y \in Z_\Lambda(\mu) \\ &\qquad \text{and } \sigma^{d(\lambda)}x = \sigma^{d(\mu)}y\},\end{aligned}$$

and

$$Z_\Lambda(\lambda *_s \mu \setminus G) := Z_\Lambda(\lambda *_s \mu) \setminus \Big(\bigcup_{\nu \in G} Z_\Lambda(\lambda\nu *_s \mu\nu)\Big).$$

The sets $Z_\Lambda(\lambda *_s \mu \setminus G)$ form a basis of compact open bisections for a second-countable, Hausdorff topology on $\mathcal{G}_\Lambda$ under which it is an ample groupoid. Further, the sets $Z_\Lambda(\lambda \setminus G)$ form a basis of compact open sets for $\mathcal{G}_\Lambda^{(0)}$.

Remark 5.3. A number of notes of this example:

(i) We think of $\mathcal{G}_\Lambda^{(0)} = \partial\Lambda$ as a subset of $\mathcal{G}_\Lambda$ under the correspondence $x \mapsto (x, 0, x)$.

(ii) In [32], Yeend defines $Z_\Lambda(\lambda \setminus G)$ and $Z_\Lambda(\lambda *_s \mu \setminus G)$ where G is finite. However, if G is exhaustive, then $Z_\Lambda(\lambda \setminus G)$ and $Z_\Lambda(\lambda *_s \mu \setminus G)$ are empty sets. Thus our definitions make sure that both $Z_\Lambda(\lambda \setminus G)$ and $Z_\Lambda(\lambda *_s \mu \setminus G)$ are non-empty.

Next we generalise [10, Proposition 4.3] as follows:

Proposition 5.4. *Let Λ be a finitely aligned k-graph and $\mathcal{G}_\Lambda$ be its boundary-path groupoid as defined in Example 5.2. Let R be a commutative ring with 1. Then there is an isomorphism $\pi_T : \mathrm{KP}_R(\Lambda) \to A_R(\mathcal{G}_\Lambda)$ such that $\pi_T(s_\lambda) = 1_{Z_\Lambda(\lambda *_s s(\lambda))}$ and $\pi_T(s_{\mu^*}) = 1_{Z_\Lambda(s(\mu) *_s \mu)}$ for $\lambda, \mu \in \Lambda$.*

The only part of the proof of Proposition 5.4 that requires much additional work is showing the surjectivity of π_T. For this, we establish the following two lemmas. These lemmas show that the characteristic function associated to a compact open set in $\mathcal{G}_\Lambda$ can be written as a sum of elements in the form $1_{Z_\Lambda(\lambda *_s \mu \setminus G)}$.

Lemma 5.5. *Let $(\lambda, \mu), (\lambda', \mu') \in \Lambda *_s \Lambda$, $G \subseteq s(\lambda)\Lambda$, and $G' \subseteq s(\lambda')\Lambda$. Define $F := \Lambda^{\min}(\lambda, \lambda') \cap \Lambda^{\min}(\mu, \mu')$. Then*

$$Z_\Lambda(\lambda *_s \mu \setminus G) \cap Z_\Lambda(\lambda' *_s \mu' \setminus G') = \bigsqcup_{(\gamma,\gamma') \in F} Z_\Lambda(\lambda\gamma *_s \mu'\gamma' \setminus [\mathrm{Ext}(\gamma; G) \cup \mathrm{Ext}(\gamma'; G')]). \quad (*)$$

Proof. We generalise the argument of [12, Example 3.2] for 1-graphs. First we show that the collection

$$\{Z_\Lambda(\lambda\gamma *_s \mu'\gamma' \setminus [\mathrm{Ext}(\gamma; G) \cup \mathrm{Ext}(\gamma'; G')]) : (\gamma, \gamma') \in F\}$$

is disjoint. It suffices to show that the collection

$$\{Z_\Lambda(\lambda\gamma *_s \mu'\gamma') : (\gamma, \gamma') \in F\}$$

is disjoint. Suppose for contradiction that there exist $(\gamma, \gamma'), (\gamma'', \gamma''') \in F$ such that $(\gamma, \gamma') \neq (\gamma'', \gamma''')$ and $V := Z_\Lambda(\lambda\gamma *_s \mu'\gamma') \cap Z_\Lambda(\lambda\gamma'' *_s \mu'\gamma''') \neq \emptyset$. Note that if $\gamma = \gamma''$, then

$$\begin{aligned} \lambda'\gamma' &= \lambda\gamma (\text{since } (\gamma, \gamma') \in \Lambda^{\min}(\lambda, \lambda')) \\ &= \lambda\gamma'' \ (\text{since } \gamma = \gamma'') \\ &= \lambda'\gamma''' \ (\text{since } (\gamma'', \gamma''') \in \Lambda^{\min}(\lambda, \lambda')) \end{aligned}$$

and $\gamma' = \gamma'''$ by the factorisation property, which contradicts $(\gamma, \gamma') \neq (\gamma'', \gamma''')$. The same argument shows that $\gamma' = \gamma'''$ implies $\gamma = \gamma''$. Hence $\gamma \neq \gamma''$ and $\gamma' \neq \gamma'''$. Meanwhile, since $(\gamma, \gamma'), (\gamma'', \gamma''') \in F$, then $d(\gamma) = d(\gamma'')$ and $d(\gamma') = d(\gamma''')$. Take $(x, m, y) \in V$. Then $x \in Z_\Lambda(\lambda\gamma)$ and $x \in Z_\Lambda(\lambda\gamma'')$. Since $d(\gamma) = d(\gamma'')$, then $d(\lambda\gamma) = d(\lambda\gamma'')$ and $\gamma = x(d(\lambda), d(\lambda\gamma)) = x(d(\lambda), d(\lambda\gamma'')) = \gamma''$, which contradicts $\gamma \neq \gamma''$. Hence the collection $\{Z_\Lambda(\lambda\gamma *_s \mu'\gamma') : (\gamma, \gamma') \in F\}$ is disjoint, and so is

$$\{Z_\Lambda(\lambda\gamma *_s \mu'\gamma' \setminus [\mathrm{Ext}(\gamma; G) \cup \mathrm{Ext}(\gamma'; G')]) : (\gamma, \gamma') \in F\}.$$

