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On a class of non-solvable groups *

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Abstract

In this paper, we use the properties of subgroups with given order to study the structure of finite groups. The main result is as follows:

Let G be a group and P be a Sylow p -subgroup of G . Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is \mathcal{M} -supplemented in G , then every non-abelian pd - G -chief factor A/B satisfies one of the following conditions:

- (1) $A/B \cong PSL(2, 7)$ and $p = 7$; $A/B \cong PSL(2, 11)$ and $p = 11$;
- (2) $A/B \cong PSL(2, 2^t)$ and $p = 2^t + 1 > 3$ is a Fermat prime;
- (3) $A/B \cong PSL(n, q)$, $n \geq 3$ is a prime, $(n, q - 1) = 1$ and $p = q^n - 1/q - 1$;
- (4) $A/B \cong M_{11}$ and $p = 11$; $A/B \cong M_{23}$ and $p = 23$;
- (5) $A/B \cong A_p$ and $p \geq 5$.

AMS classification: 20D10, 20D20

Keywords: \mathcal{M} -supplemented subgroups, chief factor, composition factor, simple groups

1 INTRODUCTION

All groups considered in this paper will be finite. We shall adhere to the notation employed in [5,9]. In particular, let π be a set of primes, π' be the

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complement in the set of all primes, $|G|$ denote the order of a group G , $\pi(G)$ denote the set of all prime divisors of $|G|$, and let $|G|_\pi$ (G_π) denote the π -part of $|G|$ (a Hall π -subgroup of G). Let S_n be the symmetric group of degree n and A_n be the alternating group of degree n . Let H_G be the core of H in G when $H \leq G$ and $M < G$ denote M is a maximal subgroup of G . Further, assume that H/K is a composition factor (chief factor) of G . If $p \in \pi(H/K)$, then H/K is called a pd -composition factor (chief factor) of G .

As we all know, composition series and chief series are basic concepts in the Theory of Finite Groups. In the literature, there are many notions related to them, for example, (p) -solvability, (p) -supersolvability, and so on. In 1937, Hall [8] proved that G is solvable if and only if each Sylow subgroup of G is complemented in G . Based on Hall's work, the relationship between the structure of a group and complementability has been considered by some authors [1, 2, 7, 14]. Recently, Monakhov and Kniahina [12] investigated the structure of composition factors by considering complementability of subgroups with prime order.

In view of the concept of the minimal supplement of a group, in 2009, Miao and Lempken [10] introduced the notion of \mathcal{M} -supplemented subgroups.

Definition 1.1 [10, Definition 1.1] A subgroup H of G is called \mathcal{M} -supplemented in G , if there exists a subgroup B of G such that $G = HB$ and $H_i B < G$ for every maximal subgroup H_i of H .

Clearly, every subgroup with prime order is complemented in G if and only if it is \mathcal{M} -supplemented in G . But, for some p -groups with certain order, this is not true.

Example 1.2 Consider the group $G = A_5 \times A_5$ and P is a Sylow 5-subgroup of G . Clearly, P is complemented in G . But it is not \mathcal{M} -supplemented in G .

Example 1.3 Consider the group $G = S_5$ and $H = \langle (1234) \rangle$. Clearly, H is not complemented in G since there is no subgroup of order 30 in G . But it is \mathcal{M} -supplemented in G since A_5 is an \mathcal{M} -supplement to H in G .

In 1980, when the classification of finite simple groups was almost completed, Wielandt proposed giving priority to the extension of these brilliant results of the theory of finite solvable groups to the more ambitious universe of all finite groups. Along this clue and previous works, we will investigate the structure of finite non-solvable groups by considering \mathcal{M} -supplemented subgroups with given order.

Main Theorem Let G be a group and P be a Sylow p -subgroup of G . Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is \mathcal{M} -supplemented in G , then every non-abelian pd - G -chief factor A/B satisfies one of the following conditions:

- (1) $A/B \cong PSL(2, 7)$ and $p = 7$; $A/B \cong PSL(2, 11)$ and $p = 11$;

- (2) $A/B \cong PSL(2, 2^t)$ and $p = 2^t + 1 > 3$ is a Fermat prime;
- (3) $A/B \cong PSL(n, q)$, $n \geq 3$ is a prime, $(n, q-1) = 1$ and $p = q^n - 1/q - 1$;
- (4) $A/B \cong M_{11}$ and $p = 11$; $A/B \cong M_{23}$ and $p = 23$;
- (5) $A/B \cong A_p$ and $p \geq 5$.

