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On the product of a π-group and a π-decomposable group

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Abstract

The main result in the paper states the following: Let π be a set of odd primes. Let the finite group $G = AB$ be the product of a π-decomposable subgroup $A = O_\pi(A) \times O_{\pi'}(A)$ and a π-subgroup B. Then $O_\pi(A) \leqslant O_\pi(G)$; equivalently the group G possesses Hall π-subgroups. In this case $O_\pi(A)B$ is a Hall π-subgroup of G. This result extends previous results of Berkovich (1966), Rowley (1977), Arad and Chillag (1981) and Kazarin (1980) where stronger hypotheses on the factors A and B of the group G were being considered. The results under consideration in the paper provide in particular criteria for the existence of non-trivial soluble normal subgroups for a factorized group G.

Keywords: Products of groups; π-Decomposable groups; π-Groups; Hall subgroups

1. Introduction and statement of results

All groups considered are finite.

A group G is called factorized by the subgroups A and B (denoted by $G = AB$) if every element $g \in G$ can be expressed in the form $g = ab$ for some $a \in A$ and some $b \in B$. A well-known

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theorem by Kegel and Wielandt asserts that G is soluble provided that A and B are nilpotent. This theorem has been the motivation for a number of results in the literature on factorized groups. Particularly some of them consider the situation when one of the factors is π-decomposable for a set of primes π. A group X is said to be *π-decomposable* if $X = O_\pi(X) \times O_{\pi'}(X)$ is the direct product of a π-subgroup and a π'-subgroup, where π' stands for the complementary of π in the set of all prime numbers.

The present paper constitutes a further contribution in this context. Berkovich [3] proved that the Kegel and Wielandt's result remains true if A is 2-decomposable, B is nilpotent of odd order and the orders of A and B are coprime. Rowley [11] generalized this result proving that in this case G is $\pi(O(A))$-separable, if B is a metanilpotent group just instead of nilpotent. Arad and Chillag [1] showed that this conclusion is true without any restriction on the nilpotent length of B. Previously Kazarin [5] had obtained that $O(A) \leqslant O(G)$ under the same hypothesis (see also [6]). Note that the proofs in [11] and [5] do not depend on the classification of finite simple groups.

Here as standard, for any group X and any set of primes π, we set $O_\pi(X)$ for the largest normal π-subgroup in the group X. In particular, $O(X)$ stands for $O_{2'}(X)$. Moreover $\pi(X)$ denotes the set of all prime divisors of $|X|$, the order of X.

We also write $G \in E_\pi$ to mean that the group G satisfies the E_π-property, that is, the group G possesses Hall π-subgroups. Finally, if n is an integer and p a prime number, we denote by n_p the greatest power of p dividing n.

The aim of this paper is to prove the following more general result.

Theorem 1. *Let π be a set of odd primes. Let the group $G = AB$ be the product of a π-decomposable subgroup A and a π-subgroup B. Then $O_\pi(A) \leqslant O_\pi(G)$.*

The following result provides an equivalent statement for Theorem 1.

Lemma 1. *Let the group $G = AB$ be the product of a π-decomposable subgroup A and a π-subgroup B. Then the following statements are equivalent*:

(i) $O_\pi(A) \leqslant O_\pi(G)$;
(ii) $G \in E_\pi$.

In this case $O_\pi(A)B$ is a Hall π-subgroup of G.

Proof. If $O_\pi(A) \leqslant O_\pi(G)$, it is clear that $O_\pi(G)B$ is a Hall π-subgroup of G, that is, $G \in E_\pi$. On the other hand, if $G \in E_\pi$, there is a Hall π-subgroup T of G. Since $AT = G$ it follows that $T \cap A = O_\pi(A)$. Consequently $O_\pi(A) \leqslant \bigcap_{g \in G} T^g = O_\pi(G)$. The equivalence between (i) and (ii) is proved.

We notice now that under these conditions $X := O_\pi(G)B = (X \cap A)B = O_\pi(A)B$ is a Hall π-subgroup of G. $\square$

As a consequence of Theorem 1 together with the results in [2] we obtain the following:

Corollary 1. *Let π be a set of odd primes. Let the group $G = AB$ be the product of a π-decomposable subgroup A and a π-subgroup B. Then the composition factors of G belong to one of the following types*:

(1) *π-groups,*
(2) *π'-groups,*
(3) *groups in the list of Arad–Fisman* [2, Theorem 1.1]*, that is*:
  (i) *A_r with $r \geqslant 5$ a prime,*
  (ii) *M_{11},*
  (iii) *M_{23},*
  (iv) *$L_2(q)$ where either $q = 29$ or $3 < q \not\equiv 1 \pmod 4$,*
  (v) *$L_r(q)$ with r an odd prime such that $(r, q-1) = 1$.*

Proof. In order to prove this result we notice first that whenever a group $X = X_\pi X_{\pi'}$ is the product of a π-subgroup X_π and a π'-subgroup $X_{\pi'}$, then $N = (N \cap X_\pi)(N \cap X_{\pi'})$ for any $N \trianglelefteqslant X$, as $N \cap X_\pi$ and $N \cap X_{\pi'}$ are Hall subgroups of N for π and π' respectively.

Now Theorem 1 implies that

$$G/O_\pi(G) = \big(O_{\pi'}(A)O_\pi(G)/O_\pi(G)\big)\big(BO_\pi(G)/O_\pi(G)\big)$$

is the product of a π-subgroup and a π'-subgroup, and this is also the case for its compositions factors by the above-mentioned fact. Simple groups with this property are described in [2, Theorem 1.1]. The conclusion is now clear. □

We finally remark that the above-mentioned results in [3], [11], [1] and [5] can be also derived from Theorem 1. This follows by considering π the set of all odd prime numbers in this theorem and taking into account the particular hypotheses in each of these results.

It is worth noticing that the results under consideration in this paper provide in particular criteria for the existence of non-trivial soluble normal subgroups for a factorized group G.

