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Localization at hyperplane arrangements: Combinatorics and $\mathcal{D}$-modules

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Abstract

We describe an algorithm deciding if the annihilating ideal of the meromorphic function $\frac{1}{f}$, where $f = 0$ defines an arrangement of hyperplanes, is generated by linear differential operators of order 1. The algorithm is based on the comparison of two characteristic cycles and uses a combinatorial description of the characteristic cycle of the $\mathcal{D}$-module of meromorphic functions with respect to f, due to Àlvarez Montaner, García López and Zarzuela.

Keywords: Characteristic cycle; Localization; Hyperplane arrangements; Logarithmic $\mathcal{D}$-modules; Gröbner bases

1. Introduction

Let $R = k[x_1, \dots, x_n]$ be the ring of polynomials in n variables where k is a field of characteristic zero. Let $A_n := R\langle\partial_1, \dots, \partial_n\rangle$ be the associated Weyl algebra, i.e. the ring extension generated by the partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$, with the relations given by $\partial_i \partial_j = \partial_j \partial_i$ and

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$\partial_i r - r\partial_i = \frac{\partial r}{\partial x_i}$, for $r \in R$. For any unexplained terminology concerning the theory of rings of differential operators we refer to [3]. For any polynomial $f \in R$, I.N. Bernšteĭn [2] proved that the localization ring $R_f = \{\frac{g}{f^m} \mid g \in R,\ m \in \mathbb{N}\}$ has a natural structure as a finitely generated left A_n-module. Moreover it has finite length so the ascending chain of A_n-submodules

$$A_n \cdot \frac{1}{f} \subseteq A_n \cdot \frac{1}{f^2} \subseteq \cdots \subseteq A_n \cdot \frac{1}{f^j} \subseteq \cdots \subseteq R_f$$

stabilizes. In particular we have

$$R_f \cong A_n \cdot \frac{1}{f^\ell} \cong \frac{A_n}{\operatorname{Ann}_{A_n}(\frac{1}{f^\ell})}$$

where the integer $-\ell$ is the smallest integer root of the Bernstein–Sato polynomial of f (see [2]) and $R_f \ncong A_n \cdot \frac{1}{f^j}$ for $j < \ell$ [26, Lemma 1.3]. T. Oaku [17], and later T. Oaku and N. Takayama [18], described algorithms to compute this presentation of R_f using the theory of Gröbner bases over rings of differential operators. The algorithm has been implemented in the package `D-modules` [15] for `Macaulay 2` [12], in the system `Kan/sm1` [22] and in `Risa/Asir` [16]. The computation of the Bernstein–Sato polynomial is often very expensive due to the use of Gröbner bases so different short cuts have been considered to describe R_f. We will follow the approach given in [7,8] where the authors use the module of logarithmic derivations introduced by K. Saito [21].

Let $\operatorname{Ann}^{(j)}_{A_n}(\frac{1}{f^\ell})$ denote the ideal generated by the set of differential operators of order $\leqslant j$ that annihilate $\frac{1}{f^\ell}$. We have the ascending chain of ideals

$$\operatorname{Ann}^{(1)}_{A_n}\left(\frac{1}{f^\ell}\right) \subseteq \operatorname{Ann}^{(2)}_{A_n}\left(\frac{1}{f^\ell}\right) \subseteq \cdots \subseteq \operatorname{Ann}_{A_n}\left(\frac{1}{f^\ell}\right).$$

Any of the terms $\operatorname{Ann}^{(j)}_{A_n}(\frac{1}{f^\ell})$ of this chain can be described without using the Bernstein–Sato polynomial since its computation reduces to the one of commutative syzygies for a set of elements involving the partial derivatives of f of order less or equal to j.

For the case $j = 1$ the annihilator can be described in terms of logarithmic derivations. Namely, let $\operatorname{Der}(-\log f)$ be the R-module of logarithmic derivations with respect to f, i.e. derivations $\delta = \sum_i a_i\partial_i \in \operatorname{Der}_k(R)$ such that $\delta(f) = af$ for some $a \in R$. Then, $\operatorname{Ann}^{(1)}_{A_n}(\frac{1}{f^\ell})$ is the ideal generated by the set

$$\left\{\delta + \ell\frac{\delta(f)}{f} \;\middle|\; \delta \in \operatorname{Der}(-\log f)\right\}.$$

Our motivation is the natural question: *When is* $\operatorname{Ann}_{A_n}(\frac{1}{f^\ell})$ *generated by operators of order less or equal to* j? In this paper we will restrict ourselves to the case $j = 1$.

The analytic counterpart to this question has been treated in [9] following ideas introduced and developed in [4]. They got an affirmative answer for the case of locally quasi-homogeneous free divisors $D \subset \mathbb{C}^n$. Namely, let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X = \mathbb{C}^n$. Let $\mathcal{D}_X$ denote the corresponding sheaf of differential operators with coefficients in $\mathcal{O}_X$. If $\operatorname{Der}_{\mathcal{O}_X}(-\log D)_x := \operatorname{Der}_{\mathcal{O}_{X,x}}(-\log f)$, where $f \in \mathcal{O}_{X,x}$ is a local reduced equation of the germ

$(D,x) \subset (\mathbb{C}^n, x)$, is a free $\mathcal{O}_{X,x}$-module at every point $x \in D$ (we say then that D is a free divisor) and f is locally quasi-homogeneous then $\mathrm{Ann}_{\mathcal{D}_{X,x}}(\frac{1}{f})$ is generated by operators of order 1 [9].

These results are closely related to the so-called *Logarithmic Comparison Theorem* (LCT) [6]. We say that LCT holds for a divisor $D \subset \mathbb{C}^n$ if the inclusion $i_D : \Omega(\log D) \hookrightarrow \Omega(\star D)$ is a quasi-isomorphism, where $\Omega(\star D)$ is the de Rham complex of differential meromorphic forms with poles along D and $\Omega(\log D)$ is its subcomplex of logarithmic forms. When LCT holds the complex $\Omega(\log D)$ computes the cohomology $H^p(X \setminus D, \mathbb{C})$ of the complementary of D by Grothendieck Comparison Theorem [14]. Locally quasi-homogeneous free divisors satisfy LCT [6].

