Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 306 (2006) 344–367
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Automorphisms of categories of free algebras of some varieties

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Received 4 March 2005
Available online 4 October 2006
Communicated by Efim Zelmanov

Abstract

Let $\mathcal{V}$ be a variety of universal algebras. We suggest a method for describing automorphisms of the category of free $\mathcal{V}$-algebras. All automorphisms of such categories are found in two cases: (1) $\mathcal{V}$ is the variety of all associative K-algebras over an infinite field K; (2) $\mathcal{V}$ is the variety of all representations of groups in unital R-modules over a commutative associative ring R with unit. We prove that all these automorphisms are close to inner automorphisms.

Keywords: Automorphisms of categories; Inner automorphisms; Free algebras; Associative algebras; Group representations; Universal algebraic geometry

1. Introduction

Let $\mathcal{V}$ be a variety of universal algebras. Consider the category $\Theta(\mathcal{V})$ whose objects are all algebras from $\mathcal{V}$ and morphisms are all homomorphisms of these algebras. Fix an infinite set X_0. Let $\Theta^0(\mathcal{V})$ be the full subcategory of $\Theta(\mathcal{V})$ defined by all free algebras from $\mathcal{V}$ over finite subsets of the set X_0. The main problem is to *describe all automorphisms of the category* $\Theta^0(\mathcal{V})$. Motivations for this research can be found in the papers [9,10,12,13]. Note that the problem under

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1 The research of the author was partially supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities—Center of Excellence Program.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.07.028

consideration is tightly connected with some problems in universal algebraic geometry, in particular, with the following question: when two algebras of a given variety have the same geometry?

The above mentioned main problem has been solved in many cases. For example, the automorphisms are known for the varieties of all groups, all semigroups, all inverse semigroups, all Lie algebras, semimodules and modules (see [4,5,9–11]). In each of these cases any automorphism of the category $\Theta^0(\mathcal{V})$ turns out to be inner or close to inner. In all these cases, the proofs are based on some reduction to the group $AutEnd(A)$, where A is a finitely generated free algebra from the variety $\mathcal{V}$. Once one can describe the group $AutEnd(A)$, the Reduction Theorem [9] allows one to describe the automorphisms of the category $\Theta^0(\mathcal{V})$. However, the problem of describing $AutEnd(A)$, interesting by its own, may happen to be even more complicated than the original one (see [3,8,11]).

In the present paper we suggest another method for describing automorphisms of categories of universal algebras. This method extends some ideas from [15]. For the reader's convenience we give below a sketch of the method.

Recall that an automorphism Φ of a category $\mathcal{C}$ is said to be inner if it is isomorphic to the identity functor $Id^{\mathcal{C}}$ in the category of all endofunctors of $\mathcal{C}$. This means that there exists a function assigning to every object A of $\mathcal{C}$ an isomorphism $\sigma_A : A \to \Phi(A)$ such that for every morphism $\mu : A \to B$ we have $\Phi(\mu) = \sigma_B \circ \mu \circ \sigma_A^{-1}$. This fact explains the term "inner."

1. Let A_0 be a monogenic free algebra in a category $\mathcal{C}$ of universal algebras, and let x_0 be a free generator of A.

Remark 1. Throughout below we assume that every automorphisms Φ of $\mathcal{C}$ takes a monogenic free algebra to an isomorphic one. This minor restriction is fulfilled in the most interesting cases.

Let A be a $\mathcal{C}$-algebra, and let $a \in A$. Denote by α_a the unique homomorphism from A_0 to A taking x_0 to a: $\alpha_a(x_0) = a$. Clearly, $a \mapsto \alpha_a$ gives a one-to-one correspondence between the sets A and $Hom(A_0, A)$. Thus every automorphism Φ of $\mathcal{C}$ fixing A_0 determines a family of bijections $(s_A^\Phi \mid A \in Ob\,\mathcal{C})$ defined by: $s_A^\Phi(a) = \Phi(\alpha_a)(x_0)$ for every $a \in A$.

It is easy to see that if Φ is the identity automorphism then $s_A^\Phi = 1_A$, if Φ is a product $\Gamma \circ \Psi$ then $s_A^\Phi = s_A^\Gamma \circ s_A^\Psi$, and if $\Psi = \Phi^{-1}$ then $s_A^\Psi = (s_A^\Phi)^{-1}$. It follows from the definition of the function $A \mapsto s_A^\Phi$ that for every homomorphism $\nu : A \to B$, where A and B are objects of $\mathcal{C}$, we have $\Phi(\nu) = s_B \circ \nu \circ s_A^{-1}$. This fact leads to the idea of introducing the notion of potentially inner automorphism.

2. Let $\mathcal{D}$ be an extension of a category $\mathcal{C}$ obtained by adding some new maps as morphisms. We say that an automorphism of $\mathcal{C}$ is *$\mathcal{D}$-inner* if Φ is a restriction to $\mathcal{C}$ of some inner automorphism of $\mathcal{D}$, that is, if there exists a function f assigning to every object A a $\mathcal{D}$-isomorphism $f_A : A \to \Phi(A)$ such that $\Phi(\mu) = f_B \circ \mu \circ f_A^{-1}$ holds for every $\mathcal{C}$-morphism $\mu : A \to B$. We say that an automorphism Φ of $\mathcal{C}$ is potentially inner if it is $\mathcal{D}$-inner for some extension $\mathcal{D}$ of $\mathcal{C}$.

This notion allows us to reformulate the problem as follows:

(1) What extension $\mathcal{D}$ of a given category $\mathcal{C}$ we have to construct in order to make all $\mathcal{C}$-automorphisms to be $\mathcal{D}$-inner?
(2) When all potentially inner automorphisms are inner?

We show that an automorphism Φ of a category $\mathcal{C}$ containing a monogenic free algebra A_0 is potentially inner if and only if $\Phi(A_0)$ is isomorphic to A_0 (Theorem 1). Thus by our assumption (Remark 1) the case of potentially inner automorphisms turns out to be general.

3. Let Φ be an arbitrary potentially inner automorphism. We can reduce the problem of describing Φ to a simpler case. The first reduction is provided by Lemma 2, which shows that we can assume that Φ fixes all free algebras in $\mathcal{C}$. The second reduction is provided by Theorem 2. It shows that Φ is a composition of two automorphisms, one of which is inner and the other has the property that the corresponding permutation s_A preserves all basis elements of every free algebra A. Therefore, in what follows we consider automorphisms satisfying these two conditions.

Using bijections s_A we can define a new algebraic structure A^* on the underlying set of every free algebra A such that $s_A : A \to A^*$ is an isomorphism. These new algebras A^* need not be objects of $\mathcal{C}$, but it turns out that A^* can be described. This is the crucial point of our argument. If A is a free algebra such that the number of its free generators is not less than arities of all its operations, then A^* is a derived algebraic structure, i.e., all its basic operations are determined by terms of the corresponding language or, in other words, are polynomial operations in the algebra A (Theorem 3).

Thus, every map s_A is an isomorphism between the source structure and the derived structure on the same set. It is worth mentioning that the source structure is a derived structure for A^*. This helps us to find the derived structures A^* and to describe the maps s_A.

4. Describing the maps s_A leads to a description of a given automorphism, that is, we know its form. However, this description is not unique, and it may happen that there exists a better one. Lemma 3 shows that the last problem can be solved by means of the so-called central functions (see Definition 3).

The paper is organized as follows. In the next section we prove the main statements outlined above. Then we apply the suggested method in order to describe all automorphisms of the category of free associative algebras (Section 3) and all automorphisms of the category of free group representations (Section 4). In both cases the automorphisms are close to inner. We also show that our method allows one to recover some known results in a simpler way.

Let us list some results of the paper. Theorem 1 gives a necessary and sufficient condition for an automorphism to be potentially inner and also shows an important property of such automorphisms. Theorem 2 reduces the problem to the case when the maps s_A fix all basis elements of every free algebra A. Theorem 3 describes the maps s_A. Theorem 4 characterizes automorphisms of the category of free associative algebras as almost inner ones,[2] and Theorem 5 gives an exact form of such automorphisms. Theorem 8 presents a similar result for the variety of group representations. This case is quite different because we deal with a two-sorted theory that causes extra difficulties.

All standard notions from category theory and universal algebra which are not defined in the text can be found in [6] and [2].

2. General approach

2.1. Preliminaries

In this section we present a general approach to the main problem. For the sake of completeness we include here some results from [15] in the form adjusted to the case of the categories of free universal algebras.

We consider arbitrary (not necessarily one-sorted) universal algebras, with homomorphisms as morphisms. In the case of one-sorted algebras the notion of homomorphism is the usual one,

[2] Other proofs of this result are given by Berzins [1] and Mashevitzky [7] using quite different approaches.

while in the many-sorted case the homomorphisms are compatible with the many-sorted structure. Obviously, one can speak about varieties of such algebras. Every variety $\mathcal{V}$ of algebras contains free algebras $F(X)$ where X is a set of free generators.

By definition every map of X to a $\mathcal{V}$-algebra A can be extended to a unique homomorphism of $F(X)$ to A.

Let $\mathcal{V}$ be a variety. We denote by $\Theta(\mathcal{V})$ the category whose objects are algebras from $\mathcal{V}$ and morphisms are presented by homomorphisms of these algebras. We also assume that we have a forgetful functor, that is, a faithful functor from $\Theta(\mathcal{V})$ to the category of all sets and maps. In the case of a one-sorted theory, this functor assigns to every algebra its underlying set and to every homomorphism the corresponding map. The many-sorted case requires additional considerations.

2.2. *Inner and potentially inner automorphisms*

All categories under consideration are full subcategories of $\Theta(\mathcal{V})$. We assume that they all contain a monogenic (one-generated) free algebra. Note that this condition is fulfilled automatically for the categories of free algebras.

Since morphisms of $\Theta(\mathcal{V})$ are maps, we can consider an extension $\mathcal{Q}$ of $\Theta(\mathcal{V})$ with the same objects but with extended sets of morphisms. These morphisms will be called quasi-homomorphisms. Let $\mathcal{C}$ be a full subcategory of the category $\Theta(\mathcal{V})$. *An extension* $\mathcal{D}$ of $\mathcal{C}$ will always mean a full subcategory of $\mathcal{Q}$ with the same objects as those of $\mathcal{C}$.

Definition 1. We say that an automorphism Φ of a category $\mathcal{C}$ is *inner* if for every object A of $\mathcal{C}$ there exists an isomorphism $\sigma_A : A \to \Phi(A)$ such that for every homomorphism $\mu : A \to B$ we have $\Phi(\mu) = \sigma_B \circ \mu \circ \sigma_A^{-1}$. That is, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\sigma_A} & \Phi(A) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \Phi(\mu)} \\ B & \xrightarrow{\sigma_B} & \Phi(B). \end{array}$$

In other words, Φ is isomorphic as a functor to the identity functor *Id* of $\mathcal{C}$.

We say that an automorphism Φ is $\mathcal{D}$-*inner* if it is the restriction to $\mathcal{C}$ of some inner automorphism of an extension $\mathcal{D}$ of the category $\mathcal{C}$ or, in other words, the isomorphisms σ_A and σ_B in the above diagram are $\mathcal{Q}$-isomorphisms (quasi-isomorphisms).