2. Preliminaries

In order to prove Theorem 1 we state in this section the following lemmas.

Lemma 2. *Let L be a finite group such that $L/Z(L)$ is a simple non-abelian group and $L = L'$. If φ is an automorphism of L such that $[L, \langle\varphi\rangle] \leqslant Z(L)$, then $\varphi = 1$.*

Proof. Since $[L, \langle\varphi\rangle] \leqslant Z(L)$, it follows $[[L, \langle\varphi\rangle], L] = 1 = [[\langle\varphi\rangle, L], L]$. Then by the three subgroups lemma $[[L, L], \langle\varphi\rangle] = [L, \langle\varphi\rangle] = 1$, that is, $\varphi = 1$. □

Lemma 3. *Let $L = L(q)$ be a classical finite simple group defined over the field $GF(q)$, $q = p^{er}$, r and p prime numbers, and let φ be its field automorphism of order r. Then $C = C_L(\varphi) \leqslant L(p^e)H$ where H is a Cartan subgroup of L normalizing $L(p^e)$.*

In particular, it follows that $C' \cong L(p^e)$.

Proof. It follows from the definition of the field automorphism φ, Bruhat decomposition and the commutator relations for the root subgroups in the corresponding groups (see [4, Chapters 5, 8 and 12]). □

Let n be a positive integer and p a prime. A prime r is said to be *primitive with respect to the pair $\{p, n\}$* if r divides $p^n - 1$ but r does not divide $p^e - 1$ for every integer e such that $1 \leqslant e < n$.

Lemma 4 *(Zsigmondy [12]). Let n be a positive integer and p a prime. Then*:

(1) *If $n \geqslant 2$, then there exists a prime r primitive with respect to the pair $\{p, n\}$ unless $n = 2$ and p is a Mersenne prime or $\{p, n\} = \{2, 6\}$.*
(2) *If the prime r is primitive with respect to the pair $\{p, n\}$, then $r - 1 \equiv 0 \pmod n$. In particular, $r \geqslant n + 1$.*

We introduce the following two families of finite simple groups. They will be of relevance in the proof of Theorem 1.

$$\mathfrak{M} = \{PSp_{2n}(q),\ n \geqslant 2;\ P\Omega_{2n+1}(q),\ n \geqslant 3;\ P\Omega^{+}_{2n}(q),\ n \geqslant 4;\\ P\Omega^{-}_{2n}(q),\ n \geqslant 4;\ G_2(q) \mid q = p^e,\ p \text{ an odd prime}\},$$

$$\mathfrak{N} = \{L_n(q),\ n \geqslant 2;\ U_n(q),\ n \geqslant 3 \mid q = p^e,\ p \text{ an odd prime}\}.$$

The notation concerning simple groups in $\mathfrak{M} \cup \mathfrak{N}$ is as in [10]. We will use information about automorphism groups and maximal factorizations of these groups from these memoirs.

The following lemma can be extracted from [9, Theorems 4, 5].

Lemma 5. *Let N be a simple group in $\mathfrak{M} \cup \{U_n(q),\ n \geqslant 3\}$ and let S be a Sylow 2-subgroup of N. If L is a S-invariant subgroup of N of odd order, then $(|L|, p) = 1$. In particular, every maximal parabolic subgroup of N is of even index.*

3. Minimal counterexample for Theorem 1

In this section we provide a description of the structure of a minimal counterexample to Theorem 1. Hence, from here on we assume the following hypotheses:

(H1) π is a set of odd primes.
(H2) G is a group of minimal order satisfying the following conditions:
  (1) $G = AB$ is the product of a π-decomposable subgroup A and a π-subgroup B.
  (2) $O_\pi(A) \nleqslant O_\pi(G)$.
  We set also $A_\pi = O_\pi(A)$ and $A_{\pi'} = O_{\pi'}(A)$.

For such a group G we have the following results.

Lemma 6. *G has a unique minimal normal subgroup which is a non-abelian simple group, say N. Hence $N = \mathrm{soc}(G) \trianglelefteq G \leqslant \mathrm{Aut}(N)$. In particular, $C_G(N) = 1$. Moreover $G = NA_\pi$ and, consequently, $A_{\pi'} \leqslant N$, $|G/N| \equiv 1 \pmod 2$ and $1 \neq A_\pi$ centralizes a Sylow 2-subgroup of N.*

Proof. We split the proof of this lemma into the following steps:

(1) G has a unique minimal normal subgroup N.

Assume that M and N are different minimal normal subgroups of G. The choice of G implies that $A_\pi M/M \leqslant O_\pi(G/M)$ and $A_\pi N/N \leqslant O_\pi(G/N)$. Let us consider the monomorphism $\gamma : G \to$

$G/M \times G/N$ defined by $(g)\gamma = (gM, gN)$ for any $g \in G$. Obviously $G \cong (G)\gamma$ and $(A_\pi)\gamma \leqslant (O_\pi(G/M) \times O_\pi(G/N)) \cap (G)\gamma \leqslant O_\pi((G)\gamma)$, which implies the contradiction $A_\pi \leqslant O_\pi(G)$.

(2) N is neither a π-group or a π'-group.

If N were a π-group, then $A_\pi N/N \leqslant O_\pi(G/N) = O_\pi(G)/N$, a contradiction.

Assume that N is a π'-group. The choice of G implies that $A_\pi N/N \leqslant F/N := O_\pi(G/N)$ and in addition F is π-separable in this case. In particular A_π is contained in some Hall π-subgroup F_π of F. Moreover $N \leqslant A_{\pi'}$ and so $A_\pi \leqslant C_{F_\pi}(N) \trianglelefteq F = F_\pi N$. Consequently $A_\pi \leqslant C_{F_\pi}(N) \leqslant O_\pi(F) \leqslant O_\pi(G)$, a contradiction.