T. Torrelli [25] gave some criteria to answer our main question and also conjectured the following link with LCT.

Conjecture 1.1. *Let $f \in \mathcal{O}_{X,x}$ is a local reduced equation of the germ $(D,x) \subset (\mathbb{C}^n, x)$. Then, the ideal $\mathrm{Ann}_{\mathcal{D}_{X,x}}(\frac{1}{f})$ is generated by operators of order* 1 *if and only if LCT holds for (D,x).*

In the present work we turn our attention to the case of arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}_k^n$. It has been proved by A. Leykin (see [26]) that R_f is generated by $\frac{1}{f}$ so we have to study when $\mathrm{Ann}_{A_n}^{(1)}(\frac{1}{f}) = \mathrm{Ann}_{A_n}(\frac{1}{f})$. If the base field $k = \mathbb{C}$, the equality holds for the union of a generic hyperplane arrangement with an hyperbolic arrangement, as it has been proved by T. Torrelli [25] giving an explicit set of generators and solving a conjecture posed by U. Walther [26]. Conjecture 1.1 is also true for free Spencer divisors [10].

It has been conjectured in [23] that LCT holds for central arrangements of hyperplanes over any field k of characteristic zero. H. Terao observed that, using the methods in [6], LCT holds for tame arrangements over $\mathbb{C}$, i.e. arrangements such that the projective dimension of the module of logarithmic p-forms $\Omega^p(\log f)$ is less or equal than p. The conjecture is also proved for arbitrary arrangements over $\mathbb{C}$ in dimension less or equal than 5 [24] and over any field k of characteristic zero for the case of tame arrangements and the case of all arrangements in dimension less or equal than 4 [27].

Our aim is to describe an indirect method to compare the ideals $\mathrm{Ann}_{A_n}^{(1)}(\frac{1}{f})$ and $\mathrm{Ann}_{A_n}(\frac{1}{f})$ using an invariant that we may attach to finitely generated A_n-modules: the characteristic cycle. We intend to use the interplay of A_n-modules and combinatorial techniques to deal with examples not covered in [25] and that cannot be treated using the methods in [8]. One of our main examples will be a non-tame arrangement such that $\mathrm{Ann}_{A_n}(\frac{1}{f})$ is generated by operators of order 1. As far as we know, for these examples a direct approach using the methods in [17] or in [18] implemented in the available Computer Algebra systems is not possible due to crashing memory problems.

The scripts of the source codes we will use in this work as well as the output in full detail of the examples are available at the web page http://www.ma1.upc.edu/~jalvz/acu.html.

2. Using the characteristic cycle

Let $R = k[x_1, \dots, x_n]$ be the ring of polynomials in n variables where k is a field of characteristic zero. The associated Weyl algebra $A_n := R\langle \partial_1, \dots, \partial_n \rangle$ has a natural increasing filtration given by the order of differential operators, the order of a monomial $x^\alpha \partial^\beta$ being $|\beta| = \sum_i \beta_i$. The corresponding associated graded ring $\mathrm{gr}(A_n)$ is isomorphic to the polynomial ring $R[\xi] = R[\xi_1, \dots, \xi_n]$ graded by the natural degree of polynomials on the variables ξ_i. The principal symbol of an operator $P = \sum_\beta p_\beta(x)\partial^\beta$ is the polynomial

$$\sigma(P) = \sum_{|\beta|=\mathrm{order}(P)} p_\beta(x)\xi^\beta.$$

A finitely generated A_n-module M has a so-called good filtration, i.e. an increasing sequence of finitely generated R-submodules such that the associated graded module $\mathrm{gr}(M)$ is a finitely generated $\mathrm{gr}(A_n)$-module. If $I \subset A_n$ is a left ideal, then $\mathrm{gr}(I)$ is nothing but the ideal of $\mathrm{gr}(A_n) \simeq R[\xi]$ generated by the set $\{\sigma(P) \mid P \in I\}$. All the A_n-modules considered here are left modules.

The *characteristic variety* of M is the closed algebraic set $\mathrm{Ch}(M)$ in the affine space $\mathbb{A}_k^{2n}$ defined by the *characteristic ideal* $J(M) := \mathrm{rad}(\mathrm{Ann}_{\mathrm{gr}(A_n)}(\mathrm{gr}(M)))$, where rad() stands for the radical ideal. It can be shown that $J(M)$ does not depend on the choice of the good filtration on M [3]. If $M = A_n/I$ for I a left ideal in A_n, the characteristic variety $\mathrm{Ch}(M)$ is just defined by the graded ideal $\mathrm{gr}(I)$. A finitely generated A_n-module M is said to be *holonomic* if $M = 0$ or $\dim \mathrm{Ch}(M) = n$. The *characteristic cycle* of a holonomic A_n-module M is defined as:

$$\mathrm{Ch}\, C(M) = \sum m_i \Lambda_i$$

where the sum is taken over all the irreducible components Λ_i of the characteristic variety $\mathrm{Ch}(M)$, and m_i is the multiplicity of $\mathrm{gr}(M)$ at a generic point of Λ_i.

For the case $k = \mathbb{C}$ the components of the characteristic cycle can be described in terms of conormal spaces (see [20, Section 10]). Namely, for each i, the irreducible variety Λ_i that appear in the characteristic cycle is the conormal space to $X_i := \pi(\Lambda_i)$ in $\mathbb{A}_{\mathbb{C}}^n$ where $\pi : \mathbb{A}_{\mathbb{C}}^n \times \mathbb{A}_{\mathbb{C}}^n \to \mathbb{A}_{\mathbb{C}}^n$ is the first projection. So we have the following description

$$\mathrm{Ch}\, C(M) = \sum m_i T_{X_i}^* \mathbb{A}_{\mathbb{C}}^n.$$

For the case of k being any field of characteristic zero we will make an abuse of notation referring to $T_{X_i}^* \mathbb{A}_k^n$ for the irreducible component Λ_i such that $X_i := \pi(\Lambda_i)$.