2 PRELIMINARIES

For the sake of convenience, we first list here some known results which will be useful in this paper.

Lemma 2.1 [10, Lemma 2.1] *Let G be a group. Then*

- (1) *If $H \leq M \leq G$ and H is \mathcal{M} -supplemented in G , then H is \mathcal{M} -supplemented in M .*
- (2) *Let $N \trianglelefteq G$ and $N \leq H$. If H is \mathcal{M} -supplemented in G , then H/N is \mathcal{M} -supplemented in G/N .*
- (3) *Let π be a set of primes. Let N be a normal π' -subgroup and H be a π -subgroup of G . If H is \mathcal{M} -supplemented in G , then HN/N is \mathcal{M} -supplemented in G/N .*

Lemma 2.2 [10, Lemma 2.2] *Let $p \in \pi(G)$, P be a p -subgroup of G having an \mathcal{M} -supplement B in G . Then*

- (1) *$P \cap B = P_1 \cap B = \Phi(P) \cap B$ and $|G : P_1 B| = p$ for any maximal subgroup P_1 of P .*
- (2) *If L is a minimal normal subgroup of G contained in P , then $|L| = p$ or $L \leq \Phi(P)$.*

Lemma 2.3[12, Theorem 1(2)] *Let G be not a p -solvable group and $p \in \pi(G)$. If every subgroup of order p is complemented in G , then every non-abelian pd -composition factor A/B of G is isomorphic to one of the following groups:*

- (1) $A/B \cong PSL(2, 7)$ and $p = 7$; $A/B \cong PSL(2, 11)$ and $p = 11$;
- (2) $A/B \cong PSL(2, 2^t)$ and $p = 2^t + 1 > 3$ is a Fermat prime;
- (3) $A/B \cong PSL(n, q)$, $n \geq 3$ is a prime, $(n, q-1) = 1$ and $p = q^n - 1/q - 1$;
- (4) $A/B \cong M_{11}$ and $p = 11$; $A/B \cong M_{23}$ and $p = 23$;
- (5) $A/B \cong A_p$ and $p \geq 5$.

Lemma 2.4 [15, Lemma 4] *If P is a Sylow p -subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p -nilpotent.*

Lemma 2.5 *Let G be a group and $\pi(G) = \{p_1, p_2, \dots, p_n\}$, $p_1 < p_2 < \dots < p_n$. Suppose that P is a Sylow p -subgroup of G , $p \in \pi(G)$, $1 < d \leq |P|$. Suppose that every subgroup H of P with $|H| = d$ is \mathcal{M} -supplemented in G .*

(1) *If G is p -solvable, then G is p -supersolvable.*

(2) *If $p \in \{p_1, p_2\}$, then G is p -supersolvable.*

Proof. (1) See [11, Theorem 3.1] and [16, Theorem 3.3].

(2) See [13, Theorem 1 and Theorem 2] and [16, Theorem 3.1 and Corollary 3.4].

Lemma 2.6 *Let G be a group and P be a Sylow p -subgroup of G . If P is \mathcal{M} -supplemented in G , then every G -chief factor A/B satisfies one of the following conditions:*

- (1) $A/B \leq \Phi(G/B)$; (2) A/B is a p' -group; (3) $|A/B|_p = p$.

Proof. Assume that the assertion is false and choose G to be a counterexample of minimal order.

Since P is \mathcal{M} -supplemented in G , we may choose a subgroup K such that $G = PK$ and $P_i K < G$ for every maximal subgroup P_i of P . If G is simple, then $G \hookrightarrow S_p$ and the claim holds for G by the structure of S_p , the symmetric group of degree p , a contradiction. Hence $G/\cap (P_i K)_G \hookrightarrow \times S_p$ and we set $T = \cap (P_i K)_G$. We assert that $T \neq 1$. If not, then the claim holds for G , a contradiction. By Lemma 2.4, T is p -nilpotent. If $T_{p'} \neq 1$, then the claim holds for $G/T_{p'}$ and for G , a contradiction. Hence T is a p -subgroup and $T \leq \Phi(G)$. Clearly, the claim holds for G/T and for G , a contradiction.

Lemma 2.7 *Let G be a group and P be a Sylow p -subgroup of G . If every minimal subgroup of P is complemented in G , then every G -chief factor A/B satisfies one of the following conditions:*

- (1) A/B is a p' -group; (2) $|A/B|_p = p$.