(3) $O_\pi(G) = O_{\pi'}(G) = 1$ and $N = N_1 \times \cdots \times N_r$, where N_i are isomorphic non-abelian simple groups for $i = 1, \ldots, r$. In particular, $C_G(N) = 1$ and $N \trianglelefteq G \leqslant \mathrm{Aut}(N)$.

This follows from step (2).

(4) G/N is a π-group.

By the choice of G we have that $A_\pi N/N \leqslant T/N := O_\pi(G/N)(BN/N)$. In particular $N \leqslant T = A_\pi(T \cap A_{\pi'})B$. If T were a proper subgroup of G, then $A_\pi \leqslant O_\pi(T) \leqslant C_G(N) = 1$, which is a contradiction.

(5) $G = A_\pi N$.

From step (4) we have that $A_{\pi'} \leqslant N$ and so $X := A_\pi N = A(B \cap X)$. If X were a proper subgroup of G we would argue as above to conclude the contradiction $A_\pi \leqslant O_\pi(X) = 1$.

(6) N is a simple group.

From steps (3) and (5) we deduce that $A_{\pi'} = (N_1 \cap A_{\pi'}) \times \cdots \times (N_r \cap A_{\pi'})$ is a Hall π'-subgroup of N and A_π acts transitively on the components $N_1, \ldots, N_r$ of N. This implies $r = 1$, that is, N is a simple group.

The result follows now easily. □

Lemma 7. $N \in \mathfrak{M} \cup \mathfrak{N}$.

Proof. We prove first that N is a group of Lie type over a field of odd characteristic. Let S be a Sylow 2-subgroup of A. Since $G = NA_\pi$ by Lemma 6, we have that $S \in \mathrm{Syl}_2(N)$. If $A_\pi \cap N \neq 1$, then $O(C_N(S)) \neq 1$ and by Theorem 7 of [9] either $N \in \mathfrak{N}$ or $N \in \{E_6(q),\ {}^2E_6(q)\}$ for some prime power q. But then Theorem B of [10] implies that $N \in \mathfrak{N}$.

Assume now that $N \notin \mathfrak{N}$ and $A_\pi \cap N = 1$. Hence, from $G = NA_\pi$ it follows that A_π is a non-trivial group of odd order outer automorphisms of N centralizing a Sylow 2-subgroup S of N. Since the outer automorphisms of the alternating and sporadic simple groups are 2-groups (see [10, Table 2.1]), N is a group of Lie type over a field $GF(q)$ of characteristic p. By [7, 1.17], we have $p > 2$.

Now from the main results of [10] we obtain that $N \in \mathfrak{M}$. The lemma is proved. □

Lemma 8. $G = BN$.

Proof. Assume that BN is a proper subgroup of G. We claim that $N = BA_{\pi'}$, $N \cap A_\pi = 1$ and $|A_\pi| = t$ for some prime t.

Let us consider $M := NB = B(NB \cap A) = BA_{\pi'}(NB \cap A_\pi)$. Then $NB \cap A_\pi \leqslant O_\pi(M) = 1$. Since $G = NA_\pi = (NB)A_\pi$, we deduce that $|N| = |NB|$ and so $B \leqslant N = BA_{\pi'}$.

Now let C be a subgroup of A_π of order t, for some prime t, and assume that $X := NC = BA_{\pi'}C$ is a proper subgroup of G. Then $C \leqslant O_\pi(X) = 1$, a contradiction. The claim follows.

From [2, Theorem 1.1] and Lemma 7 the subgroup N should be isomorphic to one of the following: $L_2(q)$ where q is odd and either $q \in \{11, 29, 59\}$ or $3 < q \not\equiv 1 \pmod 4$, or $L_r(q)$ where q is odd and r is an odd prime such that $(r, q-1) = 1$.

We check next that none of these cases is possible and prove that $G = NB$.

We notice first that for $N \cong L_2(q)$ with q a prime number, it holds that $\pi(\mathrm{Out}(N)) = \{2\}$. Consequently these cases are not possible.

For $N \cong L_2(q)$ we have from [10, Table 1] that $(B, A_{\pi'})$ should be a pair of subgroups of N among pairs of subgroups of N of type $(N_N(N_p), D_{q+1})$, where N_p is a Sylow p-subgroup of N and D_{q+1} a dihedral group of order $q+1$.

(We notice that other exceptional factorizations for $L_2(q)$ exist for q prime (see [10, Table 1]). But as we have mentioned before this is not our case.)

For $N \cong L_r(q)$ with r an odd prime we know also from [2, Theorem 1.1] that B is either cyclic or Frobenius.

For any of the cases for N under consideration we conclude that there exists $N_l \in \mathrm{Syl}_l(B) \subseteq \mathrm{Syl}_l(N)$ for some prime l with $B \leqslant N_N(N_l)$. Consequently, taking into account that $G = NN_G(N_l)$ and the initial claim in the proof, we deduce that $N_G(N_l) = BA_\pi N_{A_{\pi'}}(N_l)$, which is a proper subgroup of G. But the choice of G implies that $A_\pi B$ is a subgroup of G and so $G \in E_\pi$, which is a contradiction. □

From now on the arguments rest mainly on the analysis of the possible factorizations of groups G with $N \trianglelefteq G \leqslant \mathrm{Aut}(N)$, such that $N \in \mathfrak{M} \cup \mathfrak{N}$. For these purposes we will use the main results in [10] where the maximal factorizations of such groups are described. More exactly, we will assume the factorization $G = XY$ where X and Y are maximal subgroups of the group G with $N \trianglelefteq G \leqslant \mathrm{Aut}(N)$, not containing N. In [10, Tables 1–5] an explicit description of the "large" subgroups $X_0 \trianglelefteq X \cap N$, $Y_0 \trianglelefteq Y \cap N$ is given. We will always assume below that $A \leqslant X$ and $B \leqslant Y$. In particular, we notice that X contains a Sylow 2-subgroup of G.

