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The weak minimal condition on subgroups which fail to be close to normal subgroups

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ABSTRACT

A subgroup H of a group G is *commensurable (or close) to a normal subgroup* if there is a normal subgroup N of G such that the index $|HN : (H \cap N)|$ is finite; if further the subgroup N can be chosen to be contained in H, i.e. if H/H_G is finite, then H is called *core-finite*. We describe the structure of (generalized) soluble groups satisfying the weak minimal condition on subgroups that are not commensurable with a normal subgroup. Our results describe also (generalized) soluble groups satisfying the weak minimal condition on non-(core-finite) subgroups.

1. Introduction

In 1897 Dedekind studied groups in which all subgroups are normal. Since then the investigation of groups in which *all* subgroups have a given property χ has been a standard in the theory of groups. However, since for a subgroup of an abstract group the

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property of being normal is rather strong, in the literature one has considered generalizations of the concept of normality. In [19], B.H. Neumann considered groups G in which each subgroup H is *nearly normal* (say *nn* for short) in G, that is H has finite index in a normal subgroup of G, i.e. $|H^G : H| < \infty$. Such groups turn out to be precisely the groups G in which the derived subgroup G' is finite, i.e. *finite-by-abelian* groups. A dual property was introduced in [1]: a group G is said to be a CF-group if each subgroup H is *core-finite* in G (say *cf*), that is H contains a normal subgroup of G with finite index in H, i.e. $|H : H_G| < \infty$. Since Tarski groups are CF, a complete classification of CF-groups seems to be difficult. However, *a CF-group G such that every periodic image of G is locally finite is abelian-by-finite* (see [21]), i.e. G has an abelian subgroup with finite index.

In order to consider the properties *nn* and *cf* in a common framework in [15] one has considered subgroups H with finite normal oscillation (say *fno*), i.e. the property that either $|H^G : H|$ or $|H : H_G|$ is finite. Later in [2], one has introduced the notion of *cn-subgroup*, that is a subgroup which is commensurable with a normal subgroup. Recall that two subgroups H and K of a group G are said to be *commensurable* if $H \cap K$ has finite index in both H and K; in particular we have that H is a *cn*-subgroup of G if and only if there exists a normal subgroup N of G such that $|HN : (H \cap N)| < \infty$. Clearly, *fno* implies *cn*, but not conversely (see [6], Remark 2.15.(a)). Note that *the intersection and the product of two cn (resp. cf) -subgroups is likewise a cn (resp. cf) -subgroup* (provided the latter is a subgroup).

A group in which all subgroups are *cn* is called a CN-group. In [2] it has been shown that *a CN-group G such that every periodic image of G is locally finite is finite-by-abelian-by-finite*, that is G has a finite-by-abelian subgroup with finite index. Hence G is nilpotent-by-finite. Moreover, there are soluble CN-groups which are not CF as they are neither abelian-by-finite nor finite-by-abelian (see [2], Proposition 1.2).

Let χ be a property pertaining to subgroups. A group G is said to have the *weak minimal condition* on χ-subgroups *Min*–∞-χ if there is no infinite descending chain $H_0 > H_1 > \ldots > H_n > \ldots$ of χ-subgroups of G with each index $|H_i : H_{i+1}|$ infinite (see [18]). It is easy to see that *a group has Min–∞, i.e. the weak minimal condition on all subgroups, if and only of the poset of commensurability classes of subgroups has the full minimal condition* (commensurability is an equivalence relation).

The structure of (generalized) soluble groups with *Min*–∞ is plainly understood, as by Zaicev's celebrated results (see [23]), we have that *for groups with an ascending series with locally (soluble-by-finite) factors, the condition Min–∞ is equivalent to that of being a soluble-by-finite minimax group*, that is a group with a series of finite length for which each infinite factor is cyclic or quasicyclic (i.e. a Prüfer group). Following Zaicev's approach several authors investigated groups with the weak minimal condition on non-χ-subgroups, *Min*–∞-$\overline{\chi}$, for various properties χ which generalize normality (see for example [4,8–10,16]). In particular, in [8] (generalized) soluble groups in *Min*–∞–$\overline{nn}$ have been described.