Proof. Assume that the theorem is false and let G be a counterexample of minimal order.

Let L be a minimal subgroup of P . By the hypothesis, L is complemented in G , so there exists a subgroup K such that $G = LK$ and $L \cap K = 1$. We set $\delta = \{M_i < \cdot G \mid |G : M_i| = p\}$ and $T = \cap (M_i)_G$ where $M_i \in \delta$. We assert that $T \neq 1$. If not, then the claim holds for G by Lemma 2.2, a contradiction. If $T_p \neq 1$, then we may choose a subgroup $L \leq T$ with order p . By the hypothesis, there exists a maximal subgroup M_i where $M_i \in \delta$ such that $G = LM_i = M_i$, a

contradiction. Hence T is a p' -subgroup. Then the claim holds for G/T and for G , a contradiction.

3 MAIN RESULTS

Theorem 3.1 *Let G be a group and P be a Sylow p -subgroup of G , $p \in \pi(G)$. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is \mathcal{M} -supplemented in G , then every non-abelian pd -composition factor A/B of G satisfies one of the following conditions:*

- (1) $A/B \cong PSL(2, 7)$ and $p = 7$; $A/B \cong PSL(2, 11)$ and $p = 11$;
- (2) $A/B \cong PSL(2, 2^t)$ and $p = 2^t + 1 > 3$ is a Fermat prime;
- (3) $A/B \cong PSL(n, q)$, $n \geq 3$ is a prime, $(n, q-1) = 1$ and $p = q^n - 1/q - 1$;
- (4) $A/B \cong M_{11}$ and $p = 11$; $A/B \cong M_{23}$ and $p = 23$;
- (5) $A/B \cong A_p$ and $p \geq 5$.

Proof. Assume that the assertion is false and let G be a counterexample of minimal order. Clearly, G is not an abelian group and $O_{p'}(G) = 1$ by Lemma 2.1. Further,

- (1) G is not a non-abelian simple group.

Otherwise, we pick a subgroup H with $|H| = d$. By the hypothesis, there exists a subgroup K such that $G = HK$ and $H_i K < G$ for every maximal subgroup H_i of H . Since $|G : H_i K| = p$ by Lemma 2.2, $G \hookrightarrow S_p$, the symmetric group of degree p . Then $d = p$ and every non-abelian composition factor A/B of G satisfies one of the conditions in Theorem 3.1 by Lemma 2.3, a contradiction.

- (2) $|L_p| = p$ and $L = L_p L_{p'}$, for every minimal normal subgroup L of G .

We pick a minimal normal subgroup L of G . If $|L_p| \geq d$, then we may choose a subgroup X such that $X \leq L_p$ and $|X| = d$. By the hypothesis, X is \mathcal{M} -supplemented in G , thus there exists a subgroup B such that $G = XB$ and $X_i B < G$ for every maximal subgroup X_i of X . Since $G = L X_i B$ and $G/(X_i B)_G \hookrightarrow S_p$, $L \cong L(X_i B)_G/(X_i B)_G \hookrightarrow S_p$, the symmetric group of degree p . Then $d = p$ and every non-abelian composition factor A/B of G satisfies one of the conditions in Theorem 3.1 by Lemma 2.3, a contradiction. Hence $|L_p| < d$.

Then we may choose a subgroup E such that $L_p < E$ and $|E| = d$. By the hypothesis, E is \mathcal{M} -supplemented in G , so there exists a subgroup S such that $G = ES$ and $E_i S < G$ for every maximal subgroup E_i of E . We consider the subgroup EL . We assert that $L_p \not\leq \Phi(E)$. Otherwise, by Lemma 2.4, L is p -nilpotent. Hence L is a p -subgroup and $L \leq \Phi(G)$. Clearly, every non-abelian composition factor of G/L and hence of G satisfies one of the conditions

in Theorem 3.1 by the choice of G , a contradiction. Then we may pick a maximal subgroup E_1 of E such that $E = L_p E_1$. Similar to the discussion above, $|L_p| = p$. Further, $|G : E_1 S| = |L : L \cap E_1 S| = p$ and $L = L_p L_{p'}$.

(3) The final contradiction.

