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Lie Triple Systems, Restricted Lie Triple Systems, and Algebraic Groups

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We define a restricted structure for Lie triple systems in the characteristic $p > 2$ setting, akin to the restricted structure for Lie algebras, and initiate a study of a theory of restricted modules. In general, Lie triple systems have natural embeddings into certain canonical Lie algebras, the so-called "standard" and "universal" embeddings, and any Lie triple system can be shown to arise precisely as the -1-eigenspace of an involution (an automorphism which squares to the identity) on some Lie algebra. We specialize to Lie triple systems which arise as the differentials of involutions on simple, simply connected algebraic groups over algebraically closed fields of characteristic p. Under these hypotheses we completely classify the universal and standard embeddings in terms of the Lie algebra and its universal central extension.

Key Words: Lie triple systems; algebraic groups; restricted Lie triple systems; standard enveloping Lie algebra; universal enveloping Lie algebra; representation theory; involutions; modular Harish–Chandra modules.

1. INTRODUCTION

1.1. *Background and Motivation.* Let k be an algebraically closed field that has characteristic $p > 2$, and let G be a connected, reductive algebraic group over k. Let $\theta \in \mathrm{Aut}(G)$ be an involution of G (i.e., $\theta^2 = 1$). Then the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ possesses a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into $+1$- and -1-eigenspaces of the differential of θ, and $\mathfrak{k} = \mathrm{Lie}(K)$ for $K = G^\theta$, the group of fixed points of θ. Although not a Lie subalgebra, the -1-eigenspace $\mathfrak{p}$ of $\theta \in \mathrm{Aut}(\mathfrak{g})$ does bear the structure of a Lie triple system.

Lie triple systems arose initially in Cartan's studies of Riemannian geometry, in which he employed his classification of the real simple Lie

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algebras to classify an important subcollection of Riemannian manifolds, the symmetric spaces. Up to a factor of a Euclidean space, Cartan's classification associated to each symmetric space M a semisimple Lie group G with an involutive automorphism. More precisely, for some $n \geq 0$,

$$M \cong \mathbb{R}^n \times G/K,$$

where K is a compact subgroup of G determined by the fixed points of an involution from G to itself, and G/K possesses a G-invariant Riemannian structure. By means of this association of Lie groups to symmetric spaces and of Lie algebras to the Lie groups, Cartan was able to classify the symmetric spaces.

In a similar fashion, the objects called Lie triple systems arise upon consideration of certain subspaces of Riemannian manifolds, the totally geodesic submanifolds. Essentially, totally geodesic submanifolds are like planes in Euclidean space. Given a Riemannian globally symmetric space M, another result of Cartan's associates to each totally geodesic submanifold S of M a Lie triple system $T \subset \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the group of isometries of M, and vice versa. (See [7, Theorem 7.2, pp. 189–190].) The theory of symmetric spaces was subsequently placed in a more algebraic setting by the fundamental work of Loos [18]. Lie triple systems (and their connections with symmetric spaces and related spaces) continue to be a source of interest; see for example [3, 14, 15].

Now, maximal compact subgroups of a connected Lie group (with finite center) are utilized to study a large class of representations of the Lie group, via Harish-Chandra modules. (There is a vast literature on the subject, but see [16] as one entry point.) Informally, Harish-Chandra modules can be regarded simultaneously as modules for the complexification of the maximal compact subgroup and the complexified Lie algebra of the group, subject to certain compatibility conditions on the two actions. By replacing the compact subgroup with the fixed point group $K = G^{\theta}$ of an involution θ of G as in the first paragraph above, we develop in [8] and [9] a theory of Harish-Chandra modules for algebraic groups in characteristic $p > 2$. This spurred our initial interest in the theory of Lie triple systems, for the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is fundamental to the study of the modular Harish-Chandra modules. In some sense, the compatibility conditions linking the actions of $\mathfrak{g}$ and K, together with the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k} = \mathrm{Lie}(K)$, suggest that the resulting modular Harish-Chandra modules might be approached as modules for K and for $\mathfrak{p}$, subject as before to some compatibility conditions.

Furthermore, in the representation theory of algebraic groups in positive characteristic, the theory of restricted Lie algebras is used extensively, since every rational module for G is a restricted module for $\mathfrak{g}$ as a restricted Lie algebra. Thus, we were also drawn to develop an analogous definition and theory of restricted Lie triple systems and to begin a study of restricted modules for such Lie triple systems. As another area of applications, we have recently learned that Nagy [21] has also (independently) defined restricted Lie triple systems in the context of the theory of quasi-groups and loops. In characteristic $p = 3$, he has, for example, proved an analogue of Cartier's duality linking formal Bruck loops (of height 0) with the category of restricted Lie triple systems.

1.2. *Organization of the Paper*. Assume the notation as in the beginning of Section 1.1. In Section 2, we provide some background and structure theory results regarding Lie triple systems in the abstract, adjusted for the characteristic p setting. This material includes discussions of the associations between a Lie triple system T and certain important related Lie algebras, such as the standard enveloping Lie algebra $L_s(T)$ and the universal enveloping Lie algebra $L_u(T)$. Next, recalling, in the Lie algebra case, the significance for the module theory of G of the existence of a restricted structure on $\mathfrak{g}$, we develop a theory of restricted Lie triple systems in Section 3. In Section 4, we return to the specific instance of the Lie triple system $\mathfrak{p}$. When G is simple and simply connected over k, we completely determine the standard and universal enveloping Lie algebras $L_s(\mathfrak{p})$ and $L_u(\mathfrak{p})$ of $\mathfrak{p}$, describing them in terms of $\mathfrak{g}$ and its universal (central) covering $\mathfrak{g}^\star$. We then turn to the representation theory of Lie triple systems in Section 5. Following [6], we define modules for a Lie triple system T and relate them to special modules for the universal enveloping Lie algebra $L_u(T)$ of T. With the concept of a restricted Lie triple system in hand, we extend the notion of a Lie triple system module to a definition of a restricted module for a (restricted) Lie triple system. In the final subsection, Section 5.2, we look to the possible significance of these developments in the special case $T = \mathfrak{p}$ and propose some further directions for research.

2. LIE TRIPLE SYSTEMS

In the next two sections, we will pass from the classical setting to a consideration of Lie triple systems for an algebraically closed field k, $\mathrm{char}(k) = p > 2$. (These will be our standing assumptions on k throughout the paper.) Much of the background material in Section 2 is not new; the theory of Lie triple systems has been considered over fields of nonzero characteristic. However, we will wish to address particular aspects of the positive characteristic case and to discuss further the introductory example rising from algebraic group theory which will be of great interest later in the paper. We also include a substantial amount of material in order to make our results accessible to readers without a background in Lie triple systems and because our references for this material are fairly old and/or not readily available. The material in Section 3 is new. There, we will extend the theory of Lie triple systems by developing a restricted theory for Lie triple systems over k, analogous to the restricted theory for Lie algebras over k.

To accomplish this, we begin by explaining just what a Lie triple system is. Subsequently in this subsection, we will consider how Lie triple systems "sit" inside Lie algebras, including a more detailed discussion of two important Lie algebras related to Lie triple systems, the "standard enveloping Lie algebra" $L_s(T)$ and the "universal enveloping Lie algebra" $L_u(T)$.

DEFINITION 2.0.1. A *Lie triple system* is a k-vector space T closed under a ternary operation $[abc]$ which is trilinear and satisfies the three properties, for all a, b, c, x, y, z in T,

$$[aab] = 0, \tag{2.0.2}$$

$$[abc] + [bca] + [cab] = 0, \tag{2.0.3}$$

and

$$[ab[xyz]] = [[abx]yz] + [x[aby]z] + [xy[abz]]. \tag{2.0.4}$$

Note that by (2.0.2),

$$\begin{aligned} 0 &= [(a+b)(a+b)c] = [aac] + [abc] + [bac] + [bbc] \\ &= [abc] + [bac], \end{aligned}$$

hence

$$[abc] = -[bac]. \tag{2.0.5}$$

A *morphism of Lie triple systems* T, T' is a k-linear map $\phi\colon T \to T'$ satisfying $\phi([abc]) = [\phi(a)\phi(b)\phi(c)]$ for all $a, b, c \in T$. We let $\operatorname{Hom}_{LTS}(T, T')$ denote the Lie triple system morphisms from T to T'. For brevity, we will often refer to a Lie triple system simply by the initials LTS. In what follows, we will also assume for convenience that T is always finite-dimensional. Finally, we will let **LTS** be the category with objects the Lie triple systems and morphisms as defined above.

Following our initial description of the geometrical considerations motivating the study of Lie triple systems, the fact that the first two axioms above are reminiscent of those for the binary operation [,] in a Lie algebra should cause no surprise. If $\mathfrak{g}$ is any (finite-dimensional) Lie algebra over k, then the triple product $[xyz] = [[x, y], z]$ on $\mathfrak{g}$ satisfies the hypotheses above. Letting **LIE** denote the category of finite-dimensional Lie algebras, we immediately see that we get a functor $(-)_{triple}$: **LIE** $\to$ **LTS** by setting $\mathfrak{g}_{triple}$ to be the LTS with this triple product for any $\mathfrak{g} \in \mathrm{Ob}(\mathbf{LIE})$. Moreover, any subspace of $\mathfrak{g}$ which is closed under the ternary product $[xyz] = [[x, y], z]$ determines a Lie triple system.

In light of the well-known interpretation of the Jacobi identity for Lie algebras as a statement that the mapping $D_x = \mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is a derivation, we now consider the third axiom of a Lie triple system T. For $a, b \in T$, define a mapping $D_{a,b}: T \to T$ by

$$D_{a,b}(x) = [abx]. \tag{2.0.6}$$

Then (2.0.4) becomes, writing $D = D_{a,b}$,

$$D([xyz]) = [D(x)yz] + [xD(y)z] + [xyD(z)]. \tag{2.0.7}$$

Any k-linear endomorphism D of a Lie triple system T satisfying property (2.0.7) will be called a *derivation* in T. As a special case, the derivation $D_{a,b}$ will be called an *inner derivation*.

An important example of a Lie triple system arises by considering the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of a connected, reductive algebraic group G over k, with an involution $\theta \in \mathrm{Aut}(G)$, $\theta \neq 1$. (We will give a concrete example below; other examples of involutions are given in the Appendix.) As in the Introduction, we let $K = G^\theta$ be the group of fixed points of θ in G. (In case G is semisimple and simply connected, K is actually a connected reductive group.) Let θ also denote the associated differential of $\mathfrak{g}$. Then it is easily proved that this Lie algebra decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{2.0.8}$$

where $\mathfrak{p}$ is by definition the -1-eigenspace of θ in $\mathfrak{g}$ and the $+1$-eigenspace equals $\mathrm{Lie}(K) = \mathfrak{k}$.

EXAMPLE 2.0.9. Now, under the bracket operation on $\mathfrak{g}$, for any x, y, z in $\mathfrak{p}$ we have

$$\theta([x, y]) = [\theta(x), \theta(y)] = [-x, -y] = (-1)^2[x, y] = [x, y], \tag{2.0.10}$$

hence

$$\theta([[x, y], z]) = [\theta([x, y]), \theta(z)] = [[x, y], -z] = -[[x, y], z]. \tag{2.0.11}$$

Thus, $\mathfrak{p}$ is closed under the ternary operation $[xyz] = [[x, y], z]$, so $\mathfrak{p}$ becomes a Lie triple system.

As a concrete example, take $G = SL_n(k)$, and let θ: $SL_n(k) \to SL_n(k)$ by $\theta(x) = (x^t)^{-1}$. Then $\mathfrak{g} = \mathfrak{sl}_n(k)$ carries the associated involution given by $\theta(X) = -X^t$. (We will continue to refer to an automorphism of a Lie algebra which squares to the identity as an "involution.") The -1-eigenspace $\mathfrak{p}$ of θ on $\mathfrak{sl}_n(k)$ consists of the symmetric $n \times n$ matrices. Here, for X, Y, Z symmetric $n \times n$ matrices, one can check our assertions above by calculating that $[[X, Y], Z]$ is another symmetric matrix, whereas the bracket of any two symmetric matrices is skew-symmetric. This is what we expect from (2.0.8), for here $K = SO_n$ and $\mathfrak{l} = \mathfrak{o}_n$. This completes Example 2.0.9.

Note that at the heart of the discussion above the origin of the involution on $\mathfrak{g}$ is irrelevant. More precisely, the calculations (2.0.10) and (2.0.11) show that the -1-eigenspace of any involution θ on any Lie algebra $\mathfrak{g}$ becomes a Lie triple system in a natural way. Moreover, such a construction in essence yields all Lie triple systems, as a consequence of the following definition and theorem.

DEFINITION 2.0.12. Let T be a Lie triple system over k and $\mathfrak{g}$ any Lie algebra over k. A linear mapping φ: $T \to \mathfrak{g}$ will be called an *imbedding* if $\varphi: T \to \mathfrak{g}_{triple} \in \mathrm{Hom}_{LTS}(T, \mathfrak{g}_{triple})$; i.e., for all $a, b, c \in T$, $\varphi([abc]) = [[\varphi(a), \varphi(b)], \varphi(c)]$. For any imbedding φ: $T \to \mathfrak{g}$, let L_φ denote the Lie subalgebra of $\mathfrak{g}$ generated by $\mathrm{Im}(\varphi)$ and call L_φ the *enveloping Lie algebra of the imbedding* φ.

As in [17, Definition 1.5], $L_\varphi = \varphi(T) + [\varphi(T), \varphi(T)]$, but this need not be a direct sum. In any case, if $\dim_k(T) = n$, then we obtain the bound

$$\dim_k(L_\varphi) \leq n + \frac{n(n-1)}{2}, \tag{2.0.13}$$

for the relations $[x, x] = 0$ and $[x, y] = -[y, x]$ hold inside $[\varphi(T), \varphi(T)]$.

Now, an imbedding need not be an injection. However, the following result essentially appears in [12].

THEOREM 2.0.14. *Let T be a Lie triple system over k. Then*:

(a) *There is a Lie algebra $L_s(T)$ and an imbedding ι_T: $T \to L_s(T)$ which is one-to-one.*

(b) *As vector spaces, $L_s(T) \cong T \oplus [T, T]$.*

(c) *$L_s(T)$ is determined up to isomorphism by the following universal property*: *Suppose φ: $T \to \mathfrak{g}$ is any injective imbedding for which $L_\varphi = T \oplus [T, T]$. Then there is a unique Lie algebra homomorphism ϕ of L_φ onto*

$L_s(T)$ which is the identity on T. More precisely, ϕ satisfies the commutative diagram

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[dr, "\iota_T"'] & L_\varphi \arrow[d, "\phi"] \\ & L_s(T). \end{tikzcd}$$

The algebra $L_s(T)$ is called the *standard enveloping Lie algebra* of the LTS T.

Although we will not need the details of the proof of (2.0.14), it will be useful to us to have a construction of the Lie algebra $L_s(T)$ in hand, which we now present, following the more modern notation of [6]. First, let V_T be the quotient of the k-vector space $T \otimes_k T$ by the subspace of all elements $\sum_i a_i \otimes b_i$, for which $\sum_i [a_i b_i x] = 0$ for all $x \in T$. Now, as a vector space, let $L_s(T) = T \oplus V_T$. Equip $L_s(T)$ with a bracket operation $[x, y]$ as follows, where $a, b, c, a_i, b_i, c_i, d_i \in T$ and $\overline{a \otimes b}$ denotes the coset of $a \otimes b \in V$,

$$[a, b] = \overline{a \otimes b},$$

$$\left[\left(\sum_i [a_i, b_i]\right), c\right] = \sum_i [a_i b_i c],$$

$$\left[c, \left(\sum_i [a_i, b_i]\right)\right] = -\sum_i [a_i b_i c],$$

$$\left[\left(\sum_i [a_i, b_i]\right), \left(\sum_j [c_j, d_j]\right)\right] = \sum_{i,j} [[a_i b_i c_j], d_j] - [[a_i b_i d_j], c_j].$$

Observe that the obvious map $\iota_T\colon T \to L_s(T)$ (for $L_s(T)$ as constructed above) is an injective imbedding of T satisfying the statement of (2.0.14), subsequently termed the *standard imbedding*. Moreover, as vector spaces, $L_s(T) = T \oplus [T, T]$, identifying T with $\iota_T(T)$.

From (2.0.14) and the ensuing discussion we see that, indeed, every Lie triple system identifies with the -1-eigenspace of an involution on some Lie algebra: one can define an involution θ on $L_s(T) = T \oplus [T, T]$ by setting, for all a, b, c in T,

$$\theta(a) = -a \qquad \text{and} \qquad \theta([b, c]) = [b, c]. \tag{2.0.15}$$

Analogously, but slightly more generally, Theorem 1.1 of [17] states

THEOREM 2.0.16. *Suppose $\varphi\colon T \to \mathfrak{g}$ is a one-to-one imbedding for which $\mathfrak{g} = L_\varphi = T \oplus [T, T]$. Then T is determined as the -1-eigenspace of a unique involution in $\mathfrak{g}$.*

Proof. Write $X \in L_\varphi$ in the form $X = a + h$, $a \in T$, $h \in [T,T]$. Set $\theta(X) = h - a$. Clearly, θ is linear and $\theta^2 = 1$. Next, for $Y = a' + h' = a' + \sum_i [b_i', c_i']$,

$$[\theta(X), \theta(Y)] = [a, a'] + [h, h'] - [a, h'] - [h, a']. \quad (2.0.17)$$

Now, $[a, h'] = \sum_i [a_i, [b_i', c_i']] = -\sum_i [[b_i, c_i], a_i] = -\sum_i [b_i c_i a_i] \in T$; likewise, $[h, a'] \in T$. Moreover, by using the Jacobi identity in the form $[[U,V],W] = -[[V,W],U] - [[W,U],V]$ with $[U,V] = [b_i, c_i]$ and $W = [b_i', c_i']$, we have, for each i,

$$\begin{aligned}[[b_i, c_i],[b_i', c_i']] &= -[[c_i,[b_i', c_i']], b_i] - [[[b_i', c_i'], b_i], c_i] \\ &= [[b_i' c_i' c_i], b_i] - [[b_i' c_i' b_i'], c_i].\end{aligned}$$

From this it follows that $[h, h'] \in [T,T]$. Thus

$$\begin{aligned}\theta([X,Y]) &= \theta([a,a']) + \theta([h,h']) + \theta([a,h']) + \theta([h,a']) \\ &= [a,a'] + [h,h'] - [a,h'] - [h,a']. \quad (2.0.18)\end{aligned}$$

Together, (2.0.17) and (2.0.18) show that θ is an automorphism, whence θ is an involution. Since T generates L_φ, θ is unique. ∎

We now discuss the *universal enveloping Lie algebra* $L_u(T)$ of T, which fills a more general dual role. Precisely, there exists a Lie algebra $L_u(T)$ and an imbedding $\eta_T\colon T \to L_u(T)$ such that if $\varphi\colon T \to \mathfrak{g}$ denotes any imbedding (not necessarily injective) then φ lifts uniquely to a surjective Lie algebra homomorphism $L_u(T) \to L_\varphi$ satisfying the commutative diagram

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "\varphi"'] & L_u(T) \arrow[d, two heads, "\phi"] \\ & L_\varphi. \end{tikzcd}$$

Equivalently, taking the universal enveloping Lie algebra determines a functor L_u: **LTS** $\to$ **LIE** which is left adjoint to $(-)_{triple}$.