Remark 2.1. Let $I_i \subseteq R$ be the defining ideal of a linear variety $X_i \subseteq \mathbb{A}_k^n$ of codimension h then $T_{X_i}^* \mathbb{A}_k^n$ is the characteristic cycle of the local cohomology module $H_{I_i}^h(R)$.

2.1. Comparison using the characteristic cycle

Let $f \in R$ be any polynomial and $-\ell$ be the smallest integer root of the Bernstein–Sato polynomial associated to f. Our aim is to use the characteristic cycle to decide whether $\mathrm{Ann}_{A_n}(\frac{1}{f^\ell})$ is generated by differential operators of order 1.

Remark 2.2. If $\mathrm{Ann}_{A_n}(\frac{1}{f^\ell})$ is generated by differential operators of order 1 for any given $\ell \in \mathbb{N}$ then the smallest integer root of the Bernstein–Sato polynomial of f is strictly greater than $-\ell - 1$ [25, Proposition 1.3].

To our purpose we consider the A_n-module

$$\widetilde{M}^{\log f^\ell} := \frac{A_n}{\operatorname{Ann}_{A_n}^{(1)}(\frac{1}{f^\ell})}.$$

Notice that, although the localization module R_f is holonomic by [2] the module $\widetilde{M}^{\log f^\ell}$ may not be holonomic, e.g. $f = (xz + y)(x^4 + y^5 + xy^4)$ (see [8]). In [5] the authors prove that the divisor defined by f in $\mathbb{A}^3_{\mathbb{C}}$ is not of Spencer type.

Proposition 2.3. *Let $f \in k[x_1, \dots, x_n]$ be any polynomial and let $-\ell$ be the smallest integer root of the Bernstein–Sato polynomial associated to f. If $\widetilde{M}^{\log f^\ell}$ is holonomic, then $\operatorname{Ann}_{A_n}(\frac{1}{f^\ell})$ is generated by differential operators of order 1 if and only if $\operatorname{Ch} C(R_f) = \operatorname{Ch} C(\widetilde{M}^{\log f^\ell})$. If $\widetilde{M}^{\log f^\ell}$ is not holonomic, then $\operatorname{Ann}_{A_n}(\frac{1}{f^\ell})$ cannot be generated by differential operators of order 1.*

Proof. By [2], the localization module R_f is isomorphic to $\frac{A_n}{\operatorname{Ann}_{A_n}(\frac{1}{f^\ell})}$ so we have the short exact sequence of A_n-modules

$$0 \to K_{f^\ell} \to \widetilde{M}^{\log f^\ell} \to R_f \to 0$$

where $K_{f^\ell} := \frac{\operatorname{Ann}_{A_n}(\frac{1}{f^\ell})}{\operatorname{Ann}_{A_n}^{(1)}(\frac{1}{f^\ell})}$.

Notice that the question about $\operatorname{Ann}_{A_n}(\frac{1}{f^\ell})$ being generated by operators of order 1 reduces then to the vanishing of the kernel K_{f^ℓ}.

If $\widetilde{M}^{\log f^\ell}$ is not holonomic then $K_{f^\ell} \neq 0$. Otherwise, by the additivity of the characteristic cycle with respect to holonomic A_n-modules

$$K_{f^\ell} = 0 \quad \Leftrightarrow \quad \operatorname{Ch} C\big(\widetilde{M}^{\log f^\ell}\big) = \operatorname{Ch} C(R_f).$$

That ends the proof. □

Besides computing the Bernstein–Sato polynomial of f it may also be difficult to compute the characteristic cycles of the A_n-modules $\widetilde{M}^{\log f^\ell}$ and R_f. To compute the characteristic variety of these modules we need Gröbner basis over rings of differential operators. The software packages `D-modules` [15] for `Macaulay 2` [12] and `Kan/sm1` [22] can do the job for some examples but then, in order to get the irreducible components of the characteristic variety, we must perform primary decomposition so it is expensive to develop large examples. Algorithms for primary decomposition are defined over the field of rational numbers but this is not an issue for most examples due to the good behavior of these modules with respect to flat base changes.

2.2. The case of hyperplane arrangements

In this work we will restrict ourselves to the case of hyperplane arrangements since we can avoid the computation of Bernstein–Sato polynomials. Namely, as it was proved by A. Leykin, the smallest integer root of the Bernstein–Sato polynomial of any arrangement is -1 [26] so we have to study the A_n-module $\widetilde{M}^{\log f}$.

Let $\mathcal{A} = \{H_1, \ldots, H_r\}$ be an arrangement of hyperplanes defined by a polynomial $f = f_1 \cdots f_r \in R$, f_i being a linear form in R. We will suppose that the coefficients of f_i are in $\mathbb{Q}$. The arrangement $\mathcal{A}$ defines a partially ordered set $P(\mathcal{A})$ whose elements correspond to the intersections of irreducible components of $\mathcal{A}$ and where the order is given by inclusion, i.e. for $p, q \in P(\mathcal{A})$ we write $p < q$ if $q \subset p$.

Given $p \in P(\mathcal{A})$, we will denote by X_p the linear affine variety corresponding to p and by $I_p \subseteq R$ the radical ideal which defines X_p. Notice that the poset $P(\mathcal{A})$ is isomorphic to the poset of ideals $\{I_p\}_p$, ordered by reverse inclusion. We denote by $\mathrm{ht}(p)$ the height of the ideal I_p.

For each $\delta = \sum_i a_i(x)\partial_i \in \mathrm{Der}(-\log f)$ the principal symbol $\sigma(\delta) = \sum_i a_i(x)\xi_i \in k[x, \xi]$ defines a function on the affine space $\mathbb{A}_k^{2n}$.

We denote following K. Saito [21, (3.15)] $L_k(\log f)$ the algebraic subvariety defined in $\mathbb{A}_k^{2n}$ by $\{\sigma(\delta) \mid \delta \in \mathrm{Der}(-\log f)\}$.