Next we show the right inclusion of (∗). Write

$$U := Z_\Lambda\left(\lambda *_s \mu \setminus G\right) \cap Z_\Lambda\left(\lambda' *_s \mu' \setminus G'\right)$$

and take $(x, m, y) \in U$. We show $(x, m, y) \in Z_\Lambda\left(\lambda\gamma *_s \mu'\gamma' \setminus \left[\operatorname{Ext}(\gamma; G) \cup \operatorname{Ext}(\gamma'; G')\right]\right)$ for some $(\gamma, \gamma') \in F$. Because $x \in Z_\Lambda(\lambda)$ and $x \in Z_\Lambda(\lambda')$, then $d(x) \geq d(\lambda) \vee d(\lambda')$ and there exists $(\gamma, \gamma') \in \Lambda^{\min}(\lambda, \lambda')$ such that

$$x \in Z_\Lambda(\lambda\gamma). \tag{5.1}$$

Using a similar argument, there exists $(\gamma'', \gamma''') \in \Lambda^{\min}(\mu, \mu')$ such that

$$y \in Z_\Lambda(\mu\gamma''). \tag{5.2}$$

We claim that $\gamma = \gamma''$ and $\gamma' = \gamma'''$. To see this, note that $m = d(\lambda) - d(\mu) = d(\lambda') - d(\mu')$ and

$$\begin{aligned} d(\gamma) &= d(\lambda) \vee d(\lambda') - d(\lambda) = (d(\mu) + m) \vee (d(\mu') + m) - (d(\mu) + m) \\ &= (d(\mu) \vee d(\mu')) + m - (d(\mu) + m) = d(\mu) \vee d(\mu') - d(\mu) = d(\gamma''). \end{aligned}$$

Since $(x, m, y) \in Z_\Lambda\left(\lambda *_s \mu \setminus G\right)$, then $\sigma^{d(\lambda)}x = \sigma^{d(\mu)}y$ and

$$\gamma = \left(\sigma^{d(\lambda)}x\right)(0, d(\gamma)) = \left(\sigma^{d(\mu)}y\right)(0, d(\gamma')) = \gamma''.$$

Using a similar argument, we also get $\gamma' = \gamma'''$ proving the claim.

Next we show that $(x, m, y) \in Z_\Lambda(\lambda\gamma *_s \mu'\gamma')$. By (5.1) and (5.2), we have $x \in Z_\Lambda(\lambda\gamma)$ and $y \in Z_\Lambda(\mu\gamma'')$. Since $\gamma = \gamma''$, $\gamma' = \gamma'''$, $(\gamma'', \gamma''') \in \Lambda^{\min}(\mu, \mu')$, then $\mu\gamma'' = \mu\gamma = \mu'\gamma'$ and $y \in Z_\Lambda(\mu'\gamma')$. On the other hand, since $(x, m, y) \in Z_\Lambda\left(\lambda *_s \mu \setminus G\right)$, then $\sigma^{d(\lambda)}x = \sigma^{d(\mu)}y$ and

$$\sigma^{d(\lambda\gamma)}x = \sigma^{d(\mu\gamma)}y = \sigma^{d(\mu'\gamma')}y$$

since $\mu\gamma = \mu'\gamma'$. Since $m = d(\lambda) - d(\mu) = d(\lambda\gamma) - d(\mu'\gamma')$, then $(x, m, y) \in Z_\Lambda(\lambda\gamma *_s \mu'\gamma')$, as required.

Finally we show that $(x, m, y) \notin Z_\Lambda(\lambda\gamma\nu *_s \mu'\gamma'\nu)$ for all $\nu \in \operatorname{Ext}(\gamma; G) \cup \operatorname{Ext}(\gamma'; G')$. Suppose for a contradiction that there exists $\nu \in \operatorname{Ext}(\gamma; G) \cup \operatorname{Ext}(\gamma'; G')$ such that $(x, m, y) \in Z_\Lambda(\lambda\gamma\nu *_s \mu'\gamma'\nu)$. Without loss of generality, suppose $\nu \in \operatorname{Ext}(\gamma; G)$. Then there exists $\nu' \in G$ such that $\gamma\nu \in Z_\Lambda(\nu')$. Since $x \in Z_\Lambda(\lambda\gamma\nu)$, $y \in Z_\Lambda(\mu'\gamma'\nu) = Z_\Lambda(\mu\gamma\nu)$, and $\gamma\nu \in Z_\Lambda(\nu')$, then $x \in Z_\Lambda(\lambda\nu')$ and $y \in Z_\Lambda(\mu\nu')$ where $\nu' \in G$. This contradicts $(x, m, y) \in Z_\Lambda\left(\lambda *_s \mu \setminus G\right)$. Hence

$$(x, m, y) \in Z_\Lambda\left(\lambda\gamma *_s \mu'\gamma' \setminus \left[\operatorname{Ext}(\gamma; G) \cup \operatorname{Ext}(\gamma'; G')\right]\right)$$

and

$$U \subseteq \bigsqcup_{(\gamma,\gamma')\in F} Z_\Lambda\left(\lambda\gamma *_s \mu'\gamma' \setminus [\mathrm{Ext}(\gamma;G) \cup \mathrm{Ext}(\gamma';G')]\right).$$

Next we show the left inclusion of $(*)$. Take $(\gamma,\gamma') \in F$ and

$$(x,m,y) \in Z_\Lambda\left(\lambda\gamma *_s \mu'\gamma' \setminus [\mathrm{Ext}(\gamma;G) \cup \mathrm{Ext}(\gamma';G')]\right). \tag{5.3}$$