In particular, by setting $(\varphi, L_\varphi) = (\iota_T, L_s(T))$ in the diagram above, we conclude that η_T must be injective. Consequently, $[T,T] \cap T = 0$ in $L_s(T)$ implies $[T,T] \cap T = 0$ in $L_u(T)$ as well.

To construct the universal enveloping algebra $L_u(T)$ of T, one may be guided by the parallel relationship of Lie algebras to their universal (associative) enveloping algebras. Indeed, as described in [12], the same method employed to produce the universal enveloping algebra of a Lie algebra will produce $L_u(T)$ for the LTS T. Specifically, take $\mathscr{L}$ to be the free Lie algebra over the vector space T. By a theorem of Witt (e.g., see [23, Theorem I.3.1]), $\mathscr{L}$ can be obtained as follows. Form the free associative algebra A_T over the vector space T; subsequently setting $[x, y] = xy - yx$ determines a Lie algebra structure on A_T. Then $\mathscr{L}$ is the Lie subalgebra generated by the copy of T inside A_T. Now, inside $\mathscr{L}$, take the ideal I generated by all elements $[[a, b], c] - [abc]$, a, b, c in T. The quotient $L_u(T) = \mathscr{L}/I$ is a Lie algebra into which T imbeds (let η_T send $a \in T$ to its coset $\bar{a}$) and which has the desired universal property, induced from the universal property of $\mathscr{L}$. Moreover, by construction, $L_u(T) = \eta_T(T) + [\eta_T(T), \eta_T(T)]$, whence $L_u(T) = T \oplus [T, T]$.

In particular, there is a unique Lie algebra homomorphism

$$\Phi \colon L_u(T) \to L_s(T), \tag{2.0.19}$$

which is the identity on T. Necessarily, by our earlier observations, $\operatorname{Ker}(\Phi) \subset [T, T] \subset L_u(T)$. In fact, we can identify $\operatorname{Ker}(\Phi)$ more precisely as a consequence of the next lemma. Before this, however, let us introduce some more pieces of notation. If $L_\varphi = \varphi(T) + [\varphi(T), \varphi(T)]$ is any enveloping Lie algebra of an imbedding φ of T, we may use $[\ ,\ ]_\varphi$ to denote the bracket operation in $L_\varphi(T)$. In the particular case $L_\varphi = L_s(T)$ (resp., $L_\varphi = L_u(T)$), we will simply employ the notation $[\ ,\ ]_s$ (resp., $[\ ,\ ]_u$). Later, of course, if no confusion will result we may just use the plain brackets $[\ ,\ ]$. Finally, for any Lie algebra $\mathfrak{g}$, we will let $Z(\mathfrak{g})$ denote its center.

LEMMA 2.0.20. *Suppose $\varphi\colon T \to \mathfrak{g}$ is any injective imbedding of a LTS T into a Lie algebra $\mathfrak{g}$ for which $L_\varphi = T \oplus [T, T]$. Let $\phi\colon L_\varphi \to L_s(T)$ denote the extension for which $\phi \circ \varphi = \iota_T$, ι_T being the imbedding of T into $L_s(T)$ as before. Then* $\operatorname{Ker}(\phi) = Z(L_\varphi) \cap [T, T]_\varphi$.

Proof. By assumption, φ is an injection, hence $\operatorname{Ker}(\phi) \subset [T, T]_\varphi$. Suppose $a \in L_\varphi$. Now, observe that $a \in Z(L_\varphi)$ if and only if $[a, T]_\varphi = 0$; one direction is immediate, while the other follows from the fact that $[a, T]_\varphi = 0$ implies $[a, [T, T]_\varphi]_\varphi = 0$ by an application of the Jacobi identity. Next, by construction of $L_s(T)$, for $b \in [T, T]_s$, $[b, T]_s = 0$ implies $b = 0$. Thus, using the universal property of $L_s(T)$, for $a \in [T, T]_\varphi$, the following are equivalent:

(a) $a \in Z(L_\varphi)$,

(b) $[a, T]_\varphi = 0$,

(c) $\phi([a, T]_\varphi) = [\phi(a), T]_s = 0$,

(d) $a \in \operatorname{Ker}(\phi)$.

This completes the proof. ∎

The following result (stated without proof on p. 151 of [6]) yields the promised characterization of $\operatorname{Ker}(\Phi)$.

COROLLARY 2.0.21. *Let T be a Lie triple system over k, with universal enveloping algebra $L_u(T)$. Then*

$$\operatorname{Ker}(\Phi) = Z(L_u(T)) \cap [T,T]_u. \tag{2.0.22}$$

As an aside, we note that (with considerably more effort) one can also, in fact, first prove (2.0.21) using the universal properties of $L_u(T)$ and $L_s(T)$ and then deduce (2.0.20) as a consequence, as was done in [8]. We thank J. Faulkner for proposing the more direct approach employed above.

Later in this paper, as we consider the restricted structure for a Lie triple system $\mathfrak{p}$ arising as the -1-eigenspace of an involution induced from an involution of an algebraic group as in (2.0.9), we will again examine relationships between the universal enveloping Lie algebra $L_u(\mathfrak{p})$ and the standard enveloping Lie algebra $L_s(\mathfrak{p})$. We will, in that particular context, prove that, in many situations, $L_u(\mathfrak{p}) \cong L_s(\mathfrak{p})$. For now, we offer an example to demonstrate that $L_s(T) \not\cong L_u(T)$ in general.

EXAMPLE 2.0.23. Suppose that T has $\dim_k(T) = n$ and that $[abc] = 0$ for all a, b, c in T. Reviewing the construction of $L_s(T)$, we find that $V_T = 0$ and $L_s(T)$ is simply the n-dimensional Lie algebra with $[x,y] = 0$ for all $x, y \in L_s(T)$. On the other hand, note that the second exterior power $\wedge^2(T)$ on the vector space T is an $\frac{n(n-1)}{2}$-dimensional vector space, hence the vector space $L = T \oplus \wedge^2(T)$ becomes an $n + \frac{n(n-1)}{2}$-dimensional Lie algebra upon setting $[X,Y] = 0$ if $X \in T$, $Y \in \wedge^2(T)$ or if $X, Y \in \wedge^2(T)$, and $[X,Y] = X \wedge Y$ if $X, Y \in T$. Then $L = L_\varphi$ for the obvious imbedding of T. It now follows from the universal property of $L_u(T)$ that

$$\dim_k(L_u(T)) \geq \dim_k(L) = n + \frac{n(n-1)}{2}, \tag{2.0.24}$$

whence $L_u(T) \not\cong L_s(T)$. We can say more here. By combining (2.0.24) and (2.0.13), we see that $L_u(T) \cong L$ in the particular case under consideration. Furthermore, the extension Φ: $L_u(T) \to L_s(T)$ has $\operatorname{Ker}(\Phi) = [T,T]$, consistent with (2.0.22).

2.1. *Further Connections*. In this subsection, we will explore further connections between a Lie triple system T and certain imbeddings; the focus will be on relationships between T and the standard enveloping Lie algebra $L_s(T)$ accorded by considering the Killing form of $L_s(T)$. Later, in Section 4, we will apply the developments herein to the special case elaborated in Example 2.0.9.

Following [19], for fixed $a, b \in T$, define a k-linear map $R(a,b)$: $T \to T$ by

$$R(a,b)(c) = [cab], \qquad \text{for all } c \in T. \tag{2.1.1}$$

Next, let $\rho(a, b)$: $T \to k$ be the mapping

$$\rho(a,b) = \frac{1}{2}\operatorname{trace}[R(a,b) + R(b,a)].$$

Then ρ: $T \times T \to k$ is a symmetric bilinear form on T. Using the properties of Lie triple systems, [19, p. 59] proves that ρ is associative, in the sense that

$$\rho([abc], d) = \rho(a, [dcb]). \tag{2.1.2}$$

Moreover, it is further shown there that

$$\rho(a,b) = \operatorname{trace} R(a,b), \tag{2.1.3}$$

assuming ρ is nondegenerate. The form ρ provides an analogue for the Killing form on a Lie algebra. In fact, up to a scalar multiple, ρ arises as the restriction of the Killing form κ on $L_s(T)$ (see Chap. VI, Theorem 11, of [19]). As a corollary of this theorem, [19] presents the following result:

PROPOSITION 2.1.4. *The form ρ on a LTS T is nondegenerate if and only if the Killing form κ on the standard enveloping Lie algebra $L_s(T)$ is nondegenerate.*

We now define the *center* $Z(T)$ of a Lie triple system T by

$$Z(T) = \{x \in T \mid [xab] = 0 \text{ for all } a, b \in T\}. \tag{2.1.5}$$

Obviously, $\operatorname{Ker}(R(a, b)) = \{x \in T \mid [xab] = 0\}$ and $Z(T) = \bigcap \operatorname{Ker}(R(a, b))$, taking the intersection over all pairs $(a, b) \in T \times T$. Although much more can be said in this direction by pursuing the parallels with Lie algebra theory (see [19]), for our purposes we wish to develop here only a few more results. First, we give a lemma.

LEMMA 2.1.6. *Let φ: $T \to \mathfrak{g}$ be any injective imbedding of a LTS T. Then*

(1) *$Z(T) \neq 0$ implies $Z(L_\varphi) \neq 0$;*

(2) *if, in addition, $L_\varphi = T \oplus [T, T]$, then $0 \neq Z(L_\varphi) \not\subseteq [T, T]$ implies $Z(T) \neq 0$.*

Proof. Suppose $x \in Z(T)$, $x \neq 0$. Then for any $a, b \in T$, $0 = [xab] = [[x, a], b]$ in L_φ. Thus, by rewriting the Jacobi identity in the form $[X, [Y, Z]] = -[[X, Z], Y] + [[X, Y], Z]$ and taking $X = [x, a]$, $Y = b$, $Z = c$, we find that $[[x, a], [b, c]] = -[[xac], b] + [[xab], c] = 0 - 0 = 0$. As a consequence, $[x, a]$ commutes with every element in T and $[T, T]$, whence it follows that $[x, a] \in Z(L_\varphi)$ for all $a \in T$. Note that the calculations above yield $x \in Z(L_\varphi)$ if $[x, a] = 0$ for all $a \in T$; otherwise, for some $a \in T$, $[x, a]$ is nonzero. In either case, $Z(L_\varphi) \neq 0$, so (1) holds.

Now suppose $L_\varphi = T \oplus [T, T]$ and $Z(L_\varphi) \neq 0$. If $S = Z(L_\varphi) \cap T \neq 0$, then for any $0 \neq x \in S$ and $a, b \in T \subset L_\varphi$,

$$[xab] = [[x, a], b] = [0, b] = 0. \tag{2.1.7}$$

Thus, $Z(T) \neq 0$. Now suppose $S = 0$, so that $x \in Z(L_\varphi)$ must have the form $x = a' + \sum_i [b_i', c_i']$, $a', b_i', c_i' \in T$, with each $[b_i', c_i'] \neq 0$. Then for any $a \in T$, $0 = [x, a] = [a', a] + \sum_i [[b_i', c_i'], a] = [a', a] + \sum_i [b_i' c_i' a]$, so $[a', a] = -\sum_i [b_i' c_i' a]$. Now, by assumption $[T, T] \cap T = 0$, whence we conclude that $[a', a] = 0$ for all $a \in T$. If $a' \neq 0$, then $Z(T) \neq 0$, following the calculation (2.1.7) above. On the other hand, if $a' = 0$, we are reduced to the case $x = \sum_i [b_i', c_i']$; i.e., $Z(L_\varphi) \subset [T, T]$. This proves (2). ▮

COROLLARY 2.1.8. *The center $Z(L_s(T))$ of the standard enveloping Lie algebra of a LTS T is nonzero if and only if $Z(T)$ is nonzero.*

Proof. In light of (2.1.6), we must only demonstrate that $Z(L_s(T)) \cap [T, T] = 0$, for then $Z(L_s(T)) \neq 0$ will imply $Z(T) \neq 0$. Now, if $h = \sum_i [a_i, b_i] \in Z(L_s(T)) \cap [T, T]$, then in particular $0 = [h, c] = \sum_i [[a_i, b_i], c] = \sum_i [a_i b_i c]$ for all $c \in T$. By the construction of $L_s(T)$, this occurs only when $h = 0$. ▮

Reconsidering (2.0.23) in light of (2.1.8), we note that there $T = Z(T)$ and $L_s(T)$ was an n-dimensional abelian Lie algebra. On the other hand, $L_u(T)$ was not abelian; in fact, $Z(L_u(T)) = \wedge^2(T)$.

THEOREM 2.1.9. *Suppose T is a LTS with $Z(T) = 0$. Then $L_u(T)/Z(L_u(T)) \cong L_s(T)$.*

Proof. By (2.1.6), $Z(T) = 0$ implies $Z(L_u(T)) \subset [T, T]_u$. Hence, (2.0.21) shows that the surjective morphism $\Phi\colon L_u(T) \to L_s(T)$ has $\operatorname{Ker}(\Phi) = Z(L_u(T))$. ▮

Finally, we will conclude this section with a handy (intrinsic) condition for ensuring that a Lie triple system has a zero center.

THEOREM 2.1.10. *Suppose the bilinear form ρ on a LTS T is nondegenerate. Then $Z(T) = 0$.*

Proof. By assumption, ρ is nondegenerate, hence, by (2.1.4), κ is nondegenerate on the Lie algebra $L_s(T)$. Since (e.g., see [23, Theorem I.7.1]) any abelian ideal of $L_s(T)$ is contained in the radical of κ, the only abelian ideal in $L_s(T)$ is 0. In particular, $Z(L_s(T)) = 0$. Thus, by (2.1.8), $Z(T) = 0$. ▮

2.2. *$L_s(T)$ Once Again.* Given a Lie triple system T, [19] presents the standard enveloping Lie algebra $L_s(T)$ in terms of inner derivations. We will find this second approach instrumental to setting up the restricted theory of Lie triple systems in the next section. In order to describe this formulation of $L_s(T)$, we propose first to consider some general results regarding the derivations of a Lie triple system.

Recall that the associative algebra structure of $\mathrm{End}_k(T)$ gives rise to a Lie algebra structure in the usual way; i.e., define $[f, g] = fg - gf$. For a Lie triple system T, let $\mathrm{Der}(T) = \mathrm{Der}_k(T)$ denote the set of LTS derivations in T. Then we have

PROPOSITION 2.2.1. *Suppose T is a Lie triple system over the field k. Then $\mathrm{Der}(T)$ is a Lie algebra under the bracket operation $[D_1, D_2] = D_1 D_2 - D_2 D_1$ taken in $\mathrm{End}_k(T)$.*

Proof. The proof just follows the same steps as the proof that $\mathrm{Der}(L)$ is a Lie algebra for any Lie algebra L. For $D_1, D_2 \in \mathrm{Der}(T)$ and $\alpha \in k$, it is easy to check that $D_1 + D_2, \alpha D_1 \in \mathrm{Der}(T)$, whence $\mathrm{Der}(T)$ is a subspace of $\mathrm{End}_k(T)$. Next,

$$\begin{aligned} D_1 D_2([abc]) &= D_1([D_2(a)bc] + [aD_2(b)c] + [abD_2(c)]) \\ &= [D_1 D_2(a)bc] + [D_2(a)D_1(b)c] + [D_2(a)bD_1(c)] \\ &\quad + [D_1(a)D_2(b)c] + [aD_1 D_2(b)c] + [aD_2(b)D_1(c)] \\ &\quad + [D_1(a)bD_2(c)] + [aD_1(b)D_2(c)] + [abD_1 D_2(c)]. \end{aligned}$$

Switching indices gives an analogous expression for $D_2 D_1([abc])$; from this we see that

$$\begin{aligned} &D_1 D_2([abc]) - D_2 D_1([abc]) \\ &\quad = ([D_1 D_2(a)bc] - [D_2 D_1(a)bc]) \\ &\qquad + ([aD_1 D_2(b)c] - [aD_2 D_1(b)c]) \\ &\qquad + ([abD_1 D_2(c)] - [abD_2 D_1(c)]). \end{aligned}$$

Therefore,

$$\begin{aligned} [D_1, D_2]([abc]) &= [[D_1, D_2](a)bc] + [a[D_1, D_2](b)c] \\ &\quad + [ab[D_1, D_2](c)], \end{aligned}$$

hence $[D_1, D_2] \in \mathrm{Der}(T)$. This shows that $\mathrm{Der}(T)$ becomes a subalgebra of the Lie algebra $\mathrm{End}_k(T)$. ∎

Next, recall the LTS derivation $D_{a,b}$ in T, defined for any $a, b \in T$ as in (2.0.6). Inside $\operatorname{Der}(T)$, take the subspace $\operatorname{InnDer}(T)$ spanned by all $D_{a,b}$, a, b in T. Thus, by the trilinearity of the triple product, each $h \in \operatorname{InnDer}(T)$ has the form $h = \sum_i D_{a_i, b_i}$, for some collection $\{a_i, b_i\}$ of elements in T. The subspace $\operatorname{InnDer}(T)$ is, moreover, a Lie algebra, as a consequence of the following lemma which appears in [19].