Assume that the base field is $k = \mathbb{C}$. By [21, (3.14)–(3.18)] one has

$$L_{\mathbb{C}}(\log f) = \bigcup_{p \in P(\mathcal{A})} T^*_{X_p}\mathbb{A}^n_{\mathbb{C}}$$

where $T^*_{X_p}\mathbb{A}^n_{\mathbb{C}}$ is the conormal space to the linear variety $X_p \subset \mathbb{A}^n_{\mathbb{C}}$. Notice that $T^*_{X_p}\mathbb{A}^n_{\mathbb{C}}$ is a linear subvariety of $\mathbb{A}^{2n}_{\mathbb{C}}$ defined by a set of linear equations with rational coefficients. Let us denote by C_p (respectively $C_{p,k}$) the corresponding linear variety in $\mathbb{A}^{2n}_{\mathbb{Q}}$ (respectively $\mathbb{A}^{2n}_k$).

Proposition 2.4. [21] *Let $\mathcal{A} \subseteq \mathbb{A}^n_k$ be an arrangement of hyperplanes defined by a polynomial $f = f_1 \cdots f_r \in R$, f_i being a linear form in R with coefficients in $\mathbb{Q}$. Then, the A_n-module $\widetilde{M}^{\log f}$ is holonomic.*

Proof. The characteristic variety $\mathrm{Ch}(\widetilde{M}^{\log f})$ is defined by the graded ideal $\mathrm{gr}(\mathrm{Ann}^{(1)}_{A_n}(\frac{1}{f}))$ which contains the set $\{\sigma(\delta) \mid \delta \in \mathrm{Der}(-\log f)\}$. So, $\mathrm{Ch}(\widetilde{M}^{\log f})$ is a subset of $L_k(\log f)$. That proves the result for $k = \mathbb{C}$. As the linear forms defining f have rational coefficients, one has $L_{\mathbb{Q}}(\log f) = \bigcup_{p \in P(\mathcal{A})} C_p$. If k is a field of characteristic zero, the irreducible components of the algebraic variety $L_k(\log f)$ are $C_{k,p}$ for $p \in P(\mathcal{A})$. That proves the result for any field k of characteristic zero. $\square$

Proposition 2.4 holds for any complex locally quasi-homogeneous divisor as pointed out by T. Torrelli.

So, if k is any field of characteristic zero we have

$$\mathrm{Ch}\, C\big(\widetilde{M}^{\log f}\big) = T^*_{\mathbb{A}^n_k}\mathbb{A}^n_k + \sum n_p C_{p,k},$$

where n_p is the multiplicity of $\mathrm{gr}(\widetilde{M}^{\log f})$ at a generic point of $C_{p,k}$. Notice that $p_0 = \mathbb{A}^n_k$ is an element of $P(\mathcal{A})$ and that $C_{p_0,k} = T^*_{\mathbb{A}^n_k}\mathbb{A}^n_k$.

Localizing $\widetilde{M}^{\log f}$ at a generic point of $\mathbb{A}^n_k$ we can compute the multiplicity of $\mathrm{gr}(\widetilde{M}^{\log f})$ at a generic point of $C_{p_0,k}$. The module $\widetilde{M}^{\log f}$ is isomorphic to R in the neighborhood of any point

$x_0 \in \mathbb{A}_k^n$ such that $f(x_0) \neq 0$ and $\operatorname{Ch} C(R) = T^*_{\mathbb{A}_k^n}\mathbb{A}_k^n$. Thus, the multiplicity of the irreducible component $T^*_{\mathbb{A}_k^n}\mathbb{A}_k^n$ in $\operatorname{Ch} C(\widetilde{M}^{\log f})$ is one.

On the other hand, the characteristic cycle of the localization module R_f can be completely described using the combinatorial approach given in [1]. Let $K(> p)$ be the simplicial complex attached to the subposet

$$P(> p) := \{q \in P(X) \mid q > p\} \subseteq P(\mathcal{A}).$$

Namely, $K\ (> p)$ has as vertices the elements of $P(> p)$ and a set of vertices $p_0, \dots, p_r$ determines a r-dimensional simplex if $p_0 < \cdots < p_r$. Then, the characteristic cycle of the localization module R_f can be described in terms of the elements of the poset $P(\mathcal{A})$ and the dimensions of the reduced simplicial homology groups of $K\ (> p)$. Namely, the result we will use in this work, conveniently reformulated states the following:

Proposition 2.5. [1, Corollary 1.3] *Let $\mathcal{A} \subseteq \mathbb{A}_k^n$ be an arrangement of hyperplanes defined by a polynomial $f \in R = k[x_1, \dots, x_n]$. Then, the characteristic cycle of the localization module R_f is*

$$\operatorname{Ch} C(R_f) = T^*_{\mathbb{A}_k^n}\mathbb{A}_k^n + \sum m_p\, T^*_{X_p}\mathbb{A}_k^n,$$

where $m_p = \dim_k \widetilde{H}_{\mathrm{ht}(p)-2}(K\ (> p); k)$.

Let $f \in R$ be the defining polynomial of an hyperplane arrangement $\mathcal{A} \subseteq \mathbb{A}_k^n$. Our strategy for this family of examples boils down to the following

Algorithm.

INPUT: The defining polynomial $f \in R$ of an hyperplane arrangement $\mathcal{A} \subseteq \mathbb{A}_k^n$.

OUTPUT: **True** if $\operatorname{Ann}_{A_n}(\frac{1}{f})$ is generated by operators of order 1 and **False** otherwise.

(1) Compute the characteristic cycle of R_f.
(2) Compute the ideal $\operatorname{Ann}^{(1)}_{A_n}(\frac{1}{f})$.
(3) Compute the characteristic cycle of $\widetilde{M}^{\log f}$.
  (3.1) Compute the characteristic variety $\operatorname{Ch}(\widetilde{M}^{\log f})$.
  (3.2) Compute the irreducible components of $\operatorname{Ch}(\widetilde{M}^{\log f})$.
  (3.3) Compute the multiplicity associated to each component.
(4) Compare both characteristic cycles.

Remark 2.6. Steps (1) and (3.2) are purely combinatorial. The other steps require Gröbner basis computations in either the polynomial ring or the Weyl algebra.

The correctness of the algorithm is given by the following

Theorem 2.7. *The previous algorithm is correct, i.e. there is an algorithm based on the comparison of two characteristic cycles deciding if $\operatorname{Ann}_{A_n}(\frac{1}{f})$ is generated by differential operators of order 1, f being the defining polynomial of an arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}_k^n$.*

Proof. By Proposition 2.3 and the previous result by A. Leykin we have only to prove that each step can be achieved algorithmically. For simplicity we will denote $I = \mathrm{Ann}_{A_n}^{(1)}(\frac{1}{f})$.