In this paper we consider groups with the weak chain condition on non-(core-finite) subgroups, *Min*–∞–$\overline{cf}$, which can be regarded as a dual of *Min*–∞–$\overline{nn}$. Actually, it seems desirable to handle the weaker condition on non-*cn*-subgroups, *Min*–∞–$\overline{cn}$, since the property *cn* is compatible with commensurability and therefore *a group has Min–∞–$\overline{cn}$ if and only if the poset of commensurability classes of subgroups which contain some non-normal member has the (full) minimal condition* (see [5]).

Our results correspond to those for the property *nn* (see [8]).

Theorem A. *Let G be a group which satisfies Min–∞–$\overline{cn}$. If G has an ascending series with locally (soluble-by-finite) factors, then G is soluble-by-finite.*

As in [18], a group G is said to have $FATR$ (*finite abelian total rank*) if it has a finite series whose factors are abelian of finite total rank, i.e. are the direct product of finitely many cyclic or quasicyclic groups and a torsion-free group of finite rank. Clearly, any $FATR$-group has finite rank, where a group is said to have *finite (Prüfer) rank r* if every finitely generated subgroup can be generated by r elements and r is the least such integer.

Theorem B. *Let G be a soluble-by-finite group which is not $FATR$-by-finite. Then G has Min–∞–$\overline{cn}$ (resp. Min–∞–$\overline{cf}$) if and only if each subgroup of G is cn (resp. cf).*

Theorem C. *Let G be a soluble-by-finite group. Then G satisfies Min–∞–$\overline{cn}$ (resp. Min–∞–$\overline{cf}$) if and only if each non-minimax subgroup of G is cn (resp. cf).*

Theorem D. *Let G be a soluble-by-finite non-minimax group. Then G has Min–∞–$\overline{cn}$ (resp. Min–∞–$\overline{cf}$) if and only if either each subgroup of G is cn (resp. cf) or G contains a normal $FATR$-subgroup of finite index N such that $N/Z(N)$ is a finitely generated abelian group and for each non-minimax subgroup H of N it holds $|H^G/H_G| < \infty$.*

Notice that, if G as in the above statement is also *locally finite*, then either G is a Chernikov group or all subgroups of G are *cn* (resp *cf*). This generalizes the corresponding results for the full minimal condition which appear in [7] (resp. [13]). In the general case, we remark that G has nilpotent subgroup (of class at most 2) which has finite index and satisfies *Min*–∞–$\overline{nn}$.

Finally, notice that by Theorem D and Proposition 12 below, it follows that *soluble-by-finite groups with Min–∞–$\overline{cn}$ but not Min–∞–$\overline{cf}$ are precisely those soluble-by-finite CN-groups which are not CF-groups.*

For undefined notation and well-known results we refer to [14,18].

2. Preliminiaries

We begin by proving basic facts about *cn*-subgroups.

Proposition 1. *Let G be a group and let H be a cn-subgroup of finite rank of G, then H is a cf-subgroup. Moreover, if H^G is nilpotent and has finite rank, then H^G/H_G is finite.*

Proof. Let N be a normal subgroup of G such that $K = H \cap N$ has finite index in both H and N. Then K is an *nn*-subgroup of G and has finite rank. We may assume $H = K$ is *nn*. If $|H^G : H| = n$, then $(H^G)^{n!}$ is a G-invariant subgroup of H so that H^G/H_G has finite exponent. As H^G/H_G is clearly a residually finite group of finite rank, it follows that H^G/H_G is finite (see [12], Corollary 4.3.9), as wished.

To prove the second part of the statement, assume that H^G is nilpotent and has finite rank. Since H/H_G is finite, the nilpotent group of finite rank H^G/H_G has finite exponent. Hence, H^G/H_G is finite as wished (see [20] Part 2, p. 38). □

We prove now a technical lemma and deduce two statements that will be crucial in our investigations.