LEMMA 2.2.2. *Let* $D \in \operatorname{Der}(T)$ *and* $D_{x,y} \in \operatorname{InnDer}(T)$. *Then we have*

$$\left[D, D_{x,y}\right] = D_{D(x),y} + D_{x,D(y)}. \tag{2.2.3}$$

Proof. For any $z \in T$,

$$\begin{aligned}(D_{D(x),y} + D_{x,D(y)})(z) &= [D(x)yz] + [xD(y)z] \\ &= D([xyz]) - [xyD(z)] \\ &= DD_{x,y}(z) - D_{x,y}D(z) \\ &= \left[D, D_{x,y}\right](z),\end{aligned} \tag{2.2.4}$$

verifying the claim. ∎

Note that, in the special case in which we take $D = D_{a,b}$, (2.2.3) becomes

$$\left[D_{a,b}, D_{x,y}\right] = D_{[abx],y} + D_{x,[aby]}. \tag{2.2.5}$$

In this particular setting, the calculation (2.2.4) amounts to rearranging (2.0.4) to produce

$$[xy[abz]] - [ab[xyz]] = [[abx]yz] + [x[aby]z].$$

COROLLARY 2.2.6. *For a Lie triple system* T, $\operatorname{InnDer}(T)$ *is an ideal of* $\operatorname{Der}(T)$ (*hence a Lie algebra*).

Now, for a Lie triple system T, set $L_D(T)$ to be the k-vector space

$$L_D(T) = \operatorname{Der}(T) \oplus T. \tag{2.2.7}$$

It is proved in [19, Chap. V, Sect. 6.5] that a Lie algebra structure may be defined on $\operatorname{Der}(T)$ via setting, for any $D_1, D_2 \in \operatorname{Der}(T)$ and $a_1, a_2 \in T$,

$$\begin{aligned}[D_1 + a_1, D_2 + a_2] &= \left([D_1, D_2] + D_{a_1,a_2}\right) + \left(D_1(a_2) - D_2(a_1)\right) \\ &\in \operatorname{Der}(T) \oplus T.\end{aligned} \tag{2.2.8}$$

Under this operation, (2.2.6) shows that $\operatorname{InnDer}(T) \oplus T$ is a Lie subalgebra of $L_D(T)$.

For completeness, we now verify that the two proposed constructions of $L_s(T)$ indeed give the same Lie algebra (this does not appear in [19]).

THEOREM 2.2.9. *Let T be a Lie triple system. Then $L_s(T) \cong \mathrm{InnDer}(T) \oplus T$ as Lie algebras.*

Proof. Let $\varphi: T \to \mathrm{InnDer}(T) \oplus T$ by $\varphi(a) = 0 + a$. Then it follows from (2.2.8) that $[\varphi(T), \varphi(T)] = \mathrm{InnDer}(T)$, so by the universal mapping property of $L_s(T)$ there is a (unique) surjective Lie algebra homomorphism $\phi: \mathrm{InnDer}(T) \oplus T \to L_s(T)$, which is injective on T. By (2.0.20), $\mathrm{Ker}(\phi) = Z(L_\varphi) \cap [T, T]$. If $h = \sum_i D_{a_i, b_i} \in \mathrm{Ker}(\phi)$, then in particular for any $c \in T$, $0 = [h, c] = \sum_i [D_{a_i, b_i}, c] = \sum_i D_{a_i, b_i}(c)$, using (2.2.8). This shows that the derivation h is identically zero on T. Thus, $\mathrm{Ker}(\phi) = 0$, and ϕ is then an isomorphism. ∎

This alternate approach to $L_s(T)$ as an ideal of $L_D(T)$ will be helpful in setting up the restricted structure of a Lie triple system by enabling us to work in the broader context of T and *all* its derivations. Finally, to conclude this section, we examine a scenario in which $L_s(T)$ and $L_D(T)$ are one and the same.

THEOREM 2.2.10. *Let T be a LTS for which ρ is nondegenerate. Then $L_s(T) = L_D(T)$.*

Proof. We modify the proof of Theorem 10 (p. 57) of [19]. It will suffice to show $\mathrm{Der}(T) = \mathrm{InnDer}(T)$. By (2.1.4), κ is nondegenerate on $L_s(T)$. Under this condition, Zassenhaus's Lemma (e.g., see [23, Corollary to Theorem I.8.1]) states that any derivation of the Lie algebra $L_s(T)$ is inner. Because $L_s(T)$ is an ideal in $L_D(T)$, any $D \in \mathrm{Der}(T)$ defines a derivation $\mathrm{ad}\, D$ of $L_s(T)$. Thus, $\mathrm{ad}\, D = \mathrm{ad}(D' + a)$ for some $D' \in \mathrm{InnDer}(T)$ and $a \in T$. However, by assumption $Z(L_s(T)) = 0$ (e.g., see the proof of (2.1.10)), whence $\mathrm{ad}\, D = \mathrm{ad}(D' + a)$ implies $D = D' + a$. Finally, $\mathrm{Der}(T) \cap T = 0$, so we conclude $D = D' \in \mathrm{InnDer}(T)$, as desired. ∎

3. RESTRICTED LIE TRIPLE SYSTEMS

With the results of the last section in hand, we can make our first move toward developing a restricted structure for Lie triple systems. Seeking further inspiration from our knowledge of Lie algebras (and algebraic groups), suppose G is an algebraic group over k with coordinate algebra $k[G]$. Then $\mathfrak{g} = \mathrm{Lie}(G)$ may be defined as certain "left invariant" derivations of $k[G]$; i.e., $\mathfrak{g} = \{D \in \mathrm{Der}_k(k[G]) \mid D \circ \lambda_x = \lambda_x \circ D \text{ for all } x \in G\}$, where $\lambda_x: k[G] \to k[G]$ by $(\lambda_x f)(y) = f(x^{-1}y)$ for all $f \in k[G]$, $y \in G$.

As a consequence, $\mathfrak{g}$ is a restricted Lie algebra. In general, any Lie algebra $\mathfrak{g}$ over k is said to be a *restricted Lie algebra* (or sometimes a *p-restricted Lie algebra*) if $\mathfrak{g}$ carries the additional structure of a mapping $X \mapsto X^{[p]}$ such that for all $X, Y \in \mathfrak{g}$ and all $\alpha \in k$, the properties

$$(\alpha X)^{[p]} = \alpha^p X^{[p]}, \tag{3.1}$$

$$[X^{[p]}, Y] = (\mathrm{ad}\, X)^p(Y), \tag{3.2}$$

and

$$(X+Y)^{[p]} = X^{[p]} + Y^{[p]} + \sum_{i=1}^{p-1} s_i(X,Y), \tag{3.3}$$

where $is_i(X,Y)$ may be regarded as the coefficient of λ^{i-1} in the formal expression $(\mathrm{ad}(\lambda X + Y))^{p-1}(X)$ (e.g., see [13, Chap. 5, Sect. 7]), hold.

Now, any associative algebra $\mathscr{R}$ gives rise to a Lie algebra $\mathscr{R}_L$ by taking $[X,Y] = XY - YX$, X, $Y \in \mathscr{R}$. Since $\mathfrak{g} = \mathrm{Lie}(G)$ is a Lie subalgebra of $(\mathrm{End}_k(k[G]))_L$, the Leibniz rule implies that for any $D \in \mathfrak{g}$, D^p is another derivation which is, furthermore, left invariant since D is. Under such circumstances, setting $D^{[p]} = D^p$ determines a restricted Lie algebra structure on $\mathfrak{g}$, as a consequence of the general result below (see, for example, [23]), which we shall need later.

PROPOSITION 3.4. *Let $\mathscr{R}$ be an associative algebra over a field of positive characteristic p. Then any Lie subalgebra $\mathfrak{g}$ of $\mathscr{R}_L$ is a restricted Lie algebra if $\mathfrak{g}$ is closed under pth powers in $\mathscr{R}$.*

Our initial step, then, is to prove a "Leibniz rule" for derivations of a Lie triple system T. In the interest of conciseness, in the lemma and its proof below we will omit parentheses where this will cause no confusion, writing $[Dxyz]$ for $[D(x)yz]$, etc.

LEMMA 3.5. *Let $D \in \mathrm{Der}(T)$. Then, for any integer $n \geq 1$,*

$$D^n[xyz] = \sum_{\substack{0 \leq i,j,k \leq n \\ i+j+k=n}} \binom{n}{ijk} [D^i x D^j y D^k z] \tag{3.6}$$

(*here*, $D^0 := 1$ *and* $\binom{n}{ijk} := \frac{n!}{i!j!k!}$).

Proof. The proof is by induction. The case $n = 1$ is just the definition (2.0.7). Suppose the result holds for n. We wish to show

$$D^{n+1}[xyz] = \sum_{\substack{0 \leq i,j,k \leq n+1 \\ i+j+k=n+1}} \binom{n+1}{ijk} [D^i x D^j y D^k z].$$

We have

$$\begin{aligned} D^{n+1}[xyz] &= DD^{n}[xyz] \\ &= D \sum_{i+j+k=n} \binom{n}{ijk} [D^i x D^j y D^k z] \\ &= \sum_{i+j+k=n} \binom{n}{ijk} D[D^i x D^j y D^k z] \\ &= \sum_{i+j+k=n} \binom{n}{ijk} \big([D^{i+1} x D^j y D^k z] + [D^i x D^{j+1} y D^k z] \\ &\qquad + [D^i x D^j y D^{k+1} z] \big). \end{aligned}$$

Define $\binom{n}{lmn} = 0$ if any one of l, m, n is negative. Now, the coefficient of the term $[D^l x D^m y D^n z]$ in the expression $\sum_{i+j+k=n}\binom{n}{ijk}([D^{i+1}xD^jyD^kz] + [D^ixD^{j+1}yD^kz] + [D^ixD^jyD^{k+1}z])$ is $\binom{n}{(l-1)mn} + \binom{n}{l(m-1)n} + \binom{n}{lm(n-1)} = \binom{n+1}{lmn}$. From this, we can conclude that $D^{n+1}[xyz] = \sum_{0 \le i,j,k \le n+1, i+j+k=n+1}\binom{n+1}{ijk}[D^ixD^jyD^kz]$. By induction, the claim holds for all $n \ge 1$. ∎

COROLLARY 3.7. *Let T be a Lie triple system over k. Let D be any derivation of T. Then D^p is also a derivation of T, where the pth power is taken in the associative algebra* $\mathrm{End}_k(T)$.

Proof. By (3.5), $D^p([xyz]) = \sum_{0 \le i,j,k \le p, i+j+k=p}\binom{p}{ijk}[D^i(x)D^j(y)D^k(z)]$, for all x, y, z in T. Each trinomial coefficient $\binom{p}{ijk}$ $(0 \le i,j,k \le p)$ is divisible by p (and thus equals 0 in k) unless one of i, j, k equals p. Therefore the expression for $D^p([xyz])$ reduces to $D^p([xyz]) = [D^p(x)yz] + [xD^p(y)z] + [xyD^p(z)]$. Moreover, D^p is linear, since D is. Thus, D^p is a derivation of T. ∎

Our further development of the notion of a restricted Lie triple system will be modeled with the aim in mind that T should be restricted if $L_s(T)$ is a restricted Lie algebra. The first property (3.1) of a restricted Lie algebra is easy to emulate in the LTS setting; that is, we can begin by supposing that a restricted Lie triple system T should carry a mapping $a \mapsto a^{[p]}$ under which $(\alpha a)^{[p]} = \alpha^p a^{[p]}$ for all $\alpha \in k$ and $a \in T$. Consequently, our work will focus on adapting (3.2) and (3.3).

Beginning with (3.3), we examine in greater detail the nature of the elements $s_i(X,Y)$. Specifically, let L be any Lie algebra over k (not necessarily restricted). For any $X, Y \in L$, $s_i(X,Y)$ is defined as above;

namely, $is_i(X,Y)$ is the coefficient of λ^{i-1} in $(\mathrm{ad}(\lambda X + Y))^{p-1}(X)$. By computing $(\mathrm{ad}(\lambda X + Y))^{p-1}(X)$, one finds that[1]

- if $p = 3$,

$$\begin{aligned} s_1(X,Y) &= [[X,Y],Y], \\ 2s_2(X,Y) &= [[X,Y],X]; \end{aligned}$$

- if $p = 5$,

$$\begin{aligned} s_1(X,Y) &= [[[[X,Y],Y],Y],Y], \\ 2s_2(X,Y) &= [[[[X,Y],X],Y],Y] + [[[[X,Y],Y],X],Y] \\ &\quad + [[[[X,Y],Y],Y],X], \\ 3s_3(X,Y) &= [[[[X,Y],X],X],Y] + [[[[X,Y],X],Y],X] \\ &\quad + [[[[X,Y],Y],X],X], \text{ and} \\ 4s_4(X,Y) &= [[[[X,Y],X],X],X]. \end{aligned}$$

Likewise, the pattern continues for $p \geq 7$. If we now suppose that $L = L_s(T)$ for some Lie triple system T (or, more generally, that $L = L_\varphi$ for some injective imbedding of T) and we take $X = a$, $Y = b \in T$, then the identities above can be rewritten in terms of the triple product on T as

- $s_1(a,b) = [abb]$ and $2s_2(a,b) = [aba]$, if $p = 3$;
- $s_1(a,b) = [[abb]bb]$, $2s_2(a,b) = [[aba]bb] + [[abb]ab] + [[abb]ba]$, $3s_3(a,b) = [[aba]ab] + [[aba]ba] + [[abb]aa]$, and $4s_4(a,b) = [[aba]aa]$, if $p = 5$.

Similarly, for $p \geq 7$, the values of $s_i(a,b)$ (up to scalar multiples) can be interpreted as triple products of elements in T. In general, each summand of $is_i(a,b)$ involves $p - 1$ applications of the bracket operator in the Lie algebra; since this is an even number for $p \geq 3$, the results can be rewritten in terms of the triple product by grouping together the terms in the two innermost brackets, then the next two innermost brackets, and so on. More formally, observe that the decomposition of L associated to the eigenspaces of the unique involution of (2.0.16),

$$L = T \oplus [T,T] := L_1 \oplus L_0, \tag{3.8}$$

[1] The reader who wishes to compare this paper with [13] will note our ad acts on the left, instead of the right. However, under the assumption $p > 2$, $p - 1$ is even, hence $(\mathrm{ad}(\lambda X + Y))^{p-1}(X) = [(\lambda X + Y), [\cdots [(\lambda X + Y), X]] \cdots] = (-1)^{p-1}[[\cdots [X, (\lambda X + Y)], \dots], (\lambda X + Y)] = [[\cdots [X, (\lambda X + Y)], \dots], (\lambda X + Y)]$, which is $X(\mathrm{ad}(\lambda X + Y))^{p-1}$ is the notation of [13].

is a $\mathbb{Z}_2$-grading, in the sense that $[L_i, L_j] \subset L_{i+j}$, with subscripts read mod 2. Thus by the $\mathbb{Z}_2$-grading on $L_s(T)$, for $a, b \in T$, $is_i(a,b) \in T$, since $T = L_1$ and p is odd.

PROPOSITION 3.9. *Let T be a Lie triple system. For $a, b \in T$, define $s_i(a,b) \in L_s(T)$ by requiring that $is_i(a,b)$ be the coefficient of λ^{i-1} in $(ad(\lambda a + b))^{p-1}(a) \in L_s(T)$. Then $s_i(a,b) \in T$.*

The calculations above suggest that we would profit from some new notation. Let L be any Lie algebra. For any elements $x_1, x_2, \dots, x_n$ of L, set

$$(x_1, x_2, \dots, x_n) := [[\cdots[[x_1, x_2], x_3]\cdots], x_n] \in L. \tag{3.10}$$

For example, specializing to the setting $L = L_s(T)$ for T a Lie triple system, for any $a, b \in T$, each $is_i(a,b) = (a, b, \epsilon_1, \dots, \epsilon_n) \in T$, where ϵ_j equals one of a or b.

We are now at the point where we can adapt the properties (3.1) and (3.3) to the Lie triple system setting. At this stage, then, we propose

DEFINITION 3.11. Call a Lie triple system T over k a *restricted Lie triple system* over k if there is given a map $(-)^{[p]}\colon T \to T$ satisfying the following conditions, for all $a, b, c \in T$ and all $\alpha \in k$:

$$(\alpha a)^{[p]} = \alpha^p a^{[p]}, \tag{3.12}$$

$$(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a,b), \tag{3.13}$$

$$[abc^{[p]}] = (a, b, c, \dots, c) \ (p \text{ copies of } c), \tag{3.14}$$

$$[ab^{[p]}c] = (a, b, \dots, b, c) \ (p \text{ copies of } b). \tag{3.15}$$

Notationally, the right-hand sides of (3.14) and (3.15) may be taken in $L_s(T)$, but in fact lie back in T.

Comparing this definition for a restricted Lie triple system with the definition of a restricted Lie algebra, observe that properties (3.14) and (3.15) take the place of the remaining condition to be discussed, (3.2). Note that the comparable element $[a^{[p]}bc] \in T$ is determined by (3.14) and/or (3.15) by applying either (2.0.3) or (2.0.5). To further validate our choice of (3.14) and (3.15), let us show that (3.11) meets the original test we set for it, under some additional hypotheses.

THEOREM 3.16. *Suppose T is a LTS and that $L_s(T)$ is a restricted Lie algebra under $X \mapsto X^{[p]}$. Then T is a restricted Lie triple system if T is closed under the $[p]$-operator.*

Proof. The LTS T is a subspace of $L_s(T)$, and for each $1 \leq i \leq p-1$ and for all $a, b \in T$, $s_i(a, b) \in T$, so (3.1) and (3.3) immediately restrict to give properties (3.12) and (3.13). From (3.2), for any $X, Y \in L_s(T)$, $[X^{[p]}, Y] = (-1)^p(Y, X, \ldots, X) = -(Y, X, \ldots, X)$ (p copies of X). Thus for any $c \in T$, $[ab^{[p]}c] = [[a, b^{[p]}], c] = -[[b^{[p]}, a], c] = (-1)^2(a, b, \ldots, b, c)$. By this, (3.15) holds. Similarly, $[abc^{[p]}] = [[a, b], c^{[p]}] = -[c^{[p]}, [a, b]] = (-1)^2([a, b], c, \ldots, c) = (a, b, c, \ldots, c)$, so (3.14) holds. Thus T is a restricted LTS. ∎

Still assuming for the moment that $L_s(T)$ is restricted, we note that since T is only a subspace of $L_s(T)$, the additional condition requiring closure of T under the $[p]$-operator is not unreasonable. (In general, even Lie subalgebras of a restricted Lie algebra need not be restricted Lie algebras.) However, by using the characterization of T as the -1 eigenspace of an involution θ on $L_s(T)$ as in (2.0.15), we may ensure that T will always be closed under the $[p]$-operator of $L_s(T)$.