We show (x,m,y) belongs to both $Z_\Lambda(\lambda *_s \mu \setminus G)$ and $Z_\Lambda(\lambda' *_s \mu' \setminus G')$. Without loss of generality, it suffices to show $(x,m,y) \in Z_\Lambda(\lambda *_s \mu \setminus G)$. First we show that $(x,m,y) \in Z_\Lambda(\lambda *_s \mu)$. Note that we have $\mu\gamma = \mu'\gamma'$ and $m = d(\lambda\gamma) - d(\mu'\gamma') = d(\lambda) - d(\mu)$. On the other hand, $(x,m,y) \in Z_\Lambda(\lambda\gamma *_s \mu'\gamma')$ also implies $x \in Z_\Lambda(\lambda\gamma)$ and $y \in Z_\Lambda(\mu'\gamma') = Z_\Lambda(\mu\gamma)$. Furthermore,

$$\begin{aligned}
\sigma^{(\lambda)}x &= [x(d(\lambda), d(\lambda\gamma))]\left[\sigma^{(\lambda\gamma)}x\right] \\
&= \gamma\left[\sigma^{(\lambda\gamma)}x\right] \text{ (since } x(d(\lambda), d(\lambda\gamma)) = \gamma) \\
&= \gamma[\sigma^{(\mu'\gamma')}y] \text{ (since } \sigma^{(\lambda\gamma)}x = \sigma^{(\mu'\gamma')}y) \\
&= [y(d(\mu), d(\mu\gamma))][\sigma^{(\mu'\gamma')}y] \text{ (since } y(d(\mu), d(\mu\gamma)) = \gamma) \\
&= [y(d(\mu), d(\mu\gamma))][\sigma^{(\mu\gamma)}y] \text{ (since } \mu\gamma = \mu'\gamma') \\
&= \sigma^{(\mu)}y
\end{aligned}$$

and then $(x,m,y) \in Z_\Lambda(\lambda *_s \mu)$, as required.

To complete the proof, we have to show $(x,m,y) \notin Z_\Lambda(\lambda\nu *_s \mu\nu)$ for all $\nu \in G$. Suppose for contradiction that there exists $\nu \in G$ such that $(x,m,y) \in Z_\Lambda(\lambda\nu *_s \mu\nu)$. In particular, $x \in Z_\Lambda(\lambda\nu)$. Since $x \in Z_\Lambda(\lambda\gamma)$ and $x \in Z_\Lambda(\lambda\nu)$, then there exists $\nu' \in \mathrm{Ext}(\gamma;\{\nu\})$ such that $x \in Z_\Lambda(\lambda\gamma\nu')$. Hence

$$\begin{aligned}
\sigma^{(\lambda\gamma\nu')}x &= \sigma^{(\mu\gamma\nu')}y \text{ (since } \sigma^{(\lambda)}x = \sigma^{(\mu)}y) \\
&= \sigma^{(\mu'\gamma'\nu')}y \text{ (since } \mu\gamma = \mu'\gamma'), \\
\left(\sigma^{(\mu)}y\right)(0, d(\gamma\nu')) &= \left(\sigma^{(\lambda)}x\right)(0, d(\gamma\nu')) \text{ (since } \sigma^{(\lambda)}x = \sigma^{(\mu)}y) \qquad (5.4)\\
&= x(d(\lambda), d(\lambda\gamma\nu')) \\
&= \gamma\nu' \text{ (since } x \in Z_\Lambda(\lambda\gamma\nu')),
\end{aligned}$$

and

$$\begin{aligned}
y(0, d(\mu'\gamma'\nu')) &= y(0, d(\mu\gamma\nu')) \text{ (since } \mu\gamma = \mu'\gamma') \\
&= \mu\gamma\nu' \text{ (by (5.4))} \\
&= \mu'\gamma'\nu' \text{ (since } \mu\gamma = \mu'\gamma').
\end{aligned}$$

Furthermore,

$$\begin{aligned} d(\lambda\gamma\nu') - d(\mu'\gamma'\nu') &= d(\lambda\gamma) - d(\mu'\gamma') \\ &= d(\lambda\gamma) - d(\mu\gamma) \ \text{(since } \mu\gamma = \mu'\gamma'\text{)} \\ &= d(\lambda) - d(\mu) = m. \end{aligned}$$

Hence $(x, m, y) \in Z_\Lambda(\lambda\gamma\nu' *_s \mu'\gamma'\nu')$ for some $\nu' \in \operatorname{Ext}(\gamma; \{\nu\}) \subseteq \operatorname{Ext}(\gamma; G)$, which contradicts (5.3). The conclusion follows. □

Lemma 5.6. *Let $\{Z_\Lambda(\lambda_i *_s \mu_i \setminus G_i)\}_{i=1}^n$ be a finite collection of compact open bisection sets and*

$$U := \bigcup_{i=1}^n Z_\Lambda(\lambda_i *_s \mu_i \setminus G_i).$$

Then

$$1_U \in \operatorname{span}_R \left\{ 1_{Z_\Lambda(\lambda *_s \mu \setminus G)} : (\lambda, \mu) \in \Lambda *_s \Lambda, G \subseteq s(\lambda)\Lambda \right\}.$$

Proof. It is trivial for $n = 1$. Now let $n = 2$ and $F := \Lambda^{\min}(\lambda_1, \lambda_2) \cap \Lambda^{\min}(\mu_1, \mu_2)$. If $F = \emptyset$, then

$$1_U = 1_{Z_\Lambda(\lambda *_s \mu \setminus G)} + 1_{Z_\Lambda(\lambda' *_s \mu' \setminus G')}.$$

Otherwise, by Proposition 5.5, we have

$$1_U = 1_{Z_\Lambda(\lambda *_s \mu \setminus G)} + 1_{Z_\Lambda(\lambda' *_s \mu' \setminus G')} - \sum_{(\gamma,\gamma') \in F} 1_{Z_{\gamma,\gamma'}}$$

where $Z_{\gamma,\gamma'} := Z_\Lambda(\lambda\gamma *_s \mu'\gamma' \setminus \operatorname{Ext}(\gamma; G) \cup \operatorname{Ext}(\gamma'; G'))$, as required. For $n \geq 3$, by using the inclusion-exclusion principle and de Morgan's law, 1_U can be written as a sum of elements in the form $1_{Z_\Lambda(\lambda *_s \mu \setminus G)}$. □

Proof of Proposition 5.4. Define $T_\lambda := 1_{Z_\Lambda(\lambda *_s s(\lambda))}$. Then by [13, Theorem 6.13] (or [32, Example 7.1]), $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Kumjian–Pask Λ-family in $A_R(\mathcal{G}_\Lambda)$. Hence, there exists a homomorphism $\pi_T : \operatorname{KP}_R(\Lambda) \to A_R(\mathcal{G}_\Lambda)$ such that $\pi_T(s_\lambda) = T_\lambda$ and $\pi_T(s_{\mu^*}) = T_{\mu^*}$ for $\lambda, \mu \in \Lambda$ by Theorem 3.7(a).