We pick a minimal normal subgroup L of G and we consider G/L . By (2), for every subgroup T/L of order d/p of PL/L , $T/L = HL/L$ where H is some subgroup of P with $|H| = d$. Since $G = HB$ and $H_i B < G$ for every maximal subgroup H_i of H , $G/L = (T/L)(BL/L)$. For every maximal subgroup T_i/L of T/L , $T_i/L = H_i L/L$ where H_i is some maximal subgroup of H . Then $|(G/L) : (T_i/L)(BL/L)| = |G : T_i BL| = |G : H_i LBL| = p$ and $(T_i/L)(BL/L) < G/L$. Hence G/L satisfies the hypothesis and every non-abelian composition factor of G/L satisfies one of the conditions in Theorem 3.1 by the choice of G . We have reached a contradiction in case L is solvable. If L is non-solvable, then $|L_p| = p$ and so L is a simple group. Application of Lemma 2.3 for the group L now yields a final contradiction.

The final contradiction completes our proof.

Theorem 3.2 *Let G be a group and $r \in \pi(G)$, $\pi = \pi(G) \setminus \{r\}$. Suppose that for every $p \in \pi$ and a fixed Sylow p -subgroup P of G there exists an integer $d(p)$ depending only on p such that $1 < d(p) \leq |P|$ and every subgroup H of P with $|H| = d(p)$ is \mathcal{M} -supplemented in G . Then G is solvable and G is π -supersolvable.*

Proof. Assume that p is the smallest integer in π . By Lemma 2.5, G is p -supersolvable and G has a Hall p' -subgroup K . By Lemma 2.1 and induction on $|G|$, K satisfies the hypothesis of Theorem 3.2 and K is solvable and K is π -supersolvable. Hence G is solvable and G is π -supersolvable.

Theorem 3.3 *Let G be a group and $r, t \in \pi(G)$, $\pi = \pi(G) \setminus \{r, t\}$. Suppose that for every $p \in \pi$ and a fixed Sylow p -subgroup P of G there exists an integer $d(p)$ depending on p such that $1 < d(p) \leq |P|$ and every subgroup H of P with $|H| = d(p)$ is \mathcal{M} -supplemented in G . Then*

(a) G is solvable when $3 \notin \pi(G)$.

(b) Every non-abelian composition factor A/B of G satisfies one of the following conditions:

(1) $A/B \cong \text{PSL}(2, 5)$;

(2) $A/B \cong \text{PSL}(2, 7)$;

(3) $A/B \cong \text{PSL}(3, 3)$.

Proof. (a). By the discussion in (1) of Theorem 3.1 and [12, Theorem 3], G is not a non-abelian simple group. Then we choose a minimal normal subgroup L of

G . If $\pi(L) \cap \pi = \emptyset$, then G/L is solvable by Lemma 2.1(3) and induction on $|G|$, and G is solvable since L is solvable. Now we assume that $\pi(L) \cap \pi \neq \emptyset$. By the discussion in (2) of Theorem 3.1, $|L_p| = p$ and $L = L_p L_{p'}$ for every $p \in \pi(L) \cap \pi$.

If L is a non-abelian simple group, then $|\pi(L)| = 3$ since $PSL(2, 7)$ is the only simple group which contains two subgroups of distinct prime indices. Hence $3 \in \pi(G)$ by [12, Lemma 3], a contradiction.

Now assume that $|L| = p$. If $|L| < d(p)$, then G/L is solvable by Lemma 2.1(2) and induction on $|G|$, and so G is solvable. If $|L| = d(p)$, then $G = L \rtimes M$ by the hypothesis, M is a maximal subgroup of G . By induction on $|G|$, M is solvable and so G is solvable.

(b). Assume that the assertion is false and let G be a counterexample of minimal order. Clearly, G is not an abelian group and $\{r, t\} = \{2, 3\}$ by Lemma 2.5 and (a). Further,

(1) G is not a non-abelian simple group.

Similar to the discussion in (1) of Theorem 3.1, it is proved by [12, Theorem 3].

(2) $|L_p| = p < d(p)$ and $L = L_p L_{p'}$, for every minimal normal subgroup L of G and for every $p \in \pi(L) \cap \pi$.

Similar to the discussion in (2) of Theorem 3.1, $|L_p| = p$ and $L = L_p L_{p'}$, for every minimal normal subgroup L of G and for every $p \in \pi(L) \cap \pi$. If $|L_p| = p = d(p)$, for every $p \in \pi(L) \cap \pi$, then every composition factor A/B of G satisfies one of the conditions in Theorem 3.3, a contradiction. If $|L_p| = p = d(p)$ and $|L_q| = q = d(q)$, for $p \neq q$, $p, q \in \pi(L) \cap \pi$, then $L \cong PSL(2, 7)$ since L is a non-abelian simple group where the Sylow p -subgroup and the Sylow q -subgroup of L are complemented in L . Then it contradicts the choice of p and q . Hence $|L_p| = p < d(p)$, for every $p \in \pi(L) \cap \pi$.