THEOREM 3.17. *Suppose T is a Lie triple system for which $L_s(T)$ is a restricted Lie algebra. Then T is closed under the $[p]$-operator $X \mapsto X^{[p]}$ of $L_s(T)$.*

Proof. Viewing T as the -1-eigenspace of θ in (2.0.15), we need only show that $\theta(a^{[p]}) = -a^{[p]}$ for all $a \in T$. Since $L_s(T)$ is restricted, for any $b \in T$,

$$[a^{[p]}, b] = (\operatorname{ad} a)^p(b). \tag{3.18}$$

Now, $p \geq 3$, so

$$\begin{aligned}(\operatorname{ad} a)^p(b) &= [a, [a, [\cdots, [a, b]]] \cdots] \\ &= (-1)^p(b, a, \ldots, a) \\ &= -\big([[\cdots[[[[baa]aa]aa]\cdots]aa], a]\big).\end{aligned} \tag{3.19}$$

That is, upon writing the last expression in terms of the triple bracket in T, it becomes the bracket in $L_s(T)$ of two elements in T, namely, $-[[\cdots[[[baa]aa]aa]\cdots]aa]$ and a. Since θ fixes $[T, T] \subset L_s(T)$, applying θ to both sides of (3.18) produces

$$[\theta(a^{[p]}), \theta(b)] = (\operatorname{ad} a)^p(b), \tag{3.20}$$

whence

$$-[\theta(a^{[p]}), b] = [a^{[p]}, b]. \tag{3.21}$$

From (3.21), we find that $[-\theta(a^{[p]}), b] = [a^{[p]}, b]$ for all $b \in T$.

Now, replacing b with an element of the form $[b,c] \in [T,T]$, we may repeat the calculations (3.19)–(3.21) for

$$\left[a^{[p]},[b,c]\right] = (\operatorname{ad} a)^p([b,c]). \tag{3.22}$$

This time, $(\operatorname{ad} a)^p([b,c]) = (-1)^p(b,c,a,\dots,a) \in T$, hence applying θ to (3.22) produces

$$\left[\theta(a^{[p]}),[b,c]\right] = -(\operatorname{ad} a)^p([b,c]). \tag{3.23}$$

Thus, we find $[-\theta(a^{[p]}),[b,c]] = [a^{[p]},[b,c]]$. Consequently, we may conclude that $[-\theta(a^{[p]}),\Sigma_i[b_i,c_i]] = [a^{[p]},\Sigma_i[b_i,c_i]]$ for an arbitrary element $\Sigma_i[b_i,c_i] \in [T,T]$. Therefore, it follows that $[-\theta(a^{[p]}),X] = [a^{[p]},X]$ for all $X \in L_s(T)$.

The calculation above shows that $-\theta(a^{[p]}) - a^{[p]} \in Z(L_s(T))$. Since $\theta(-\theta(a^{[p]}) - a^{[p]}) = -\theta(a^{[p]}) - a^{[p]}$, we see also that $-\theta(a^{[p]}) - a^{[p]} \in [T,T]$. However (e.g., as shown in the proof of (2.1.8)), $Z(L_s(T)) \cap [T,T] = 0$. Thus, $\theta(a^{[p]}) = -a^{[p]}$, as desired. The theorem follows. ∎

COROLLARY 3.24. *Suppose T is a Lie triple system. If $L_s(T)$ is a restricted Lie algebra, then T is a restricted LTS. Furthermore, if $Z(T) = 0$, the involution θ defined above is an automorphism of the restricted structure on $L_s(T)$.*

Proof. The first statement arises simply by combining (3.17) and (3.16). The proof of (3.17) shows that the $[p]$-operator $(-)^{[p]}$ of $L_s(T)$ sends T back to itself. We need only check that $(-)^{[p]}$ likewise sends $[T,T]$ to itself. However, repeating the argument in (3.17) mutatis mutandis shows, for any $X \in [T,T]$, $\theta(X^{[p]}) - X^{[p]} \in Z(L_s(T))$, and also $\theta(X^{[p]}) - X^{[p]} \in T$. Therefore, by (2.1.8), if $Z(T) = 0$ then we conclude $\theta(X^{[p]}) = X^{[p]}$, as desired. ∎

The next goal of this section will be to look for converses to (3.16) and (3.24), that is, to determine whether, under some hypotheses on a restricted Lie triple system T, the Lie algebra $L_s(T)$ bears the structure of a restricted Lie algebra. Our approach will examine the question for the (possibly larger) Lie algebra $L_D(T)$ of (2.2.7). We will be able to prove the following theorem:

THEOREM 3.25. *Let T be a restricted Lie triple system with $Z(T) = 0$. Then $L_D(T)$ is a restricted Lie algebra, with a restricted structure extending that of T.*

After establishing (3.25), we will examine the implications regarding the existence of a restricted structure on related Lie algebras such as $L_s(T)$.

The proof of (3.25) will require a sequence of lemmas, designed to enable us to meet the requirements of the following criterion, presented in [13, Chap. V, Theorem 11].

THEOREM 3.26. *Let L be a Lie algebra over k with basis $\{u_i\}$ such that for every u_i, $(\operatorname{ad} u_i)^p$ is an inner derivation. For each u_i, let $u_i^{[p]}$ be an element of L such that $(\operatorname{ad} u_i)^p = \operatorname{ad} u_i^{[p]}$. Then there exists a unique mapping $X \mapsto X^{[p]}$ of L into itself such that $u_i^{[p]}$ is as given and L is a restricted Lie algebra relative to the mapping $X \mapsto X^{[p]}$.*

We now embark upon the necessary lemmas. The proof of the first, which will be used to establish (3.29) below, was provided by J. Faulkner.

LEMMA 3.27. *Let L be any Lie algebra over k, and let $X, Y, Z \in L$. Consider the expression $(Y, X, \dots, X, Z, X, \dots, X)$, in which X appears $p - 1$ times. Then*

$$-\sum_{i=1}^{p} (Y, X, \dots, X, Z, X, \dots, X) = (Z, X, \dots, X, Y).$$

Here, in the ith summand on the left, Z appears in the position $i + 1$ (identifying each summand with a $(p + 1)$-tuple of elements of L).

Proof. Observe that, for any positive integer n,

$$\begin{aligned}\sum_{i=1}^{n} &(Y, X, \dots, X, Z, X, \dots, X)\\ &= \sum_{j=0}^{n-1} (\operatorname{ad}(-X))^j\big[(\operatorname{ad}(-X))^{n-1-j}(Y), Z\big].\end{aligned}$$

Applying the Leibniz rule to the right-hand side shows

$$\begin{aligned}\sum_{i=1}^{n} &(Y, X, \dots, X, Z, X, \dots, X)\\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{j} \binom{j}{k}\big[(\operatorname{ad}(-X))^{n-1-k}(Y), (\operatorname{ad}(-X))^{k}(Z)\big]\\ &= \sum_{k=0}^{n-1} c_{k,n}\big[(\operatorname{ad}(-X))^{n-1-k}(Y), (\operatorname{ad}(-X))^{k}(Z)\big],\end{aligned}$$

where by definition $c_{k,n} = \sum_{j=k}^{n-1}\binom{j}{k}$. For k fixed, induction on n shows $\sum_{j=k}^{n-1}\binom{j}{k} = \binom{n}{k+1}$. (The base case $k = n$ is trivial. The induction step to $n + 1$ is almost as easy: it becomes $\sum_{j=k}^{n}\binom{j}{k} = \sum_{j=k}^{n-1}\binom{j}{k} + \binom{n}{k} = \binom{n}{k+1} +$

$\binom{n}{k} = \binom{n+1}{k+1}$, by induction hypothesis.) Thus, switching indices set $i = k + 1$),

$$\sum_{i=1}^{n} (Y, X, \dots, X, Z, X, \dots, X)$$
$$= \sum_{i=1}^{n} \binom{n}{i} [(Y, X, \dots, X), (Z, X, \dots, X)]. \qquad (3.28)$$

In the particular case that $n = p$, this yields

$$\sum_{i=1}^{n} (Y, X, \dots, X, Z, X, \dots, X) = [Y, (Z, X, \dots, X)]$$
$$= -(Z, X, \dots, X, Y),$$

since $p \mid \binom{n}{i}$ for all $1 \leq i < p$. This is the result we wished to establish. ∎

LEMMA 3.29. *Let T be a restricted Lie triple system over k, with $[p]$-operator $a \mapsto a^{[p]}$. Assume, in addition, that $Z(T) = 0$. Let $D \in \mathrm{Der}(T)$, regarded as an element of $L_D(T)$. Then for any $a \in T$,*

$$[a^{[p]}, D] = (\mathrm{ad}\, a)^p (D). \qquad (3.30)$$

Proof. By definition (2.2.8) of the bracket operation in $L_D(T)$, the left-hand side of (3.30) equals $-D(a^{[p]})$. On the other hand, using the notation (3.10), the right-hand side is simply $(-1)^p(D(a), a, \dots, a) = -(D(a), a, \dots, a)$ (p copies of a). Thus, it suffices to show that

$$D(a^{[p]}) = (D(a), a, \dots, a). \qquad (3.31)$$

By assumption, the center of T is trivial, that is, $[abc] = 0$ for all $b, c \in T$ only if $a = 0$. Thus, to verify (3.31) it is enough to show that, for all $b, c \in T$,

$$[D(a^{[p]}) bc] = (D(a), a, \dots, a, b, c). \qquad (3.32)$$

To arrive at the equality (3.32), we begin by applying the derivation D to $[a^{[p]}bc]$,

$$D([a^{[p]}bc]) = [D(a^{[p]}) bc] + [a^{[p]} D(b) c] + [a^{[p]} b D(c)],$$

hence

$$[D(a^{[p]}) bc] = D([a^{[p]}bc]) - [a^{[p]} D(b) c] - [a^{[p]} b D(c)]$$
$$= -D([ba^{[p]}c]) + [D(b) a^{[p]} c] + [ba^{[p]} D(c)], \quad (3.33)$$

by (2.0.5). Using property (3.15) of a restricted Lie triple system and applying the derivation D, the right-hand side of (3.33) now becomes

$$-\Bigg[(D(b),a,\ldots,a,c) + \sum_{i=1}^{p}(b,a,\ldots,D(a),a,\ldots,a,c) + (b,a,\ldots,a,D(c))\Bigg] + (D(b),a,\ldots,a,c) + (b,a,\ldots,a,D(c)), \tag{3.34}$$

where a appears p times in all but the second term, $\sum_{i=1}^{p}(b,a,\ldots,D(a),\ldots,a,c)$. There, the ith summand is determined by the appearance of $D(a)$ in position $i+1$ (identifying each summand with a $(p+2)$-tuple) and with a appearing $p-1$ times. Cancelling like terms in (3.34) and substituting the result into (3.32), we see that we need only show that

$$(D(a),a,\ldots,a,b,c) = -\sum_{i=1}^{p}(b,a,\ldots,a,D(a),a,\ldots,a,c), \tag{3.35}$$

where, as before, in the ith term on the right, $D(a)$ occurs i places down among the a's. Letting d denote $D(a)$, we see that we need

$$(d,a,\ldots,a,b,c) = -\sum_{i=1}^{p}(b,a,\ldots,a,d,a,\ldots,a,c). \tag{3.36}$$

By bracketing both sides of the equation in (3.27) on the right with $c \in T$, we get exactly the form of (3.36). Thus (3.31) holds, whence (3.30) follows. ∎

LEMMA 3.37. *Let T be a restricted Lie triple system, with $[p]$-operator $a \mapsto a^{[p]}$. Then, for all $a, b \in T$,*

$$[a^{[p]}, b] = (\operatorname{ad} a)^{p}(b) \tag{3.38}$$

in the Lie algebra $L_D(T)$.

Proof. Write $D := (\operatorname{ad} a)^p$. Then, by the Leibniz rule, $D \in \operatorname{Der}(L_D(T))$ as the pth power of $\operatorname{ad} a \in \operatorname{Der}(L_D(T))$. Thus, for any $b, c \in T$,

$$[D(b),c] = D([b,c]) - [b,D(c)] \tag{3.39}$$

holds inside $L_D(T)$. Since p is odd, rewriting the right-hand side of (3.39) using (3.10) produces the equivalent equation

$$[D(b),c] = -(b,c,a,\ldots,a) - (c,a,\ldots,a,b). \tag{3.40}$$

The left-hand side of (3.40) is $[(\operatorname{ad} a)^p(b), c]$, by construction. On the other hand, by first applying properties (3.14) and (3.15) of a restricted LTS and then using (2.0.3), we find

$$\begin{aligned} -(b, c, a, \ldots, a) - (c, a, \ldots, a, b) &= -[bca^{[p]}] - [ca^{[p]}b] \\ &= [a^{[p]}bc]. \end{aligned} \tag{3.41}$$

Together (3.40) and (3.41) produce

$$\left[(\operatorname{ad} a)^p(b), c\right] = \left[[a^{[p]}, b], c\right]. \tag{3.42}$$

Again, p is odd, so by using the $\mathbb{Z}_2$-grading (3.8), both $(\operatorname{ad} a)^p(b)$ and $[a^{[p]}, b]$ are in $\operatorname{Der}(T)$ and hence are equal. ∎

A final lemma in our sequence follows.

LEMMA 3.43. *Let T be a Lie triple system over k. Then* $\operatorname{Der}(T)$ *is a restricted Lie algebra under the $[p]$-operator given by setting $D^{[p]} = D^p$, the pth power of $D \in$* $\operatorname{Der}(T)$. *In particular,* $[D^p, D'] = (\operatorname{ad} D)^p(D')$ *for all* $D, D' \in \operatorname{Der}(T)$.

Proof. By (3.7), $\operatorname{Der}(T)$ is closed under pth powers. Taking $[D, D'] = DD' - D'D$ turns $\operatorname{Der}(T)$ into a Lie subalgebra of $(\operatorname{End}_k(T))_L$. Thus, by (3.4), $\operatorname{Der}(T)$ is a restricted Lie algebra. ∎

We are now in a position to establish Theorem 3.25.

Proof of (3.25). For any $D \in \operatorname{Der}(T)$, set $D^{[p]} = D^p$ as in (3.4.3). For $a \in T \subset L_D(T)$, take $a^{[p]}$ to be the image of a under the p-operator on the restricted Lie triple system T. Now, a k-basis B for $L_D(T)$ consists of a basis B_D for $\operatorname{Der}(T)$ together with a basis B_T for T. By (3.26), to ensure that $L_D(T)$ is a restricted Lie algebra, it suffices to show that $[X^{[p]}, Y] = (\operatorname{ad} X)^p(Y)$ for all $Y \in L_D(T)$ and all $X \in B$. Let $Y = D' + b$, $D' \in \operatorname{Der}(T)$, $b \in T$.

First, suppose $X = D \in B_D$. Recall that, by definition (2.2.8) of the Lie algebra bracket in $L_D(T)$, $[D, b] = D(b)$. Thus, $(\operatorname{ad} D)^p(b) = [D, [D, [\cdots [D, b]]] \cdots] = D^p(b)$. Employing this, together with (3.43), we find

$$\begin{aligned} [D^p, D' + b] &= [D^p, D'] + D^p(b) \\ &= (\operatorname{ad} D)^p(D') + (\operatorname{ad} D)^p(b) \\ &= (\operatorname{ad} D)^p(D' + b). \end{aligned}$$

Now suppose $X = a \in B_T$. From (3.29) and (3.37), it follows that

$$\begin{aligned}[a^{[p]}, D' + b] &= [a^{[p]}, D'] + [a^{[p]}, b] \\ &= (\operatorname{ad} a)^p(D') + (\operatorname{ad} a)^p(b) \\ &= (\operatorname{ad} a)^p(D' + b).\end{aligned}$$

From this it follows that $[X^{[p]}, Y] = (\operatorname{ad} X)^p(Y)$ for all $Y \in L_D(T)$ and $X \in B$, hence $L_D(T)$ possesses the structure of a restricted Lie algebra. Moreover, this structure restricts to give the restricted Lie triple system structure of T. ∎

As we know, $L_s(T)$ is a Lie subalgebra of $L_D(T)$. Suppose T is restricted and $Z(T) = 0$, so $L_D(T)$ is a restricted Lie algebra. Inside $L_D(T)$, take the $[p]$-closure $L_s(T)_p$ of $L_s(T)$. By definition, $L_s(T)_p$ is the smallest Lie subalgebra of $L_D(T)$ containing $L_s(T)$ for which $X \in L_s(T)_p$ implies $X^{[p]} \in L_s(T)_p$. Note that since T is closed under the $[p]$-operator,

$$L_s(T)_p = \operatorname{InnDer}(T)_p \oplus T, \tag{3.44}$$

taking the $[p]$-closure $\operatorname{InnDer}(T)_p$ of $\operatorname{InnDer}(T)$ inside the restricted Lie algebra $\operatorname{Der}(T)$. The Lie algebra $L_s(T)_p$ forms a restricted Lie subalgebra of $L_D(T)$ which contains $L_s(T)$.

With this in mind, we shall now see that, under the stronger hypothesis that the restricted LTS T supports a nondegenerate form ρ (as in (2.1.3)), the Lie algebras $L_D(T)$, $L_s(T)_p$, and $L_s(T)$ are isomorphic restricted Lie algebras.

THEOREM 3.45. *Suppose T is a restricted Lie triple system with nondegenerate form ρ. Then $L_D(T) = L_s(T)_p = L_s(T)$ as restricted Lie algebras, and the restricted structure is unique.*

Proof. From (2.1.10), $Z(T) = 0$, hence by (3.25) $L_D(T)$ is a restricted Lie algebra. The nondegeneracy of ρ also implies $L_D(T) = L_s(T)$, as demonstrated in (2.2.10). Thus $L_D(T) = L_s(T)_p = L_s(T)$.