We shall adhere also to the notation in [10]. In particular, if $\hat{L}$ is a classical linear group on the vector space V with center Z (so that $L = \hat{L}/Z$ is a classical simple group) and $\hat{L} \trianglelefteq \hat{G} \leqslant GL(V)$, for any subgroup X of G we will denote by $\hat{}X$ the subgroup $(XZ \cap \hat{L})/Z$ of L. Also we will use the notation P_i, $1 \leqslant i \leqslant m-1$, $P_{m,m-1}$, N_i, N_i^ϵ $(\epsilon = \pm)$, for stabilizers of subspaces as described in [10, 2.2.4].

Lemma 9. $A_\pi \leqslant O_\pi(X)$.

Proof. We have $X = A(X \cap B) \neq G$. Hence $A_\pi \leqslant O_\pi(X)$ as claimed. □

Lemma 10. $N \ncong G_2(q)$.

If $N \in \mathfrak{M} \setminus \{G_2(q)\}$, then $A_\pi \cap N = 1$, $e > 2$, $|A_\pi|$ divides e and every element of A_π is conjugate to a field automorphism of N.

Proof. We prove first that $N \cong G_2(q)$ cannot occur. By [10, Theorem B, Table 5] we would have in that case that $q = 3^e$ and all possible factorizations $G = XY$ (not only the maximal ones) with subgroups X, Y not containing N are known. From this result we would have that

$$\{A \cap N, B \cap N\} \subseteq \{SL_3(q), SL_3(q).2, SU_3(q), SU_3(q).2, {}^2G_2(q)\}.$$

Since one of the factors in our decomposition of the group has odd order, this is not the case.

Assume now that $N \in \mathfrak{M} \setminus \{G_2(q)\}$. Then the odd part of $|\mathrm{Out}(N)|$ divides e (see [10, Table 2.1]). In particular, $|A_\pi N/N|$ divides e. Now from the description of the automorphism groups of these groups (see [4, Chapter 12]) it follows that $A_\pi N/N$ is a group of field automorphisms of N. On the other hand, by [9, Theorem 7] we have that $A_\pi \cap N = 1$ for all groups in $\mathfrak{M}$. Thus $1 < |A_\pi|$ divides e, which implies $e > 2$. The final assertion follows from [8, 7-2, p. 81]. Now the result follows. □

After Lemma 10 the remaining cases are among classical groups. The next lemma shows that we can in fact restrict ourselves to linear groups.

Lemma 11. $N \cong L_n(q)$.

Proof. We will prove now that the possibility $N \in \mathfrak{M} \cup \{U_n(q)\}$ cannot occur. We recall that in all cases q is odd by Lemma 7. We will consider each case according Tables 1–4 in [10]. We will refer to Table i in [10] as Table i.

Case. $N \cong PSp_{2n}(q),\ n \geqslant 2$.

There is just one possibility for the subgroups $X_0 \trianglelefteq X \cap N$, $Y_0 \trianglelefteq Y \cap N$ in Table 1, namely $\{X_0, Y_0\} = \{K, P_1\}$ where $K = PSp_{2a}(q^b).b$, $ab = n$, b a prime number, and P_1 is a maximal parabolic subgroup of N. Moreover, from the proof of Theorem A in [10, 3.2.1], we deduce also that $X \cap N = X_0$ and $Y \cap N = Y_0$. Since $X \geqslant A$ and A contains a Sylow 2-subgroup of N, it follows from Lemma 5 that $X \cap N \neq P_1$. Therefore $Y \cap N = P_1$. Since $a < n$, it is easy to check from the order formula of $K = X \cap N$ that this subgroup does not contain a Sylow 2-subgroup of N. Hence this case cannot occur. Notice that Table 2 gives no additional possibilities for this case since q is odd. The subcases in Table 3 are excluded by Lemma 10 since q is not a prime number.

Case. $N \cong P\Omega_{2n+1}(q), n \geqslant 3$.

There is only one possibility in Table 1 for the subgroups $X_0 \trianglelefteq X \cap N$, $Y_0 \trianglelefteq Y \cap N$, namely $\{X_0, Y_0\} = \{K, P_n\}$, where P_n is a maximal parabolic subgroup of N and $K = N_1^-$ is a stabilizer of a non-singular one-dimensional subspace of a natural module of $O_{2n+1}(q)$. By Lemma 5 we know that P_n does not contain a Sylow 2-subgroup of N. Therefore $Y_0 = Y \cap N = P_n$ and $X_0 = K$. In fact, from [10, 2.2.5], it is known that $X_0 = X \cap N = K$. Note that in this case $(X \cap N)'$ has a unique composition factor isomorphic to $P\Omega^-_{2n}(q)$ and $O((X \cap N)') = O_\pi((X \cap N)') = 1$. By Lemma 9, it follows $[(X \cap N)', A_\pi] \leqslant [(X \cap N)', O_\pi(X)] \leqslant O_\pi(X) \cap (X \cap N)' = 1$. Hence A_π centralizes $(X \cap N)'$. If a is an element of prime order r in A_π, then $(X \cap N)' \leqslant C_N(a)$, and so

$$\left|C_N(a)\right|_p \equiv 0 \left(\mathrm{mod}\ \left|P\Omega^-_{2n}(q)\right|_p\right) \equiv 0 \left(\mathrm{mod}\ q^{n^2-n}\right).$$

On the other hand, it follows from Lemmas 3 and 10 that

$$\left|C_N(a)\right|_p = \left|P\Omega_{2n+1}\left(q^{1/r}\right)\right|_p = q^{n^2/r}.$$

Since $n^2 - n > n^2/r$, for $r > 2$, $n > 2$, this gives a contradiction.