Step 1: By Proposition 2.5 we only have to compute the simplicial homology groups associated to the poset $\mathcal{P}(\mathcal{A})$. It can be algorithmically achieved.

Step 2: Since the ideal I is generated by the operators of the form $\delta + \frac{\delta(f)}{f}$ for $\delta \in \mathrm{Der}(-\log f)$ it is enough to compute a finite generating system of the R-module $\mathrm{Der}(-\log f)$. This last R-module is in fact isomorphic to the module of syzygies of $(f_1, \ldots, f_n, -f)$ where $f_i = \frac{\partial f}{\partial x_i}$. More precisely, if $\{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ is a generating systems for $Syz(f_1, \ldots, f_n, -f)$, where $\mathbf{s}_i = (s_{i1}, \ldots, s_{in}, s_{i(n+1)})$, then I is generated by the family $\{\sum_j s_{ij}\partial_j + s_{i(n+1)} \mid i = 1, \ldots, r\}$ of operators in A_n. Therefore this step is solved using commutative Gröbner basis computations in R.

Step 3.1: It is performed by a direct computation of a system of generators of the graded ideal $\mathrm{gr}(I)$ which uses Gröbner basis computation in A_n.

Step 3.2: The components for the case of arrangements of hyperplanes are described in terms of the associated poset $\mathcal{P}(\mathcal{A})$ (see Section 2.2) so this step is solved by using commutative Gröbner basis computations in R.

Step 3.3: For simplicity we will denote $X = \mathbb{A}_k^n$. We have to compute for each $p \in \mathcal{P}(\mathcal{A})$ the generic multiplicity of the component $T_{X_p}^* X$, i.e. the multiplicity of $\mathrm{gr}(I)$ at a generic point of $T_{X_p}^* X$. A point $(a, b) \in T_{X_p}^* X$ is generic if $(a, b) \notin T_{X_q}^* X$ for $q \in \mathcal{P}(\mathcal{A})$ and $q \neq p$ since each $T_{X_p}^* X$ is an affine linear variety in $\mathbb{A}_k^{2n}$. Once a generic point (a, b) has been chosen we consider a n-dimensional linear affine variety Y through (a, b) and transversal to $T_{X_p}^* X$. The generic multiplicity we are looking for is just the dimension of the vector space $(\frac{R[\xi]}{J+\mathrm{gr}(I)})_{(a,b)}$ where $R[\xi] = R[\xi_1, \ldots, \xi_n] = \mathrm{gr}(A_n)$, J is the defining ideal of Y and $()_{(a,b)}$ stands for the localization at the maximal ideal of the point $(a, b) \in \mathbb{A}_k^{2n}$. $\square$

2.3. Some tips on the implementation

The algorithm we present uses Gröbner basis computations in both the commutative ring R and the non-commutative ring A_n so the complexity is very high. It does not prevent us from developing some relatively large examples that cannot be treated by other known algorithms. In the sequel we will point out the major bottlenecks we may find in the process as well as the Computer Algebra systems we have used to develop our examples.

The general version of Proposition 2.5, namely the computation of the characteristic cycle of local cohomology modules supported on arrangements of linear varieties, has been implemented by J. Pfeifle in the software package `polymake` [11]. The script interfaces with the computer algebra system `Singular` to construct the poset associated to the arrangement. Since we have to compute the homology groups of simplicial complexes it becomes very difficult to develop arrangements having a lot of components in a large-dimensional affine space.

Computing a set of generators of $I = \mathrm{Ann}_{A_n}^{(1)}(\frac{1}{f})$ can be done in almost any Computer Algebra system such as `Macaulay 2` [12] or `Singular` [13] since we have to use Gröbner basis over R. We have done our examples using `Macaulay 2`. However, we should point out that we can speed up the computations using in an iterative way on the components of our arrangement $f = f_1 \cdots f_r \in R$ the following straightforward property.

Lemma 2.8. *Let $f, g \in R$ be polynomials with no common factors. Then*

$$\mathrm{Der}\bigl(-\log(f \cdot g)\bigr) = \mathrm{Der}(-\log f) \cap \mathrm{Der}(-\log g).$$

To compute the characteristic variety of $\widetilde{M}^{\log f}$, in particular the computation of $\mathrm{gr}(I)$, we have to use the theory of Gröbner basis over A_n so it can be implemented in the software package `D-modules` [15] for `Macaulay 2` or `Kan/sm1` [22]. The components of the characteristic variety are described from the combinatorial data given by the poset associated to the arrangement. We recall that to construct the poset we must compute the sums of ideals in the minimal primary decomposition of the defining ideal of the arrangement so it can be done using Gröbner basis over R. For each irreducible component $T^*_{X_p}X$ of the characteristic variety $\mathrm{Ch}(\widetilde{M}^{\log f})$ we produce a generic point and a linear variety Y containing this point that is transversal to $T^*_{X_p}X$ using `Maple`. After a suitable change of variables that translates the generic point to the origin we compute the multiplicity of $\mathrm{gr}(I)$ at the origin using `Maple` or `Singular`.

3. A family of examples

We include in this section some examples that can be treated with our method and are not covered by Torrelli's result [25, Theorem 5.2.], i.e. they are not the union of a generic hyperplane arrangement with an hyperbolic arrangement. Recall that a (central) hyperplane arrangement defined by $f = f_1 \cdots f_r$ with $r \geqslant n$ is *generic* if any subarrangement defined by $f_{i_1} \cdots f_{i_s}$ has dimension $n - s$ for all $s \leqslant n$ and is *hyperbolic* if $f_i \in kf_1 + kf_2$ for $i \geqslant 3$. They live in dimension greater than 4 so we do not know whether LCT holds for these kind of examples but we are able to prove that the corresponding annihilator is generated by operators of order 1. We first describe with details the case of the arrangement $\mathcal{A} \subseteq \mathbb{A}^5_k$ defined by the polynomial in $R = k[x, y, z, t, u]$

$$f = xyztu(x + y + z + t + u)(x + y)(z + t).$$

The hyperbolic subarrangements of $\mathcal{A}$ are defined by $xy(x + y)$ and $zt(z + t)$. It is clear that respectively $ztu(x + y + z + t + u)(z + t)$ and $xyu(x + y + z + t + u)(x + y)$ are not generic, so this example is not covered by [25, Theorem 5.2].