Definition 2. An automorphism Φ of the category $\mathcal{C}$ is said to be *potentially inner* if it is $\mathcal{D}$-inner for some extension $\mathcal{D}$ of $\mathcal{C}$.

Under our assumption, every considered category has a monogenic free algebra. Let A_0 be a free monogenic algebra in a category $\mathcal{C}$ over a fixed element x_0. There is a bijection between the underlying set of any $\mathcal{C}$-algebra A and the set of all homomorphisms from A_0 to A, namely, to each element $a \in A$, there corresponds a homomorphism α_a^A defined by

$$\alpha_a^A(x_0) = a. \tag{2.1}$$

The next result gives a necessary and sufficient condition for an automorphism of a category of universal algebras to be potentially inner.

Theorem 1. *An automorphism Φ of a category $\mathcal{C}$ is potentially inner if and only if $\Phi(A_0)$ is isomorphic to A_0.*

If Φ is potentially inner, i.e., if there is a function $A \mapsto s_A$ such that $\Phi(\mu) = s_B \circ \mu \circ s_A^{-1}$ for every $\mu : A \to B$, then $\Phi(F)$ is isomorphic to F for every free algebra F in the category $\mathcal{C}$, and the quasi-isomorphism s_F maps a basis of F onto a basis of $\Phi(F)$.

Proof. Let $\sigma : A_0 \to \Phi(A_0)$ be an isomorphism. Let A be an arbitrary $\mathcal{C}$-algebra, and let $a \in A$. The formula (2.1) implies that there exists a unique element $\bar{a} \in \Phi(A)$ such that $\Phi(\alpha_a^A) \circ \sigma = \alpha_{\bar{a}}^{\Phi(A)}$. Since $\alpha_a^A = \Phi^{-1}(\alpha_{\bar{a}}^{\Phi(A)} \circ \sigma^{-1})$, we obtain a bijection $s_A : A \to \Phi(A)$ setting for every $a \in A$:

$$s_A(a) = \Phi\big(\alpha_a^A\big) \circ \sigma(x_0). \tag{2.2}$$

This gives rise to a family of bijections $(s_A : A \to \Phi(A) \mid A \in Ob\mathcal{C})$.

Let $\nu : A \to B$ be a homomorphism. According to the definition above we have $s_B(\nu(a)) = \Phi(\alpha_{\nu(a)}^B) \circ \sigma(x_0)$. Since $\alpha_{\nu(a)}^B = \nu \circ \alpha_a^A$, we obtain $(s_B \circ \nu)(a) = \Phi(\nu) \circ \Phi(\alpha_a^A) \circ \sigma(x_0) = (\Phi(\nu) \circ s_A)(a)$. Hence,

$$\Phi(\nu) = s_B \circ \nu \circ s_A^{-1}. \tag{2.3}$$

Let us add to the category $\mathcal{C}$ new isomorphisms (bijections) $s_A : A \to \Phi(A)$ and their inverses $s_A^{-1} : \Phi(A) \to A$. Denote the obtained category by $\mathcal{D}$. By definition, the automorphism Φ is $\mathcal{D}$-inner.

Suppose that an automorphism Φ of $\mathcal{C}$ is potentially inner, and let $F = F(X)$ be a free algebra in the category $\mathcal{C}$ over a set X. Denote $A = \Phi(F)$ and $B = \Phi^{-1}(F)$. We have the following diagram:

$$B \xrightarrow{\ s_B\ } F \xrightarrow{\ s_F\ } A.$$

Set $\widetilde{X} = s_F(X)$ and take into account that s_F determines a bijection between X and $\widetilde{X}$. Denote by σ the unique homomorphism from F to $A = \Phi(F)$ that extends $s_F|_X$, that is, $\sigma(x) = s_F(x)$ for every $x \in X$. In the same way, we have a homomorphism $\tau : F \to B$ such that $\tau(x) = s_B^{-1}(x)$ for every $x \in X$. We have

$$\Phi(\tau) \circ \sigma(x) = s_B \circ \tau \circ s_F^{-1} \circ s_F(x) = s_B \circ s_B^{-1}(x) = x.$$

Hence $\Phi(\tau) \circ \sigma = 1_F$. Replacing Φ with Φ^{-1} and vice versa, we switch σ and τ, therefore $\Phi^{-1}(\sigma) \circ \tau = 1_F$. Hence $\sigma \circ \Phi(\tau) = 1_{\Phi(F)}$. Thus σ is an isomorphism from F to $\Phi(F)$. The statement is proved. □

The above result shows that if an automorphisms takes a monogenic free algebra to an isomorphic one, it does the same with all free algebras in $\mathcal{C}$.

The next fact is simple but useful.

Lemma 1. *Let $\mathcal{C}$ be a subcategory of a category $\mathcal{D}$, and let $\Phi : \mathcal{C} \to \mathcal{D}$ be a functor. Let $\mathcal{E}$ be a subcategory of $\mathcal{C}$ satisfying the following conditions*:

(1) *Φ acts on the class of $\mathcal{E}$-algebras as a permutation*;
(2) *for every $\mathcal{E}$-algebra A there exists a quasi-isomorphism* (*a $\mathcal{D}$-isomorphism*) $\sigma_A : A \to \Phi(A)$ *such that $\Phi(\nu) = \sigma_B \circ \nu \circ \sigma_A^{-1}$ for every $\mathcal{E}$-morphism $\nu : A \to B$.*

Then Φ is a composition of two functors $\Phi = \Psi \circ \Gamma$, where $\Gamma : \mathcal{C} \to \mathcal{D}$ is an identity on $\mathcal{E}$ (*preserves all objects and all morphisms of $\mathcal{E}$*), *and the functor Ψ is an inner automorphism of the category $\mathcal{D}$.*

Proof. By the hypotheses of the lemma, we have a family $(\sigma_A : A \to \Phi(A) \mid A \in Ob\,\mathcal{E})$ of $\mathcal{D}$-isomorphisms. We construct an inner automorphism Ψ of $\mathcal{D}$ in the following way. For every $\mathcal{D}$-object U, we set $\Psi(U) = U$ if U is not an object of $\mathcal{E}$, and $\Psi(A) = \Phi(A)$ for every object A of $\mathcal{E}$. We define a $\mathcal{D}$-isomorphism $\tau_U : U \to \Psi(U)$ in the following way: $\tau_U = 1_U$ if U is not an object of $\mathcal{E}$, and $\tau_A = \sigma_A$, where $A \in Ob(\mathcal{E})$. For every $\mathcal{D}$-morphism $\mu : U \to V$, let $\Psi(\mu) = \tau_V \circ \mu \circ \tau_U^{-1}$. According to the given construction, Ψ is an inner automorphism of $\mathcal{D}$.

It is clear that $\Phi = \Psi \circ \Psi^{-1} \circ \Phi$. Let $\Gamma = \Psi^{-1} \circ \Phi$. By definition, we have $\Gamma(A) = (\Psi^{-1} \circ \Phi)(A) = A$ for all $A \in Ob(\mathcal{E})$ and $\Gamma(\nu) = (\Psi^{-1} \circ \Phi)(\nu) = \sigma_B^{-1} \circ \Phi(\nu) \circ \sigma_A = \sigma_B^{-1} \circ \sigma_B \circ \nu \circ \sigma_A^{-1} \circ \sigma_A = \nu$ for every $\mathcal{E}$-morphism $\nu : A \to B$. □

The next result is a variation of the above one.

Lemma 2. *Let Φ be an automorphisms of a category $\mathcal{C}$. Suppose that for some class* $\mathbf{E}$ *of $\mathcal{C}$-objects $\Phi(A)$ is isomorphic to A for every $A \in \mathbf{E}$. Then Φ is a composition of two $\mathcal{C}$-automorphisms $\Phi = \Psi \circ \Gamma$, where Γ fixes all objects from* $\mathbf{E}$ *and Ψ is an inner automorphism.*

Proof. We apply the previous lemma by setting $\mathcal{D} = \mathcal{C}$. Suppose that a class $\mathbf{E}$ is closed under Φ and Φ^{-1}. Let $\mathcal{E}$ be the subcategory of $\mathcal{C}$ whose class of objects is $\mathbf{E}$ and whose only morphisms are identity morphisms (i.e., a discrete subcategory). The previous lemma gives us the required statement. If $\mathbf{E}$ is not closed, we can consider its Φ- and Φ^{-1}-closure that clearly has the same property as $\mathbf{E}$. □

2.3. *The main function*

As we have assumed, in all our categories $\mathcal{C}$ any automorphism Φ of $\mathcal{C}$ takes a monogenic free algebra to an isomorphic one. Applying Lemma 2, we restrict our consideration to the case when the automorphism Φ fixes all free algebras in $\mathcal{C}$. This implies that the maps s_A are permutations of all free algebras in $\mathcal{C}$ and simplifies the formula (2.2):

$$s_A(a) = \Phi\big(\alpha_a^A\big)(x_0). \tag{2.4}$$

Note that the function $A \mapsto s_A$ that we call a *main function* is not a unique function realizing Φ as a $\mathcal{D}$-inner automorphism for some category $\mathcal{D}$.

Definition 3. Suppose that for every $\mathcal{C}$-algebra A we are given a permutation c_A of its underlying set. The function $A \mapsto c_A$, $A \in Ob\,\mathcal{C}$, is said to be *central* if for every homomorphism $\nu : A \to B$ the following equation is satisfied: $c_B \circ \nu \circ c_A^{-1} = \nu$. In other words, a function $A \mapsto c_A$ is a central function if it determines the identity automorphism of the category $\mathcal{C}$.

It is obvious that the function $A \mapsto \sigma_A$, $A \in Ob\mathcal{C}$, from Definition 1 is determined for a potentially inner automorphism Φ up to a central function. Thus, if we find out that an automorphism Φ is $\mathcal{D}$-inner, Φ may be $\mathcal{D}'$-inner for some subcategory $\mathcal{D}'$ of $\mathcal{D}$, or even inner. Central functions are known for many varieties, for example, for varieties of semigroups, groups and associative algebras, and we will use them below.

The next result shows how to use central functions. It is clear that the bijections s_A allow one to define a new algebraic structure A^* on the underlying set of every algebra $\Phi(A)$ so that $s_A : A \to A^*$ is an isomorphism.

Lemma 3. *An automorphism Φ of $\mathcal{C}$ is $\mathcal{D}$-inner for an extension $\mathcal{D}$ of $\mathcal{C}$ (by adding some kind of quasi-homomorphisms) if and only if there exists a central function $A \mapsto c_A$, $A \in Ob\mathcal{C}$, such that every $c_{\Phi(A)}$ is a quasi-isomorphism of $\Phi(A)$ onto A^*. In particular, Φ is inner if and only if every $c_{\Phi(A)}$ in the condition above is an isomorphism.*

Proof. If Φ is $\mathcal{D}$-inner, $\Phi(\nu) = \sigma_B \circ \nu \circ \sigma_A^{-1}$ for all $\nu : A \to B$, where $\sigma_A : A \to \Phi(A)$ are $\mathcal{D}$-isomorphisms for every A. Consider $c_{\Phi(A)} = s_A \circ \sigma_A^{-1}$. Clearly, $c_{\Phi(A)}$ is a quasi-isomorphism of $\Phi(A)$ onto A^*. Since the two considered functions $A \mapsto s_A$ and $A \mapsto \sigma_A$ define the same automorphism, the function $A \mapsto c_A$ is central.