To see that π_T is injective, first we show that π_T is graded. Take $\lambda, \mu \in \Lambda$. Then $s_\lambda s_{\mu^*} \in \operatorname{KP}_R(\Lambda)_{d(\lambda)-d(\mu)}$ and

$$\pi_T(s_\lambda s_{\mu^*}) = 1_{Z_\Lambda(\lambda *_s \mu)} = 1_{\{(x, d(\lambda)-d(\mu), y) : (\lambda,\mu) \in \Lambda *_s \Lambda, z \in s(\lambda)\partial\Lambda\}} \in A_R(\mathcal{G}_\Lambda)_{d(\lambda)-d(\mu)}.$$

Since for every $n \in \mathbb{Z}^k$, $\operatorname{KP}_R(\Lambda)_n$ is spanned by elements in the form $s_\lambda s_{\mu^*}$ (Theorem 3.7.(c)), then for $n \in \mathbb{Z}^k$, $\pi_T(\operatorname{KP}_R(\Lambda)_n) \subseteq A_R(\mathcal{G}_\Lambda)_n$ and π_T is graded. Since

$\pi_T(rs_v) = r1_{Z_\Lambda(v *_s v)} \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$, and π_T is graded, then by Theorem 4.1, π_T is injective, as required.

Finally we show the surjectivity of π_T. Take $f \in A_R(\mathcal{G}_\Lambda)$. By [9, Lemma 2.2], f can be written as $\sum_{U\in F} a_U 1_U$ where $a_U \in R$, each U is in the form $\bigcup_{i=1}^n Z_\Lambda(\lambda_i *_s \mu_i \setminus G_i)$ for some $n \in \mathbb{N}$, and F is finite set of mutually disjoint elements. Hence, to show $f \in \operatorname{im}(\pi_T)$, it suffices to show

$$1_U \in \operatorname{im}(\pi_T)$$

where $U := \bigcup_{i=1}^n Z_\Lambda(\lambda_i *_s \mu_i \setminus G_i)$ for some $n \in \mathbb{N}$ and collection $\{Z_\Lambda(\lambda_i *_s \mu_i \setminus G_i)\}_{i=1}^n$. By Lemma 5.6, 1_U can be written as the sum of elements in the form $1_{Z_\Lambda(\lambda *_s \mu \setminus G)}$. On the other hand, for $(\lambda, \mu) \in \Lambda *_s \Lambda$ and finite $G \subseteq s(\lambda)\Lambda$, we have

$$\begin{aligned}
T_\lambda\Big(\prod_{\nu\in G}\big(T_{s(\lambda)} - T_\nu T_{\nu^*}\big)\Big)T_{\mu^*} &= 1_{Z_\Lambda(\lambda *_s s(\lambda))}\Big(\prod_{\nu\in G}\big(1_{Z_\Lambda(s(\lambda) *_s s(\lambda))} - 1_{Z_\Lambda(\nu *_s \nu)}\big)\Big)1_{Z_\Lambda(s(\mu) *_s \mu)} \qquad (5.5)\\
&= 1_{Z_\Lambda(\lambda *_s s(\lambda))}\Big(\prod_{\nu\in G}\big(1_{Z_\Lambda(s(\lambda) *_s s(\lambda)\setminus\{\nu\})}\big)\Big)1_{Z_\Lambda(s(\mu) *_s \mu)}\\
&= 1_{Z_\Lambda(\lambda *_s s(\lambda))}\big(1_{\prod_{\nu\in G} Z_\Lambda(s(\lambda) *_s s(\lambda)\setminus\{\nu\})}\big)1_{Z_\Lambda(s(\mu) *_s \mu)}\\
&= 1_{Z_\Lambda(\lambda *_s s(\lambda))}\big(1_{Z_\Lambda(s(\lambda) *_s s(\lambda)\setminus G)}\big)1_{Z_\Lambda(s(\mu) *_s \mu)}\\
&= 1_{Z_\Lambda(\lambda *_s \mu \setminus G)}
\end{aligned}$$

since $s(\lambda) = s(\mu)$. Hence, $1_{Z_\Lambda(\lambda *_s \mu \setminus G)}$ belongs to $\operatorname{im}(\pi_T)$ and then so does 1_U, as required. Therefore, π_T is surjective and then is an isomorphism. □

Remark 5.7. Finitely aligned k-graphs include 1-graphs and row-finite k-graphs with no sources. Further, in these cases, the boundary path groupoid $\mathcal{G}_\Lambda$ of Example 5.2 coincides with $\mathcal{G}_E$ of [12] and $\mathcal{G}_\Lambda$ of [10]. Thus, we have generalised Example 3.2 of [12] and Proposition 4.3 of [10]. For locally convex row-finite k-graphs, our construction gives a Steinberg algebra model of the Kumjian–Pask algebras of [11].

6. Aperiodic higher-rank graphs and effective groupoids

In this section and Section 7, we investigate the relationship between a k-graph Λ and its boundary-path groupoid $\mathcal{G}_\Lambda$ as constructed in Example 5.2. We expect the Cuntz–Krieger uniqueness theorem (Theorem 8.1) to apply only to *aperiodic* finitely aligned k-graphs (definition below). On the other hand, *effective* groupoids (definition below) are needed in the hypothesis of the Cuntz–Krieger uniqueness theorem for Steinberg algebras (Theorem 8.2). In this section, our main result is Proposition 6.3 which says that a finitely aligned k-graph Λ is aperiodic if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is effective.

We say a boundary path x is *aperiodic* if for all $\lambda, \mu \in \Lambda r(x)$, $\lambda \neq \mu$ implies $\lambda x \neq \mu x$. We say a finitely aligned k-graph Λ is *aperiodic* if for each $v \in \Lambda^0$, there exists an aperiodic boundary path x with $r(x) = v$.

Remark 6.1. There are several equivalent ways to define the aperiodicity condition for finitely aligned k-graphs (see [13,18,22,26]). However, those definitions are equivalent by [18, Proposition 3.6] and [26, Proposition 2.11]. The definition we use is called Condition (B′) in [13, Remark 7.3] and [26, Definition 2.1.(ii)].

Remark 6.2. For 1-graphs, the aperiodicity condition is known as Condition (L), which, using our conventions, says that every cycle has an entry (see [1,3,7,16,23,29,30]).