There are the following subcases in Table 2 for the groups $N = P\Omega_7(q)$, $P\Omega_{13}(3^e)$ or $P\Omega_{25}(3^e)$, where the possibilities for $\{X \cap N, Y \cap N\} = \{C, D\}$ are given as follows:

N	C	D	Remark
$P\Omega_7(q)$	$G_2(q)$	P_1	
	$G_2(q)$	N_1^ϵ	$\epsilon = \pm$
	$G_2(q)$	N_2^ϵ	$\epsilon = \pm$, $q > 3$ if $\epsilon = +$
$P\Omega_{13}(3^e)$	$PSp_6(3^e).a$	N_1^-	$a \leqslant 2$
$P\Omega_{25}(3^e)$	$F_4(3^e)$	N_1^-	

Since $X \cap N$ contains a Sylow 2-subgroup of N, it is easy to check that $X \cap N \neq C$ in all cases, and also $X \cap N \neq P_1$.

Whenever $X \cap N = N_1^-$ we obtain a contradiction as in the preceding subcase (for factorizations in Table 1). Moreover, we can use similar arguments to show that $X \cap N = N_1^+$ cannot occur, since in this case $(X \cap N)'$ has a unique non-abelian composition factor, which is isomorphic to $P\Omega_{2n}^+(q)$.

It remains to consider the case $N = P\Omega_7(q)$, $Y \cap N = G_2(q)$ and $X \cap N = N_2^\epsilon$. We notice that $|N|_p = p^{9e}$, $|X \cap N|_p = p^{4e}$, $|Y \cap N|_p = p^{6e}$. From $G = AN = BN = AB$ we deduce that

$$|N||A \cap B| = |G : N||N \cap A||N \cap B|.$$

We claim that p divides both $|A \cap N|$ and $|B \cap N|$.

Assume that p does not divide $|A \cap N|$. Then p^{3e} divides $|G : N|$ because p^{9e} divides $|G : N|_p|N \cap A|_p|N \cap B|_p$ and $|N \cap B|_p$ divides $|N \cap Y|_p = p^{6e}$. But this is not possible because $|G : N|$ divides e by Lemma 10. We consider now that $|N \cap A|_p$ divides $|N \cap X|_p = p^{4e}$ and argue in an analogous way to conclude that p divides also $|B \cap N|$.

But $A \cap N$ is a π'-group by Lemma 10 and $B \cap N$ is a π-group. Therefore we have obtained a contradiction showing that the case under consideration is not possible.

Finally, from Table 3 there is only the possibility $N = P\Omega_7(3)$ but this contradicts Lemma 10. Hence $N \ncong P\Omega_{2n+1}(q)$.

Case. $N \cong P\Omega_{2n}^-(q)$, $n \geqslant 4$.

There are two possibilities for the subgroups $X_0 \trianglelefteq X \cap N$, $Y_0 \trianglelefteq Y \cap N$ in Table 1, namely

$$\{X_0, Y_0\} = \left\{P_1, \hat{\ }GU_n(q)\right\} \quad \text{or} \quad \{X_0, Y_0\} = \left\{N_1, \hat{\ }GU_n(q)\right\},$$

where $n \geqslant 4$ is odd in both cases, P_1 is a maximal parabolic subgroup of N, $\hat{\ }GU_n(q)$ is a factor group of $GU_n(q)$ by its central subgroup and N_1 is a stabilizer of some non-singular one-dimensional subspace of a natural module for $\Omega_{2n}^-(q)$. Moreover, from [10, 2.2.5] we know that for all these cases it holds $X_0 = X \cap N$ and $Y_0 = Y \cap N$. Again $X \cap N \neq P_1$ from Lemma 5. It holds also that $X \cap N \neq \hat{}GU_n(q)$, since from the order formulas of the corresponding groups it can be deduced that the index $|N : \hat{}GU_n(q)|$ is even.

Therefore the only possibility we have is $X_0 = X \cap N = N_1$ and $Y_0 = \hat{}GU_n(q)$. As in the preceding case it turns out that $O_\pi((X \cap N)') = 1$, hence A_π centralizes $(X \cap N)'$ and so

$$\left|C_N(A_\pi)\right|_p \equiv 0 \left(\bmod \left|P\Omega^-_{2n-1}(q)\right|_p\right) \equiv 0 \left(\bmod q^{(n-1)^2}\right).$$

On the other hand, from Lemma 3 we deduce that for an element a of prime order s in A_π, $|C_N(a)|_p = q^{n(n-1)/s}$. Since $(n-1)^2 > n(n-1)/s$, for $s > 2$, $n > 2$, this gives a contradiction.

Tables 2–3 give no other possibilities. Hence $N \cong P\Omega^-_{2n}(q)$ cannot occur.

Case. $N \cong U_n(q)$, $n \geqslant 3$.

Note that the possibility to have a factorization $G = XY$ for this case in Table 1 appears only when n is even, say $n = 2m$. Moreover, since q is odd the only possibilities for the subgroups $X_0 \trianglelefteq X \cap N$, $Y_0 \trianglelefteq Y \cap N$ are:

$$\{X_0, Y_0\} = \{N_1, P_m\} \quad \text{or} \quad \{X_0, Y_0\} = \left\{N_1, PSp_{2m}(q)\right\}.$$

For the cases $X_0, Y_0 \in \{N_1, P_m\}$ we know by [10, 2.2.5] that $X_0 = X \cap N$ and $Y_0 = Y \cap N$. Moreover $|N : P_m|$ is even, by Lemma 5, and this is also the case for $|N : N_1| = q^{2m-1}(q^{2m} - 1)/(q + 1)$. Finally we notice that $PSp_{2m}(q)$ is maximal in N and again $|N : PSp_{2m}(q)|$ is even, to obtain a contradiction.

The subcases $N \cong U_3(3)$ and $N \cong U_4(3)$ are excluded because $|\operatorname{Out}(N)|$ is even. Finally the factorization $U_3(5) = A_7 P_1$ in Table 3 is also excluded because both subgroups have even index. Thus the case $N = U_n(q)$ cannot occur.

Case. $N \cong P\Omega^+_{2n}(q)$, $n \geqslant 4$.