Conversely, if there exists a central function $A \mapsto c_A$, $A \in Ob\mathcal{C}$, such that every $c_{\Phi(A)}$ is a quasi-isomorphism of $\Phi(A)$ onto A^*, then we set $\sigma_A = c_{\Phi(A)}^{-1} \circ s_A$. Clearly σ_A is a $\mathcal{D}$-isomorphism of A onto $\Phi(A)$, and $\sigma_B \circ \nu \circ \sigma_A^{-1} = c_{\Phi(B)}^{-1} \circ s_B \circ \nu \circ s_A^{-1} \circ c_{\Phi(A)} = c_{\Phi(B)}^{-1} \circ \Phi(\nu) \circ c_{\Phi(A)} = \Phi(\nu)$ for all $\nu : A \to B$. Hence Φ is $\mathcal{D}$-inner. □

Therefore in order to describe an automorphism of a given category, the first step is to describe maps s_A defined by (2.4). Then one has to find a suitable central function or to prove that it does not exist. The formula (2.4) can be rewritten in the form:

$$\Phi(\alpha)(x_0) = (s_A \circ \alpha)(x_0) \tag{2.5}$$

for every $\alpha : A_0 \to A$.

The following simple fact shows that we can reduce the problem to the case when the maps s_A satisfy very nice conditions.

Theorem 2. *For every free algebra A in $\mathcal{C}$ fix some basis X_A. Suppose that an automorphism Φ of $\mathcal{C}$ preserves all free algebras. Then Φ is a composition of two automorphisms $\Phi = \Psi \circ \Gamma$, where Ψ is an inner automorphism and $s_A^\Gamma(x) = x$ for all $x \in X_A$ for every free algebra A.*

Proof. According to the second part of Theorem 1, we have an automorphism σ_A for every free algebra A such that $\sigma_A(x) = s_A(x)$ for all $x \in X_A$. Now we apply Lemma 1, where $\mathcal{D} = \mathcal{C}$ and $\mathcal{E}$ is the subcategory of $\mathcal{C}$ containing all free algebras A in $\mathcal{C}$ with set of morphisms $\{\alpha_x^A,\ x \in X_A\}$ from the A_0 to the other ones and, of course, the identities 1_{A_0} and 1_A. Since $\Phi(\alpha_x^A)(x_0) = s_A \circ \alpha_x^A \circ s_{A_0}^{-1}(x_0) = s_A \circ \alpha_x^A(x_0) = s_A(x) = \sigma_A \circ \alpha_x^A(x_0) = \sigma_A \circ \alpha_x^A \circ 1_{A_0}(x_0)$, we obtain that $\Phi(\alpha_x^A) = \sigma_A \circ \alpha_x^A \circ 1_{A_0}$ for all $x \in X_A$ and all objects A of the category $\mathcal{E}$. Thus the restriction of Φ to $\mathcal{E}$ acts according to the conditions of Lemma 1.

Applying this lemma, we obtain that Φ is a composition of two automorphisms $\Phi = \Psi \circ \Gamma$, where Ψ is an inner automorphism and $\Gamma(\alpha_x^A) = \alpha_x$ for all $x \in X_A$. The last condition means that $s_A^\Gamma(x) = x$ for all $x \in X_A$ for every free algebra A. □

Corollary 1. *Let Φ be an automorphism of the category $\mathcal{C}$ fixing the free algebra A over a set X. Suppose that $\Phi(\alpha_x)=\alpha_x$ for all $x\in X$. Let $f:X\to A$. Denote by θ_f the unique endomorphism of A such that $\theta_f(x)=f(x)$ for all $x\in X$. Then $\Phi(\theta_f)=\theta_{s_A\circ f}$.*

Proof. The condition $\theta_f(x)=f(x)$ can be expressed by the equality $\alpha_{f(x)}=\theta_f\circ\alpha_x$. Applying Φ to this equality we obtain $\alpha_{s_A(f(x))}=\Phi(\theta_f)\circ\alpha_x$. Hence $s_A(f(x))=\Phi(\theta_f)(x)$. Since the last equality is valid for all $x\in X$, we have $\Phi(\theta_f)=\theta_{s_A\circ f}$. $\square$

Let $X=\{x_1,\dots,x_n\}$, $f(x_1)=a_1,\dots,f(x_n)=a_n$. In this situation we write $\theta_{a_1,\dots,a_n}$ instead of θ_f. We thus have the following result:

$$\Phi(\theta_{a_1,\dots,a_n})=\theta_{s_A(a_1),\dots,s_A(a_n)}. \tag{2.6}$$

2.4. Derived algebras

From now on we consider the category $\Theta^0(\mathcal{V})$ defined in Section 1, that is, the full subcategory of $\Theta(\mathcal{V})$ formed by all free in $\mathcal{V}$ algebras $A=A(X)$, where finite sets X are subsets of a fixed infinite set X_0. This restriction is motivated only by future applications. According to results obtained in the previous section, we can reduce our consideration to the case when the automorphisms Φ of $\Theta^0(V)$ satisfy the following conditions:

(1) $\Phi(A)=A$ for every algebra A,
(2) for every algebra $A=A(X)$, $s_A^{\Phi}(x)=x$ for each element x of X.

Let Φ be an automorphism of such kind. It determines two new structures on the underlying set of every algebra A. These structures have the same type as the source structure. The first one is already defined. It gives the algebra A^* induced by the permutation s_A, thus $s_A:A\to A^*$ is an isomorphism. The second one will be defined below.

Let ω stand for the symbol of a basic k-ary operation. Denote by ω_A the corresponding k-ary operation of the algebra A. Consider an algebra A whose set $X_A=\{x_1,\dots,x_n\}$ of free generators contains at least k elements. Fix the term $\omega(x_1,\dots,x_k)$ and the corresponding element $w=\omega_A(x_1,\dots,x_k)$ in A. For every collection $a_1,\dots,a_k\in A$ we have:

$$\omega_A(a_1,\dots,a_k)=\theta_{a_1,\dots,a_k,a_{k+1},\dots,a_n}\bigl(\omega_A(x_1,\dots,x_k)\bigr), \tag{2.7}$$

where $a_{k+1}=\cdots=a_n=a_k$.

Let us now apply Φ and consider the element $\widetilde{w}=s_A(w)=s_A(\omega_A(x_1,\dots,x_k))$. Being an element of the free algebra A, $\widetilde{w}$ is also a term. By means of $\widetilde{w}$ we can define a new k-ary operation $\widetilde{\omega}_A$ by the rule:

$$\widetilde{\omega}_A(a_1,\dots,a_k)=\theta_{a_1,\dots,a_k,a_k,\dots,a_k}(\widetilde{w}). \tag{2.8}$$

Being defined by means of a term, the new operation is a derived operation which is often called a polynomial operation. Let B be another algebra. The term $\widetilde{w}$ determines a new operation $\widetilde{\omega}_B$ in the usual way. Let $b_1,\dots,b_k\in B$. Consider a homomorphism $\nu:A\to B$ defined by $\nu(x_1)=b_1,\dots,\nu(x_k)=b_k$, $\nu(x_{k+1})=\cdots=\nu(x_n)=b_k$ and set

$$\widetilde{\omega}_B(b_1,\dots,b_k)=\nu\bigl(\widetilde{\omega}_A(x_1,\dots,x_k)\bigr). \tag{2.9}$$

The next result completes our general description of automorphisms of the category $\Theta^0(\mathcal{V})$. By the way this result shows that the operation defined by (2.9) does not depend on the choice of an algebra A.

Theorem 3. *For every algebra B the derived operation $\widetilde{\omega}_B$ defined by* (2.9) *coincides with the induced operation ω_B^*, that is,*

$$s_B\big(\omega_B(b_1,\dots,b_k)\big)=\widetilde{\omega}_B\big(s_B(b_1),\dots,s_B(b_k)\big)$$

for every $b_1,\dots,b_k\in B$.

Proof. The homomorphism $\nu : A\to B$ in (2.9) is defined by $\nu(x_1)=b_1,\dots,\nu(x_k)=b_k$, $\nu(x_{k+1})=\dots=\nu(x_n)=b_k$. Further, $\Phi(\nu)=s_B\circ\nu\circ s_A^{-1}$ and hence $\Phi(\nu)(x_1)=s_B(b_1),\dots,$ $\Phi(\nu)(x_k)=s_B(b_k)$, $\Phi(\nu)(x_{k+1})=\dots=\Phi(\nu)(x_n)=s_B(b_k)$. Thus we obtain

$$\begin{aligned}s_B\big(\omega_B(b_1,\dots,b_k)\big)&=s_B\big(\nu\big(\omega_A(x_1,\dots,x_k)\big)\big)=\Phi(\nu)\circ s_A\big(\omega_A(x_1,\dots,x_k)\big)\\&=\Phi(\nu)\big(\widetilde{\omega}_A(x_1,\dots,x_k)\big)=\widetilde{\omega}_B\big(s_B(b_1),\dots,s_B(b_k)\big).\end{aligned}$$

This shows that $\omega_B^*=\widetilde{\omega}_B$ is a derived operation on B. □

Remark 2. Note that the above facts are also valid for every full subcategory of $\Theta(\mathcal{V})$ containing a monogenic free algebra and a free algebra whose set of free generators contains not less elements than the arities of all basic operations.

For the category $\Theta^0(\mathcal{V})$ we conclude that every term $s_A(\omega_A(x_1,\dots,x_k))$ is also built from the same free generators. Indeed, for every set $X=\{x_1,\dots,x_k\}$ there is only one $\Theta^0(\mathcal{V})$-algebra A such that $A=A(X)$. Since the term $s_A(\omega_A(x_1,\dots,x_k))$ belongs to A, it is built from the same free generators and determines the corresponding derived operation. This fact will be used in the next sections.

In view of the previous result we would like to emphasize the situation we come to. For every algebra A of our category we have a derived algebra $\widetilde{A}$ of the same type. All derived operations are defined by terms (polynomials) of the source structure, the algebra $\widetilde{A}$ is isomorphic to A, and the isomorphism $s_A : A\to\widetilde{A}$ preserves all free generators from X_A which, of course, are free generators for $\widetilde{A}$. Thus the problem is reduced to finding the derived operations. Once they are found, we will know what quasi-homomorphisms we need to add in order to make Φ a $\mathcal{D}$-inner automorphism for some extension $\mathcal{D}$ of our category.

The following observation may help in some cases. As indicated above, every basic operation of each of free algebras A and $\widetilde{A}$ can be expressed as a term in the language of the other one, i.e., it is a polynomial operation with respect to the other algebra. Let $\widetilde{w}$ be a word in the language of the algebra $\widetilde{A}$ such that a k-ary operation ω of the algebra A can be expressed as follows: $\omega(x_1,\dots,x_k)=\widetilde{w}$. In exactly the same way, every basic operation appearing in $\widetilde{w}$ can be expressed by means of words in the language of the first algebra A. Replacing the basic operations in $\widetilde{w}$ by their expressions in the language of the algebra A, we get an identity of the kind $\omega(x_1,\dots,x_k)=w$, where w is the word obtained by replacement.