Next let $\mathcal{G}$ be a topological groupoid. Define $\mathrm{Iso}(\mathcal{G})$ the *isotropy groupoid* of $\mathcal{G}$ by

$$\mathrm{Iso}(\mathcal{G}) := \{a \in \mathcal{G} : s(a) = r(a)\}.$$

We then say $\mathcal{G}$ is *effective* if the interior of $\mathrm{Iso}(\mathcal{G})$ is $\mathcal{G}^{(0)}$. See [8, Lemma 3.1] for some equivalent characterisations.

Proposition 6.3. *Let Λ be a finitely aligned k-graph. Then Λ is aperiodic if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is effective.*

Proof. ($\Rightarrow$) First suppose that Λ is aperiodic. We trivially have $\mathcal{G}_\Lambda^{(0)}$ belongs to the interior of $\mathrm{Iso}(\mathcal{G}_\Lambda)$. Now we show the reverse inclusion. Take a an interior point of $\mathrm{Iso}(\mathcal{G}_\Lambda)$. Then there exists $Z_\Lambda(\lambda *_s \mu \backslash G)$ such that $Z_\Lambda(\lambda *_s \mu \backslash G) \subseteq \mathrm{Iso}(\mathcal{G}_\Lambda)$ and $a \in Z_\Lambda(\lambda *_s \mu \backslash G)$. We show $\lambda = \mu$.

Note that since $a \in Z_\Lambda(\lambda *_s \mu \backslash G)$, then $Z_\Lambda(\lambda *_s \mu \backslash G)$ is not empty and by Remark 5.3.(ii), G is not exhaustive. Hence, there exists $\nu \in s(\lambda)\Lambda$ such that $\Lambda^{\min}(\nu, \gamma) = \emptyset$ for $\gamma \in G$. Because Λ is aperiodic, there exists an aperiodic boundary path $x \in s(\nu)\partial\Lambda$.

We claim that the boundary path νx is also aperiodic. Suppose for contradiction that there exists $\lambda', \mu' \in \Lambda r(\nu x)$ such that $\lambda' \neq \mu'$ and

$$\lambda'(\nu x) = \mu'(\nu x). \tag{6.1}$$

Since $\lambda', \mu', \nu \in \Lambda$, by the unique factorisation property we have $\lambda' \neq \mu'$ implies $\lambda'\nu \neq \mu'\nu$. Now because x is aperiodic, $\lambda'\nu \neq \mu'\nu$ implies $\lambda'\nu(x) \neq \mu'\nu(x)$, which contradicts (6.1). Hence, νx is aperiodic, as claimed.

Since $\lambda\nu x \in Z_\Lambda(\lambda) \backslash Z_\Lambda(\lambda\gamma)$ and $\mu\nu x \in Z_\Lambda(\mu) \backslash Z_\Lambda(\mu\gamma)$ for $\gamma \in G$, we have

$$(\lambda\nu x, d(\lambda) - d(\mu), \mu\nu x) \in Z_\Lambda(\lambda *_s \mu \backslash G).$$

Thus $Z_\Lambda(\lambda *_s \mu \backslash G) \subseteq \mathrm{Iso}(\mathcal{G}_\Lambda)$, and hence $\lambda\nu x = \mu\nu x$. Since νx is aperiodic, we have $\lambda(\nu x) = \mu(\nu x)$ which implies $\lambda = \mu$. Therefore, $\mathcal{G}_\Lambda$ is effective.

($\Leftarrow$) Now suppose that Λ is not aperiodic. Then there exists $v \in \Lambda^0$ such that for all boundary path $x \in v\partial\Lambda$, x is not aperiodic.

Claim 6.4. *For $x \in v\partial\Lambda$, we have $x\mathcal{G}_\Lambda x \neq \{x\}$.*

Proof of Claim 6.4. Take $x \in v\partial\Lambda$. Since x is not aperiodic, then there exist $\lambda, \mu \in \Lambda r(x)$ such that $\lambda \neq \mu$ and $\lambda x = \mu x$. If $d(\lambda) = d(\mu)$, then

$$\lambda = (\lambda x)(0, d(\lambda)) = (\mu x)(0, d(\mu)) = \mu,$$

which contradicts with $\lambda \neq \mu$.

So suppose $d(\lambda) \neq d(\mu)$. Note that for $1 \leq i \leq k$ such that $d(\lambda)_i \neq d(\mu)_i$, we have $d(x)_i = \infty$ (since $\lambda x = \mu x$). Hence

$$((d(\lambda) \vee d(\mu)) - d(\lambda)) \vee ((d(\lambda) \vee d(\mu)) - d(\mu)) \leq d(x).$$

Write $p := (d(\lambda) \vee d(\mu)) - d(\lambda)$ and $q := (d(\lambda) \vee d(\mu)) - d(\mu)$. Then

$$\begin{aligned} \sigma^p x = \sigma^p \left(\sigma^{d(\lambda)}(\lambda x)\right) &= \sigma^{d(\lambda)\vee d(\mu)}(\lambda x) \\ &= \sigma^{d(\lambda)\vee d(\mu)}(\mu x) \ \text{ (since } \lambda x = \mu x) \\ &= \sigma^q \left(\sigma^{d(\mu)}(\mu x)\right) = \sigma^q x \end{aligned}$$

and $p \neq q$ (since $d(\lambda) \neq d(\mu)$). This implies $(x, p - q, x) \in \mathcal{G}_\Lambda \setminus \mathcal{G}_\Lambda^{(0)}$ and $x\mathcal{G}_\Lambda x \neq \{x\}$. □ Claim 6.4

Since $x\mathcal{G}_\Lambda x \neq \{x\}$ for all $x \in v\partial\Lambda$, then

$$Z_\Lambda(v) \cap \{z \in \mathcal{G}_\Lambda^{(0)} : z\mathcal{G}_\Lambda z = \{z\}\} = \emptyset$$

and $\{z \in \mathcal{G}_\Lambda^{(0)} : z\mathcal{G}_\Lambda z = \{z\}\}$ is not dense in $\mathcal{G}_\Lambda^{(0)}$. Since $\mathcal{G}_\Lambda$ is locally compact, second-countable, Hausdorff and étale, then by [24, Proposition 3.6.(b)], $\mathcal{G}_\Lambda$ is not effective, as required. □

Remark 6.5. In fact, for a finitely aligned k-graph Λ, the following five conditions are equivalent:

(a) $\mathcal{G}_\Lambda$ is effective.
(b) $\mathcal{G}_\Lambda$ is *topologically principal* in that the set of units with trivial isotropy is dense in $\mathcal{G}^{(0)}$.
(c) $\mathcal{G}_\Lambda$ satisfies Condition (1) of Theorem 5.1 of [25].
(d) Λ has *no local periodicity* as defined in [26].
(e) Λ is aperiodic.