Lemma 2. *Let G be a group with Min–∞–$\overline{cn}$ and a section H/K which is the direct product of an infinite family $(H_\lambda/K)_{\lambda\in\Lambda}$ of non-trivial groups. If L is a subgroup of G such that $L \cap H \le K \le L$ and $\langle H_\lambda, L\rangle = H_\lambda L$ for all $\lambda \in \Lambda$, then L is a cn-subgroup of G.*

Proof. There is no loss of generality in assuming that Λ is countable and

$$H/K = \operatorname*{Dr}_{n\in\mathbb{N}} H_n/K.$$

Then if $U = \langle H_{2n} | n \in \mathbb{N}\rangle$, $V = \langle H_{2n+1} | n \in \mathbb{N}\rangle$, and $U_k = \langle H_{2^k n} | n \in \mathbb{N}\rangle$ for all $k \in \mathbb{N}$ we have $\langle U_k, L\rangle = U_k L$ and the index $|U_k : U_{k+1}|$ is infinite. Therefore, since $L\cap H \le K$, we have that

$$U_1 L > U_2 L > \cdots$$

is an infinite descending chain such that the index $|U_i L : U_{i+1} L|$ is infinite for all i, so that $U_m L$ is a *cn*-subgroup of G for some $m \in \mathbb{N}$. Similarly there exists a subgroup V_* of V such that $\langle V_*, L\rangle = V_* L$ is a *cn*-subgroup of G. Since $L \cap H \le K = U_m \cap V_* \le L$, it can be easily checked that $L = U_m L \cap V_* L$ and hence L is a *cn*-subgroup of G. □

Lemma 3. *Let G be a group with Min–∞–$\overline{cn}$ and let A be a subgroup which is the direct product of infinitely many non-trivial G-invariant subgroups. If L is a subgroup of G such that $L \cap A = \{1\}$, then L is a cn-subgroup of G.*

Proof. This follows immediately from Lemma 2. □

Lemma 4. *Let G be a group with Min–∞–$\overline{cn}$ and let H/K be any section of G which is the direct product of infinitely many non-trivial subgroups. Then both H and K are cn-subgroups of G.*

Proof. Write $H/K = X/K \times Y/K$ where X/K and Y/K are both a direct product of infinitely many non-trivial subgroups. Since $[X, Y] \leq X \cap Y = K$, Lemma 2 yields that X and Y are *cn*-subgroups of G. It follows that $H = XY$ and $K = X \cap Y$ are *cn*-subgroups of G. $\square$

Recall here that a well-know result of Kulikov states that *any subgroup of a direct product of cyclic subgroups is likewise a direct product of cyclic subgroups* (see [14], Theorem 5.7). In what follows we make use of this result also without further reference.

Lemma 5. *Let G be a group with Min–∞–$\overline{cn}$. If G has a subgroup which is the direct product of infinitely many non-trivial cyclic subgroups, then:*

(i) *G has a normal abelian subgroup A which is the direct product of infinitely many non-trivial cyclic subgroups.*
(ii) *A has a finite subgroup A_0 such that all subgroups of A containing A_0 are normal in G.*
(iii) *all cyclic subgroups of G are cf.*

Proof. Let A_* be a subgroup of G which is the direct product of infinitely many non-trivial cyclic subgroups, then A_* is a *cn*-subgroup of G by Lemma 4. Let N be a normal subgroup of G such that $A_* \cap N$ has finite index in both A_* and N. Since it is well-know that any abelian-by-finite group has a characteristic abelian subgroup of finite index, it follows that N contains a G-invariant subgroup N_* of finite index. Clearly, A_* and N_* are commensurable and then N_* is likewise a direct product of infinitely many non-trivial cyclic subgroups (see [14], Theorem 5.7 and Exercise 8 p. 99). Replacing A_* with N_* it can be supposed that A_* is a normal subgroup of G.

If X is any subgroup of A_*, then X is a direct product of cyclic subgroups. If X has infinitely many direct factors, then X is a *cn*-subgroup of G by Lemma 4. Otherwise, X is finitely generated and we may write $A_* = H \times K$ where H is a finitely generated subgroup containing X and K is the direct product of infinitely many non-trivial cyclic subgroup, so that Lemma 3 yields that X is a *cn*-subgroup of G. Therefore all subgroups of A_* are *cn*-subgroups of G and hence it follows from Theorem 2.2 in [2] that there are G-invariant subgroups $A_1 \leq A_2$ of A_* with A_1 finite, A_*/A_2 finite and either all subgroup of A_2 containing A_1 are G-invariant or there is a finitely generated G-invariant subgroup V such that $A_1 \leq V \leq A_2$ and A_2/V is periodic. In the first case, if we put $A = A_2$ and $A_0 = A_1$, statements (i) and (ii) follows, so that suppose that the second case hold and write $A_2 = B \times C$ where $V \leq B$ and C is the direct product of infinitely many finite cyclic groups. As before for A_*, it can be supposed that C is a normal subgroup of G and that each subgroup of C is a *cn*-subgroup of G, so that, since C is periodic and reduced, Lemma 2.8 in [2] yields that C contains G-invariant subgroups $C_1 \leq C_2$ with C_1 finite, C/C_2 finite and all subgroup of C_2 containing C_1 are G-invariant. Therefore if now we put $A = C_2$ and $A_0 = C_1$, statements (i) and (ii) hold.