If we know what identities of such kind are satisfied in the considered variety, we can understand what kind of derived operation we have to treat. For example, in the variety of all semigroups there is only one identity of such kind: $xy = xy$. In the variety of all commutative semigroups, there are only two such identities: $xy = xy$ and $xy = yx$. For the unary operation $^{-1}$ in the variety of all inverse semigroups all such identities are of the form: $x^{-1} = x^{-1}(xx^{-1})^m$. It is clear that in many cases one cannot hope to get such a simple solution.

2.5. Some simple examples

Now we present three examples illustrating our method. The results we obtain here are known and published [3,9,10], so we can compare different ways leading to the same results.

Semigroups

For the simplest example, consider the variety **SEM** of all semigroups. As mentioned above, there is a unique identity of the form $xy = w$ satisfied in **SEM**, i.e., the identity $xy = xy$. Hence if $x \bullet y = v$ defines a new operation, the word v contains the same letters x and y and is of length 2. This fact immediately implies that there are no suitable polynomial operations except the original and dual ones. Thus we have only one new derived operation $x \bullet y = yx$. Therefore all automorphisms of the category $\mathbf{SEM}^0$ are $\mathcal{D}$-inner, where the category $\mathcal{D}$ is the category whose objects are objects of $\mathbf{SEM}^0$ and morphisms are both homomorphisms and anti-homomorphisms of semigroups.

If an automorphism Φ determines the same derived operation, then Φ is inner. If it determines the dual operation, it is not inner because in this case there are no central maps which are isomorphisms of the two-generated free semigroup onto the dual semigroup. In the case of the variety of commutative semigroups, all automorphisms are inner.

Groups

Let **Grp** be the category of all groups, viewed as algebras with one 0-ary, one unary and one binary operations. The derived 0-ary operation coincides with the original one. Consider a 1-generated free group and the derived unary operation $x^* = s(x^{-1}) = w$. The word w is built from one variable x. Hence $x^* = x^m$, where m is a nonzero integer. Because of the identity $(x^*)^* = x$, we have $m = 1$ or $m = -1$. It is clear that only the last case is suitable.

Let G be a free group generated by two free generators x and y. Let $s(xy) = w(x, y)$, that is, the term $w(x, y)$ determines a derived binary operation that gives an isomorphic free group with the same free generators. Denote the new product by $a \bullet b = w(a, b)$. The term $w(x, y)$ is of the following form: $w(x, y) = x^{i_1} y^{j_1} x^{i_2} y^{j_2} \ldots x^{i_k} y^{j_k}$, where $i_1, j_1, \ldots, i_k, j_k$ are integers.

The identities $w(x, e) = x$, $w(e, y) = y$ imply that $i_1 + \cdots + i_k = j_1 + \cdots + j_k = 1$, and the identity $w(x, y)^{-1} = w(y^{-1}, x^{-1})$ implies that $i_1 = j_k, \ldots, i_k = j_1$. Thus $w(x, y) = x^{i_1} y^{i_k} x^{i_2} y^{i_{k-1}} \ldots x^{i_k} y^{i_1}$. Therefore, $w(a, a) = a^2$ for all $a \in G$. By induction, we obtain that $a^{\bullet n} = a \bullet a \bullet \cdots \bullet a$ (n times) is equal to a^n.

Since the source operation has a similar expression in terms of $w(x, y)$, we have an identity of the form $xy = x^{i_1} \bullet y^{i_k} \bullet x^{i_2} \bullet y^{i_{k-1}} \ldots x^{i_k} \bullet y^{i_1}$. This identity holds if only if $w(x, y)$ is equal to xy or to yx.

Unlike the category $\mathbf{SEM}^0$, all automorphisms of the category $\mathbf{Grp}^0$ are inner because the map $g \mapsto g^{-1}$ is an isomorphism of a group onto the dual one, and the function assigning to every group such a map is a central function.

Lie algebras

The case when $\mathcal{V}$ is the variety of Lie algebras over an infinite field $\mathbf{K}$ is more complicated since there are two binary operations, "+" and "[ ]," and the set of unary operations

$a \mapsto ka$ for every $k \in \mathbf{K}$. To obtain the derived unary operations, we consider the monogenic free Lie algebra, which is a one-dimensional linear space $L = \mathbf{K}x$ with trivial multiplication. Thus $s_L(kx) = \varphi(k)x$. The map $\varphi : \mathbf{K} \to \mathbf{K}$ is, of course, a bijection preserving multiplication in $\mathbf{K}$. Hence $\varphi(0) = 0$ and $\varphi(1) = 1$.

To find the derived binary operations, we consider the two-generated free Lie algebra $F = F(x, y)$. We have $x \perp y = s_F(x + y) = P(x, y)$, where $P(x, y)$ is a polynomial in F. Since $x \perp y$ is homogeneous of degree 1, the polynomial $P(x, y)$ is linear: $x \perp y = ax + by$. It is clear that $a = b = 1$ because of commutativity and the condition $x \perp 0 = x$.

We conclude that s_F is an additive isomorphism such that $s_F(kw) = \varphi(k)s_F(w)$, where φ is an automorphism of $\mathbf{K}$, $k \in \mathbf{K}$ and $w \in F$.

Further, since $x * y = s_F([xy])$ is a homogeneous polynomial of degree 2, $x * y = a[xy]$, where $a \in \mathbf{K}$. Consider a function assigning to every free Lie algebra $F(X)$ a permutation $\mathbf{c}_{F(X)}$ of $F(X)$ as follows: $\mathbf{c}_{F(X)}(w) = \frac{w}{a}$. This function is clearly central. Hence every automorphism of the category $\Theta^0(\mathcal{V})$ is semi-inner according to the definition given in [10], that is, in Definition 1, the bijections σ_A are additive and multiplicative isomorphisms of the corresponding Lie rings but $\sigma_A(kw) = \varphi(k)\sigma_A(w)$ for every $k \in \mathbf{K}$ and $w \in A$.

In the next two sections, we apply the method to the variety of all associative linear algebras over a fixed infinite field and to the variety of all representations of groups in unital R-modules over a associative commutative ring R with unit.

3. Associative linear algebras

In this section, $\mathbf{K}$ will always denote an infinite field. Let A be an associative ring with unit, and let $f : \mathbf{K} \to A$ be a ring homomorphism of $\mathbf{K}$ to the center of A. We view such a homomorphism as an associative (linear) algebra over $\mathbf{K}$, or shortly, a $\mathbf{K}$-algebra. Let two $\mathbf{K}$-algebras $f : \mathbf{K} \to A$ and $g : \mathbf{K} \to B$ be given. A homomorphism of the first algebra to the second one is a ring homomorphism $\nu : A \to B$ such that $g = \nu \circ f$. As usual, we identify elements of $\mathbf{K}$ viewed as symbols of 0-ary operations (constants) with their images in A and write ka instead of $f(k)a$, if this does not lead to misunderstanding.

The category of all $\mathbf{K}$-algebras and their homomorphisms will be denoted by Ass-$\mathbf{K}$. Let (Ass-$\mathbf{K})^0$ denote the full subcategory of Ass-$\mathbf{K}$ consisting of all free $\mathbf{K}$-algebras over finite subsets of an infinite set X_0. We fix a monogenic free algebra F_1 in (Ass-$\mathbf{K})^0$ with free generator x_0 and a two-generated free algebra F_2 with free generators x and y.

Let Φ be an automorphism of the category (Ass-$\mathbf{K})^0$. It is clear that $\Phi(F_1)$ is isomorphic to F_1, hence Φ is potentially inner (Theorem 1), and we can assume that Φ fixes all the objects. According to Theorem 2, we can assume that the permutations s_A act as identities on the bases X_A of the objects A of this category. Every map s_A is an isomorphism of A onto the derived algebra $\widetilde{A}$. According to the method suggested in the previous section, we have to find the derived algebraic structure $F_2^* = \widetilde{F}_2$, which is, of course, an associative free two-generated $\mathbf{K}$-algebra isomorphic to F_2.

Lemma 4. *Every map s_A acts on $\overline{\mathbf{K}} = f_A(\mathbf{K})$ as a permutation.*

Proof. Let $u \in A$. Evidently, $u \in \overline{\mathbf{K}}$ if and only if for every endomorphism ν of A we have $\nu \circ \alpha_u = \alpha_u$. We apply the automorphism Φ and obtain that for every endomorphism ν of A we have $\Phi(\nu) \circ \alpha_{s_A(u)} = \alpha_{s_A(u)}$. Since $\Phi(\nu)$ runs over all endomorphisms, this means that $u \in \overline{\mathbf{K}}$ if and only if $s_A(u) \in \overline{\mathbf{K}}$. $\square$

This result shows that the automorphism Φ determines a permutation $\tilde{s}$ of the field **K**. The derived 0-ary structure on the set A is given by the map $f_A^*: \mathbf{K} \to A^*$, where $f_A^* = s_A \circ f_A = f_A \circ \tilde{s}$. We denote $\tilde{0} = \tilde{s}(0)$ and $\tilde{1} = \tilde{s}(1)$. Remind that the same symbols denote the corresponding elements in **K**-algebras.

Remark. The same result can be obtained using the following arguments. The free algebra in Ass-**K** over the empty set of generators is **K**. Thus $s_{\mathbf{K}}$ is a permutation of **K**, and according to Definition 2.9 of derived operations and Theorem 3, $f_A^* = \tilde{f}_A = f_A \circ s_{\mathbf{K}} = s_A \circ f_A$. Here $s_{\mathbf{K}}$ coincides with $\tilde{s}$ above.

The next step is to make the permutation $\tilde{s}$ of the field **K** more convenient for finding the derived structure.

Lemma 5. *Every automorphism of the category* Ass-**K** *is a composition of an inner automorphism and an automorphism Γ satisfying the following conditions*:

(1) *Γ fixes all objects of the category*;
(2) *the corresponding permutations s_A^Γ preserve all elements of $X_A \cup \{0, 1\}$ for every objects A of this category.*

Proof. The permutation $\tilde{s}$ of the field **K** takes the null element 0 to an element $\tilde{0}$ and the unit 1 to an element $\tilde{1}$. Clearly, $\tilde{1} \neq \tilde{0}$. Consider endomorphisms ψ_A and ξ_A defined as follows: $\psi_A(x) = (\tilde{1} - \tilde{0})x + \tilde{0}$ and $\xi_A(x) = \frac{x-\tilde{0}}{\tilde{1}-\tilde{0}}$ for all $x \in X_A$. Since $\psi_A \circ \xi_A(x) = \xi_A \circ \psi_A(x) = x$, both endomorphisms are automorphisms of A.

Let us construct an inner automorphism Ψ of our category as follows: $\Psi(A) = A$ for every object A and $\Psi(\nu) = \psi_B \circ \nu \circ \psi_A^{-1}$ for every homomorphism $\nu: A \to B$. The main function s_A^Ψ has the following properties:

$$s_A^\Psi(x) = \Psi(\alpha_x)(x_0) = \psi_A \circ \alpha_x \circ \psi_{F_1}^{-1}(x_0) = \psi_A \circ \alpha_x\left(\frac{x_0 - \tilde{0}}{\tilde{1} - \tilde{0}}\right) = \psi_A\left(\frac{x - \tilde{0}}{\tilde{1} - \tilde{0}}\right) = x$$

for all $x \in X_A$ and

$$s_A^\Psi(\tilde{0}) = \Psi(\alpha_{\tilde{0}})(x_0) = \psi_A \circ \alpha_{\tilde{0}} \circ \psi_{F_1}^{-1}(x_0) = \psi_A \circ \alpha_{\tilde{0}}\left(\frac{x_0 - \tilde{0}}{\tilde{1} - \tilde{0}}\right) = \psi_A\left(\frac{\tilde{0} - \tilde{0}}{\tilde{1} - \tilde{0}}\right) = 0.$$

In the same way, $s_A^\Psi(\tilde{1}) = 1$.