Finally, the Killing form κ on the finite-dimensional Lie algebra $L_s(T)$ is nondegenerate by (2.1.4). In such a setting, a corollary to (3.26) [13, p. 191] states that the restricted structure on $L_s(T)$ is unique. The theorem now follows. ∎

To summarize, in this section we have defined a restricted Lie triple system and have examined the relations between this restricted structure and the presence of a restricted structure on related Lie algebras, the standard Lie algebra in particular. A priori, for T a restricted LTS, an enveloping Lie algebra L_φ of an imbedding φ of T need not be restricted as a Lie algebra. This leads to the question of whether, in the spirit of (3.26), there exist intrinsic conditions which characterize an enveloping Lie algebra as a Lie algebra of a restricted Lie triple system. Upon fixing T, such a characterization may be obtained by modifying the results of [13, pp. 189–192]; the fundamental result here is as follows.

THEOREM 3.46. *Let T be a LTS and $L_\varphi = T \oplus [T,T]$ an enveloping Lie algebra of T. Suppose there exists a basis $\{u_1, \dots, u_r\}$ of T such that, for $1 \le i \le r$, there exists a $v_i \in T$ with the property that*

$$(\operatorname{ad} u_i)^p = \operatorname{ad} v_i : L_\varphi \to L_\varphi.$$

Then there exists a restricted structure $x \mapsto x^{[p]}$ on T such that $u_i^{[p]} = v_i$, $1 \le i \le r$.

Applied to $L_\varphi = L_s(T)$, this approach can also be used to streamline the definition of a restricted Lie triple system. In any case, the arguments involved are straightforward, but the interested reader will find some details in [10], where these ideas are employed to analyze the restricted representation theory of T (see Section 5) and to create a restricted cohomology theory for T.

4. ALGEBRAIC GROUPS AND LIE TRIPLE SYSTEMS

Having tackled Lie triple systems and important imbeddings of them largely in the abstract, in this section we shall return to the more specific setting of Example 2.0.9, raised in the Introduction to this paper and in further detail in Section 2. Thus, we suppose G is a connected, reductive algebraic group over k, with involution $\theta \in \operatorname{Aut}(G)$. Subsequently, $\mathfrak{g} = \operatorname{Lie}(G)$ decomposes as the direct sum $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ of the -1-eigenspace $\mathfrak{p} \neq 0$ and the $+1$-eigenspace $\mathfrak{k}$ of the corresponding involution $\theta \in \operatorname{Aut}(\mathfrak{g})$. As we have seen, $\mathfrak{p}$ is a Lie triple system; the obvious map $\varphi_\mathfrak{p}$: $\mathfrak{p} \to \mathfrak{g}$ is an injective LTS imbedding of $\mathfrak{p}$. Our next lemma shows that $L_{\varphi_\mathfrak{p}}$ is more than just a Lie subalgebra of $\mathfrak{g}$.

PROPOSITION 4.1. *With the above notation, the enveloping Lie algebra $L_{\varphi_\mathfrak{p}}$ is an ideal in $\mathfrak{g}$.*

Proof. The Lie algebra $L_{\varphi_\mathfrak{p}} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is of course a vector space, so to show that $L_{\varphi_\mathfrak{p}}$ is an ideal of $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ we need only show that $[\mathfrak{g}, L_{\varphi_\mathfrak{p}}] \subset L_{\varphi_\mathfrak{p}}$. By the calculation (2.0.10), $[\mathfrak{p}, \mathfrak{p}]$ is a subset of the $+1$-eigenspace $\mathfrak{k}$ of θ. Similarly, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Subsequently, for any $X \in \mathfrak{k}$, Y, $Z \in \mathfrak{p}$,

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]] \in [\mathfrak{p}, \mathfrak{p}],$$

so $[\mathfrak{k},[\mathfrak{p}, \mathfrak{p}]] \subset [\mathfrak{p}, \mathfrak{p}]$. From this, it follows that $[\mathfrak{g}, L_{\varphi_\mathfrak{p}}] \subset L_{\varphi_\mathfrak{p}}$. ∎

Note that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ implies $L_{\varphi_\mathfrak{p}} = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$. Furthermore, under an additional hypothesis, $\mathfrak{g} = L_{\varphi_\mathfrak{p}}$.

LEMMA 4.2. *Suppose* $\mathfrak{g}$ *is simple. Then* $\mathfrak{g}$ *is the enveloping Lie algebra* $L_{\varphi_\mathfrak{p}}$ *of the obvious LTS imbedding* $\varphi_\mathfrak{p}\colon \mathfrak{p} \to \mathfrak{g}$ (*i.e., in* $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$).

Proof. From (4.1), $L_{\varphi_\mathfrak{p}} = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple and $L_{\varphi_\mathfrak{p}} \neq 0$, $\mathfrak{g} = L_{\varphi_\mathfrak{p}}$, with $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$. ∎

More generally, the proof shows that $L_{\varphi_\mathfrak{p}} = \mathfrak{g}$ if an analysis of the ideals of $\mathfrak{g}$ eliminates the possibility that $L_{\varphi_\mathfrak{p}} \neq 0$ can be proper. To this end, as well as to carry out further analysis in this section, we will need the following theorem, along with some properties of a Chevalley basis and an analysis of the possible involutions of certain simple, simply connected algebraic groups. In order not to break up the flow, we have included the determination of involutions as an appendix, along with a brief reminder of the necessary Chevalley basis material for the reader's ease.

THEOREM 4.3 (Hogeweij [11]). *Suppose G is a simple, simply connected algebraic group over an algebraically closed field k* (*recall* $\operatorname{char}(k) = p > 2$). *Then* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. *Also,* $\mathfrak{g}$ *has no nontrivial ideals, except in the following cases*:

(1) A_n, $p \mid (n+1)$,

(2) E_6, $p = 3$, *and*

(3) G_2, $p = 3$.

In Cases (1) *and* (2), *the only nontrivial ideal of* $\mathfrak{g}$ *is the center* $Z(\mathfrak{g})$ *of* $\mathfrak{g}$, *with* $\dim_k(Z(\mathfrak{g})) = 1$. *In Case* (3), $\mathfrak{g}$ *has a single nontrivial ideal, and it is of the form* $\mathfrak{h}_S + \mathfrak{e}_S$, *spanned by the elements* h_α, e_α *in a Chevalley basis* (6.1.4) *for which* α *is a short root. Thus,* $\mathfrak{g}$ *is simple except in Cases* (1), (2), *and* (3).

With our next result, we begin in earnest to link the structure of an algebraic group under an involution with the form of the associated Lie triple system by understanding the relationships between the Lie algebra of the group and enveloping Lie algebras of the Lie triple system.

THEOREM 4.4. *Suppose G is a simple, simply connected algebraic group over k, and* $\theta \in \operatorname{Aut}(G)$ *is an involution. Then* $\mathfrak{g} = [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$; *i.e.,* $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$.

Proof. From (4.2) and (4.3), the theorem holds as long as G is not among the types

(1) A_n, $p \mid (n+1)$,

(2) E_6, $p = 3$, and

(3) G_2, $p = 3$.

In these remaining cases, the nonzero ideal $L_{\varphi_\mathfrak{p}} = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$ must equal $\mathfrak{g}$ if $L_{\varphi_\mathfrak{p}}$ is not the unique nontrivial ideal of $\mathfrak{g}$. Note that, in Cases (1) and (2), this amounts to $\mathfrak{p} = Z(\mathfrak{g})$ if $L_{\varphi_\mathfrak{p}} \neq \mathfrak{g}$, since the one-dimensional ideal $Z(\mathfrak{g})$ is the unique nontrivial ideal in $\mathfrak{g}$. We will now rule out the possibility that $L_{\varphi_\mathfrak{p}} \neq \mathfrak{g}$ by examining each of the cases (1)–(3).

Suppose, in Case (1) or (2), $\mathfrak{p} = Z(\mathfrak{g})$. Since $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, it now follows easily that $\mathfrak{k}$ is an ideal in $\mathfrak{g}$, contradicting the fact that $Z(\mathfrak{g})$ is the only nontrivial ideal.

Finally, in Case (3), recall that, up to conjugacy, (6.2.2) shows that there is only one involution $\theta = \operatorname{Int} t$, and under the associated involution $\theta = \operatorname{Ad}(t)$ of $\mathfrak{g}$, $e_{\alpha_2} \mapsto \alpha_2(t)e_{\alpha_2} = -e_{\alpha_2}$, for α_2 the long simple root. This shows that $\mathfrak{p}$ must contain the root space $\mathfrak{g}_{\alpha_2}$. However, the unique nontrivial ideal $\mathfrak{h}_S + \mathfrak{e}_S$ of $\mathfrak{g}$ contains only short root spaces, so $\mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$ cannot be this ideal. ∎

Thus in the case G is simple and simply connected, we can apply the results developed earlier in this paper to relate $\mathfrak{g} = L_{\varphi_\mathfrak{p}}$ with the two important enveloping Lie algebras $L_s(\mathfrak{p})$ and $L_u(\mathfrak{p})$. Such relations constitute the content of (4.5) and (4.6) below.

THEOREM 4.5. *Assume the hypotheses of* (4.4). *Then* $\mathfrak{g} \cong L_s(\mathfrak{p})$, *except when* $\theta \in Aut(G)$ *is an inner automorphism and* G *is of type*

(1) A_n, $p \mid (n+1)$, *or*

(2) E_6, $p = 3$.

In these exceptional cases, $L_s(\mathfrak{p}) \cong \mathfrak{g}/Z(\mathfrak{g})$.

Proof. From (4.4), $\mathfrak{g} = L_{\varphi_\mathfrak{p}} = \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$. Thus, $\varphi_\mathfrak{p}$ lifts to a surjective Lie algebra homomorphism $\phi\colon \mathfrak{g} \to L_s(\mathfrak{p})$. Except in Cases (1) and (2) of (4.4) above, $Z(\mathfrak{g}) = 0$. According to (2.0.20), $\operatorname{Ker}(\phi) = Z(\mathfrak{g}) \cap [\mathfrak{p}, \mathfrak{p}] = 0 \cap [\mathfrak{p}, \mathfrak{p}] = 0$. Thus, ϕ is an isomorphism.

Our analysis of the remaining possibilities for G will tackle the cases θ is inner and θ is outer separately. First, suppose θ is an inner automorphism of G, in either case (1) or (2), with T, as usual, a fixed maximal torus of G. We will now show that $Z(\mathfrak{g}) \subset [\mathfrak{p}, \mathfrak{p}]$; then, arguing as immediately above, $\operatorname{Ker}(\phi) = Z(\mathfrak{g})$. As in (6.2.2), we may take $\theta = \operatorname{Int} t$ for some semisimple element $t \in T$. Now, T is abelian, so $\operatorname{Ad}|_T$ acts trivially on $\operatorname{Lie}(T)$. Since $Z(\mathfrak{g}) \subset \operatorname{Lie}(T)$, this shows that the associated involution $\theta = \operatorname{Ad}(t)$ on $\mathfrak{g}$ fixes $Z(\mathfrak{g})$ pointwise. Thus, $Z(\mathfrak{g}) \subset [\mathfrak{p}, \mathfrak{p}]$, whence $L_s(\mathfrak{p}) \cong \mathfrak{g}/Z(\mathfrak{g})$ (via ϕ).

Next, suppose θ is an outer automorphism of G, in Case (1) or Case (2). Our argument will proceed as follows. If $\theta' \in \operatorname{Aut}(G)$ is any other involution which is also outer, we will prove that $\theta|_{Z(\mathfrak{g})} = \theta'|_{Z(\mathfrak{g})}$. Using this, it will then be sufficient to show that $Z(\mathfrak{g}) \subset \mathfrak{p}$ for a particular outer involution θ (depending upon Case (1) or (2)). It will then follow that ϕ: $\mathfrak{g} \to L_s(\mathfrak{p})$ is an isomorphism.

Let θ, θ' be as above. Then $\theta' = \theta \circ \operatorname{Int} x$ for some $x \in G$. If $x \in T$, then we have already argued that $\operatorname{Ad}(t)$ fixes $Z(\mathfrak{g})$ pointwise. More generally, any semisimple element belongs to some maximal torus T' and $\operatorname{Lie}(T') \supset Z(\mathfrak{g})$, so if x is semisimple, $\operatorname{Ad}(x)$ fixes $Z(\mathfrak{g})$ identically. Since G is by assumption semisimple and simply connected, the semisimple elements of G are dense in G (see [24]). As a continuous map, $\operatorname{Ad}(x)$ must then fix $Z(\mathfrak{g})$ pointwise, for all $x \in G$. Therefore, the induced involutions θ, θ' on $\mathfrak{g}$ must agree on $Z(\mathfrak{g})$.

We now consider Case (1). For $G = SL_{n+1}(k)$, take θ to be the outer automorphism with $\theta(x) = (x^t)^{-1}$. The associated involution $\theta(X) = -X^t$ on $\mathfrak{g} = \mathfrak{sl}_{n+1}(k)$ takes scalar matrices to their negatives. Since $p \mid (n+1)$, $Z(\mathfrak{g})$ consists of all scalar matrices, so this shows $Z(\mathfrak{g}) \subset \mathfrak{p}$. Thus, $L_s(\mathfrak{p}) \cong \mathfrak{g}$.

Finally, we consider Case (2). The Dynkin diagram associated to $\mathfrak{g}$ is

$$\begin{array}{ccccccccc} & & & & \circ\ \ \alpha_2 & & & & \\ & & & & | & & & & \\ \circ & — & \circ & — & \circ & — & \circ & — & \circ \\ \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \end{array} .$$

As in (6.1.3) let h_i be the element in a Chevalley basis corresponding to α_i. Define $z = -h_1 + h_3 - h_5 + h_6$; z is nonzero by the linear independence of the h_i. Consider now the effect of α_i on z. Recall that $\alpha_i(h_j) = (\alpha_i, \alpha_j^\vee)$, so, for example, reading off the Coxeter matrix of E_6 (e.g., see p. 262 of [1]), we get $\alpha_i(h_i) = 2$, while for $i \neq j$, $\alpha_i(h_j) = -1$ if i and j are adjacent and 0 otherwise. Thus, $\alpha_3(z) = 3 = \alpha_6(z)$, $\alpha_1(z) = -3 = \alpha_5(z)$, and $\alpha_2(z) = 0 = \alpha_4(z)$. However, by assumption $\operatorname{char}(k) = 3$ here, whence in fact $\alpha_i(z) = 0$ for all $i = 1, \ldots, 6$. Thus, $z \in Z(\mathfrak{g})$ and hence is a basis vector for the one-dimensional space $Z(\mathfrak{g})$.

We can assume θ is the involution which corresponds to the graph automorphism which exchanges α_1 and α_6, α_3 and α_5, and fixes α_2 and α_4. Thus, the effect of θ on z is to carry z to its negative. As a consequence, $Z(\mathfrak{g}) \subset \mathfrak{p}$. Therefore, $\mathfrak{g} \cong L_s(\mathfrak{p})$ in this case. The proof is now complete. ∎

COROLLARY 4.6. *Assume the hypotheses of* (4.4). *Unless* $\mathfrak{g}$ *is either of type* A_n, $p \mid (n+1)$, *or of type* E_6, $p = 3$, *we have* $\mathfrak{g} \cong L_s(\mathfrak{p}) \cong L_u(\mathfrak{p})/Z(L_u(\mathfrak{p}))$ (*with* $Z(L_u(\mathfrak{p})) \subset [\mathfrak{p}, \mathfrak{p}]$).

Proof. With the exceptions noted above, $Z(\mathfrak{g}) = 0$, so (4.2) and (2.1.8) guarantee that $Z(\mathfrak{p}) = 0$. Consequently, $L_s(\mathfrak{p}) \cong L_u(\mathfrak{p})/Z(L_u(\mathfrak{p}))$ by (2.1.9), so $\mathfrak{g} = L_s(\mathfrak{p}) \cong L_u(\mathfrak{p})/Z(L_u(\mathfrak{p}))$, by (4.5). ∎

Up to this point, with the exception of some analysis of the exceptional cases (1)–(3) listed in (4.3), the developments in this section have followed simply as special cases of the theory in Section 2, largely independent of the context of $\mathfrak{g} = \mathrm{Lie}(G)$. Employing the theory of central extensions of the Lie algebras of algebraic groups, discussed below, we can in fact establish a stronger relationship between $L_u(\mathfrak{p})$ and $\mathfrak{g}$ than is afforded by (4.6). Let us introduce the material regarding central extensions which we will need. Our references for this material will be [20] and [26]. By definition, a *central extension* of any given Lie algebra $\mathfrak{g}$ is a short exact sequence of Lie algebras

$$0 \to \mathfrak{c} \to \mathfrak{e} \to \mathfrak{g} \to 0, \tag{4.7}$$

for which $\mathfrak{c} \subset Z(\mathfrak{e})$. Central extensions form the objects of a category for which the morphisms are pairs (ψ_0, ψ) of Lie algebra homomorphisms ψ_0: $\mathfrak{c} \to \mathfrak{c}'$, ψ: $\mathfrak{e} \to \mathfrak{e}'$ making the diagram

$$\begin{tikzcd} \mathfrak{c} \arrow[r] \arrow[d, "\psi_0"'] & \mathfrak{e} \arrow[d, "\psi"'] \arrow[dr] & \\ & & \mathfrak{g} \\ \mathfrak{c}' \arrow[r] & \mathfrak{e}' \arrow[ur] & \end{tikzcd}$$

commute. A central extension (4.7) is a *covering* of $\mathfrak{g}$ if $\mathfrak{e} = [\mathfrak{e}, \mathfrak{e}]$; i.e., $\mathfrak{e}$ is *perfect*. A covering (4.7) of $\mathfrak{g}$ is *universal* if for every central extension of $\mathfrak{g}$ there exists a unique morphism from the covering to the central extension. The reader may find a proof of the following theorem in, e.g., [20, Propositions 2 and 3 in Section 1.9].

THEOREM 4.8. *Let $\mathfrak{g}$ be any perfect Lie algebra* (*i.e.*, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$). *Then a universal covering*

$$0 \to \mathfrak{c} \to \mathfrak{e} \to \mathfrak{g} \to 0$$

of $\mathfrak{g}$ exists. Further, the Lie algebra $\mathfrak{e}$ is perfect and isomorphic to its own universal covering.

In such a situation, we will write $\mathfrak{g}^\star$ for $\mathfrak{e}$; $\mathfrak{g}^\star$ is also called the *universal* (*central*) *extension* of $\mathfrak{g}$. Recalling (4.3), we see that for G semisimple and simply connected, $\mathfrak{g} = \mathrm{Lie}(G)$ is perfect and thus possesses a universal central extension $\mathfrak{g}^\star$.