To show (iii), let L be any cyclic subgroup of G. Since A_0 is finite and by Proposition 1, it suffices to prove that LA_0/A_0 is a *cn*-subgroups of G/A_0. On the other hand, it is clear that A/A_0 contains a subgroup which is likewise the direct product of infinitely many non-trivial cyclic groups, so that there is no loss of generality if we suppose that $A_0 = \{1\}$ and that all subgroups of A are normal in G. Write $A = B \times C$ where B is a finitely generated subgroup containing $L \cap A$ and C is the direct product of infinitely many non-trivial cyclic subgroups. Then $L \cap C = \{1\}$ and so application of Lemma 3 gives that L is a *cn*-subgroup. The Lemma is proved. □

3. The periodic case

Our goal now is to prove Proposition 7 below and settle the locally finite case of Theorem A. To this aim we prove a lemma about an elementary property of abelian groups (additively written).

Lemma 6. *Let A be an abelian group which is not the sum of two subgroups of infinite index. Then A contains a finitely generated subgroup F such that A/F is either trivial or a Prüfer group. In particular, A is minimax.*

Proof. Assume first that A is periodic. Clearly, A has only finitely many primary components and only one of them can be infinite, so that we may further assume that A is a p-group for some prime number p. Since A/pA is elementary abelian and cannot be the sum of two subgroups of infinite index, it must be finite. Thus, if $A = D \oplus R$ with D divisible and R reduced, it follows easily that R must be finite (as for example in [6], Theorem 4.3.(e)) and D a Prüfer group.

In the general case, let F be a free subgroup of A such that A/F is periodic. It remains to show that F is finitely generated. To this aim, note that for each positive integer n, the group A/nF is periodic and cannot be the sum of two subgroups of infinite index. By the periodic case, A/nF has finite rank. Thus its bounded subgroup F/nF is finite. Hence F is finitely generated. □

Proposition 7. *Let G be a group with Min–∞–$\overline{cn}$ whose periodic images are locally finite. If G contains a subgroup which is a direct product of infinitely many non-trivial cyclic subgroups, then G is a CN-group.*

Proof. The group G has normal subgroups A and A_0 as in Lemma 5. Since A_0 is finite, it is enough to prove that G/A_0 is CN, and hence we may assume $A_0 = \{1\}$ and that all subgroups of A are G-invariant.

Let, by contradiction, H be a subgroup of G which is non-*cn*. Because of *Min*–∞–$\overline{cn}$, we may assume that all subgroups of H with infinite index are *cn*-subgroups of G. Thus H is in fact a CN-group and, by the quoted result of [2], we have that H contains a normal subgroup K of finite index such that K' is finite. Since K has finite index in H, also K is not a *cn*-subgroup of G. Then, as remarked in the introduction, K cannot be the product of two *cn*-subgroups of G so that K cannot be the product of two subgroups of infinite index and hence Lemma 6 yields that K/K' is minimax; hence H is likewise minimax. Then $A \cap H$ is minimax. Since $A \cap H$ is a direct product of cyclic subgroups we have that $A \cap H$ is finitely generated, hence A contains an infinite direct factor which intersects trivially H and so application of Lemma 3 gives us the wished contradiction. □

Now we can settle the periodic case of Theorem A.

Proposition 8. *Let G be a locally finite group with Min–∞–$\overline{cn}$. Then G is either a Chernikov group or a CN-group.*

Proof. If G is not a Chernikov group, then G contains an abelian subgroup with infinite socle (see [22]) and hence Proposition 7 yields that G is a CN-group. □

4. Proofs of Theorems

We split the proof of Theorem A (in the non periodic case) into Lemmas.