Thus the composition $\Gamma = \Psi \circ \Phi$ possesses the properties assumed for Φ, but in addition the permutation determined by this composition preserves 0 and 1 of the field **K**. Since Ψ is an inner automorphism, our statement is proved. □

Thus we can assume that our source automorphism Φ has the properties mentioned above (Lemma 5).

Lemma 6. *Suppose that an automorphism Φ of the category* Ass-**K** *satisfies conditions* (1) *and* (2) *of the previous lemma. Then the corresponding permutations s_A are ring automorphisms* (*or anti-automorphisms*) *of every object A.*

Proof. We shorten the notation s_{F_2} to s. Our aim is to find derived operations: $x \perp y = s(x+y)$ and $x \odot y = s(xy)$, which are polynomials in two non-commuting variables x, y. Suppose

$$x \perp y = a_1 + P_1(x) + Q_1(y) + R_1(x, y),$$
$$x \odot y = a_2 + P_2(x) + Q_2(y) + R_2(x, y),$$

where $a_1, a_2 \in \mathbf{K}$, $P_1(x)$, $P_2(x)$, $Q_1(y)$, $Q_2(y)$ are the polynomials without constant terms depending only on x and y, respectively, and $R_1(x, y)$, $R_2(x, y)$ are the polynomials whose each summand contains both variables x and y.

Because of the identities $x \perp 0 = 0 \perp x = x$, $x \odot 0 = 0 \odot x = 0$, it follows that $a_1, a_2 = 0$, $P_1(x) = x$, $Q_1(y) = y$, $P_2(x) = Q_2(y) = 0$. Hence we have

$$x \perp y = x + y + R_1(x, y),$$
$$x \odot y = R_2(x, y).$$

Denote the full degree of the polynomial $R_2(x, y)$ by p. Take a constant $k \in \mathbf{K}$ and consider $k \odot x = R_2(k, x) = T_k(x)$, where $T_k(x)$ is a polynomial in one variable x. It is clear that the degree of such a polynomial can be at most p for all k. Let $m = k \odot k$. We have the identity

$$T_m(x) = m \odot x = (k \odot k) \odot x = k \odot (k \odot x) = T_k\big(T_k(x)\big).$$

The degree of the polynomial in the left-hand side is still at most p, and the degree of the polynomial in the right-hand side is equal to the square of the degree of the polynomial $T_k(x)$. Repeating this process, we arrive at a contradiction if the degree of $T_k(x)$ is not 1.

Thus $k \odot x = R_2(k, x) = T_k(x) = ax$ for some $a \in \mathbf{K}$. Since $k \odot 1 = k$, we obtain $a = k$ and hence $k \odot x = kx$. In view of homogeneity of the polynomials $R_1(x, y)$, $R_2(x, y)$, we conclude that $R_1(x, y) = 0$ and $R_2(x, y)$ is equal to xy or to yx. Therefore

$$x \perp y = x + y \quad \text{and} \quad x \odot y = xy \quad (\text{or } x \odot y = yx).$$

This gives us the required statement: $s_A(u + v) = s_A(u) + s_A(v)$ and $s_A(uv) = s_A(u)s_A(v)$ ($s_A(uv) = s_A(v)s_A(u)$) for all $u, v \in A$. $\square$

Consider the case when the maps s_A are ring automorphisms. Note that in general these maps are not isomorphisms of $\mathbf{K}$-algebras because for $k \in \mathbf{K}$ we have

$$s_A(ku) = s_A(k)s_A(u) = \tilde{s}(k)s_A(u).$$

The permutation $\tilde{s} = s_{\mathbf{K}} : \mathbf{K} \to \mathbf{K}$ is an automorphism of $\mathbf{K}$. This leads to the following definition.

Definition 4. Let $f_A : \mathbf{K} \to A$ and $f_B : \mathbf{K} \to B$ be $\mathbf{K}$-algebras. A map $\sigma : A \to B$ is called a twisted homomorphism if it is a ring homomorphism of A to B and $\sigma \circ f_A = f_B \circ \varphi$ for some automorphism φ of $\mathbf{K}$, in other words, $\sigma(aw) = \varphi(a)\sigma(w)$ for every $a \in \mathbf{K}$ and $w \in A$. In this case we say that σ is a φ-homomorphism.

We now define the category $R(\text{Ass-}\mathbf{K})^0$ as the extension of the source category obtained by adding all twisted homomorphisms and anti-homomorphisms. Then every automorphism Φ of the category $(\text{Ass-}\mathbf{K})^0$ is an $R(\text{Ass-}\mathbf{K})^0$-inner automorphism.

Let us now formulate our final result more precisely. We say that an automorphism Φ is *semi-inner* if it acts on morphisms of the category $(\text{Ass-}\mathbf{K})^0$ as follows:

$$\Phi(\nu) = \tau_B \circ \nu \circ \tau_A^{-1}, \tag{3.1}$$

where $\tau_A : A \to \Phi(A)$ is a φ-isomorphism of $\mathbf{K}$-algebras and φ is an automorphism of the field $\mathbf{K}$ determined by Φ.

Now consider the case when all maps s_A are ring anti-automorphisms. It is known [9] that in this case we deal with a special automorphism of our category called the mirror automorphism. We recall its construction.

Given a free $\mathbf{K}$-algebra A with the set X_A of free generators, consider the free semigroup X_A^+. For every word $w = x_{i_1} \ldots x_{i_n}$ define the *inverse* word $\overline{w} = x_{i_n} \ldots x_{i_1}$ where all letters of w are written in the inverse order. The map $w \mapsto \overline{w}$ is an anti-automorphism of X_A^+. This map can be uniquely extended to an anti-automorphism η_A of the $\mathbf{K}$-algebra A.

It is obvious that for every homomorphism $\nu : A \to B$ of free algebras the map $\eta_B \circ \nu \circ \eta_A^{-1}$ is also a homomorphism from A to B. Hence we have an automorphism Υ of the category $(\text{Ass-}\mathbf{K})^0$ defined by: $\Upsilon(A) = A$ and $\Upsilon(\nu) = \eta_B \circ \nu \circ \eta_A^{-1}$ for every $\nu : A \to B$. This automorphism Υ is called the *mirror* automorphism. Note that every η_A fixes all elements of X_A and acts identically on the field $\mathbf{K}$.

Now we can give the following final description.

Theorem 4. *Every automorphism Φ of* $(\text{Ass-}\mathbf{K})^0$ *is either semi-inner* (*in particular, inner*) *or a composition of a semi-inner automorphism and the mirror automorphism.*

Proof. Every automorphism Φ of $(\text{Ass-}\mathbf{K})^0$ is a composition $\Phi = \Omega \circ \Psi$ of an inner automorphism Ω and an automorphism Ψ satisfying the conditions of Lemma 5. According to Lemma 6, every map s_A^Ψ is either a ring automorphism or a ring anti-automorphism of the corresponding object A. In the first case, Ψ is semi-inner. In the second case, the automorphism $\Gamma = \Psi \circ \Upsilon$ satisfies the same conditions as Ψ (Lemma 5), but unlike Ψ, the maps s_A^Γ are ring automorphisms and hence Γ is semi-inner. Since $\Psi = \Gamma \circ \Upsilon$, we obtain $\Phi = \Omega \circ \Gamma \circ \Upsilon$ and the required description. □

It is useful to present the above description in a more convenient form. Every automorphism φ of the field $\mathbf{K}$ can be uniquely extended to a twisted automorphism φ_A for every free $\mathbf{K}$-algebra A which identically acts on X_A^+. These twisted automorphisms determine a semi-inner automorphism $\hat{\varphi}$ of the category $(\text{Ass-}\mathbf{K})^0$ that we call the *standard φ-automorphism*:

$$\hat{\varphi}(\nu) = \varphi_B \circ \nu \circ \varphi_A^{-1}$$

for every $\nu : A \to B$.

Theorem 5. *Every automorphism Φ of the category* $(\text{Ass-}\mathbf{K})^0$ *can be represented as a composition of three automorphisms*:

$$\Phi = \Upsilon \circ \hat{\varphi} \circ \Psi,$$

where Ψ is inner, $\hat{\varphi}$ is the standard φ-automorphism, and Υ is the mirror or the identity automorphism.

Proof. Let Φ be a semi-inner automorphism of the category $(\text{Ass-}\mathbf{K})^0$, and let φ be the corresponding automorphism of $\mathbf{K}$. It is obvious that the composition $\Phi \circ \hat{\varphi}^{-1}$ is an inner automorphism of $(\text{Ass-}\mathbf{K})^0$. Thus we can assert that Φ is a composition of an inner automorphism and the standard φ-automorphism. Combining this result with Theorem 4, we complete the proof. $\square$

4. Category of group representations

4.1. Basic definitions

R will always denote a commutative associative ring with unit 1. All R-modules under consideration are assumed to be unital. A representation of a group G in an R-module A is an arbitrary group homomorphism $\rho : G \to Aut_R(A)$, where $Aut_R(A)$ is the group of all R-module automorphisms of A.

To define such a representation is the same as to define an *action* of the group G on A, that is, a map $(a, g) \mapsto a \cdot g$ from $A \times G$ to A satisfying the following conditions:

(1) for every $g \in G$ the map $a \mapsto a \cdot g$ is an automorphism of the module A;
(2) $(a \cdot g_1) \cdot g_2 = a \cdot (g_1 g_2)$ for every $g_1, g_2 \in G$ and $a \in A$.

A group representation is viewed as a triple $(A, G, \cdot)$, where A is an R-module, G is a group and "$\cdot$" denotes an action of G on A. We regard the theory of group representations as a two-sorted theory [14]. Therefore, homomorphisms of representations are defined as homomorphisms of two-sorted algebras.

Let two group representations $(A, G, \cdot)$ and $(B, H, \bullet)$ be given. A homomorphism $\mu : (A, G, \cdot) \to (B, H, \bullet)$ is a pair of maps $(\mu^{(1)}, \mu^{(2)})$ of the form $\mu^{(1)} : A \to B$, $\mu^{(2)} : G \to H$ satisfying the following conditions:

(1) $\mu^{(1)} : A \to B$ is a homomorphism of R-modules;
(2) $\mu^{(2)} : G \to H$ is a homomorphism of groups;
(3) $\mu^{(1)}$ and $\mu^{(2)}$ are related by:

$$\mu^{(1)}(a \cdot g) = \mu^{(1)}(a) \bullet \mu^{(2)}(g),$$

for every $a \in A$ and every $g \in G$.

The last condition means that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\tilde{g}"] \arrow[d, "\mu^{(1)}"'] & A \arrow[d, "\mu^{(1)}"] \\ B \arrow[r, "\widetilde{\mu^{(2)}(g)}"] & B, \end{tikzcd}$$

where $\tilde{g}$ and $\widetilde{\mu^{(2)}(g)}$ denote the corresponding actions.