(3) The final contradiction.

Similar to the discussion in (3) of Theorem 3.1.

The final contradiction completes our proof.

Next, we shall study the construction of chief factors of a group. Firstly, we prove the following Theorem which is important to the Main Theorem in this paper.

Theorem 3.4 *Let G be a group and P be a Sylow p -subgroup of G . Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is \mathcal{M} -supplemented in G , then every G -chief factor A/B satisfies one of the following conditions:*

- (1) $A/B \leq \Phi(G/B)$;
- (2) A/B is a p' -group;
- (3) $|A/B|_p = p$.

Proof. Assume that the assertion is false and let G be a counterexample of minimal order.

For every subgroup H of P with $|H| = d$, by the hypothesis, H is \mathcal{M} -supplemented in G , thus there exists a subgroup K_H depending only on H such that $G = HK_H$ and $H_i K_H < G$ for every maximal subgroup H_i of H . Further, $\delta_H := \{H_i K_H \mid H_i < \cdot H\}$ and we set $\delta = \cup \delta_H$, $T = \cap (H_i K_H)_G$ for every H of P with $|H| = d$ and every $H_i K_H \in \delta$. By Lemma 2.2, $G/(H_i K_H)_G \hookrightarrow S_p$ for some $(H_i K_H)_G$ and $G/T \hookrightarrow \times S_p$, S_p is the symmetric group of degree p . We assert that $T \neq 1$. If not, then $G \hookrightarrow \times S_p$ and the claims of the Theorem holds for G by the structure of S_p , a contradiction.

If $|T_p| \geq d$, then we may choose a subgroup X such that $X \leq T_p$ and $|X| = d$. By the hypothesis, X is \mathcal{M} -supplemented in G , so there exists a subgroup B such that $G = XB$ and $X_i B < G$ for every maximal subgroup X_i of X . Then $X_i B \in \delta$ and $G = TX_i B = X_i B < G$, a contradiction.

If $|T_p| < d$, then we may choose a subgroup E such that $T_p < E$ and $|E| = d$. By the hypothesis, E is \mathcal{M} -supplemented in G , thus there exists a subgroup S such that $G = ES$ and $E_i S < G$ for every maximal subgroup E_i of E . We assert that $T_p \leq \Phi(E)$. Otherwise, we may pick a maximal subgroup E_1 of E such that $E = T_p E_1$. Then $E_1 S \in \delta$ and $G = ES = T_p E_1 S = E_1 S < G$, a contradiction. We consider the subgroup ET . By Lemma 2.4, T is p -nilpotent. If $T_p \neq 1$, then the claims of the Theorem holds for G/T_p , and for G , a contradiction. Hence T is a p -subgroup and $T \leq \Phi(G)$. Clearly, every G/T -chief factor satisfies one of the conditions in the Theorem and so the claims of the Theorem hold for G , a contradiction.

The final contradiction completes our proof.

Corollary 3.5 *Let G be a p -solvable group and P be a Sylow p -subgroup of G where p is a prime divisor of $|G|$. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is \mathcal{M} -supplemented in G , then G is p -supersolvable.*

By Theorem 3.1 and Theorem 3.4, we obtain the Main Theorem:

Theorem 3.6 *Let G be a group and P be a Sylow p -subgroup of G . Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is \mathcal{M} -supplemented in G , then every non-abelian pd - G -chief factor A/B satisfies one of the following conditions:*

- (1) $A/B \cong PSL(2, 7)$ and $p = 7$; $A/B \cong PSL(2, 11)$ and $p = 11$;
- (2) $A/B \cong PSL(2, 2^t)$ and $p = 2^t + 1 > 3$ is a Fermat prime;
- (3) $A/B \cong PSL(n, q)$, $n \geq 3$ is a prime, $(n, q-1) = 1$ and $p = q^n - 1/q - 1$;

- (4) $A/B \cong M_{11}$ and $p = 11$; $A/B \cong M_{23}$ and $p = 23$;
 (5) $A/B \cong A_p$ and $p \geq 5$.