We say $\mathfrak{g}$ equals its own universal cover if $\mathfrak{g}^\star \cong \mathfrak{g}$. As a consequence of our earlier remarks and [26], we have

THEOREM 4.9. *Let G be semisimple and simply connected over k, $p > 2$. Then $\mathfrak{g} = \mathfrak{g}^\star$ except when $p = 3$ and G is either of type A_2 or G_2.*

Moreover, [26] shows that G acts on $\mathfrak{g}^\star$ by an action compatible with the surjective morphism π: $\mathfrak{g}^\star \to \mathfrak{g}$. Let $\mathfrak{z}^\star = \mathrm{Ker}(\pi)$. Then the G-structure of $\mathfrak{z}^\star$ is known and can be used to determine the dimension of $\mathfrak{g}^\star$ in the two exceptional cases above in which $\mathfrak{g}^\star \not\cong \mathfrak{g}$, a fact we will employ just a little later.

We now turn back to our examination of the relations between $L_s(\mathfrak{p})$, $L_u(\mathfrak{p})$, and $\mathfrak{g}$. Having fully characterized the relationship between $L_s(\mathfrak{p})$ and $\mathfrak{g}$ in (4.5), our next goal will be to strengthen our understanding of the one between $L_u(\mathfrak{p})$ and $\mathfrak{g}$, beyond (4.6).

THEOREM 4.10. *Suppose G is a simple, simply connected algebraic group over k. Then $\mathfrak{g} \cong L_u(\mathfrak{p})$, unless $p = 3$ and $\mathfrak{g}$ is either of type A_2 or of type G_2.*

In Section 5, we will employ the isomorphism $\mathfrak{g} \cong L_u(\mathfrak{p})$ to relate modules for the LTS $\mathfrak{p}$ and modules for $\mathfrak{g}$, providing an additional source of interest in (4.10).

Proof of (4.10). Since $\mathfrak{g} = \mathrm{Lie}(G)$ has the form $\mathfrak{g} = L_{\varphi_\mathfrak{p}}$ by (4.2), from the universal property of $L_u(\mathfrak{p})$ there is a surjective morphism $\zeta\colon L_u(\mathfrak{p}) \to \mathfrak{g}$. Moreover, $\Phi\colon L_u(\mathfrak{p}) \to L_s(\mathfrak{p})$ in (2.0.19) factors through ζ. Thus, (2.0.21) implies that

$$0 \to \mathfrak{c}' \to L_u(\mathfrak{p}) \xrightarrow{\zeta} \mathfrak{g} \to 0 \tag{4.11}$$

is a central extension, with $\mathfrak{c}' := \mathrm{Ker}(\zeta) \subset Z(L_u(\mathfrak{p})) \cap [\mathfrak{p}, \mathfrak{p}]_u$.

Moreover, no smaller subalgebra of $L_u(\mathfrak{p})$ maps onto $\mathfrak{g}$. To see this, let $\mathfrak{m}$ be a subalgebra of $L_u(\mathfrak{p})$ such that $\zeta(\mathfrak{m}) = \mathfrak{g}$. For $x \in \mathfrak{p}$ (as a subset of $\mathfrak{g}$), there exists $c_x \in \mathfrak{c}'$ for which $x + c_x \in \mathfrak{m}$. Since $\mathfrak{c}'$ is central, $[x, y]_u = [x + c_x, y + c_y]_u \in [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$. Therefore $[\mathfrak{p}, \mathfrak{p}]_u \subset \mathfrak{m}$. As we already know, $\mathfrak{c}' \subset [\mathfrak{p}, \mathfrak{p}]_u$, thus $\mathfrak{c}' \subset \mathfrak{m}$, whence $\mathfrak{p} \subset \mathfrak{m}$. Therefore $\mathfrak{m} \supset \mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]_u = L_u(\mathfrak{p})$, so we conclude that $\mathfrak{m} = L_u(\mathfrak{p})$.

By (4.8) and the observations following it, a universal cover

$$0 \to \mathfrak{z}^\star \to \mathfrak{g}^\star \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{4.12}$$

of $\mathfrak{g}$ exists, and there is a unique morphism (ψ_0, ψ) from (4.12) to (4.11),

$$\begin{tikzcd} \mathfrak{z}^\star \arrow[r] \arrow[d, "\psi_0"'] & \mathfrak{g}^\star \arrow[d, "\psi"'] \arrow[dr, "\pi"] & \\ \mathfrak{c}' \arrow[r] & L_u(\mathfrak{p}) \arrow[r, "\zeta"'] & \mathfrak{g}. \end{tikzcd}$$

From our comments above, ψ is necessarily surjective. However, by (4.9), $\mathfrak{g}^\star \cong \mathfrak{g}$. Thus, identifying $\mathfrak{g}^\star$ and $\mathfrak{g}$ via π, the commutativity of the triangle in the diagram above produces a splitting ψ of the morphism ζ: $L_u(\mathfrak{p}) \to \mathfrak{g}$. Therefore $\mathfrak{g} \cong L_u(\mathfrak{p})$, as claimed. ∎

Thus, almost always $\mathfrak{g} \cong L_u(\mathfrak{p})$. Now, assume moreover that G is defined and split over $\mathbb{F}_p$ and that θ is defined over $\mathbb{F}_p$. Then, by employing more involved arguments based on the material in [26], we may show that $L_u(\mathfrak{p}) \not\cong \mathfrak{g}$, if G is of type A_2 or G_2 and $p = 3$. Moreover, our methods will provide a unifying description of $L_u(\mathfrak{p})$ for any type, and for value of $p > 2$. Let us sketch the approach and results before proceeding with the formal details in (4.13)–(4.20) below.

In the A_2 case, we can describe $L_u(\mathfrak{p})$ as a nonsplit central extension of $\mathfrak{g}$ of dimension either 10 or 11; while in the G_2 case, $L_u(\mathfrak{p})$ is a nonsplit central extension of $\mathfrak{g}$ of dimension 17. In each of these cases, $\mathfrak{g}^\star$ is no longer isomorphic to $\mathfrak{g}$; but by using fact that $L_u(\mathfrak{p})$ is still a central extension of $\mathfrak{g}$, we can show that $L_u(\mathfrak{p})$ arises as a quotient of $\mathfrak{g}^\star$. By this means we can determine $\dim_k(L_u(\mathfrak{p}))$. In fact, for $\mathfrak{z}^\star$ as in (4.12), $L_u(\mathfrak{p}) \cong \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$, where $\mathfrak{p}^\star$ is the -1-eigenspace of an involution $\theta^\star$ on $\mathfrak{g}^\star$ induced from the involution θ. More generally, we will see that $L_u(\mathfrak{p}) \cong \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$ for G of any type with no additional restrictions on p; the result (4.10) may subsequently be reworked to follow as a corollary. Carrying out the program above will provide a complete characterization of $L_u(\mathfrak{p})$ and its relationship with $\mathfrak{g}$.

We now return to the process of identifying $L_u(\mathfrak{p})$. Our first proof will be somewhat terse, and the interested reader will wish to consult [26]. By [26], $\mathfrak{g}^\star$ possesses a rational G-action compatible with the surjective morphism $\pi: \mathfrak{g}^\star \to \mathfrak{g}$. Let T be a fixed maximal torus of G, $X(T)^+$ the dominant weights, and $L(\lambda)$ the (up to isomorphism) irreducible rational G-module with high weight $\lambda \in X(T)^+$. For such a module $L(\lambda)$, $L(\lambda)^{(1)}$ will denote the "twist" of $L(\lambda)$ via the Frobenius automorphism of G.

We may now record the resulting G-module structure of $\mathfrak{z}^\star$ (in the nontrivial cases) as follows.

LEMMA 4.13. *Let G be a simple, simply connected algebraic group over k, $p = 3$, of type G_2 or A_2. Assume also that G is defined and split over $\mathbb{F}_p$. In the G_2 case, $\mathfrak{z}^\star \cong L(\lambda_1)^{(1)} \cong L(3\lambda_1)$ (λ_1 the fundamental dominant weight associated to the short simple root α_1 of G_2). In the A_2 case, $\mathfrak{z}^\star \cong L(\lambda_1)^{(1)} \oplus L(\lambda_2)^{(1)} \cong L(3\lambda_1) \oplus L(3\lambda_2)$ (λ_i associated to the ith simple root α_i in the usual labelling). Consequently,* $\dim_k(\mathfrak{z}^\star) = 7$ *in the G_2 case and* 6 *in the A_2 case.*

Proof. We first explore the G_2 case. Note that, in this proof, all (numbered) references should be understood to be from [26], unless otherwise noted. That said, by Corollary 3.14(iv)d, the zero weight space $\mathfrak{z}_0^\star$ of the G-action on $\mathfrak{z}^\star$ is one-dimensional; in particular, $\mathfrak{z}_0^\star \neq 0$. Thus, Parts (i)–(iii) of Proposition 5.2 and Table 1 of [26] imply $L(\lambda_1)^{(1)}(\cong L(3\lambda_1)) \subset \mathfrak{z}^\star$. Now, by (ii) of Corollary 3.14 again, the multiplicity of $3\lambda_1$ in $\mathfrak{g}^\star$ is one, so there exists only one copy of $L(3\lambda_1)$ inside $\mathfrak{z}^\star$. As follows from the description of $L(\lambda_1)$ (e.g., see [2]), $\dim_k(L(3\lambda_1)) = 7$, arising from a one-dimensional zero weight space, with remaining weights given by taking the short roots α_1, $\alpha_1 + \alpha_2$, and $2\alpha_1 + \alpha_2$ and their negatives, the orbit of $3\lambda_1 = \alpha_1$ under the Weyl group. However, by (iii) of Corollary 3.14 and Table 1, the only possible nonzero weights in $\mathfrak{z}^\star$ are precisely the elements of this orbit. Consequently, $\mathfrak{z}^\star$ must be exactly $L(3\lambda_1)$, as claimed.

For the A_2 case, (iv)a of Corollary 3.14 posits that $\mathfrak{z}_0^\star = 0$. In this case, consulting Table 1 and Proposition 5.2, we find that the only possible weights of $\mathfrak{z}^\star$ are in the orbits of $3\lambda_1$ and $3\lambda_2$ under the action of the Weyl group. Also, by Corollary 3.14(ii), the multiplicity of each of these weights in $\mathfrak{g}^\star$ is one. In this case, $3\lambda_1 = 2\alpha_1 + \alpha_2$ and $3\lambda_2 = \alpha_1 + 2\alpha_2$, so neither is a root, i.e., a nonzero weight of the action of G on $\mathfrak{g}$. Since $\mathfrak{g}^\star$ is a central extension of $\mathfrak{g}$, we conclude that $3\lambda_1$ and $3\lambda_2$ must appear in the weight space decomposition of $\mathfrak{z}^\star$. Subsequently, $\mathfrak{z}^\star \cong L(3\lambda_1) \oplus L(3\lambda_2)$. From this, $\dim_k(\mathfrak{z}^\star) = 6$ follows. ▮

COROLLARY 4.14. *Let G be a simple, simply connected algebraic group over k, $p = 3$, of type G_2 or A_2. Assume G is also defined and split over $\mathbb{F}_p$. Then* $\dim_k(\mathfrak{g}^\star) = 21$ *if G is of type G_2, and* $\dim_k(\mathfrak{g}^\star) = 14$ *if G is of type A_2.*

Proof. As is well known, the dimension of $\mathfrak{g}$ in type G_2 (resp., A_2) is 14 (resp., 8). The result is then an immediate consequence of (4.13) and the exactness of the sequence (4.12). ▮

By introducing an appropriate involution $\theta^\star$ on $\mathfrak{g}^\star$, we may use the knowledge of the G-module structure of $\mathfrak{z}^\star$, developed above, to express $L_u(\mathfrak{p})$ as a quotient of $\mathfrak{g}^\star$. Suppose G is any semisimple, simply connected algebraic group over k, with involution θ. From the universal mapping property of $\mathfrak{g}^\star$, there must be a morphism $\theta^\star\colon \mathfrak{g}^\star \to \mathfrak{g}^\star$ making the diagram

$$\begin{tikzcd}
\mathfrak{z}^\star \arrow[r] \arrow[d] & \mathfrak{g}^\star \arrow[d, "\theta^\star"'] \arrow[dr, "\pi"] & \\
\mathfrak{z}^\star \arrow[r] & \mathfrak{g}^\star \arrow[r, "\theta\pi"'] & \mathfrak{g}
\end{tikzcd}$$

commute.

From the commuting triangle above, $\theta\pi\theta^\star = \pi$. Since $\theta = \theta^{-1}$, we get $\pi\theta^\star = \theta\pi$. Now, this yields

$$\begin{aligned}
\pi\theta^{\star 2} &= (\pi\theta^\star)\theta^\star \\
&= (\theta\pi)\theta^\star \\
&= \theta(\theta\pi) \\
&= \theta^2\pi = \pi.
\end{aligned}$$

Therefore

$$\begin{tikzcd} \mathfrak{z}^\star \arrow[r] \arrow[dd, "1_{\mathfrak{z}^\star}"'] & \mathfrak{g}^\star \arrow[dd, "\theta^{\star 2}"'] \arrow[dr, "\pi"] & \\ & & \mathfrak{g} \\ \mathfrak{z}^\star \arrow[r] & \mathfrak{g}^\star \arrow[ur, "\pi"'] & \end{tikzcd}$$

commutes. However, replacing $\theta^{\star 2}$ with the identity map $1_{\mathfrak{g}^\star}$ trivially gives a commutative diagram. Since $\mathfrak{g}^\star$ is universal, the morphism $(1_{\mathfrak{z}^\star}, 1_{\mathfrak{g}^\star})$ is unique, hence $\theta^{\star 2} = 1_{\mathfrak{g}^\star}$, so $\theta^\star$ is an involution.

Suppose we consider the special case that $\theta \in \mathrm{Aut}(G)$ is inner, i.e., $\theta = \mathrm{Int}\, t$. As previously noted, G acts on $\mathfrak{g}^\star$ compatibly with the morphism $\pi\colon \mathfrak{g}^\star \to \mathfrak{g}$. Therefore, $\theta^\star$ here is given by the action of t on $\mathfrak{g}^\star$ lifting the action $\theta = \mathrm{Ad}(t)$ on $\mathfrak{g}$. In this case, it is easy to see, without our calculation above, that $\theta^\star$ must be an involution.

In any case, the involution $\theta^\star$ on $\mathfrak{g}^\star$ determines a decomposition

$$\mathfrak{g}^\star \cong \mathfrak{p}^\star \oplus \mathfrak{k}^\star \tag{4.15}$$

into a -1-eigenspace $\mathfrak{p}^\star$ and a $+1$-eigenspace $\mathfrak{k}^\star$.

LEMMA 4.16. *Let G be any semisimple, simply connected algebraic group G over k with involution θ. Assume the notation above. Then $\pi\colon \mathfrak{g}^\star \to \mathfrak{g}$ maps $\mathfrak{p}^\star$ onto $\mathfrak{p}$, and $\mathfrak{k}^\star = [\mathfrak{p}^\star, \mathfrak{p}^\star]$ (i.e., $\mathfrak{g}^\star = L_{\varphi_{\mathfrak{p}^\star}}$ for $\varphi_{\mathfrak{p}^\star}\colon \mathfrak{p}^\star \to \mathfrak{g}^\star$, the obvious LTS imbedding).*

Proof. Let $Y \in \mathfrak{p}^\star$. Then $\theta\pi(Y) = \pi\theta^\star(Y) = \pi(-Y) = -\pi(Y)$, therefore $\pi(\mathfrak{p}^\star) \subset \mathfrak{p}$. *Likewise, $\pi(\mathfrak{k}^\star) \subset \mathfrak{k}$. However, $\mathfrak{p} \cap \mathfrak{k} = 0$, and π is surjective, hence π maps $\mathfrak{p}^\star$ onto $\mathfrak{p}$ (and $\mathfrak{k}^\star$ onto $\mathfrak{k}$). Thus $\mathfrak{g}^\star = \mathfrak{p}^\star \oplus [\mathfrak{p}^\star, \mathfrak{p}^\star]$ since $\mathfrak{g}^\star$ has no proper subalgebras mapping onto $\mathfrak{g}$.* ∎

We may now proceed with the promised characterization of $L_u(\mathfrak{p})$.

THEOREM 4.17. *Let G be a semisimple, simply connected algebraic group over k, with involution θ. Then $L_u(\mathfrak{p}) \cong \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$.*

Proof. As in the proof of (4.10), we have the central extensions (4.11) and (4.12) and a morphism (ψ_0, ψ) from (4.12) to (4.11). In particular, $\psi\colon \mathfrak{g}^\star \to L_u(\mathfrak{p})$ with $\zeta \circ \psi = \pi$. Since ζ and π both commute with the action of θ, so does ψ. Consequently, ψ maps $\mathfrak{p}^\star$ onto $\mathfrak{p}$, and likewise $\mathfrak{k}^\star = [\mathfrak{p}^\star, \mathfrak{p}^\star]$ maps onto $[\mathfrak{p}, \mathfrak{p}]_u$. Now, setting $\mathfrak{c} := \mathrm{Ker}(\psi)$, we have $\mathfrak{c} \subset \mathfrak{z}^\star$. Thus, $\psi^{-1}(\mathfrak{p}) = \mathfrak{p}^\star + \mathfrak{c} \subset \mathfrak{p}^\star + \mathfrak{z}^\star$. Let $\mathfrak{m}$ be any subalgebra of $\mathfrak{g}^\star$ containing $\psi^{-1}(\mathfrak{p})$. Then arguing just as in the proof of (4.10), we conclude that $\mathfrak{m} = \mathfrak{p}^\star \oplus [\mathfrak{p}^\star, \mathfrak{p}^\star] = \mathfrak{g}^\star$. Therefore ψ is surjective, and no smaller subalgebra maps onto $L_u(\mathfrak{p})$.

Since $\zeta \circ \psi = \pi$ and ζ is one-to-one on $\mathfrak{p}$, necessarily $\mathfrak{z}^\star \cap \mathfrak{p}^\star \subset \mathfrak{c}$. This inclusion gives rise in the usual manner to a surjective morphism $\overline{\psi}$: $\mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star) \to L_u(\mathfrak{p})$ for which $\psi = \overline{\psi} \circ q$, for q: $\mathfrak{g}^\star \to \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$ the natural quotient map. Consequently $\dim_k(L_u(\mathfrak{p})) \leq \dim_k(\mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star))$.