Denote by Rep-R the category whose objects are group representations and morphisms are homomorphisms defined above. The forgetful functor can be defined according to the two-sorted case. This means that this functor assigns to every representation $(A, G, \cdot)$ the pair of sets (A, G) and to every homomorphism $(\mu^{(1)}, \mu^{(2)})$ the pair of corresponding maps. Therefore, we have a notion of free object in this category, i.e., we can define free representations. We recall a realization of such free objects.

A representation $(W, F, \star)$ is called a free representation over a pair of sets (Y, X) if the set Y generates the RF-module W, where RF is the group algebra over the ring R, the set X generates the group G, and for every representation $(A, G, \cdot)$ and every two maps $f^{(1)} : Y \to A$ and $f^{(2)} : X \to G$, there exists a unique homomorphism $\mu : (W, F, \star) \to (A, G, \cdot)$ such that $\mu^{(1)}$ extends $f^{(1)}$ and $\mu^{(2)}$ extends $f^{(2)}$.

Since the forgetful functor defined above is not a functor to the category of sets and maps, we cannot apply the results from Section 2 in a straightforward way.

In order to apply the results from previous sections, we need to consider the following forgetful functor from Rep-R to the category of sets and maps. This functor assigns to every object $(A, G, \cdot)$ the set $A \times G$ as the underlying set, and it assigns to every homomorphism $\mu : (A, G, \cdot) \to (B, H, \bullet)$ the map $|\mu| = \mu^{(1)} \times \mu^{(2)} : A \times G \to B \times H$. Note that free objects with respect to such forgetful functor do not coincide with free objects with respect to the two-sorted theory. This forces us to make some modifications in definitions and to check in some cases if the results from Section 2 are valid.

In the same way as in the previous section, we consider the category $(\text{Rep-}R)^0$ of all free representations over pairs (Y, X), where X and Y are finite subsets of the fixed infinite sets Y_0 and X_0, respectively. Let (W_1, F_1) denote a monogenic free representation. This means that F_1 is the infinite cyclic group $\{x^n \mid n \in \mathbb{Z}\}$, $W_1 = RF_1$ can be identified with the group algebra over the ring R, and the action of this group on the group algebra is the group algebra multiplication.

Denote by $e = x^0$ the unit of the group F_1 and by 1_R the unit of the ring R. For every $r \in R$ we identify re with r, thus R is embedded into W_1. The pair of singular sets $(\{1_R\}, \{x\})$ is a basis of the representation (W_1, F_1).

Let $(A, G, \cdot)$ be a group representation. We denote it for short by AG. According to our usual definitions, denote by $\alpha^{AG}_{(a,g)}$ the unique homomorphism from (W_1, F_1) to $(A, G, \cdot)$ that takes 1_R to $a \in A$ and x to $g \in G$.

As mentioned above, we view the product $A \times G$ as the underlying set of the representation $(A, G, \cdot)$. Since this forgetful functor is represented by (W_1, F_1), the first part of Theorem 1 is valid. This means that an automorphism Φ of the category $(\text{Rep-}R)^0$ is potentially inner if and only if it takes the representation (W_1, F_1) to an isomorphic one. The second part of this theorem will be considered later.

Let Φ be an automorphism of the category $(\text{Rep-}R)^0$ which fixes (W_1, F_1). Thus the main function $(A, G, \cdot) \mapsto s_{AG}$ is defined in the same way as in Section 2:

$$s_{AG}(a, g) = (\bar{a}, \bar{g}) \quad \Leftrightarrow \quad \Phi\bigl(\alpha^{AG}_{(a,g)}\bigr) = \alpha^{\Phi(AG)}_{(\bar{a}\bar{g})}, \tag{4.1}$$

and we have the usual form of Φ-action, namely, for every homomorphism $\mu : (A, G, \cdot) \to (B, H, \bullet)$:

$$\Phi(\mu) = s_{BH} \circ \mu \circ s_{AG}^{-1}. \tag{4.2}$$

Consider the free representation $(W, F, \cdot)$ with the basis (Y, X). Let $f^{(1)}: Y \to A$ and $f^{(2)}: X \to G$. We denote by $\theta_{(f^{(1)}, f^{(2)})}$ the unique endomorphism extending these maps. Hence

$$(\forall y \in Y,\ x \in X) \quad \theta_{(f^{(1)}, f^{(2)})}(y, x) = \bigl(f^{(1)}(y), f^{(2)}(x)\bigr).$$

To apply the method from Section 2, we have to study the monoid End_1 of endomorphisms of the free monogenic representation (W_1, F_1).

4.2. Some invariants of category automorphisms

Denote elements of the monoid End_1 by $\nu_{(w,g)}$, where $w = \nu^{(1)}_{(w,g)}(1_R)$ and $g = \nu^{(2)}_{(w,g)}(x)$.

Lemma 7. *The endomorphism $\nu_{(0,e)}$, where 0 is the zero of the ring R, is the zero element of the monoid End_1. The endomorphism $\nu_{(1_R,x)}$ is the unit of this monoid. The set T_e of all endomorphisms $\nu_{(w,e)}$, where $w \in W_1$, is a minimal prime ideal in this monoid. Every prime ideal different from T_e contains T_e or the ideal $T_0 = \{\nu_{(0,g)} \mid g \in F_1\}$.*

Proof. The first and second statements are obvious. It is also obvious that the sets T_e and T_0 are ideals in End_1. Suppose that for some $\nu, \mu \in End_1$ we have $\nu \circ \mu \in T_e$. This means that $\nu^{(2)} \circ \mu^{(2)}(x) = e$ and therefore $\mu^{(2)}(x) = e$ or $\nu^{(2)}(x) = e$. Hence T_e is a prime ideal. Suppose now that I is another prime ideal and there exists $w \in W_1$ such that $\nu_{(w,e)} \notin I$. But $\nu_{(w,e)} \circ \nu_{(0,g)} = \nu_{(0,e)}$ for all $g \in F_1$. Hence $\nu_{(0,g)} \in I$ for all $g \in F_1$, that is, $T_0 \subseteq I$. So $T_e \subseteq I$ or $T_0 \subseteq I$. Since $T_0 \cap T_e = \{\nu_{(0,e)}\}$, T_e is a minimal prime ideal. □

Denote by T_x the set of all endomorphisms of (W_1, F_1) of the form $\nu_{(w,x)}$, where w is an arbitrary element of W_1.

Corollary 2. *T_x is a submonoid of End_1 which is multiplicatively isomorphic to the group algebra RF_1.*

Proof. It is clear that T_x is a subsemigroup of End_1 containing the unit $\nu_{(1_R,x)}$. For every $u, v \in W_1$ we have:

$$\nu^{(1)}_{(v,x)} \circ \nu^{(1)}_{(u,x)}(1_R) = \nu^{(1)}_{(v,x)}(u) = vu,$$

therefore $\nu_{(v,x)} \circ \nu_{(u,x)} = \nu_{(vu,x)}$. □

Lemma 8. *T_e is a unique minimal prime ideal in End_1 that contains more than 2 right units.*

Proof. Let $w = w(x)$ be an element of the group algebra such that $w(e) = e$. Since $w = \sum r_n x^n$, where $n \in \mathbb{Z}$ and $r_n = 0$ for almost all n, this condition means that $\sum r_n = 1$. Consider the endomorphisms $\nu_{(w,e)}$ with w just defined. For every $\nu \in T_e$ we have:

$$\nu^{(1)} \circ \nu^{(1)}_{(w,e)}(1_R) = \nu^{(1)}(w) = \sum r_n \nu^{(1)}\bigl(1_R x^n\bigr) = \sum r_n \nu^{(1)}(1_R)e = \nu^{(1)}(1_R).$$

This gives $\nu \circ \nu_{(w,e)} = \nu$, i.e., $\nu_{(w,e)}$ is a right unit in T_e. Suppose that I is another minimal prime ideal and $\nu_{(u,g)}$ is its right unit. According to Lemma 7, $T_0 \subseteq I$. Hence $\nu_{(0,x)} \in I$, and we obtain: $\nu_{(0,x)} = \nu_{(0,x)} \circ \nu_{(u,g)} = \nu_{(0,g)}$. This equation gives $g = x$. Thus our right unit is of the form: $\nu_{(u,x)}$. Since $\nu_{(u,x)} \circ \nu_{(v,x)} = \nu_{(uv,x)}$ and $uv = vu$ for every two elements $u, v \in W_1$, every two right units in I coincide. $\square$

Remark 3. The ideal T_0 can be determined in the considered monoid because $T_0 = \{\nu \mid (\forall \mu \in T_e)\ \nu \circ \mu = \mu \circ \nu = \nu_{(0,e)}\}$. The set T_x can be described as follows: $\nu \in T_x \Leftrightarrow (\forall \mu \in T_0)\ \nu \circ \mu = \mu$. The element $\nu_{(0,x)}$ is a unique common element of T_0 and T_x.

In the next corollaries we assume that automorphisms preserve the object (W_1, F_1).

Corollary 3. *Suppose that an automorphism Φ of the category* (Rep-R)^0 *fixes the object* (W_1, F_1)*. Then Φ preserves the sets T_e, T_0 and T_x, and hence it preserves the endomorphism $\nu_{(0,x)}$.*

Proof. Φ acts on the monoid End_1 as an automorphism. Lemma 8 implies that Φ preserves T_e. Φ preserves the sets T_0 and T_x since they are uniquely determined (Remark 3), and hence Φ preserves the endomorphism $\nu_{(0,x)}$ in view of the same remark. $\square$

Corollary 4. *Under hypotheses of the previous corollary, Φ induces a multiplicative automorphism of the group algebra RF_1, and hence it induces an automorphism φ of the multiplicative monoid of the ring R.*

Proof. Using Corollary 2, define $\varphi(r) = t \Leftrightarrow \Phi(\nu_{(r,x)}) = \nu_{(t,x)}$. Then $\varphi(rt) = \varphi(r)\varphi(t)$. $\square$

4.3. The main function

In this section, we assume that an automorphism Φ fixes the monogenic free representation (W_1, F_1). Thus we have the formula (4.2), and we start with studying maps s_{AG}.

Let $AG = (A, G, \cdot)$, let 0 be the zero of the module A, and let e be the unit of the group G. Denote by T_e^{AG} and T_0^{AG} the sets of all homomorphisms from (W_1, F_1) to $(A, G, \cdot)$ of the form $\alpha_{(w,e)}$ and $\alpha_{(0,g)}$, respectively, where $w \in A$, $g \in G$.