In [24, Proposition 3.6], Renault shows that for a locally compact, second-countable, Hausdorff, étale $\mathcal{G}$, $\mathcal{G}$ is effective if and only if it is topologically principle. Since the boundary-path groupoid $\mathcal{G}_\Lambda$ is locally compact, second-countable, Hausdorff and étale, then (a)$\Leftrightarrow$(b). Meanwhile, in [32, Theorem 5.2], Yeend proves (b)$\Leftrightarrow$(c). [Note that Yeend uses notion "*essentially free*" instead of "topologically principal".] Lemma 5.6 of [25] gives (c)$\Leftrightarrow$(d). Finally, (d)$\Leftrightarrow$(e) follows from [26, Proposition 2.11].

7. Cofinal higher-rank graphs and minimal groupoids

In this section, we show that a finitely aligned k-graph Λ is cofinal if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is minimal (Proposition 7.1). Later, we use this relationship to study the simplicity of Kumjian–Pask algebras in Section 9.

Recall from [27, Definition 8.4] that we say a k-graph Λ is *cofinal* if for all $v \in \Lambda^0$ and $x \in \partial\Lambda$, there exists $n \leq d(x)$ such that $v\Lambda x(n) \neq \emptyset$.

In a groupoid $\mathcal{G}$, a subset $U \subseteq \mathcal{G}^{(0)}$ is called *invariant* if $s(a) \in U$ implies $r(a) \in U$ for all $a \in \mathcal{G}$. Note that U is invariant if and only if $\mathcal{G}^{(0)} \setminus U$ is invariant. We then say a topological groupoid $\mathcal{G}$ is *minimal* if $\mathcal{G}^{(0)}$ has no nontrivial open invariant subsets. Equivalently, $\mathcal{G}$ is minimal if for each $x \in \mathcal{G}^{(0)}$, the orbit $[x] := s(x\mathcal{G})$ is dense in $\mathcal{G}^{(0)}$.

Proposition 7.1. *Let Λ be a finitely aligned k-graph. Then Λ is cofinal if and only if the boundary-path groupoid $\mathcal{G}_\Lambda$ is minimal.*

Proof. ($\Rightarrow$) Suppose that Λ is cofinal. Take $x \in \mathcal{G}_\Lambda^{(0)}$. We have to show that $[x]$ is dense in $\mathcal{G}_\Lambda^{(0)}$. Take a non-empty open set $Z_\Lambda(\lambda \setminus G)$ and we claim that $Z_\Lambda(\lambda \setminus G) \cap [x] \neq \emptyset$. Since $Z_\Lambda(\lambda \setminus G)$ is non-empty, we have that G is not exhaustive (see Remark 5.3.(i)). Then there exists $\nu \in s(\lambda)\Lambda$ such that $\Lambda^{\min}(\nu, \gamma) = \emptyset$ for $\gamma \in G$. Now consider the vertex $s(\lambda\nu)$ and the boundary path x. Since Λ is cofinal, then there exists $n \leq d(x)$ such that $s(\lambda\nu)\Lambda x(n) \neq \emptyset$. Take $\mu \in s(\lambda\nu)\Lambda x(n)$. Because x is a boundary path, so is $\sigma^n x$. Hence,

$$y := \lambda\nu\mu[\sigma^n x]$$

is also a boundary path. It is clear that $y \in Z_\Lambda(\lambda)$ and since $\Lambda^{\min}(\nu, \gamma) = \emptyset$ for $\gamma \in G$, we have $y \notin Z_\Lambda(\lambda\gamma)$ for $\gamma \in G$. Hence, $y \in Z_\Lambda(\lambda \setminus G)$.

On the other hand, since $y = \lambda\nu\mu[\sigma^n x]$, then $(x, n - d(\lambda\nu\mu), y) \in \mathcal{G}_\Lambda$ and $y \in [x]$. Therefore, $Z_\Lambda(\lambda \setminus G) \cap [x] \neq \emptyset$. Thus, $[x]$ is dense in $\mathcal{G}_\Lambda^{(0)}$ and $\mathcal{G}_\Lambda$ is minimal.

($\Leftarrow$) Suppose that Λ is not cofinal. Then there exist $v \in \Lambda^0$ and $x \in \partial\Lambda$ such that for all $n \leq d(x)$, we have $v\Lambda x(n) = \emptyset$. We claim $Z_\Lambda(v) \cap [x] = \emptyset$. Suppose for contradiction that $Z_\Lambda(v) \cap [x] \neq \emptyset$. Take $y \in Z_\Lambda(v) \cap [x]$. Because $y \in [x]$, then there exist $p, q \in \mathbb{N}^k$ such that $(x, p - q, y) \in \mathcal{G}_\Lambda$. This implies $\sigma^p x = \sigma^q y$. Since $y \in Z_\Lambda(v)$, then $r(y) = v$. Hence, $\sigma^p x = \sigma^q y$ and $r(y) = v$ imply that $y(0, q)$ belongs to $v\Lambda x(p)$, which contradicts with $v\Lambda x(n) = \emptyset$ for all $n \leq d(x)$. Therefore, $Z_\Lambda(v) \cap [x] = \emptyset$, as claimed, and $[x]$ is not dense in $\mathcal{G}_\Lambda^{(0)}$. Thus, $\mathcal{G}_\Lambda$ is not minimal. $\square$

8. The Cuntz–Krieger uniqueness theorem

Throughout this section, Λ is a finitely aligned k-graph and R is a commutative ring with identity 1.

Theorem 8.1 *(The Cuntz–Krieger uniqueness theorem). Let Λ be an aperiodic finitely aligned k-graph, R be a commutative ring with 1. Suppose that $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$. Then π is injective.*

We show Theorem 8.1 by using the Cuntz–Krieger uniqueness theorem for Steinberg algebras [9, Theorem 3.2]. First we verify an alternate formulation of the Cuntz–Krieger uniqueness theorem for Steinberg algebras that will be useful.