Characteristic cycle of R_f

For the arrangement of hyperplanes $\mathcal{A}$ we are considering, the characteristic cycle of the localization module could be computed with the script developed by J. Pfeifle in 17.6 seconds in an Intel Pentium IV 3 GHz computer with 500 MB RAM. The components are the conormal bundles relative to the elements of the associated poset $P(\mathcal{A})$ that consists of 8 (respectively 24, 31, 15, 1) elements of height 1 (respectively 2, 3, 4, 5). We edited the output of the script where we get the multiplicities corresponding to each component as follows:

```
COMPONENTS AND MULTIPLICITIES

<x>: 1            <y>: 1            <z>: 1            <t>: 1            <u>: 1

<x+y+z+t+u>: 1    <x+y>:  1         <z+t>: 1


<y,x>: 2          <z,x>: 1          <t,x>: 1          <u,x>: 1          <y+z+t+u,x>: 1

<z+t,x>: 1        <z,y>: 1          <t,y>: 1          <u,y>: 1          <y,x+z+t+u>: 1

<z+t,y>: 1        <t,z>: 2          <u,z>: 1          <z,x+y+t+u>: 1    <z,x+y>: 1
```

```
<u,t>: 1        <t,x+y+z+u>: 1   <t,x+y>: 1       <u,x+y+z+t>: 1   <u,x+y>: 1
<u,z+t>: 1      <z+t+u,x+y>: 1   <z+t,x+y+u>: 1   <z+t,x+y>: 1

<z,y,x>: 2      <t,y,x>: 2       <u,y,x>: 2       <z+t+u,y,x>: 2   <z+t,y,x>: 2
<t,z,x>: 2      <u,z,x>: 1       <z,y+t+u,x>: 1   <u,t,x>: 1       <t,y+z+u,x>: 1
<u,y+z+t,x>: 1  <u,z+t,x>: 1     <z+t,y+u,x>: 1   <t,z,y>: 2       <u,z,y>: 1
<z,y,x+t+u>: 1  <u,t,y>: 1       <t,y,x+z+u>: 1   <u,y,x+z+t>: 1   <u,z+t,y>: 1
<z+t,y,x+u>: 1  <u,t,z>: 2       <t,z,x+y+u>: 2   <t,z,x+y>: 2     <u,z,x+y+t>: 1
<u,z,x+y>: 1    <t+u,z,x+y>: 1   <u,t,x+y+z>: 1   <u,t,x+y>: 2     <t,z+u,x+y>: 1
<u,z+t,x+y>: 3

<t,z,y,x>: 4    <u,z,y,x>: 2     <t+u,z,y,x>: 2   <u,t,y,x>: 2     <t,z+u,y,x>: 2
<u,z+t,y,x>: 6  <u,t,z,x>: 2     <t,z,y+u,x>: 2   <u,z,y+t,x>: 1   <u,t,y+z,x>: 1
<u,t,z,y>: 2    <t,z,y,x+u>: 2   <u,z,y,x+t>: 1   <u,t,y,x+z>: 1   <u,t,z,x+y>: 6

<u,t,z,y,x>: 12
```

Characteristic cycle of $\widetilde{M}^{\log f}$

First we compute the ideal $\mathrm{Ann}^{(1)}_{A_5}(\frac{1}{f})$ using the method in Section 2. We use the following script for `Macaulay 2`

```
i1 : load "D-modules.m2";

i2 : R=QQ[x,y,z,t,u]; W=makeWA R;

i3 : f=x*y*z*t*u*(x+y+z+t+u)*(x+y)*(z+t);

i5 : kernel matrix{{f,diff(x,f),diff(y,f),diff(z,f),diff(t,f),
     diff(u,f)}};

i6 : matrix{{-1,dx,dy,dz,dt,du}}*gens o5;

i7 : Ann1=ideal o6;
```

The characteristic ideal of $\widetilde{M}^{\log f}$ is:

```
i8 : charideal:= charIdeal Ann1

o8 = ideal(x*dx+y*dy+z*dz+t*dt+u*du,
     x*u*du+y*u*du+z*u*du+t*u*du+u^2*du,
     z*u*dz+t*u*dt-z*u*du-t*u*du,z*t*dz-z*t*dt,
     x*z*dz+y*z*dz+z^2*dz+x*t*dt+y*t*dt+2*z*t*dt+t^2*dt+z*u*du+t*u*du,
     x*y*dy+y^2*dy+y*z*dz+y*t*dt+y*u*du)
```

For each component of the characteristic variety we have to compute the multiplicity at a generic point. Here we present a sample in a `Maple` session with the computation of the multiplicity at a generic point of the conormal space to the origin $T_0\mathbb{A}_k^5$. For the other components we proceed in an analogous way.

```
> with(Groebner):

> multg:=proc(a,b,c,d,e) local section,hil,mult;

section:=subs(dx=a,dy=b,dz=c,dt=d,du=e,charideal);

section:=`minus`(section,{0});

hil:=hilbertdim(section,tdeg(x,y,z,t,u));

if hil=0  then mult:=subs(s=1,hilbertseries(section,tdeg(x,y,z,t,u),s))

else error "the point is not generic" end if end;
```

Namely, for a point $(0,0,0,0,0,a,b,c,d,e) \in T_0\mathbb{A}_k^5$ we check out if the point is generic and if it is the case we compute the corresponding multiplicity.

```
> multg(1,2,3,4,5);
                                12
> multg(1,1,1,1,1);
Error, (in multg) the point is not generic

> multg(3,5,4,7,11);
                                12
```

Computing the multiplicities of each component we check the coincidence between the characteristic cycles of $\widetilde{M}^{\log f}$ and R_f so we conclude that $\mathrm{Ann}_{A_n}(\frac{1}{f})$ is generated by operators of order 1 by Theorem 2.7. We should point out the following short cut to the method for this particular example.