Lemma 9. *Let Φ take $AG = (A, G, \cdot)$ to $BH = (B, H, \odot)$. Then $\Phi(T_e^{AG}) = T_e^{BH}$ and $\Phi(T_0^{AG}) = T_0^{BH}$.*

Proof. Let $\mathfrak{M}$ be the set of all homomorphisms from (W_1, F_1) to $AG = (A, G, \cdot)$. Then $\mathfrak{N} = \Phi(\mathfrak{M})$ is the set of all homomorphisms from (W_1, F_1) to $BH = (B, H, \odot)$. Since $T_e^{AG} = \mathfrak{M} \circ T_e$ and $T_0^{AG} = \mathfrak{M} \circ T_0$, using Corollary 3 we obtain $\Phi(T_e^{AG}) = \mathfrak{N} \circ T_e = T_e^{BH}$ and $\Phi(T_0^{AG}) = \mathfrak{N} \circ T_0 = T_0^{BH}$. $\square$

The map s_{AG} is a subdirect product of two maps $s_{AG}^{(1)}$ and $s_{AG}^{(2)}$, and $s_{AG}(u, g) = (\bar{u}, \bar{g}) \Leftrightarrow s_{AG}^{(1)}(u, g) = \bar{u}\ \&\ s_{AG}^{(2)}(u, g) = \bar{g}$. These two maps are two-place functions. We can replace them by two one-place maps. Indeed, according to the previous result, we have $s_{AG}^{(1)}(0, g) = 0$ and $s_{AG}^{(2)}(u, e) = e$. Thus we can define:

$$\pi_{AG}(a) = s_{AG}^{(1)}(a, e), \qquad \varrho_{AG}(g) = s_{AG}^{(2)}(0, g) \tag{4.3}$$

and obtain

$$\pi_{AG}(0) = s^{(1)}_{AG}(0, e) = 0, \qquad \varrho_{AG}(e) = s^{(2)}_{AG}(0, e) = e. \tag{4.4}$$

Lemma 10. *If* $\nu : (A, G, \cdot) \to (B, H, \bullet)$ *is a homomorphism and* $\mu = \Phi(\nu)$*, then*

$$\mu^{(1)} = \pi_{BH} \circ \nu^{(1)} \circ \pi^{-1}_{AG},$$
$$\mu^{(2)} = \varrho_{BH} \circ \nu^{(2)} \circ \varrho^{-1}_{AG}.$$

Proof. Denote $\Phi(A, G, \cdot)$ by $\widetilde{A}\widetilde{G}$ and $\Phi(B, H, \bullet)$ by $\widetilde{B}\widetilde{H}$. Take $\tilde{b} \in \widetilde{B}$ and calculate:

$$\begin{aligned}\mu(\tilde{b}, e) &= s_{BH} \circ \nu \circ s^{-1}_{AG}(\tilde{b}, e) = s_{BH} \circ \nu\big(\pi^{-1}_{AG}(\tilde{b}), e\big)\\ &= s_{BH}\big(\nu^{(1)} \circ \pi^{-1}_{AG}(\tilde{b}), e\big) = \big(\pi_{BH} \circ \nu^{(1)} \circ \pi^{-1}_{AG}(\tilde{b}), e\big).\end{aligned}$$

Hence

$$\mu^{(1)}(\tilde{b}) = \pi_{BH} \circ \nu^{(1)} \circ \pi^{-1}_{AG}(\tilde{b}).$$

The proof of the second statement is similar. □

It is possible to strengthen the last result. Consider the evident equality:

$$\alpha_{(w,g)} \circ \nu_{(0,x)} = \alpha_{(0,g)}.$$

Applying Φ, we obtain $\Phi(\alpha_{(w,g)}) \circ \nu_{(0,x)} = \Phi(\alpha_{(0,g)})$. Hence $\alpha_{s(w,g)} \circ \nu_{(0,x)} = \alpha_{s(0,g)}$. By the definition of composition of homomorphisms, we get $s^{(2)}(w, g) = (0, s^{(2)}(0, g)) = (0, \varrho(g))$. Finally,

$$s(w, g) = \big(s^{(1)}(w, g), \varrho(g)\big).$$

Now we are in a position to apply the second part of Theorem 1 and to conclude that Φ takes every free representation to an isomorphic one. Therefore, one can assume that Φ fixes all objects of the category $(\text{Rep-}R)^0$. Then we repeat all arguments from the proof of Theorem 2 and obtain the analogous result.

Lemma 11. *Φ is a composition of two automorphisms $\Phi = \Psi \circ \Gamma$, where Ψ is an inner automorphism and Γ fixes all objects of the category* $(\text{Rep-}R)^0$. *Moreover, the permutation π^{Γ}_{AG} of A and the permutation ϱ^{Γ}_{AG} of G fix the corresponding basis elements of every $(A, G, \cdot)$ in* $(\text{Rep-}R)^0$.

Proof. If $\nu : (A, G, \cdot) \to (B, H, \bullet)$ is a homomorphism and $\mu = \Phi(\nu)$, then by Lemma 10 we have

$$\mu^{(1)} = \pi_{BH} \circ \nu^{(1)} \circ \pi^{-1}_{AG},$$
$$\mu^{(2)} = \varrho_{BH} \circ \nu^{(2)} \circ \varrho^{-1}_{AG}.$$

Let $y_1, \ldots, y_n$ be a basis of the KG-module A, and let $x_1, \ldots, x_m$ be a basis of the group G. We define two endomorphisms σ_{AG} and τ_{AG} of the representation (A, G) as usual: σ_{AG} acts on generators as π_{AG} and ϱ_{AG}, and τ_{AG} acts by the inverse permutations. The proof that σ_{AG} and τ_{AG} are automorphisms is literally the same as in Theorem 2.

Define a new automorphism Ψ of our category as follows. Let $\Psi(\nu)$ be equal to

$$\sigma_{BH} \circ \nu \circ \sigma_{AG}^{-1}$$

for every $\nu : (A, G) \to (B, H)$.

By definition, Ψ is an inner automorphism. Now consider a new automorphism: $\Gamma = \Psi^{-1} \circ \Phi$ and calculate the corresponding π and ϱ. According to (4.3), we have to calculate $\Gamma(\alpha^{AG}_{(a,e)})$ and $\Gamma(\alpha^{AG}_{(0,g)})$. Let us start with the first one. We have

$$\Gamma\big(\alpha^{AG}_{(a,e)}\big) = \sigma_{AG}^{-1} \circ \Phi\big(\alpha^{AG}_{(a,e)}\big) \circ \sigma_{W_1F_1},$$

hence

$$s^{\Gamma}_{AG}(a, e) = \sigma_{AG}^{-1} \circ \Phi\big(\alpha^{AG}_{(a,e)}\big) \circ \sigma_{W_1F_1}(1, x),$$

and

$$\big(s^{\Gamma}_{AG}(a, e)\big)^{(1)} = \big(\sigma^{(1)}_{AG}\big)^{-1} \circ \big(\Phi\big(\alpha^{AG}_{(a,e)}\big)\big)^{(1)} \circ \sigma^{(1)}_{W_1F_1}(1).$$

The last equality means that

$$\pi^{\Gamma}_{AG}(a) = \big(\sigma^{(1)}_{AG}\big)^{-1} \circ \pi_{AG} \circ \big(\alpha^{AG}_{(a,e)}\big)^{(1)} \circ \pi^{-1}_{W_1F_1} \circ \sigma^{(1)}_{W_1F_1}(1).$$

Since by definition $\sigma^{(1)}_{W_1F_1}(1) = \pi_{W_1F_1}(1)$ holds, we have

$$\pi^{\Gamma}_{AG}(a) = \big(\sigma^{(1)}_{AG}\big)^{-1} \circ \pi_{AG} \circ \big(\alpha^{AG}_{(a,e)}\big)^{(1)}(1) = \big(\sigma^{(1)}_{AG}\big)^{-1} \circ \pi_{AG}(a).$$

Now set $a = y_i$,

$$\pi^{\Gamma}_{AG}(y_i) = \big(\sigma^{(1)}_{AG}\big)^{-1} \circ \pi_{AG}(y_i) = \big(\sigma^{(1)}_{AG}\big)^{-1} \circ \sigma^{(1)}_{AG}(y_i) = y_i.$$

Thus, π^{Γ}_{AG} fixes all generators $y_1, \ldots, y_n$. The same arguments are valid for ϱ. Since $\Phi = \Psi \circ \Gamma$ and Ψ is inner, we have the required statement. □

From now on we assume that Φ is an automorphism of $(\text{Rep-}R)^0$ such that the permutations π_{AG} of A and the permutation ϱ_{AG} of G fix the basis of $(A, G, \cdot)$. This gives

Lemma 12. $s_{AG} = \pi_{AG} \times \varrho_{AG}$.

Proof. Indeed,

$$\begin{aligned} s_{AG}(w, g) &= \Phi(\alpha_{(w,g)})(1, x) = (\pi_{AG} \times \varrho_{AG}) \circ \alpha_{(w,g)} \circ (\pi_{W_1F_1} \times \varrho_{W_1F_1})^{-1}(1, x) \\ &= (\pi_{AG} \times \varrho_{AG}) \circ \alpha_{(w,g)}(1, x) = (\pi_{AG} \times \varrho_{AG})(w, g) = \big(\pi_{AG}(w), \varrho_{AG}(g)\big). \end{aligned}$$ □

4.4. Derived binary operations

According to our method, we have to find the derived operations such that the permutation s is an isomorphism onto the derived structure. Since the biggest arity of operations is 2, we consider the two-generated free representation (W_2, F_2), where F_2 is the free group with two free generators x_1, x_2 and W_2 is the free RF_2-module with the basis $Y = \{y_1, y_2\}$, i.e., $W_2 = y_1RF_2 \oplus y_2RF_2$. Denote, for short, $\pi_{W_2F_2}$ by π and $\varrho_{W_2F_2}$ by ϱ.

Theorem 6. *The map ϱ is the identity or the mirror map on F_2.*

Proof. Since $\varrho(e) = e$, the derived binary operation $x_1 \diamond x_2 = \varrho(x_1x_2)$ has the same unit e and hence the same inverses. This implies that $x_1 \diamond x_2 = x_1x_2$ or $x_1 \diamond x_2 = x_2x_1$. □

Consider now the derived additive operation on W_2 defined by

$$y_1 \perp y_2 = \pi(y_1 + y_2).$$

Being an element of the free module W_2, $y_1 \perp y_2$ is of the form $y_1P_1 + y_2P_2$, where P_1 and P_1 are elements of the group algebra RF_2. Because of commutativity of the considered operation, $P_1 = P_2 = P$ and $y_1 \perp y_2 = (y_1 + y_2)P$. We know that $\pi(0) = 0$. Therefore $y_1 = y_1P$, and hence $P = 1$. Thus $y_1 \perp y_2 = y_1 + y_2$, and hence π is an automorphism of the additive group of the module W_2.

Lemma 13. *The permutation s of the set $W_2 \times F_2$ is of the form: $s = \pi \times \rho$, where π is an automorphism of the additive group of the module W_2 and ρ is the identity map or the mirror permutation of the group F_2. For all $r \in R$, $w \in W_2$ we have $\pi(ru) = \varphi(r)\pi(u)$, where φ is an automorphism of the ring R* (*it is the same map as defined in Corollary* 4).