Theorem 8.2. *Let $\mathcal{G}$ be an effective, Hausdorff, ample groupoid, and R be a commutative ring with 1. Let $\mathcal{B}$ be a basis of compact open bisection for the topology on $\mathcal{G}$. Let $\phi : A_R(\mathcal{G}) \to A$ be a ring homomorphism. Suppose that $\ker(\phi) \neq 0$. Then there exist $r \in R\setminus\{0\}$ and $B \in \mathcal{B}$ such that $B \subseteq \mathcal{G}^{(0)}$ and $\phi(r1_B) = 0$.*

Proof. Since $\ker(\phi) \neq 0$, then by [9, Theorem 3.2], there exist $r \in R\setminus\{0\}$ and a non-empty compact open subset $K \subseteq \mathcal{G}^{(0)}$ such that $\phi(r1_K) = 0$. Since K is open, then there is $B \in \mathcal{B}$ such that $B \subseteq K$. Hence, $B \subseteq \mathcal{G}^{(0)}$ and

$$0 = \phi(r1_K)\,\phi(1_B) = \phi(r1_{KB}) = \phi(r1_{K\cap B}) = \phi(r1_B). \qquad \square$$

Proof of Theorem 8.1. First note that $\mathcal{G}_\Lambda$ is a Hausdorff and ample groupoid that is effective by Proposition 6.3. Thus it satisfies the hypothesis of Theorem 8.2. Now recall the isomorphism $\pi_T : \mathrm{KP}_R(\Lambda) \to A_R(\mathcal{G}_\Lambda)$ as in Proposition 5.4. Then $\pi_T(s_\lambda) = 1_{Z_\Lambda(\lambda *_s s(\lambda))}$ and $\pi_T(s_{\mu^*}) = 1_{Z_\Lambda(s(\mu) *_s \mu)}$ for $\lambda, \mu \in \Lambda$. Define $\phi := \pi \circ \pi_T^{-1}$. To show the injectivity of π, it suffices to show that ϕ is injective. Suppose for contradiction that ϕ is not injective. By Theorem 8.2, there exist $r \in R\setminus\{0\}$ and $Z_\Lambda(\lambda\setminus G)$ such that $\phi\big(r1_{Z_\Lambda(\lambda\setminus G)}\big) = 0$. Since $1_{Z_\Lambda(\lambda\setminus G)}$ can be identified as $1_{Z_\Lambda(\lambda *_s \lambda\setminus G)}$ (Remark 5.3.(i)), then by following the argument of (5.5), we get

$$\phi\big(r1_{Z_\Lambda(\lambda\setminus G)}\big) = \pi\Big(rs_\lambda\Big(\prod_{\nu\in G}\big(s_{s(\lambda)} - s_\nu s_{\nu^*}\big)\Big)s_{\lambda^*}\Big)$$

and then

$$\pi\Big(rs_\lambda\Big(\prod_{\nu\in G}\big(s_{s(\lambda)} - s_\nu s_{\nu^*}\big)\Big)s_{\lambda^*}\Big) = 0. \tag{8.1}$$

On the other hand, since $\pi(rs_v) \neq 0$ for all $r \in R\setminus\{0\}$ and $v \in \Lambda^0$, and G is finite non-exhaustive, then by Proposition 3.3.(d),

$$\pi\big(rs_\lambda\big(\prod_{\nu\in G}\big(s_{s(\lambda)}-s_\nu s_{\nu^*}\big)\big)s_{\lambda^*}\big)\neq 0,$$

which contradicts (8.1). The conclusion follows. □

One application of Theorem 8.1 is:

Corollary 8.3. *Let Λ be finitely aligned k-graph and R be a commutative ring with 1. Then Λ is aperiodic if and only if the boundary-path representation $\pi_S : \mathrm{KP}_R(\Lambda) \to \mathrm{End}(\mathbb{F}_R(\partial\Lambda))$ is injective.*

To show Corollary 8.3, we establish some results and notation.

Following [26, Definition 2.3], for a finitely aligned k-graph Λ, we say Λ has *no local periodicity* if for every $v \in \Lambda^0$ and every $n \neq m \in \mathbb{N}^k$, there exists $x \in v\partial\Lambda$ such that either $d(x) \not\geq n \vee m$ or $\sigma^n x \neq \sigma^m x$. If no local aperiodicity fails at $v \in \Lambda^0$, then there are $n \neq m \in \mathbb{N}^k$ such that $\sigma^n x = \sigma^m x$ for all $x \in v\partial\Lambda$. In this case, we say Λ has *local periodicity n, m at $v \in \Lambda^0$*.

Lemma 8.4 *([26, Lemma 2.9]). Let Λ be a finitely aligned k-graph which has local periodicity n, m at $v \in \Lambda^0$. Then $d(x) \geq n \vee m$ and $\sigma^n x = \sigma^m x$ for every $x \in v\partial\Lambda$. Fix $x \in v\partial\Lambda$ and set $\mu = x(0, m)$, $\alpha = x(m, m \vee n)$, and $\nu = (\mu\alpha)(0, n)$. Then $\mu\alpha y = \nu\alpha y$ for every $y \in s(\alpha)\partial\Lambda$.*

Proof of Corollary 8.3. ($\Rightarrow$) Suppose that Λ is aperiodic. By Proposition 3.6, we have $\pi_S(rs_v) \neq 0$ for all $r \in R\backslash\{0\}$ and $v \in \Lambda^0$. Since Λ is aperiodic, then by Theorem 8.1, π_S is injective.

($\Leftarrow$) Suppose that Λ is not aperiodic. We are following the argument of [5, Lemma 5.9]. Since Λ is not aperiodic, by [26, Proposition 2.11], there exist $v \in \Lambda^0$ and $n \neq m \in \mathbb{N}^k$ such that Λ has local periodicity n, m at $v \in \Lambda^0$. Let μ, ν, α be as in Lemma 8.4 and define $a := s_{\mu\alpha}s_{(\mu\alpha)^*} - s_{\nu\alpha}s_{(\mu\alpha)^*}$. We claim that $a \in \ker(\pi_S)\backslash\{0\}$.