Lemma 9. *Let G be a locally soluble group with Min–∞–$\overline{cn}$. If G has finite rank, then G is soluble.*

Proof. Since G is a locally soluble group of finite rank, there is a positive integer n such that the n-th term $G^{(n)}$ of the derived series of G is periodic (see [20] Part 2, Lemma 10.39). Thus it follows from Proposition 8 that $G^{(n)}$, and hence also G, is soluble. □

Lemma 10. *Let G be a locally soluble group with Min–∞–$\overline{cn}$. Then G is soluble.*

Proof. Let G be a counterexample. Then G has infinite rank by Lemma 9. Moreover, since both properties soluble and finite rank are countably recognizable, G must contain a countable subgroup S which is insoluble and a countable subgroup R which has infinite rank. Therefore we may assume that $G = \langle R, S\rangle$ is countable.

We will reach a contradiction by showing that G has a non-*cn*-subgroup of infinite index K which is not soluble and has infinite rank, because from this fact it follows that there exists a descending chain of non-*cn*-subgroups $K_1 > K_2 > \dots$ with all the indices $|K_i : K_{i+1}|$ infinite.

To prove our claim, note that Proposition 7 yields that all abelian subgroups of G have finite total rank so that all soluble subgroups of G are $FATR$ by a result of Charin (see [18], 6.2.5). Hence, in particular, all finitely generated subgroups of G are minimax (see [20] Part 2, Theorem 10.38). Since G is countable, there exists an ascending chain of finitely generated subgroups $F_1 < F_2 < \cdots$ such that $G = \bigcup_{n\in\mathbb{N}} F_n$; in particular, each F_n is soluble and minimax. Then (see [17], Lemma 3) for any $n \in \mathbb{N}$ there exist normal subgroups H_n and K_n of F_n such that the following four properties hold

(i) *H_n is locally nilpotent,*
(ii) *$H_n \leq K_n$, K_n/H_n is abelian and F_n/K_n is finite,*
(iii) $K_{n+1} \cap F_n \leq K_n$,

moreover if we set $K = \langle K_n : n \in \mathbb{N}\rangle$, then we have

(iv) $K \cap F_n = K_1K_2 \cdots K_n$.

We claim that we may also assume that

(v) $F_n^{(n)} \nleq K_1K_2 \cdots K_n \qquad \forall n \in \mathbb{N}$.

To prove the claim, note that any factor F_n/K_n is finite and hence soluble. If there is a bound for the derived length of the F_n/K_n's, we have that there is a positive integer t such that $F_n^{(t+1)} \leq H_n$ for any $n \in \mathbb{N}$, so that (i) yields that $F_n^{(t+1)}$ is locally nilpotent for any $n \in \mathbb{N}$; it follows that $G^{(t+1)}$ is locally nilpotent. Since all abelian subgroups of G have finite total rank, it follows that $G^{(t+1)}$ is soluble (see [20] Part 2, p. 38), and this is clearly a contradiction. Passing to an appropriate subsequence of the F_n's if necessary, property (v) holds.

We have now also the following facts:

(vi) *the index $|G : K|$ is infinite.*

If not, the factor G/K_G is finite and so soluble. Then there exists a positive integer s such that $G^{(s)} \leq K_G$, hence (iv) yields that

$$F_s^{(s)} \leq K_G \cap F_s \leq K \cap F_s = K_1K_2 \cdots K_s$$

which is a contradiction being (v) true.

(vii) *K is not a cn-subgroup of G.*

Assume, for a contradiction, that there is a normal subgroup N of G such that $K \cap N$ has finite index in both K and N, then also $L = (K \cap N)_N$ has finite index in both K and N. Let g be any element of G. Then g belongs to some F_n and hence some power of g lies in K_n by (ii); moreover, $K_n \leq K$ and the index $|K : L|$ is finite, so that we have that there exists a positive integer t such that $g^t \in L$. But $L \leq N$, so G/N is periodic and hence soluble by Proposition 8. On the other hand N/L is finite and hence soluble, therefore it follows that there exists a positive integer s such that $G^{(s)} \leq L$. Then $G^{(s)} \leq K$ which is a contradiction, as before.