Suppose first that $N \cong P\Omega^+_{2n}(q)$, $n \geqslant 5$, as in Table 1. By Lemma 10 there are only the following possibilities for $X_0 \trianglelefteq X \cap N$, $Y_0 \trianglelefteq Y \cap N$:

(a) $\{X_0, Y_0\} = \{N_1, P\}$;
(b) $\{X_0, Y_0\} = \{N_1, \hat{}GU_n(q).2\}$, n even;
(c) $\{X_0, Y_0\} = \{N_1, PSp_2(q) \otimes PSp_n(q)\}$, n even;
(d) $\{X_0, Y_0\} = \{N_2^-, P\}$;
(e) $\{X_0, Y_0\} = \{P, \hat{}GU_n(q).2\}$, n even;
(f) $\{X_0, Y_0\} = \{N_1, \hat{}GL_n(q).2\}$.

Recall that here P is some maximal parabolic subgroup of N (see [10] for the exact description); $\hat{}GU_n(q)$, $\hat{}GL_n(q)$ are factor groups of $GU_n(q)$ and $GL_n(q)$, respectively, by central subgroups, and N_1, N_2^- are certain stabilizers of subspaces.

It follows from Lemma 5 that maximal parabolic subgroups do not contain a Sylow 2-subgroup of N. Hence $X \cap N \neq P$ in any case. The index of the subgroup $X = N_1$ is counted in (3.6.1) of [10] and it is $|G : X| = (1/2)q^{n-1}(q^n - 1)$; notice that if n is even, then this is an even number. On the other hand, it is easy to see that the index of the group $\hat{}GU_n(q).2$ in N is also even. These facts exclude cases (b) and (e). Again, the index of the group $\hat{}GL_n(q).2$ in N is even and so case (f) is excluded.

Moreover, since P has even index in G and

$$\left|G : N_2^-\right| = (1/2)q^{2n-2}\left(q^n - 1\right)\left(q^{n-1} - 1\right)/(q+1)$$

is also an even number, the subcase (d) is excluded.

The subcase (c) is also impossible because the maximal subgroup of N containing $PSp_2(q) \otimes PSp_n(q)$ is $(PSp_2(q) \otimes PSp_n(q)).c$ where $c = 2$ if $n \equiv 0 \pmod 8$ and $c = 1$ in any other case (see [8, 4.4.12]), and this subgroup has even index for even n.

It remains to consider the subcase (a), where $X_0 = N_1$, $Y_0 = Y \cap N = P$, P is a maximal parabolic subgroup and $|G : X| = (1/2)q^{n-1}(q^n - 1)$. By Lemma 10 we have that A_π induces a field automorphism group of N and (as in the case of $P\Omega_{2n}^-(q)$) the unique non-abelian composition factor of $X \cap N$ is $P\Omega_{2n-1}(q)$. This implies that A_π centralizes $(X \cap N)'$. Hence $|C_N(A_\pi)|_p \equiv 0 \pmod{q^{(n-1)^2}}$.

By Lemma 3 we have also that for an element a of prime order s in A_π, $|C_N(a)|_p = q^{n(n-1)/s}$. Since $(n-1)^2 > n(n-1)/s$, for $s > 2$, $n > 2$, this gives a contradiction.

Now we may assume that $n = 4$ and $N \cong P\Omega_8^+(q)$. In this case there exist factorizations by maximal subgroups $X \geqslant A$, $Y \geqslant B$ as in Table 4. The possibilities are as follows:

(a) $\{X \cap N, Y \cap N\} = \{\Omega_7(q), \Omega_7(q)\}$;
(b) $\{X \cap N, Y \cap N\} = \{\Omega_7(q), P\}$;
(c) $\{X \cap N, Y \cap N\} = \{\Omega_7(q), \hat{}((q+1)/2 \times \Omega_6^-(q)).2^2\}$;
(d) $\{X \cap N, Y \cap N\} = \{\Omega_7(q), \hat{}((q-1)/2 \times \Omega_6^+(q)).2^2\}$;
(e) $\{X \cap N, Y \cap N\} = \{\Omega_7(q), \Omega_8^-(q^{1/2})\}$;
(f) $\{X \cap N, Y \cap N\} = \{\Omega_7(q), (PSp_2(q) \otimes PSp_4(q)).2\}$;
(g) $\{X \cap N, Y \cap N\} = \{P, \hat{}((q+1)/2 \times \Omega_6^-(q)).2^2\}$.

(Recall that from Lemma 10 $q = p^e$ is odd and it cannot be a prime number, so no more factorizations in Table 4 can occur.)

Remark that P is a maximal parabolic subgroup of N so it has even index by Lemma 5, hence $X \cap N \neq P$. On the other hand, from the order formulas of the corresponding groups it can be seen that $|N : \Omega_7(q)|$ is even, and so $Y \cap N = \Omega_7(q)$ in each case. Thus, cases (a) and (b) can be excluded.

By Lemma 10, A_π induces a field group of automorphisms of N. Moreover, by Lemma 3, for every element a of prime order r in A_π it holds:

$$\left|C_N(a)\right|_p = \left|\Omega_8^+\left(q^{1/r}\right)\right|_p = q^{12/r}.$$

On the other hand, A_π centralizes again $(X \cap N)'$ in each of the cases (c)–(g). It is easy to see that this leads to a contradiction since $|C_N(a)|_p > q^{12/r}$ for all possible groups.

Thus this case cannot occur and the lemma is proved. □

4. Proof of Theorem 1

Assume that the result is false and let G be a counterexample of minimal order to the result. The structure of G is described in the previous section. In particular, we will keep here the same notation. It follows from Lemma 11 that $N \cong L_n(q)$ where $q = p^e$ and p an odd prime.

We claim here that $p \notin \pi(A \cap N)$. We prove first that $p \in \pi(A \cap N)$ implies $p \in \pi'$ and $A_\pi = 1$, a contradiction which will prove the claim.