First we show that $a \neq 0$. Suppose for contradiction that $a = 0$. Then $s_{\mu\alpha}s_{(\mu\alpha)^*} = s_{\nu\alpha}s_{(\mu\alpha)^*}$. Note that $d\left(s_{\mu\alpha}s_{(\mu\alpha)^*}\right) = d(\mu\alpha) - d(\mu\alpha) = 0$ and

$$d\left(s_{\nu\alpha}s_{(\mu\alpha)^*}\right) = d(\nu\alpha) - d(\mu\alpha) = d(\nu) + d(\alpha) - d(\mu) - d(\alpha) = n - m \neq 0.$$

Hence $s_{\mu\alpha}s_{(\mu\alpha)^*} = s_{\nu\alpha}s_{(\mu\alpha)^*} = 0$. Thus, $0 = s_{(\mu\alpha)^*}\left(s_{\mu\alpha}s_{(\mu\alpha)^*}\right)s_{\mu\alpha} = s^2_{s(\mu\alpha)} = s_{s(\mu\alpha)}$, which contradicts Theorem 3.7.(b). Hence $a \neq 0$.

Now we show that $a \in \ker(\pi_S)$. Take $y \in \partial\Lambda$, and it suffices to show $\pi_S(a)(y) = 0$. Recall that $\pi_S(s_\lambda) = S_\lambda$ and $\pi_S(s_{\mu^*}) = S_{\mu^*}$ where

$$S_\lambda(y) = \begin{cases} \lambda y & \text{if } s(\lambda) = r(y); \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } S_{\mu^*}(y) = \begin{cases} \sigma^{d(\mu)}y & \text{if } y(0, d(\mu)) = \mu; \\ 0 & \text{otherwise.} \end{cases}$$

First suppose that $y(0, d(\mu\alpha)) \neq \mu\alpha$. Then $S_{(\mu\alpha)^*}(y) = 0$ and $\pi_S(a)(y) = S_{\mu\alpha}S_{(\mu\alpha)^*}(y) - S_{\nu\alpha}S_{(\mu\alpha)^*}(y) = 0$. Next suppose that $y(0, d(\mu\alpha)) = \mu\alpha$. Then

$$\pi_S(a)(y) = (S_{\mu\alpha} - S_{\nu\alpha})\left(\sigma^{d(\mu\alpha)}y\right).$$

Since $y \in \partial\Lambda$, then $\sigma^{d(\mu\alpha)}y \in s(\alpha)\partial\Lambda$ and by Lemma 8.4, $\mu\alpha\left(\sigma^{d(\mu\alpha)}y\right) = \nu\alpha\left(\sigma^{d(\mu\alpha)}y\right)$ and hence $\pi_S(a)(y) = 0$. Thus, $a \in \ker(\pi_S)\setminus\{0\}$, as claimed, and π_S is not injective. □

9. Basic simplicity and simplicity

As in [30], we say an ideal I in $\mathrm{KP}_R(\Lambda)$ is *basic* if whenever $r \in R\setminus\{0\}$ and $v \in \Lambda^0$, we have $rs_v \in I$ implies $s_v \in I$. We also say that $\mathrm{KP}_R(\Lambda)$ is *basically simple* if its only basic ideals are $\{0\}$ and $\mathrm{KP}_R(\Lambda)$.

In this section, we investigate necessary and sufficient conditions for $\mathrm{KP}_R(\Lambda)$ to be basically simple (Theorem 9.3) and to be simple (Theorem 9.4). We show that both results can be viewed as consequences of basic simplicity and simplicity characterisations of Steinberg algebras. Therefore, we state necessary and sufficient conditions for the Steinberg algebra $A_R(\mathcal{G})$ to be basically simple and to be simple in the following two theorems.

Theorem 9.1 *([9, Theorem 4.1]). Let $\mathcal{G}$ be a Hausdorff, ample groupoid and R be a commutative ring with 1. Then $A_R(\mathcal{G})$ is basically simple if and only if $\mathcal{G}$ is effective and minimal.*

Theorem 9.2 *([9, Corollary 4.6]). Let $\mathcal{G}$ be a Hausdorff, ample groupoid and R be a commutative ring with 1. Then $A_R(\mathcal{G})$ is simple if and only if R is a field and $\mathcal{G}$ is effective and minimal.*

Now we are ready to prove our results in this section.

Theorem 9.3. *Let Λ be a finitely aligned k-graph and let R be a commutative ring with 1. Then $\mathrm{KP}_R(\Lambda)$ is basically simple if and only if Λ is aperiodic and cofinal.*

Proof. ($\Rightarrow$) First suppose that $\mathrm{KP}_R(\Lambda)$ is basically simple. By Proposition 5.4, $A_R(\mathcal{G}_\Lambda)$ is also basically simple and then by Theorem 9.1, $\mathcal{G}_\Lambda$ is effective and minimal. On the other hand, $\mathcal{G}_\Lambda$ is effective implies that Λ is aperiodic (Proposition 6.3), and $\mathcal{G}_\Lambda$ is minimal implies that Λ is cofinal (Proposition 7.1). The conclusion follows.

($\Leftarrow$) Next suppose that Λ is aperiodic and cofinal. By Proposition 6.3 and Proposition 7.1, $\mathcal{G}_\Lambda$ is effective and minimal and then by Theorem 9.1, $A_R(\mathcal{G}_\Lambda)$ is basically simple. Since $A_R(\mathcal{G}_\Lambda)$ is isomorphic to $\mathrm{KP}_R(\Lambda)$ (Proposition 5.4), then $\mathrm{KP}_R(\Lambda)$ is also basically simple, as required. □

Theorem 9.4. *Let Λ be a finitely aligned k-graph and let R be a commutative ring with 1. Then $\mathrm{KP}_R(\Lambda)$ is simple if and only if R is a field and Λ is aperiodic and cofinal.*

Proof. ($\Rightarrow$) First suppose that $\mathrm{KP}_R(\Lambda)$ is simple. Then $\mathrm{KP}_R(\Lambda)$ is also basically simple and Theorem 9.3 implies that Λ is aperiodic and cofinal. On the other hand, since $\mathrm{KP}_R(\Lambda)$ is simple, then by Proposition 5.4, $A_R(\mathcal{G}_\Lambda)$ is also simple and by Theorem 9.2, R is a field, as required.

($\Leftarrow$) Next suppose that R is a field and Λ is aperiodic and cofinal. By Proposition 6.3 and Proposition 7.1, $\mathcal{G}_\Lambda$ is effective and minimal. Hence, by Theorem 9.2, $A_R(\mathcal{G}_\Lambda)$ is simple and by Proposition 5.4, so is $\mathrm{KP}_R(\Lambda)$. □

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