Claim. *In order to compare the characteristic cycles of $\widetilde{M}^{\log f}$ and R_f for the polynomial $f = xyztu(x+y+z+t+u)(x+y)(z+t)$ it is enough to compare the multiplicities of both A_5-modules at a generic point of the conormal space to the origin $T_0\mathbb{A}_k^5$. In fact $\widetilde{M}^{\log f}$ and R_f are isomorphic in the neighborhood of any point $p \in \mathbb{C}^5$, $p \neq 0$.*

Proof. The divisor $\mathcal{A}$ defined by f is free at any $p \in \mathcal{A}$ in the complement of the plane $z = x+y = t+u = 0$ in $\mathbb{C}^5$. To this end, we prove first that $\mathrm{Der}(-\log f)$ is generated by the family of vector fields

- $\delta_1 = x\partial_x + y\partial_y + z\partial_z + t\partial_t + u\partial_u$,
- $\delta_2 = -tu\partial_t + tu\partial_u$,
- $\delta_3 = -z(t+u)\partial_z + zt\partial_t + zu\partial_u$,
- $\delta_4 = z(t+u)\partial_z + t(x+y+t+2u)\partial_t + u(x+y+u)\partial_u$,

- $\delta_5 = z(x + y + z + t + u)\partial_z$,
- $\delta_6 = y(x + y)\partial_y + yz\partial_z + yt\partial_t + yu\partial_u$.

Let us denote by M the 6×5-matrix of the coefficients of the δ_i with respect to the partial derivatives of the variables. We denote by $M(j)$ the matrix M with its j-row removed. It is easy to prove that $\det(M(3)) = zf$, $\det(M(4)) = (t + u)f$ and $\det(M(2)) = (x + y + t + u)f$. Applying K. Saito's criterion (see [21]) we deduce that the divisor $\mathcal{A}$ is free at any point $p \in \mathbb{C}$ in the complement of the plane L defined by $z = x + y = t + u = 0$. By [9, Theorem 5.2.1] $\widetilde{M}^{\log f}$ and R_f are isomorphic in the neighborhood of these points p.

Let us now consider a point $p \in L$. Assume $p = (a, -a, 0, b, -b) \neq 0$.

If $ab \neq 0$ then the germ $(\mathcal{A}, p)$ is isomorphic to $(D, 0)$ where D is the divisor in $\mathbb{C}^5$ defined by the polynomial $z(x + y + z + t + u)(x + y)(t + u)$ and this last divisor is isomorphic to $(D' \times \mathbb{C}^2, 0)$ where D' is the divisor in $\mathbb{C}^3$ defined by $f' = x'y'z'(x' + y' + z')$. It is easy to prove—using for example `Macaulay 2`—that $\mathrm{Ann}_{A_g}(\frac{1}{f'})$ is generated by operators of order 1 so the same condition is satisfied by $(\mathcal{A}, p)$.

If $ab = 0$ assume $a \neq 0$ and $b = 0$ (the case $a = 0$ and then necessarily $b \neq 0$ is analogous). Then $(\mathcal{A}, p)$ is isomorphic to the divisor $(E, 0)$ defined by the polynomial $ztu(x + y + z + t + u)(x + y)(t + u)$. We can rewrite this polynomial as $g = xztu(x + z + t + u)(t + u)$ for suitable coordinates in $\mathbb{C}^5$ and view this divisor as a divisor in $\mathbb{C}^4$. In a similar way as we did above, one may prove that the divisor E defined by g is free at any point $q \in \mathbb{C}^4$ in the complement of the line $x = z = t + u = 0$ in $\mathbb{C}^4$. By [9, Theorem 5.2.1] the claim follows for these points.

If $q = (0, 0, c, -c)$ and $c \neq 0$ then (E, q) is isomorphic to the divisor $(E', 0)$ defined in $(\mathbb{C}^4, 0)$ by the polynomial $g' = x'z't'(x' + z' + t')$ and then $\mathrm{Ann}_{A_4}(\frac{1}{g'})$ is generated by operators of order 1 so the same condition is satisfied by $(\mathcal{A}, p)$ and the claim follows for the points q.

To finish the proof of the claim we have to treat the case of the origin for the divisor E defined by g (i.e. we have to treat the germ $(E, 0)$). The computation of the ideal $\mathrm{Ann}_{A_4[s]}(g^s)$ can be done using for example `Macaulay 2` [15]. The least integer root of g is -1 (see [26]) so we can replace s by -1 on the generators of $\mathrm{Ann}_{A_4[s]}(g^s)$ in order to get a set of generators of $\mathrm{Ann}_{A_4}(\frac{1}{g})$. It follows that $\mathrm{Ann}_{A_4}(\frac{1}{g})$ is generated by operators of order 1. □

To end this section we want to remark that, when trying to test the capabilities of our method, we have been also able to treat (among others) the following examples:

In $\mathbb{A}_k^6$:

$$f = xyztuv(x + y)(x + y + z + t + u + v),$$
$$f = xyztuv(x + y)(z + t + u)(x + y + z + t + u + v).$$

In $\mathbb{A}_k^7$:

$$f = xyztuvw(x + y)(x + y + z + t + u + v + w),$$
$$f = xyztuvw(x + y)(z + t + u + v)(x + y + z + t + u + v).$$

In $\mathbb{A}_k^8$:

$$f = xyztuvws(x + y + z + t + u + v + w + s)(x + y)(z + t + u),$$
$$f = xyztuvws(x + y + z + t + u + v + w + s)(x + y + z)(t + u + v + w).$$

Remark 3.1. We have tried the computer algebra system `Risa/Asir` [16] to compare our algorithm with the direct computation of the annihilating ideal of $\frac{1}{f}$, through the computation of the annihilating ideal of f^s over the ring $A_n[s]$. In a PC Pentium IV, 1 Gb RAM and 3.06 GHz running under Windows XP the example $xyzuvw(x+y+z+u+v+w)$ collapsed the memory.