Assume that $p \in \pi(A \cap N)$. Let S be a Sylow 2-subgroup of N and set $C = C_N(S)$. It is easily proved that C is 2-decomposable.

Assume first that $p \in \pi$. Then, by [9, Theorem 7] it follows that $1 \neq A_\pi \cap N \leqslant O(C) = C_1 \times \cdots \times C_{r-1}$, where $n = 2^{t_1} + \cdots + 2^{t_r}$ for some r and $t_1, \ldots, t_r$, $r \geqslant 2$ and $0 \leqslant t_1 < \cdots < t_r$, and $C_1, \ldots, C_{r-2}$ are cyclic groups of orders $(q-1)_{2'}$ and C_{r-1} is a cyclic group of order $((q-1)/(q-1,n))_{2'}$. In particular p divides $q-1$, which is a contradiction. Hence $p \in \pi'$. Now this implies that N contains a Sylow p-subgroup of G because $G/N = A_\pi N/N$ is a π-group. But for any prime $r \in \pi(A_\pi)$, an element of A of order r defines an automorphism of order r of the simple group N of Lie type of characteristic p, which centralizes a Sylow p-subgroup of N. By [7, 1.17] it is an inner automorphism of N and $p = r$, the final contradiction. The claim is proved.

As in the previous section, we consider again the possible factorizations for G described in [10]. Tables 1 and 3 of [10] give the following possibilities (notice that $|G : N|$ is odd and so G does not contain a graph automorphism):

- From Table 1:
  (1) $\{X_0, Y_0\} = \{\hat{\ }GL_a(q^b).b, P_1 \text{ or } P_{n-1}\}$, $ab = n$, b prime, $(n,q) \neq (4,2)$.
  (2) $\{X_0, Y_0\} = \{PSp_n(q), P_1 \text{ or } P_{n-1}\}$, n even, $n \geqslant 4$.
- From Table 3:

$$N \cong L_2(q) \quad \text{where } q \in \{11, 19, 29, 59, 7, 23\}.$$

We prove next that each of these cases leads to a contradiction, which will conclude the proof.

Case. $X_0 \cong P$ where $P = P_1$ or P_{n-1}, $Y_0 \cong \hat{\ }GL_a(q^b).b$.

We claim first that $|N|_p$ divides $|G : N|_p |N \cap Y|_p$.

From $G = AN = BN = AB$ we have that

$$|N||A \cap B| = |G : N||N \cap A||N \cap B|.$$

Consequently

$$|N|_p |A \cap B|_p = |G : N|_p |N \cap A|_p |N \cap B|_p.$$

Since $|N \cap A|_p = 1$ by the initial claim and $|G : N|_p = |B : N \cap B|_p$, it follows that

$$|N|_p |A \cap B|_p = |B|_p.$$

In particular, $|N|_p$ divides $|B|_p$ and so also $|Y|_p$.

On the other hand, in this case, and taking into account [10, 2.2.5], we have that $X_0 = X \cap N = P$ is a maximal parabolic subgroup of index $(q^n - 1)/(q - 1)$. From $G = XY = NX$ we deduce now that

$$|G : X| = |N : N \cap X| = |Y : X \cap Y| \quad \text{is not divided by } p.$$

In particular,

$$|Y|_p = |X \cap Y|_p \quad \text{and} \quad |N|_p = |N \cap X|_p.$$

Again as before, the fact that $G = XN = YN = XY$ implies that

$$|N||X \cap Y| = |G : N||N \cap X||N \cap Y|.$$

Therefore $|N|_p$ divides $|X \cap Y|_p = |G : N|_p|N \cap Y|_p$ as claimed.

Now, in this case it holds also $Y_0 = Y \cap N$ by [10, 2.2.5], and moreover $|G : N|_p$ divides $|\operatorname{Out}(N)|_p = e_p$ by [10, 2.1]. Consequently $\frac{|N|_p}{|N \cap Y|_p}$ divides e which is not true.

Case. $Y_0 \cong P$ where $P = P_1$ or P_{n-1}, $X_0 \cong \hat{}GL_a(q^b).b$.

We have here that $X_0 = X \cap N$ and $Y_0 = Y \cap N$. We claim that 2 divides $|N : N \cap X|$, which is a contradiction.

For all cases except for $n = 2$, $a = 1$ and $b = 2$ the claim follows by a straightforward computation of the orders of the subgroups. We notice that for $n = 4$, $a = 2$ and $b = 2$, a maximal subgroup of N containing X_0 has order $(q + 1)q^4(q^4 - 1)2/(q - 1, 4)$ (see [8, Proposition 4.36]).

Assume now that $n = 2$, $a = 1$ and $b = 2$. In this case $X_0 = X \cap N \cong D_{q+1}$ and the maximal parabolic subgroup $Y_0 = Y \cap N$ has order $q(q - 1)/2$ by [10, 5.1.1].

If $q \equiv 1 \pmod 4$, then $|N : X \cap N| = q(q - 1)/2$ is even, a contradiction.

Assume that $q \equiv 3 \pmod 4$. In particular notice that $(q + 1, (q - 1)/2) = 1$.

Since $n = 2$ we have from [9, Theorem 7] that $A_\pi \cap N \leqslant O(C_N(S)) = 1$ for any Sylow 2-subgroup S of N. Hence $A_\pi \leqslant C_G(A_{\pi'}) = C_G(A \cap N)$. Moreover A_π is a subgroup of field automorphisms of N. This is because in this case $|\operatorname{Out}(N)| = 2e$ and so $1 \neq |A_\pi| = |A_\pi N : N| = |G : N|$ divides $|\operatorname{Out}(N)|_{2'} = e_{2'}$. Let a be an element of A_π of order a prime number r. From Lemma 3 it follows now that $C_N(a) \leqslant L_2(q^{1/r}).H$ where H is a Cartan subgroup of N of order $(q - 1)/2$ here. Consequently $|A \cap N|$ divides

$$\bigl(\bigl|L_2\bigl(q^{1/r}\bigr).H\bigr|, |X \cap N|\bigr) = \bigl(q^{1/r}\bigl(q^{2/r} - 1\bigr)(q - 1)/4, q + 1\bigr) = \bigl(\bigl(q^{2/r} - 1\bigr)/2, q + 1\bigr).$$

In particular $|A \cap N|$ divides $q^{2/r} - 1$.