We now check that the reverse inequality holds. Since $q(\mathfrak{p}^\star) = \mathfrak{p}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star) \cong \mathfrak{p}$ (as vector spaces), we find $\mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star) \cong q(\mathfrak{p}^\star) \oplus [q(\mathfrak{p}^\star), q(\mathfrak{p}^\star)]$ has the form $\mathfrak{p} \oplus [\mathfrak{p}, \mathfrak{p}]$, taking the bracket operation in the quotient algebra. By the universal property of $L_u(\mathfrak{p})$, $L_u(\mathfrak{p})$ surjects onto $q(\mathfrak{p}^\star) \oplus [q(\mathfrak{p}^\star), q(\mathfrak{p}^\star)]$. Thus $\dim_k(L_u(\mathfrak{p})) \geq \dim_k(\mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star))$. Therefore $\overline{\psi}$: $\mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star) \to L_u(\mathfrak{p})$ must be an isomorphism, as desired. ∎

As previously noted, in case G is defined and split over $\mathbb{F}_p$, (4.10) can alternately be derived as a corollary to (4.17), for except when $p = 3$ and G is of type G_2 or A_2, $\mathfrak{z}^\star = 0$ by (4.9). To complete our identification of $L_u(\mathfrak{p})$, we need now to examine the structure of $\mathfrak{z}^\star \cap \mathfrak{p}^\star$ when $p = 3$ and G is of type G_2 or A_2.

LEMMA 4.18. *Suppose $p = 3$ and G is simple, simply connected of type G_2, with involution θ. Assume also that G is defined and split over $\mathbb{F}_p$ and that θ is defined over $\mathbb{F}_p$. Then* $\dim_k(\mathfrak{z}^\star \cap \mathfrak{p}^\star) = 4$.

Proof. By (6.2.2), we may assume $\theta = \operatorname{Int} t$, where $\alpha_1(t) = \alpha_2(t) = -1$ and this determines the values of α_1 and α_2. Now, by (4.13), the nonzero weights associated to $\mathfrak{z}^\star$ are $\pm 3\alpha_1$, $\pm 3(\alpha_1 + \alpha_2)$, and $\pm 3(2\alpha_1 + \alpha_2)$. Observe that $3\alpha_1(t) = (-1)^3 = -1$, $3(\alpha_1 + \alpha_2)(t) = ((-1)(-1))^3 = 1$, and likewise $3(2\alpha_1 + \alpha_2)(t) = ((-1)^2(-1))^3 = -1$. From this, we see that the weight spaces corresponding to $\pm 3\alpha_1$ and $\pm 3(2\alpha_1 + \alpha_2)$ alone lie in the -1-eigenspace $\mathfrak{p}^\star$ of $\theta^\star$. Thus, $\dim_k(\mathfrak{z}^\star \cap \mathfrak{p}^\star) = 4$. ∎

LEMMA 4.19. *Suppose $p = 3$ and G is simple, simply connected of type A_2, with involution θ. Assume also that G is defined and split over $\mathbb{F}_p$ and that θ is defined over $\mathbb{F}_p$. Then* $\dim_k(\mathfrak{z}^\star \cap \mathfrak{p}^\star) = 3$ *if θ is outer and* 4 *if θ is inner.*

Proof. In this case, $\mathfrak{z}^\star \cong L(\lambda_1)^{(1)} \oplus L(\lambda_2)^{(1)}$ by (4.13). Suppose θ is outer. Then θ must induce a graph automorphism of the Dynkin diagram associated to A_2. Therefore, the action of $\theta^\star$ is to interchange $L(\lambda_1)^{(1)}$ and $L(\lambda_2)^{(1)}$. In this case, one can see that $\mathfrak{z}^\star \cap \mathfrak{p}^\star = \{(x, -\theta^\star(x)) \mid x \in L(\lambda_1)^{(1)}\}$. Thus, $\dim_k(\mathfrak{z}^\star \cap \mathfrak{p}^\star) = 3$.

Now suppose θ is inner. As in (6.2.2) we can assume $\theta = \operatorname{Int} t$, with $\alpha_1(t) = -1 = \alpha_2(t)$. Since $3\lambda_1 = 2\alpha_1 + \alpha_2$, the other weights of $L(3\lambda_1) \subset \mathfrak{z}^\star$ are $-\alpha_1 + \alpha_2$ and $-\alpha_1 - 2\alpha_2$. Checking the values of these weights at t, we see that $2\alpha_1 + \alpha_2$ and $-\alpha_1 - 2\alpha_2$ have value -1. Thus, the weight spaces in $L(3\lambda_1)$ corresponding to $2\alpha_1 + \alpha_2$ and $-\alpha_1 - 2\alpha_2$ lie in $\mathfrak{p}^\star$. A similar calculation yields another two-dimensional contribution to $\mathfrak{z}^\star \cap \mathfrak{p}^\star$ from $L(3\lambda_2)$. Thus, $\dim_k(\mathfrak{z}^\star \cap \mathfrak{p}^\star) = 4$. ∎

By combining (4.14), (4.17), (4.18), and (4.19), we immediately get

COROLLARY 4.20. *Let G be simple and simply connected over k, of type G_2 or A_2, $p = 3$, with involution θ. Assume also that G is defined and split over $\mathbb{F}_3$ and that θ is defined over $\mathbb{F}_3$. Then*

(1) $\dim_k(L_u(\mathfrak{p})) = 17$ *if G is of type G_2*;

(2) $\dim_k(L_u(\mathfrak{p})) = 11$ *if G is of type A_2 and θ is outer*;

(3) $\dim_k(L_u(\mathfrak{p})) = 10$ *if G is of type A_2 and θ is inner.*

In particular, in these cases $L_u(\mathfrak{p}) \not\cong \mathfrak{g}$, since their dimensions differ.

In combination together with (4.5), our analysis above produces a complete description of $L_s(\mathfrak{p})$, $L_u(\mathfrak{p})$, and their relationships with $\mathfrak{g}$ for any simple, simply connected algebraic group G over k, defined and split over $\mathbb{F}_p$, with involution θ defined over $\mathbb{F}_p$. This we summarize in the following theorem.

THEOREM 4.21. *Let G be a simple, simply connected algebraic group over k with involution θ. Assume also that G is defined and split over $\mathbb{F}_p$ and that θ is defined over $\mathbb{F}_p$. Then, following the previous notation, $L_u(\mathfrak{p}) \cong \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$ and $L_s(\mathfrak{p}) \cong \mathfrak{g}/(Z(\mathfrak{g}) \cap \mathfrak{k}) = \mathfrak{g}/(Z(\mathfrak{g}) \cap [\mathfrak{p}, \mathfrak{p}])$. In particular,*

(1) *$L_u(\mathfrak{p}) \cong \mathfrak{g} \cong L_s(\mathfrak{p})$ unless θ is inner and G is either of type A_n, $p \mid (n+1)$, $n > 2$, or of type E_6, $p = 3$, or unless θ is any involution and G is either of type A_2 or of type G_2, and $p = 3$.*

(2) *$L_u(\mathfrak{p}) \cong \mathfrak{g}$, while $L_s(\mathfrak{p}) \cong \mathfrak{g}/Z(\mathfrak{g})$ if G is either of type A_n, $n > 2$ and $p \mid (n+1)$, or of type E_6, $p = 3$, and θ is inner. In this case,* $\dim_k(L_s(\mathfrak{p})) = n^2 + 2n - 1$ *for G of type A_n, and* $\dim_k(L_s(\mathfrak{p})) = 77$ *if G is of type E_6.*

(3) *$L_u(\mathfrak{p}) \cong \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$ and $L_s(\mathfrak{p}) \cong \mathfrak{g}$, if G is of type G_2, $p = 3$. In this case,* $\dim_k(L_u(\mathfrak{p})) = 17$, $\dim_k(\mathfrak{g}) = 14$.

(4) *$L_u(\mathfrak{p}) \cong \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$ and $L_s(\mathfrak{p}) \cong \mathfrak{g}/Z(\mathfrak{g})$, if G is of type A_2, $p = 3$, and θ is inner. Here,* $\dim_k(L_u(\mathfrak{p})) = 10$, $\dim_k(\mathfrak{g}) = 8$, *and* $\dim_k(L_s(\mathfrak{p})) = 7$.

(5) *$L_u(\mathfrak{p}) \cong \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$ and $L_s(\mathfrak{p}) \cong \mathfrak{g}$, if G is of type A_2, $p = 3$, and θ is outer. In this case,* $\dim_k(L_u(\mathfrak{p})) = 11$.

5. MODULES FOR LIE TRIPLE SYSTEMS

In the first subsection of Section 5, we step back a bit from the analysis of the LTS $\mathfrak{p}$ and the related Lie algebras of the last section. Returning to the more general setting of an arbitrary LTS T, we discuss modules for T and the passage from modules for T to modules for $L_u(T)$. Under the additional assumption that T is a restricted LTS, we then introduce the category of restricted T-modules. We conclude in Section 5.2 by swinging back to the case of Lie triple systems arising from involutions on algebraic groups, combining the material of Section 5.1 with that of Section 4 to relate $\mathfrak{p}$-modules and $\mathfrak{g}$-modules. There we also raise some questions for further research.

5.1. *Modules and Restricted Modules for Lie Triple Systems*. As we have a number of times already in this paper, we first look to the Lie algebra setting for inspiration. Although a module V for a Lie algebra L over k perhaps is most succinctly defined via a Lie algebra homomorphism ξ: $L \to \mathfrak{gl}(V)$, one may also define the vector space V to be a module for L if there is a Lie algebra structure on $L \oplus V$ for which

(1) L is a subalgebra;

(2) V is an ideal, i.e., for $X, Y \in L \oplus V$, $[X, Y] \in V$ if one of $X, Y \in V$; and

(3) $[X, Y] = 0$ if both $X, Y \in V$.

In the latter case, for $X \in L$ and $Y \in V$, defining $\xi(X)(Y) = [X, Y]$ determines a homomorphism ξ: $L \to \mathfrak{gl}(V)$, returning us to our first approach. One may similarly check that the two approaches are equivalent.

With this perspective in mind, following [6], we define a module for a Lie triple system as below.

DEFINITION 5.1.1. Let T be a Lie triple system over k. A k-vector space M is a *module* for T if $E_M = T \oplus M$ is a Lie triple system for which

(1) T is a subsystem of E_M;

(2) for $a, b, c \in E_M$, $[abc] \in M$ if any one of a, b, c lies in M;

(3) $[abc] = 0$ if any two of a, b, c are elements of M.

Given two T-modules M, N, a linear map ψ: $M \to N$ is a T-module *morphism* if the induced map $1_T \oplus \psi$: $E_M \to E_N$ is a morphism of Lie triple systems. The resulting category will be denoted T-Mod, with full subcategory T-mod which has as its objects all the finite-dimensional T-modules.

Now, each one-to-one imbedding $\varphi: T \to \mathfrak{g}$ for which $L_{\varphi} = T \oplus [T, T]$ produces a realization of T as the -1-eigenspace of an involution $\theta \in \mathrm{Aut}(L_{\varphi})$, defined just as in (2.0.16). In a similar fashion, $M \in \mathrm{Ob}(T\text{-Mod})$ may be described in terms of Lie algebra modules and involutions.

DEFINITION 5.1.2. Let L be a Lie algebra over k with involution θ. Define a Lie algebra module V for L to be an (L, θ)-*module* if θ also acts on V and satisfies

$$\theta(X.v) = \theta(X).\theta(v), \tag{5.1.3}$$

for all $X \in L$, $v \in V$.

In other words, there is an automorphism of the k-vector space V, also denoted θ, satisfying $\theta^2 = 1$ and (5.1.3). Of course, a morphism of (L, θ)-modules is a morphism of L-modules which commutes with the action of θ. The notations (L, θ)-Mod and (L, θ)-mod will have the usual meanings.

In [6], an (L, θ)-module is called a (Lie algebra) *module with involution*. For a LTS T, define $\theta \in \mathrm{Aut}(L_u(T))$ as in (2.0.15). Then, given $M \in \mathrm{Ob}(T\text{-Mod})$, [6] essentially proves

LEMMA 5.1.4. *Let M be a module for a Lie triple system T. Then there exists an $(L_u(T), \theta)$-module $N_s(M)$ for which $N_s(M) = M \oplus [T, M]$ and $M = \{n \in N_s(M) \mid \theta(n) = -n\}$.*

Termed the *standard extension* of M, the module $N_s(M)$ is analogous to the standard enveloping Lie algebra $L_s(T)$ of a LTS T. To define $N_s(M)$, follow the procedure for defining $L_s(T)$ as related in Section 2, but make the following changes and/or substitutions.

(1) Replace the quotient space V_T (of $T \otimes_k T$ by the subspace of all $\Sigma a_i \otimes b_i$ for which $\Sigma[a_i b_i x] = 0$ for all $x \in T$) by the quotient space U_M of $T \otimes_k M$ modulo the subspace of all $\Sigma a_i \otimes m_i$ for which $\Sigma[a_i m_i b] = 0$ for all $b \in T$.

(2) Set $N_s(M) = M \oplus U_M$.

(3) Equip $N_s(M)$ with an $L_u(T)$-module structure. Do this by defining a bracket $[\ ,\ ]$ on $L_u(T) \oplus N_s(M)$ as follows: First set $[a, m] = \overline{a \otimes m}$, $a \in T \subset L_u(T)$, $m \in M \subset N_s(M)$. Note that this is the analogue of the first (of four) defining relations for $L_s(T)$, discussed below (2.0.14). In order to define the bracket for the remaining combinations (e.g., $[[T, T]_u, M], [T, [T, M]]$), follow the obvious analogues of the three remaining relations defining $L_s(T)$.

To ensure that the rules above are well-defined, [6] describes $N_s(M)$ as an ideal in $L_s(E_M)$, as we now discuss. Note that U_M above injects naturally into the quotient V_{E_M} of $E_M \otimes_k E_M \subset L_s(E_M)$. Hence, $N_s(M)$

inherits its structure as an $L_u(T)$-module from the Lie algebra morphism $L_u(T) \to L_s(E_M)$ induced by the imbedding $T \to E_M \to L_s(E_M)$ (and the universal property of $L_u(T)$). Pictorially,

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr] & L_u(T) \arrow[d] \\ & L_s(E_M). \end{tikzcd}$$

In this context, the action of θ on $L_s(E_M)$, defined as usual (i.e., by (2.0.15)), restricts to an action of θ on $N_s(M)$, satisfying the compatibility criterion of (5.1.3). By this means, $N_s(M)$ becomes an $(L_u(T), \theta)$-module. (We caution the reader not to be lulled by the notation into thinking that $N_s(M)$ must be a module for $L_s(T)$, although the construction of $N_s(M)$ parallels that of $L_s(T)$.)

By construction, as pointed out in [6], the standard extension $N_s(M)$ possesses the same kind of universal property carried by the standard enveloping Lie algebra $L_s(T)$. Namely, if N is any $(L_u(T), \theta)$-module for which $N = M \oplus [T, M]$ and $M = \{n \in N \mid \theta(n) = -n\}$, then there exists a surjective morphism χ: $N \to N_s(M)$ completing the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & N \arrow[d, two heads, "\chi"] \\ & N_s(M) \end{tikzcd}$$

for the obvious inclusions $M \to N$ and $M \to N_s(M)$.

Suppose ψ: $M \to M'$ is a morphism of T-modules. Then, as previously discussed, ψ gives rise to a LTS morphism $E_M \to E_{M'}$ whence, by the construction above, to a morphism $N_s(\psi)$: $N_s(M) \to N_s(M')$ of $(L_u(T), \theta)$-modules. To be more precise, regard $N_s(M)$ as $M \oplus U_M$, and observe that, given $m_i \in M$ and $a_i \in T$, $\sum_i [a_i m_i b]_{E_M} = 0$ for all $b \in T$ implies $\sum_i [a_i \psi(m_i) b]_{E_{M'}} = 0$ for all $b \in T$. Thus, sending $U_M \to U_{M'}$ by $\sum_i a_i \otimes m_i \mapsto \sum_i a_i \otimes \psi(m_i)$ yields a well-defined linear map ψ_U. Then $N_s(\psi) = \psi \oplus \psi_U$. Note that $N_s(\psi)$ commutes with the action of θ.

Our next two results describe a few properties of the functor N_s, the first of which was noted in [6].

LEMMA 5.1.5. *Let T be a Lie triple system. Let $M, M' \in$ Ob(T-Mod) and let ψ: $M \to M'$ be a morphism. Then $N_s(\psi)$: $N_s(M) \to N_s(M')$ is one-to-one* (*resp.*, *onto*) *if ψ is one-to-one* (*resp.*, *onto*).

Proof. If ψ: $M \to M'$ is a one-to-one mapping of T-modules, then the morphism of Lie triple systems $1_T \oplus \psi$: $E_M \to E_{M'}$ is one-to-one as well. From this and our comments above, it follows that ψ_U: $U_M \to U_{M'}$ is also an injection. As a consequence, $N_s(\psi) = \psi \oplus \psi_U$ is an injection. Likewise, an analogous argument shows that $N_s(\psi)$ is onto if ψ is. ∎

PROPOSITION 5.1.6. *Let T be a Lie triple system. Then the functor N_s: T-Mod $\to (L_u(T), \theta)$-Mod is a full embedding.*

Proof. Let $M, M' \in \mathrm{Ob}(T\text{-Mod})$. Suppose $N_s(M) = N_s(M')$. Then $M = M'$, for each is the -1-eigenspace of θ on $N_s(M)$. Now suppose ψ, ξ: $M \to M'$ are two LTS-module maps for which $N_s(\psi) = N_s(\xi)$. Then $\psi = N_s(\psi)|_M = N_s(\xi)|_M = \xi$. This shows N_s is faithful and one-to-one on objects. Finally, suppose ζ: $N_s(M) \to N_s(M')$. Then $\zeta \circ \theta = \theta \circ \zeta$, so $\zeta|_M$: $M \to M'$. Observe that $N_s(\zeta|_M) = \zeta$, whence $N_s(\zeta)$ is a full functor. This completes the proof. ∎

With some knowledge now of the category T-Mod, we proceed to define the category res T-Mod of restricted modules for a restricted LTS T.