The proof will be complete once we prove that

(viii) *K is not soluble and has infinite rank.*

By contradiction, assume that K is soluble. Since all soluble subgroups of G are $FATR$, the subgroup K has finite rank r, say. Let X be any free abelian subgroup of G, then X is finitely generated and hence $X \leq F_m$ for some positive integer m. Since F_m/K_m has finite order t by (ii) and $K_m \leq K$ it follows that X^t has rank at most r, so that X has rank at most r. Therefore any abelian subgroup of G has finite rank and its torsion-free rank is bounded by r, so that G itself has finite rank (see [11], Corollary 3.7), a contradiction. Thus K is not soluble and hence it has infinite rank by Lemma 9. □

Lemma 11. *Let G be a locally (soluble-by-finite) group with Min–∞–$\overline{cn}$. Then G is soluble-by-finite.*

Proof. If G has infinite rank, then G contains a locally soluble subgroup G_1 with infinite rank (see [11]) which is actually soluble by Lemma 10. Thus G_1 contains an abelian subgroup having no finite total rank (see [18], 6.2.5), hence G is a CN-group by Proposition 7 and so it is soluble-by-finite. On the other hand, if G has finite rank, then G is (locally soluble)-by-finite (see [3]) and so G is soluble-by-finite by Lemma 10. □

Proof of Theorem A. If G is locally (soluble-by-finite) apply Lemma 11. In the general case, note that a soluble-by-finite group is readily seen to have a characteristic soluble subgroup with finite index. Then one can use a standard transfinite induction argument to prove the theorem. □

Proposition 12. *Let G be a group with Min–∞–$\overline{cf}$ whose periodic images are locally finite. If G is a CN-group, then G is a CF-group.*

Proof. Let H be any subgroup of G. If $H_1 > H_2 > \cdots$ is a descending chain of subgroups of H containing H_G such that the index $|H_{i+1} : H_i|$ is infinite for all $i \in \mathbb{N}$, then there exists a positive integer n such that $H_n/(H_n)_G$ is finite; as $(H_n)_G \leq H_G$ it follows that H_n/H_G is finite. This contradiction proves that H/H_G satisfies the weak minimal condition. Since G is a CN-group whose periodic images are locally finite, G is soluble-by-finite, and hence it follows that H/H_G is minimax by the above quoted result of Zaicev.

If $xH_G \in H/H_G$, then $\langle x \rangle$ is a *cf*-subgroup by Proposition 1 so that some power of x belongs to $\langle x \rangle_G \leq H_G$ and hence the coset xH_G has finite order. Therefore the factor H/H_G is periodic and so it is a Chernikov group. Let J/H_G be the finite residual of H/H_G. Since any quasicyclic *cn*-subgroup is obviously normal, the subgroup J is normal in G, so that $J = H_G$ and hence H/H_G is finite. □

Proof of Theorem B. If G has *Min*–∞–$\overline{cn}$, as G is not a $FATR$-by-finite group, then a result of Charin (see [18], 6.2.5) yields that G contains a subgroup which is a direct product of infinitely many non-trivial cyclic subgroups and hence G is a CN-group by Proposition 7. The converse is obvious.

The corresponding result with *Min*–∞–$\overline{cf}$ follows from the first part of this proof together with Proposition 12. □

Proposition 13. *Let G be a group with Min–∞–$\overline{cn}$ whose periodic images are locally finite. If G contains an abelian subgroup A which is not minimax, then either G is a CN-group or there exists a finitely generated torsion-free G-invariant subgroup N of A such that G/N is a CN-group.*

Proof. If G is not a CN-group, then Proposition 7 yields that A has finite total rank so that, if B a free subgroup of A of maximal rank, then B is finitely generated and A/B is a periodic group with infinitely many non-trivial primary components. In particular, B is a *cn*-subgroup of G by Lemma 4 and so $N = B_G$ has finite index in B by Proposition 1. Then also A/N has infinitely many primary components and so G/N is a CN-group by Proposition 7. □

Proof of Theorem C. Suppose first that G satisfies *Min*–∞–$\overline{cn}$. Let H be a subgroup of G which is not minimax and assume, for a contradiction, that H is not a *cn*-subgroup of G; in particular, G contains a subgroup of finite index with $FATR$ by Theorem B and so G has finite rank. As H is soluble-by-finite and it is not minimax, H contains an abelian subgroup A which is not minimax (see [20], Part 2, Theorem 10.35) and so Proposition 13 yields that there exists a finitely generated torsion-free G-invariant subgroup N of A such that G/N is a CN-group. Then Proposition 1 yields that H/N is a *cf*-subgroup of G/N and hence H/H_G is finite. This contradiction proves that H either is minimax or a *cn*-subgroup of G.