4 APPLICATIONS

For a prime p , we define a class \mathcal{F}_1 containing every group G whose every chief factor A/B satisfies one of the following conditions:

- (1) $A/B \leq \Phi(G/B)$; (2) A/B is a p' -group; (3) $|A/B|_p = p$.

Clearly, \mathcal{F}_1 is a class containing all p -supersolvable groups and some non-solvable groups (for example, A_p , the alternating group of degree p . For $p = 5$, $S_6(2)$, $PSL(4, 3)$ et al. For $p = 7$, $PSL(2, 7)$, M_{22} , J_2 et al. And the direct product of them). Clearly, \mathcal{F}_1 is not a formation. Further, let \mathcal{F}_2 denote the class of all groups whose every chief factor A/B satisfies one of (2) or (3) above. Clearly, \mathcal{F}_2 is a subgroup hereditary formation. But \mathcal{F}_2 is not saturated.

In [6], Guo et al introduced the concept of boundary factors of subgroups of finite groups.

Definition 4.1[6, Definition 1.1] *Let A be a proper subgroup of G . Then any chief factor H/A_G of G is called a G -boundary factor or simply boundary factor of A . For any G -boundary factor H/A_G of A , the subgroup $H \cap A/A_G$ of G/A_G is called a G -trace or simply a trace of A .*

Here we give a characterateration on \mathcal{F}_1 by considering the properties of boundary factors.

Theorem 4.2 *Let G be a group. $G \in \mathcal{F}_1$ if and only if there exists a G -boundary factor H/M_G of M for every maximal subgroup M of G satisfying one of the following conditions:*

- (1) H/M_G is a p' -group; (2) $|H/M_G|_p = p$.

Proof. Necessity. For every maximal subgroup M of G , if there exists a G -boundary factor H/M_G of M such that $H/M_G \leq \Phi(G/M_G)$, then $H/M_G \leq M/M_G$ and so $H \leq M_G$, which is impossible by the definition of the G -boundary factor. Further, necessity is easy to be proved since $G \in \mathcal{F}_1$.

Next, we consider the sufficiency. Assume that the assertion is false and let G be a counterexample of minimal order.

- (1) G is not a non-abelian simple group.

Otherwise, by the definition of \mathcal{F}_1 , $G \in \mathcal{F}_1$, a contradiction.

- (2) For every minimal normal subgroup N of G , $G/N \in \mathcal{F}_1$ and $N \neq O_{p'}(G)$.

For every maximal subgroup M/N of G/N , there exists a G -boundary factor $(H/N)/(M/N)_{G/N}$ where $(H/N)/(M/N)_{G/N} = (H/N)/(M_G/N) \cong H/M_G$

satisfies (1) or (2) in Theorem 4.2. Hence $G/N \in \mathcal{F}_1$ by the choice of G and $N \neq O_{p'}(G)$ by the definition of \mathcal{F}_1 .

(3) If $O_p(G) \neq 1$, then $O_p(G) \cap \Phi(G) = 1$.

Assume that $O_p(G) \cap \Phi(G) \neq 1$. We may choose a minimal normal subgroup L of G such that $L \leq O_p(G) \cap \Phi(G)$. By (2), $G/L \in \mathcal{F}_1$ and $G \in \mathcal{F}_1$, a contradiction.

(4) The final contradiction.

We assert that G has the unique minimal normal subgroup N . Otherwise, there exists two different minimal normal subgroups N_1 and N_2 such that $G/N_j \in \mathcal{F}_1$ by (2) and the choice of G where $j = 1, 2$. If $N_1N_2/N_j \leq \Phi(G/N_j)$, then N_1 and N_2 are p -subgroups of G by (2). By [3, A. Lemma 9.11], $N_1N_2 \leq \Phi(G)N_2$. Since $N_1N_2 \leq O_p(G)$, $N_1N_2 \leq O_p(G) \cap \Phi(G)N_2 = (O_p(G) \cap \Phi(G))N_2 = N_2$ by (3), a contradiction. Hence $N_1 \cong N_1N_2/N_2$ satisfies the condition (2) or (3) in \mathcal{F}_1 , then $G \in \mathcal{F}_1$, a contradiction. By (2), $N \not\leq \Phi(G)$ and then there exists a maximal subgroup M of G such that $G = NM$. Further, by the hypothesis, N satisfies the condition (1) or (2) in Theorem 4.2. Hence $G \in \mathcal{F}_1$ by the definition of \mathcal{F}_1 , a contradiction.

The final contradiction completes our proof.

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