DEFINITION 5.1.7. Let T be a restricted Lie triple system over k, with $[p]$-operator $a \mapsto a^{[p]}$. If M is a T-module, we will define M to be *restricted* provided that

$$[abc^{[p]}] = (a, b, c, \dots, c) \ (p \text{ copies of } c), \tag{5.1.8}$$

for all $c \in T$ and $a, b \in E_M$ and

$$[ab^{[p]}c] = (a, b, \dots, b, c) \ (p \text{ copies of } b) \tag{5.1.9}$$

for all $b \in T$ and $a, c \in E_M$. Note that the right-hand sides of (5.1.8) and (5.1.9) may be taken in $L_s(E_M)$, but both lie back in E_M. Moreover, as is the case for restricted Lie triple systems, conditions (5.1.8) and/or (5.1.9) determine $[a^{[p]}bc]$ under similar hypotheses. A morphism ψ: $M \to M'$ of restricted T-modules is just a linear map which defines a morphism in T-Mod.

We have chosen the definition of a restricted module for a restricted LTS by way of analogy. Recall that if ψ: $L \to \mathfrak{gl}(V)$ determines a module structure on V for a restricted Lie algebra L, then V becomes a restricted module if, for all $X \in L$, $\psi(X^{[p]}) = (\psi(X))^p$, taking the pth iterate of the endomorphism $\psi(X)$ of V.

5.2. *Algebraic Groups and LTS Modules*: *Closing Remarks*. Assume the setup as at the beginning of Section 4, so that $\mathfrak{p}$ is a Lie triple system arising from an involution θ on the connected, reductive algebraic group G. Our first lemma provides us with a large class of restricted Lie triple systems and pulls together our earlier work classifying the standard enveloping algebra $L_s(\mathfrak{p})$.

LEMMA 5.2.1. *Let G be simple and simply connected, with involution $\theta \in \mathrm{Aut}(G)$. Then $\mathfrak{p}$ is a restricted Lie triple system.*

Proof. With the exception of the cases $p \mid (n+1)$ and G is of type A_n or $p = 3$ and G is of type E_6, (4.5) yields $\mathfrak{g} \cong L_s(\mathfrak{p})$. Letting $X \mapsto X^{[p]}$ represent the usual $[p]$-operation on $\mathfrak{g}$, we see that $\mathfrak{p}$ is a restricted Lie triple system under $a \mapsto a^{[p]}$, by (3.24).

In these exceptional cases, $L_s(\mathfrak{p}) \cong \mathfrak{g}/Z(\mathfrak{g})$. However, by Table 1 of [11], $Z(\mathfrak{g})$ is a $[p]$-ideal of $\mathfrak{g}$, so $L_s(\mathfrak{p})$ inherits a restricted structure from $\mathfrak{g}$. Thus, as before, $\mathfrak{p}$ becomes a restricted Lie triple system. ∎

Thus, it makes sense to discuss restricted modules for $\mathfrak{p}$, as in Section 5.1. We now point out one way in which restricted $\mathfrak{p}$-modules arise. First of all, since $\mathfrak{p}$ imbeds in $\mathfrak{g}$, we should expect appropriate modules for $\mathfrak{g}$ to give rise to modules for $\mathfrak{p}$. More precisely, it is easy to see that the restriction functor

$$\mathrm{res}_{\mathfrak{p}}^{(\mathfrak{g},\theta)}: (\mathfrak{g},\theta)\text{-Mod} \to \mathfrak{p}\text{-Mod} \tag{5.2.2}$$

associates to each restricted $(\mathfrak{g},\theta)$-module a restricted $\mathfrak{p}$-module. Furthermore, by analogy with the category of $(\mathfrak{g},\theta)$-modules, the category of (G,θ)-modules can be defined. (A classification of the irreducible (G,θ)-modules may be found in [8].) In general, if G is any algebraic group with involution θ, then one sees immediately that a (G,θ)-module M is also a $(\mathfrak{g},\theta)$-module. In addition, M is, compatibly with the θ-action of $\mathfrak{g}$, of course a restricted $\mathfrak{g}$-module. Thus (5.2.2) associates to each (G,θ)-module a restricted $\mathfrak{p}$-module.

We now consider the reverse situation, that is, whether we can produce modules for $\mathfrak{g}$ by beginning with modules for $\mathfrak{p}$. Our next fundamental result accomplishes this by utilizing our knowledge of $L_u(\mathfrak{p})$.

THEOREM 5.2.3. *Let G be a simple, simply connected algebraic group with involution θ. Let M be a module for the LTS $\mathfrak{p}$. Then aside from the possible exceptions of when $p = 3$ and G is of type A_2 or G_2, $N_s(M)$ is a $(\mathfrak{g},\theta)$-module.*

Proof. Excluding the exceptional cases $p = 3$, $\mathfrak{g}$ of type A_2 or G_2, (4.10) shows that $\mathfrak{g} = L_u(\mathfrak{p})$. The usual involution on $L_u(\mathfrak{p})$ is the unique involution determining $\mathfrak{p}$ as the -1-eigenspace by (2.0.16) and hence agrees with the involution θ on $\mathfrak{g}$ arising from the involution θ on G. The theorem follows. ∎

Now suppose G is simple, simply connected of type not A_2 or G_2 for $p = 3$, so that $\mathfrak{g} = L_u(\mathfrak{p})$. Since the restriction functor (5.2.2) carries the category of restricted $(\mathfrak{g}, \theta)$-modules to res $\mathfrak{p}$-Mod, we are led to wonder whether the functor N_s: $\mathfrak{p}$-Mod $\to$ $(\mathfrak{g}, \theta)$-Mod carries restricted $\mathfrak{p}$-modules to restricted $(\mathfrak{g}, \theta)$-modules (and, if so, under what circumstances the resulting modules arise from (G, θ)-modules). In fact, by considering irreducible modules for the so-called "reduced enveloping algebras" $\mathscr{U}_\chi(\mathfrak{g})$ associated to $\mathfrak{g}$, we can show that the functor N_s need no carry a restricted $\mathfrak{p}$-module to a restricted $\mathfrak{g}$-module. (More details on N_s and related functors will appear in a forthcoming paper [10].)

In general, then, one might consider the question of the effect of N_s on restricted modules for an abstract restricted Lie triple system T. Under what circumstances will N_s send restricted T-modules to restricted $L_u(T)$-modules? Note that if we suppose the form ρ on T is nondegenerate, then (3.45) guarantees that $L_s(T) = L_D(T)$ is a restricted Lie algebra. In this situation, however, we are only assured that $L_u(\mathfrak{p})/Z(L_u(T)) \cong L_s(T)$ (see (2.1.10) and (2.1.9)), so we do not yet know if $L_u(T)$ itself even has a restricted structure. Let us point out that, in the case $T = \mathfrak{p}$ above, we do have the following result.

THEOREM 5.2.4. *Let G be a simple, simply connected algebraic group with involution θ. Then the universal Lie algebra $L_u(\mathfrak{p})$ bears a restricted Lie algebra structure, compatible with the restricted structure of* $\mathfrak{g}$.

Proof. By (4.17), $L_u(\mathfrak{p}) \cong \mathfrak{g}^\star/(\mathfrak{z}^\star \cap \mathfrak{p}^\star)$. Now, Proposition 6.2 and Corollary 10.2 of [26] show that $\mathfrak{g}^\star$ has a unique restricted structure, compatible with the restricted structure of $\mathfrak{g}$. Moreover, from the proof of Proposition 6.2, for $X \in \mathfrak{z}^\star$, $X^{[p]} = 0$. Thus, $\mathfrak{z}^\star \cap \mathfrak{p}^\star$ is a $[p]$-ideal of $\mathfrak{g}^\star$, hence $L_u(\mathfrak{p})$ inherits a restricted structure, compatible with the restricted structure of $\mathfrak{g}$, as claimed. ∎

Suppose under some hypotheses $L_u(T)$ is restricted and, for perhaps some collection of restricted T-modules, N_s does produce restricted $L_u(T)$-modules. Then another project of interest (on which we are currently at work) is the development of a restricted cohomology theory for T and its connections with the restricted cohomology of $L_u(T)$. Here, one should keep in mind Harris's investigation of (ordinary) cohomology of T in [6]. We are especially curious to know whether there are restricted $\mathfrak{p}$-cohomology versions of restricted Lie algebra cohomology results of Friedlander and Parshall (e.g., [4, 5]) and others. For example, will the nilpotent elements of $\mathfrak{p} \subset \mathfrak{g}$ be isomorphic as a scheme to the spectrum of $H^{ev}_{res}(\mathfrak{p}, k)$, the restricted cohomology of the LTS $\mathfrak{p}$ in even degrees, suitably defined? Will there be a meaningful theory of support varieties for $\mathfrak{p}$-modules?

Other questions regarding the behavior of the functor N_s remain. This functor is neither left nor right exact. Still, one can consider its homological properties. What are its derived functors? Suitably restricted, does it fit into some setting as an adjoint functor? One may phrase other such homological questions.

What is the structure of T-Mod, resp., res T-Mod, and/or T-mod and res T-mod? For example, if $T = \mathfrak{p}$, is there a classification of irreducible restricted $\mathfrak{p}$-modules, perhaps linked to that of the irreducible (G, θ)-modules? Are these categories highest weight categories (allowing for infinitely many irreducibles), or are some "sections" of them highest weight categories with finitely many irreducibles (and thus module categories for quasi-hereditary algebras)? If not, are there still some interesting related finite-dimensional algebras for which sections of them are the module categories (e.g., Frobenius algebras)?

In any case, we are interested in understanding to what extent, for algebraic groups G, questions regarding rational G-modules and/or $\mathfrak{g}$-modules might be reduced to questions about modules for $\mathfrak{p}$. In conclusion, our original interest in Lie triple systems arose when considering modular Harish-Chandra modules, modules carrying compatible $\mathfrak{g}$- and K-actions, $K = G^{\theta}$, including that the action of $\mathfrak{k} = \mathrm{Lie}(K) \subset \mathfrak{g}$ on such a module should agree with the differential of the action of K. In some sense, then, the action of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ on a modular Harish-Chandra module is determined by the action of K and the action of $\mathfrak{p}$. Although it is yet an unrealized goal to ascertain how profitably we may (if we may) pass from these modules to certain $\mathfrak{p}$-modules (or vice versa), the results of this paper have raised a platform from which we may dive into these questions, among many others.

6. APPENDIX

6.1. *Chevalley Basis*. The following material may be found, for example, in [25]. The Lie algebra $\mathfrak{g}$ of a semisimple, simply connected algebraic group G associated to a root system Φ may be obtained from Chevalley's $\mathbb{Z}$-form for $\mathfrak{g}_{\mathbb{C}}$ (the complex semisimple Lie algebra with root system Φ and root space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\mathbb{C}\alpha}$ for $\mathfrak{h}_{\mathbb{C}}$ a fixed Cartan subalgebra) via $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. Here $\mathfrak{g}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module with "Chevalley basis" denoted $H_i = H_{\alpha_i}$, $\alpha_i \in \Pi$ (a set of simple roots for Φ), and $X_\alpha \in \mathfrak{g}_{\mathbb{C}\alpha}$, $\alpha \in \Phi$. In particular, let as usual

$$\beta^{\vee} = \frac{2\beta}{(\beta, \beta)} \qquad \text{for all } \beta \in \Phi. \tag{6.1.1}$$

Then for any $\alpha \in \Phi$, $H_\alpha \in \mathfrak{h}_{\mathbb{C}}$ is defined by $\beta(H_\alpha) = (\alpha, \beta^{\vee})$.

LEMMA 6.1.2. *Let Φ be a fixed root system in a Euclidean space with inner product (,) invariant under the action of the Weyl group of G. Let $\alpha, \beta \in \Phi$ and take $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ to be a set of simple roots. Then the structure constants of $\mathfrak{g}_{\mathbb{C}}$ with respect to the Chevalley basis $\{H_i, X_\alpha\}$ are*

(a) $[H_i, H_j] = 0$;

(b) $[H_i, X_\alpha] = (\alpha, \alpha_i^\vee) X_\alpha$;

(c) $[X_\alpha, X_{-\alpha}] = H_\alpha = \sum_i c_i H_i$ *for* $c_i \in \mathbb{Z}$;

(d) $[X_\alpha, X_\beta] = \pm(r+1) X_{\alpha+\beta}$ *if* $\alpha + \beta$ *is a root, where* $\beta - r\alpha, \ldots, \beta, \ldots, \beta + (q+1)\alpha$ *is the* α*-string of roots through* β;

(e) $[X_\alpha, X_\beta] = 0$ *if* $\alpha + \beta \neq 0$ *and* $\alpha + \beta$ *is not a root.*

For G simply connected, semisimple of type Φ, the corresponding basis

$$\{h_i = h_{\alpha_i}, e_\alpha \mid i = 1, \ldots, l, \ \alpha \in \Phi\} \tag{6.1.3}$$

of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, given by setting

$$h_\alpha = H_\alpha \otimes 1 \qquad \text{and} \qquad e_\alpha = X_\alpha \otimes 1, \tag{6.1.4}$$

will have a similar multiplication table.

6.2. *Some Involutions.* It may be easily checked that the following proposition holds (or see Proposition 2.1.2 of [8]).

PROPOSITION 6.2.1. *An involution* $\theta \in \mathrm{Aut}(G)$ *is a semisimple automorphism.*

EXAMPLE 6.2.2 (Some Involutions).

(A) Take $G = SL_n(k)$, and let $\theta\colon G \to G$ be the automorphism $x \mapsto (x^t)^{-1}$.

(B) Let $\mathrm{Int}(G)$ denote the group of inner automorphisms of any algebraic group G and $\mathrm{Int}\, x \in \mathrm{Int}(G)$ the inner automorphism associated to $x \in G$. Using Proposition 6.2.1, one may easily check that $\mathrm{Int}\, x$ is an involution if and only if $x \neq 1$ is a semisimple element for which $x^2 \in Z(G)$, the center of G.

(C) Let G be simple and simply connected, of type G_2. Then, up to conjugacy by an element G, we claim G has a unique involution θ. Furthermore, let α_1 (resp., α_2) denote the short (resp., long) simple root of G_2. Then θ can be taken to be $\mathrm{Int}\, t$ as in (B) above, where t is a semisimple element, $t^2 = 1$, and $\alpha_1(t) = -1 = \alpha_2(t)$. Let us now explain why.

In this case, $\mathrm{Aut}(G) = \mathrm{Int}(G) \cong G$ (as abstract groups). Furthermore, the assumption of simplicity implies $Z(G) = 1$, so $\mathrm{Int}\, t$ is an involution only if $t^2 = 1$ by (B). As another consequence of $Z(G) = 1$, $\mathrm{Int}\, t$ is determined by the values $\alpha_1(t)$ and $\alpha_2(t)$ (having fixed a maximal torus T containing t and having a fixed Borel subgroup $B \supset T$).

We claim that the values $\alpha_i(t)$, $i = 1, 2$, can be only ± 1. The involution $\mathrm{Int}\, t$ has differential $\mathrm{Ad}(t)$, so $(\mathrm{Ad}(t))^2 = 1$. However, for the Chevalley basis elements e_{α_i} as in (6.1.4), $\mathrm{Ad}(t)e_{\alpha_i} = \alpha_i(t)e_{\alpha_i}$; by this, $(\mathrm{Ad}(t))^2 e_{\alpha_i} = (\alpha_i(t))^2 e_{\alpha_i}$, so $\alpha_i^2(t) = 1$. At this point, we have three possibilities for pairs of values $(\alpha_1(t), \alpha_2(t))$: $(-1, -1)$, $(-1, 1)$, and $(1, -1)$. These values completely determine the effect of θ. Let t_j, $j = 1, 2, 3$, be the unique element associated to each of these three possibilities, so that $\theta = \mathrm{Int}\, t_j$ for some j. We wish to show that, up to conjugacy, we may take $t = t_1$.

Let s_{α_i} denote the simple reflection associated to α_i, $s_{\alpha_i} \in W \cong N(T)/T$. By definition, for any root β, $s_{\alpha_i}(\beta) = \beta - (\beta, \alpha_i^\vee)\alpha_i$. Using the fact that $(\alpha_i, \alpha_i^\vee) = 2$, $(\alpha_2, \alpha_1^\vee) = -3$, and $(\alpha_1, \alpha_2^\vee) = -1$, one checks that $\alpha_1(s_{\alpha_1}(t_2)) = s_{\alpha_1}(\alpha_1)(t_2) = -\alpha_1(t_2) = (-1)^{-1} = -1$, and $\alpha_2(s_{\alpha_1}(t_2)) = (3\alpha_1 + \alpha_2)(t_2) = (-1)^3(1) = -1$. Likewise, $\alpha_1(s_{\alpha_2}(t_3)) = -1 = \alpha_2(s_{\alpha_2}(t_3))$. Identifying s_{α_i} with a representative $n_{\alpha_i} \in N(T) \subset G$, it now follows from the isomorphism $G \cong \mathrm{Int}(G)$ that, up to conjugacy in G, G_2 has a unique involution $\theta = \mathrm{Int}\, t$ for which $t^2 = 1$ and $\alpha_1(t) = -1 = \alpha_2(t)$.

(D) Let us consider another rank-two case, in which G is simple and simply connected, of type A_2, when $\mathrm{char}(k) = 3$. The center of $SL_3(k)$ consists precisely of the $n \times n$ scalar matrices A for which $\det(A) = 1$ and thus identifies with the cube roots of unity in k. Since 1 is the unique cube root of unity in k, $Z(G) = 1$. Thus, if $\theta \in \mathrm{Aut}(G)$ is inner, θ must have the form $\theta = \mathrm{Int}\, t$ for some semisimple element t with $t^2 = 1$, as in the G_2 case examined above. Proceeding as in the G_2 case, we can similarly show that there is only one inner involution θ up to conjugacy by an element of the Weyl group W, where we may take $\theta = \mathrm{Int}\, t$ for $\alpha_1(t) = -1 = \alpha_2(t)$.

As for outer involutions, there is a unique graph automorphism σ of the Dynkin diagram which switches the two nodes. Thus, up to conjugacy in W, every other outer involution of G has the form $\theta = \mathrm{Int}\, t \circ \sigma$.

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