Proof. We only have to prove the second statement. It is clear that $\alpha_{(ru,x_1)} = \alpha_{(u,x_1)} \circ \nu_{(r,x)}$. Applying the automorphism Φ, we obtain $\alpha_{(\pi(ru),x_1)} = \alpha_{(\pi(u),x_1)} \circ \nu_{(\varphi(r),x)}$. Hence $\pi(ru) = \varphi(r)\pi(u)$. By Corollary 4, φ is an automorphism of the multiplicative monoid of the ring R. Further, $\pi((r + t)y_1) = \varphi(r + t)y_1$ and $\pi((r + t)y_1) = \pi(ry_1 + ty_1) = \varphi(r)y_1 + \varphi(t)y_1$. Hence $\varphi(r + t)y_1 = (\varphi(r) + \varphi(t))y_1$, and we finally get $\varphi(r + t) = \varphi(r) + \varphi(t)$. □

Let φ be an automorphism of the ring R. It can be extended to a *twisted* automorphism of every free RF-module W for a group F as follows:

$$\varphi_{WF}\left(y\sum_g r_g g\right) = y\sum_g \varphi(r_g)g.$$

Definition 5. Consider the function assigning to every free representation $(W, F, \cdot)$ the pair $(\varphi_{WF}, 1_F)$ of permutations, where φ_{WF} is the twisted automorphism of W defined above and 1_F is the identity on F. This function determines an automorphism of the category $(\text{Rep-}R)^0$. We call this automorphism the *standard twisted* automorphism determined by φ and denote it by $\hat{\varphi}$.

4.5. Derived action

Now consider an action of the group F_2 on the module W_2. We assume that the ring R has no zero divisors. The action is determined by the term $y_1 \cdot x_1$. The derived structure is determined by the term $y_1 \bullet x_1 = \pi(y_1 \cdot x_1)$, and the permutation $s = \pi \times \rho$ is an isomorphism of two structures. Let $\nu_{(x,e)}$ be the endomorphism of (W_1, F_1) defined as usual by $\nu_{(x,e)}(1, x) = (x, e)$. We have

$$\alpha_{(y_1,x_1)} \circ \nu_{(x,e)} = \alpha_{(y_1 \cdot x_1, e)}.$$

We apply the automorphism Φ and obtain

$$\alpha_{(y_1,x_1)} \circ \nu_{(w,e)} = \alpha_{(y_1 \bullet x_1, e)},$$

where w is an element of the group algebra RF_1, that is, $w = \sum r_i x_1^i$, $i \in \mathbb{Z}$. Hence

$$y_1 \bullet x_1 = y_1 \cdot \sum r_i x_1^i,$$

and for every $u \in W_2$ and every $g \in F_2$ the equality

$$u \bullet g = u \cdot \sum r_i g^i$$

holds.

Since the derived structure is isomorphic to the source one, we have $y_1 \bullet e = y_1$ which gives $\sum r_i = 1$. We write the associativity law of the derived action and obtain

$$(y_1 \bullet x_1) \bullet x_2 = y_1 \bullet (x_1 x_2),$$

if ρ is the identity map, and

$$(y_1 \bullet x_1) \bullet x_2 = y_1 \bullet (x_2 x_1),$$

if ρ is the mirror map. Consider the first case. We obtain the following identity in the variety of all group representation:

$$\Bigl(\sum r_i x_1^i\Bigr)\Bigl(\sum r_i x_2^i\Bigr) = \sum r_i (x_1 x_2)^i.$$

Both sides of this equality can have only two pairs of common words: one of degree 0 and the other of degree 2, say, $x_1 x_2$. This implies $r_i = 0$ for all $i \neq 1$ and $r_1 = 1$. Hence $y_1 \bullet x_1 = y_1 \cdot x_1$.

In the case where ρ is the mirror map, the same arguments lead to $y_1 \bullet x_1 = y_1 \cdot x_1^{-1}$.

Corollary 5. *Let $u \in W_2$ and $g \in F_2$. Then $\pi(u \cdot g) = \pi(u) \cdot g$ if ρ is the identity, and $\pi(u \cdot g) = \pi(u) \cdot \bar{g}^{-1}$, if ρ is the mirror map.*

Proof. By definition, $\pi(u \cdot g) = \pi(u) \bullet \rho(g)$. Thus in the case ρ is the identity, we get the first statement. If ρ is the mirror map, $\pi(u) \bullet \rho(g) = \pi(u) \cdot \bar{g}^{-1}$. $\square$

Let us introduce a new kind of quasi-homomorphisms. For every free group F, the map $g \mapsto \bar{g}$ can be extended to a permutation of the group algebra RF as follows: $\overline{rg} = r\bar{g}$ for every $r \in R$ and $g \in F$. In the same way, the map $g \mapsto g^{-1}$ can be extended to RF. Clearly, these two maps can be extended to every free RF-module W as follows: $(yP)^{-1} = yP^{-1}$ for every free generator y and $P \in RF$. We do the same for the map "bar."

Definition 6. Let $(W, F, \cdot)$ be a free representation. A pair $\delta = (\delta^{(1)}, \delta^{(2)})$, where $\delta^{(1)}(w) = \overline{w}^{-1}$ for $w \in W$ and $\delta^{(2)}(g) = \bar{g}$ for $g \in F$, is called a mirror automorphism of $(W, F, \cdot)$. Assigning to every free representation $(W, F, \cdot)$ the mirror automorphism δ_{WF}, we obtain the *mirror* automorphism Δ of the category $(\text{Rep-}R)^0$.

4.6. Final

Let R be a ring without zero divisors. Similarly to the previous section, we can describe the automorphisms of the category $(\text{Rep-}R)^0$ taking the regular representation (W_1, F_1) to an isomorphic one.

Theorem 7. *Suppose that an automorphism Φ of $(\text{Rep-}R)^0$ takes the regular representation (W_1, F_1) to an isomorphic one. Then Φ can be represented in the form $\Phi = \hat{\varphi} \circ \Psi$ or in the form $\Phi = \Delta \circ \hat{\varphi} \circ \Psi$, where Ψ is inner and $\hat{\varphi}$ is the standard twisted φ-automorphism for some automorphism φ of the ring R determined by Φ.*

Proof. Φ is a composition of an inner automorphism Ψ and an automorphism Φ_1 fixing all objects and their bases. According to Lemma 13, Φ_1 determines an automorphism φ of the ring R and hence the standard twisted automorphism $\hat{\varphi}$ of our category. Then the automorphism $\Gamma = \Phi_1 \circ \hat{\varphi}^{-1}$ determines the identity automorphism of the ring R, and according to Corollary 5, it is the identity automorphism or the mirror automorphism Δ. □

However, this description can be simplified. Indeed, consider the function $\mathbf{c} : (W, F, \cdot) \mapsto c_{WF}$ assigning to every free group representation $(W, F, \cdot)$ the permutation c_{WF} as follows:

$$c_{WF}(w, g) = \left(w, g^{-1}\right)$$

for every $w \in W$ and $g \in F$.

Lemma 14. *The function $\mathbf{c}$ is central for the category $(\text{Rep-}R)^0$, and every map c_{WF} is an isomorphism of $(W, F, \cdot)$ onto the derived structure $(W, F^*, \bullet)$, where $w \bullet g = w \cdot g^{-1}$ and F^* is the group dual to F.*

Proof. Let $\mu : (A, G, \cdot) \to (B, H, *)$ be a homomorphism in the category $(\text{Rep-}R)^0$. For every $a \in A$ and $g \in G$ we have:

$$\begin{aligned} \mu_{AG} \circ c_{AG}(a, g) = \mu_{AG}\left(a, g^{-1}\right) &= \left(\mu_{AG}^{(1)}(a), \left(\mu_{AG}^{(2)}(g)\right)^{-1}\right) \\ &= c_{BH}\left(\mu_{AG}^{(1)}(a), \mu_{AG}^{(2)}(g)\right) = c_{AG} \circ \mu_{BH}(a, g). \end{aligned}$$

Thus the function $\mathbf{c}$ is central for the category $(\text{Rep-}R)^0$.

Then, since the map $c_{WF}^{(1)}$ is the identity and the map $c_{WF}^{(2)}: g \mapsto g^{-1}$ is an isomorphism of the group F onto the dual group F^*, we only have to prove that c_{WF} is a homomorphism with respect to group actions. For every $w \in W$ and $g \in F$ we have: $c_{WF}^{(1)}(w \cdot g) = w \cdot g = w \bullet g^{-1} = c_{WF}^{(1)}(w) \bullet c_{WF}^{(2)}(g)$. Thus c_{WF} is an isomorphism of $(W, F, \cdot)$ onto the derived structure $(W, F^*, \bullet)$. $\square$

Applying Lemma 3, we obtain that the mirror automorphism Δ of the category $(\text{Rep-}R)^0$ is in fact inner. Thus the final description looks as follows.

Theorem 8. *Suppose that an automorphism Φ of* $(\text{Rep-}R)^0$ *takes the regular representation (W_1, F_1) to an isomorphic one. Then Φ is a semi-inner automorphism, that is, it can be represented in the form $\Phi = \hat{\varphi} \circ \Psi$, where Ψ is inner and $\hat{\varphi}$ is the standard twisted φ-automorphism for some automorphism φ of the ring R.*

The above description is given for the case when the automorphisms of $(\text{Rep-}R)^0$ take the regular representation to an isomorphic one. This property depends on the ring R. Since this condition is satisfied if R is an infinite field $\mathbf{K}$, all automorphisms of the category $(\text{Rep-}\mathbf{K})^0$ are semi-inner.

Acknowledgments

The authors are thankful to E. Plotkin for the stimulating discussions and assistance in preparing the final version of the paper.

References

[1] A. Berzins, The group of automorphisms of the category of free associative algebras, arXiv: math.RA/0502426, 2005.
[2] G. Grätzer, Universal Algebra, D. Van Nostrand Company, 1968.
[3] E. Formanek, A question of B. Plotkin about semigroup of endomorphisms of a free group, Proc. Amer. Math. Soc. 130 (2002) 935–937.
[4] Y. Katsov, R. Lipyanski, B. Plotkin, Automorphisms of categories of free modules, free semimodules and free Lie algebras, Comm. Algebra, in press.
[5] R. Lipyanski, B. Plotkin, Automorphisms of categories of free modules and free Lie algebras, arXiv: math.RA/0502212, 2005.
[6] S. MacLane, Categories for the Working Mathematicians, Springer, 1971.
[7] G. Mashevitzky, Automorphisms of the category of free associative algebras, preprint.
[8] G. Mashevitzky, B.M. Schein, Automorphisms of the endomorphism semigroup of a free monoid or a free semigroup, Proc. Amer. Math. Soc. 131 (2003) 1655–1660.
[9] G. Mashevitzky, B. Plotkin, E. Plotkin, Automorphisms of categories of free algebras of varieties, Electron. Res. Announc. Amer. Math. Soc. 8 (2002) 1–10.
[10] G. Mashevitzky, B. Plotkin, E. Plotkin, Automorphisms of the category of free Lie algebras, J. Algebra 282 (2004) 490–512.
[11] G. Mashevitzky, B.M. Schein, and G.I. Zhitomirski, Automorphisms of the semigroup of endomorphisms of free inverse semigroups, Comm. Algebra, in press.
[12] B. Plotkin, Seven lectures on the universal algebraic geometry, arXiv: math.GM/0204245, 2002.
[13] B. Plotkin, Algebras with the same (algebraic) geometry, Proc. Steklov Inst. Math. 242 (2003) 176–207.
[14] B.I. Plotkin, S.M. Vovsi, Varieties of Group Representations. General Theory, Connections and Applications, Zinatne, Riga, 1983 (in Russian).
[15] B. Plotkin, G. Zhitomirski, On automorphisms of categories of universal algebras, Internat. J. Algebra Comput., in press, arXiv: math.CT/0411408, 2004.