Since $G = NA = NB = AB$ we consider the equation

$$|N||A \cap B| = |G : N||N \cap A||N \cap B|$$

and in addition a primitive divisor q_2 of $p^{2e} - 1$. It is known that $q_2 \equiv 1 \pmod{2e}$. In particular q_2 does not divide e and so neither $|G : N|$. Moreover q_2 does not divide $|N \cap B|$ as $|N \cap Y| = q(q - 1)$. Since q_2 divides $|N|$ we deduce that q_2 divides $|N \cap A|$ and so also $q^{2/r} - 1$, a contradiction.

Case. $X_0 \cong P$ where $P = P_1$ or P_{n-1}, $Y_0 \cong PSp_n(q)$, n even, $n \geqslant 4$.

Again we have here that $X_0 = X \cap N$ is a maximal parabolic subgroup of N of index $(q^n - 1)/(q-1)$, which is divided by 2 as n is even, a contradiction.

Case. $Y_0 \cong P$ where $P = P_1$ or P_{n-1}, $X_0 \cong PSp_n(q)$, n even, $n \geqslant 4$.

We recall that $X = N_G(X_0)$ and so $X \cap N = N_N(X_0)$. But from [8, Proposition 4.8.3] a maximal subgroup X_0^* of N containing $X_0 \cong PSp_n(q)$ verifies $X_0 \trianglelefteq X_0^* = X \cap N \cong PSp_n(q).[\frac{(q-1,2)(q-1,\frac{n}{2})}{(q-1,n)}]$.

If $n > 4$ again a computation of the orders of the subgroups gives that 2 divides $|N : X_0|$, which is not possible.

Assume now that $n = 4$. We notice that in this case $|X \cap N|_p = |PSp_4(q)|_p = q^4$. Moreover A_π centralizes $X \cap N$ because $A_\pi \leqslant O_\pi(X)$, by Lemma 9, and $2 \notin \pi$, which implies

$$[A_\pi, X \cap N] \leqslant \big[O_\pi(X), X \cap N\big] \leqslant O_\pi(X) \cap X \cap N = O_\pi(X \cap N) = 1.$$

In addition we claim that A_π contains a field automorphism a of prime order r. This is because $|A_\pi : A_\pi \cap N|$ divides $|\operatorname{Out}(N)|_{2'} = e_{2'}$. Moreover $A_\pi \cap N \leqslant O(C_N(S)) = 1$ for any Sylow 2-subgroup S of N, from [9, Theorem 7] as here $n = 4 = 2^2$.

Consequently $|X \cap N|_p = q^4$ divides $|C_N(a)|_p$. But we recall that $C_N(a) \leqslant L_n(q^{1/r}).H$ where H is a Cartan subgroup of N normalizing $L_n(q^{1/r})$, by Lemma 3. It follows now that q^4 divides $|L_4(q^{1/r})|_p = q^{6/r}$ and so $4 \leqslant \frac{6}{r}$, but this is a contradiction since $r > 2$.

Case. $N \cong L_2(q)$ where $q \in \{11, 19, 29, 59, 7, 23\}$.

In this case $|\operatorname{Out}(N)| = 2$ but this is not possible because $|G : N|$ is odd.

Now the theorem is proved.

5. Final examples

(1) Let $G = X \wr Y$ be the regular wreath product of $X \cong C_3$ a cyclic group of order 3 with $Y \cong C_2$ a cyclic group of order 2. Let $X^\natural = X_1 \times X_2$ with $X = X_1 = X_2$, the base group of G, and set $D(X) = \{(x, x) \mid x \in X\}$. Then $G = AB$ with $A = D(X) \times Y = O_\pi(A) \times O_{\pi'}(A)$, if $\pi = \{3\}$, and $B = X_1$ a π-group, and π a set of odd primes, but $O_{\pi'}(A) \nleqslant O_{\pi'}(G)$. Obviously $G \in E_\pi$ and $G \in E_{\pi'}$.

(2) Let $G = Y \wr X$ be the regular wreath product of $Y \cong C_2$ with $X \cong C_3$, as above. Let $Y^\natural = Y_1 \times Y_2 \times Y_3$ with $Y = Y_1 = Y_2 = Y_3$, the base group of G, and set $D(Y) = \{(y, y, y) \mid y \in Y\}$. Then $G = AB$ with $A = D(Y) \times X = O_{\pi'}(A) \times O_\pi(A)$, with $\pi' = \{2\}$, and $B = Y_1 \times Y_2$ a π'-group. We notice that $O_\pi(A) \nleqslant O_\pi(G)$. Here also $G \in E_\pi$ and $G \in E_{\pi'}$.

(3) Let G be a group isomorphic to $L_2(2^n)$ where n is a positive integer such that $2^n + 1$ is divisible by two distinct primes (this happens if $n \neq 3$ and $2^n + 1$ is not a Fermat prime). Set $q = 2^n$. Then $G = AB$ where $A \cong C_{q+1}$ is a cyclic group of order $q + 1$ and $B = N_G(G_2)$, $G_2 \in \mathrm{Syl}_2(G)$. Let r be a prime dividing $q + 1$ and take $\pi' = \pi(N_G(G_2)) \cup \{r\}$. Then $A = O_\pi(A) \times O_{\pi'}(A)$ and B is a π'-group, but $O_{\pi'}(A) \nleqslant O_{\pi'}(G)$, being $2 \in \pi'$. In particular here, $G \notin E_{\pi'}$.

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