Conversely, suppose that all non-minimax subgroups of G are *cn*-subgroups and let $H_1 > H_2 > \cdots$ be an infinite descending chain of subgroups of G such that all the indices $|H_i : H_{i+1}|$ are infinite. Then each H_i is not a minimax group by the above quoted theorem of Zaicev, thus all H_i are *cn*-subgroups of G. Hence G satisfies *Min*–∞–$\overline{cn}$.

Suppose now that G has *Min*–∞–$\overline{cf}$. Proposition 12 allow us to suppose that G is not a CN-group. Since G is soluble-by-finite, it follows from Theorem B that G has a subgroup of finite index with $FATR$. Thus if H is any non-minimax subgroup of G, then H is a *cn*-subgroup of G by the first part of the proof and so even a *cf*-subgroup by Proposition 1.

Conversely, if all non-minimax subgroups of G are *cf*, as for the corresponding result with *cn* instead of *cf*, it is easy to obtain that G satisfies *Min*–∞–$\overline{cf}$. □

Recall now that *if a group G has finite rank and G' is polycyclic-by-finite, then $G/Z(G)$ is polycyclic-by-finite*, as shown in [4], Lemma 6.

Proof of Theorem D. Assume that G has Min–∞–$\overline{cn}$ and that G neither is a minimax group nor a CN-group. By Theorem B the group G is $FATR$-by-finite, hence G has finite rank. Then, from Proposition 1, it follows that any section of G which is a CN-group is a CF-group, hence it is abelian-by-finite. On the other hand, since G is a soluble-by-finite non-minimax group, it contains an abelian non-minimax subgroup (see [20] Part 2, Theorem 10.35). Then applying Proposition 13 we get that there exist normal subgroups N_1 and G_1 of G such that $G_1' \leq N_1 \leq G_1$, where N_1 is abelian and finitely generated, and the index $|G : G_1|$ is finite. By the above recalled Lemma 6 in [4], we have that $G_1/Z(G_1)$ is polycyclic. Thus $Z(G_1)$ is not a minimax group.

If G_1 is a CN-group, then it is abelian-by-finite as noticed above. Thus it has a characteristic abelian subgroup of finite index N which satisfies the requirements in the statement.

Assume then that G_1 is not a CN-group. Then application of Proposition 13 again yields that there are subgroups $N_2 \leq Z(G_1)$ and G_2 such that

$$G_2' \leq N_2 \leq G_2 \lhd G_1$$

where N_2 is finitely generated and G_1/G_2 is finite. Then G_2 is nilpotent of class at most 2 since

$$G_2' \leq N_2 \leq Z(G_1) \cap G_2 \leq Z(G_2).$$

As G_2 has finite index in G, the core N of G_2 in G is a normal subgroup of finite index of G which is nilpotent of class at most 2; moreover, since $N' \leq G_2' \leq N_2$ and G is $FATR$-by-finite we have that N' and $N/Z(N)$ are finitely generated (again by the recalled Lemma 6 of [4]).

Finally, if H is any non-minimax subgroup of N, then H is a *cn*-subgroup by Theorem C and so H^G/H_G is finite by Proposition 1.

Conversely, if G is not a CN-group and X is any non-minimax subgroup of G, then $X \cap N$ is not minimax; hence $(X \cap N)_G$ has finite index in X and so X is *cf*. Therefore G has Min–∞–$\overline{cf}$ by Theorem C.

The case with Min–∞–$\overline{cf}$ instead of Min–∞–$\overline{cn}$ follows now from Proposition 12. □

Finally, we remark that, by the above results about CN-groups, CF-groups and groups with Min–∞–$\overline{nn}$, we have the following.

Corollary 14. *Let G be soluble-by-finite group with Min–∞–$\overline{cn}$ (resp. Min–∞–$\overline{cf}$). Then either G is finite-by-abelian-by-finite (resp. abelian-by-finite) or G contains an abelian normal subgroup A which is minimax-by-(torsion-free abelian of rank 1) and G/A is polycyclic-by-finite.*

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