4. Orlik–Terao's hyperplane arrangement

Our test example throughout this section is going to be the hyperplane arrangement $\mathcal{A} \subseteq \mathbb{A}_k^4$ considered by P. Orlik and H. Terao in [19] defined by the polynomial in $R = k[x, y, z, t]$

$$f = xyzt(x+y)(x+z)(x+t)(y+z)(y+t)(z+t)(x+y+z)(x+y+t)(x+z+t) \times (y+z+t)(x+y+z+t).$$

This is the smallest known example of a non-tame arrangement however it satisfies Logarithmic Comparison Theorem since this is a central arrangement in dimension 4 [27]. We want to support Torrelli's conjecture proving that $\mathrm{Ann}_{A_n}(\frac{1}{f})$ is generated by operators of order 1. Notice that this example is not a generic arrangement or the union of a generic and an hyperbolic arrangement so they are not covered by the results of [25].

Characteristic cycle of R_f

For the arrangement of hyperplanes $\mathcal{A}$ we are considering, the characteristic cycle of the localization module could be computed in 2 minutes and 32 seconds. The components are the conormal bundles relative to the elements of the associated poset $P(\mathcal{A})$ that consists of 15 (respectively 55, 45, 1) elements of height 1 (respectively 2, 3, 4). We edited the output of the script where we get the multiplicities corresponding to each component as follows:

```
COMPONENTS AND MULTIPLICITIES

<x>:1           <y>:1           <z>:1           <t>:1           <x+y>:1
<x+z>:1         <x+t>:1         <y+z>:1         <y+t>:1         <z+t>:1
<x+y+z>:1       <x+y+t>:1       <x+z+t>:1       <y+z+t>:1       <x+y+z+t>:1


<y,x>:2         <z,x>:2         <t,x>:2         <y+z,x>:2       <y+t,x>:2
<z+t,x>:2       <y+z+t,x>:2     <z,y>:2         <t,y>:2         <y,x+z>:2
<y,x+t>:2       <z+t,y>:2       <y,x+z+t>:2     <t,z>:2         <z,x+y>:2
<z,x+t>:2       <z,y+t>:2       <z,x+y+t>:2     <t,x+y>:2       <t,x+z>:2
<t,y+z>:2       <t,x+y+z>:2     <y-z,x+z>:1     <y-t,x+t>:1     <y+z,x-z>:1
<y+t,x-t>:1     <z+t,x+y>:2     <y-z-t,x+z+t>:1 <y+z+t,x-z-t>:1 <z-t,x+t>:1
<y+z,x+z>:1     <y+t,x+z>:2     <z+t,x-t>:1     <y-z+t,x+z>:1   <y+z+t,x+z>:1
<y+z,x+t>:2     <y+t,x+t>:1     <z+t,x+t>:1     <y+z-t,x+t>:1   <y+z+t,x+t>:1
<z-t,y+t>:1     <z+t,y-t>:1     <y+z,x-z+t>:1   <y+z,x+z+t>:1   <z+t,y+t>:1
```

```
<y+t,x+z-t>:1      <y+t,x+z+t>:1      <z+t,x+y-t>:1      <z+t,x+y+t>:1      <z-t,x+y+t>:1
<y-t,x+z+t>:1      <y+z+t,x-t>:1      <y-z,x+z+t>:1      <y+z+t,x-z>:1      <y+z+t,x+z+t>:1


<z,y,x>:9          <t,y,x>:9          <z+t,y,x>:9        <t,z,x>:9          <z,y+t,x>:9
<t,y+z,x>:9        <z-t,y+t,x>:4      <z+t,y-t,x>:4      <z+t,y+t,x>:4      <t,z,y>:9
<z,y,x+t>:9        <t,y,x+z>:9        <z-t,y,x+t>:4      <z+t,y,x-t>:4      <z+t,y,x+t>:4
<t,z,x+y>:9        <z,y-t,x+t>:4      <z,y+t,x-t>:4      <z,y+t,x+t>:4      <t,y-z,x+z>:4
<t,y+z,x-z>:4      <t,y+z,x+z>:4      <z-t,y-t,x+t>:1    <z+t,y+t,x-t>:4    <2z+t,2y+t,2x-t>:1
<z+t,y-t,x+t>:4    <z+2t,y-t,x+t>:1   <z-t,y+t,x-t>:1    <2z+t,2y-t,2x+t>:1 <z+2t,y+t,x-t>:1
<z-t,y+t,x+t>:4    <z-t,y+2t,x+t>:1   <z+t,y-t,x-t>:1    <2z-t,2y+t,2x+t>:1 <z+t,y+2t,x-t>:1
<z+t,y+t,x+t>:1    <z-2t,y+t,x+t>:1   <z+t,y-2t,x+t>:1   <z-t,y+t,x+2t>:1   <z+t,y-t,x+2t>:1
<z+t,y+t,x-2t>:1   <z-t,y-t,x+2t>:1   <z-t,y+2t,x-t>:1   <z+2t,y-t,x-t>:1   <2z+t,2y+t,2x+t>:1


<t,z,y,x>:104
```

Characteristic cycle of $\widetilde{M}^{\log f}$

First we compute the ideal $\mathrm{Ann}^{(1)}_{A_4}(\frac{1}{f})$ using the same script for `Macaulay 2` we used in the previous example.

The output of the script where we get the characteristic ideal of $\mathrm{Ann}^{(1)}(\frac{1}{f})$ is too big to be edited here (the output file has a size of 7 KB). As in the previous example, a tedious but straightforward application of our method allow us to compute the multiplicity of each component of the characteristic variety at a generic point. However it is worthwhile to point out the following short cut for this particular example.

Claim. *In order to compare the characteristic cycles of $\widetilde{M}^{\log f}$ and R_f for the Orlik–Terao's polynomial it is enough to compare the multiplicities of both A_4-modules at a generic point of the conormal space to the origin $T_0\mathbb{A}^4_k$.*

Proof. From the computation of $\mathrm{Der}(-\log f)$ we deduce that the divisor $\{f=0\}$ is free at any point different from the origin since there is a set of four logarithmic vector fields $\{\delta_1,\dots,\delta_4\}$ of the form $\delta_i=\sum_j a_{ij}\partial_j$ such that the determinant of the matrix (a_{ij}) equals $(z+t)f$. By symmetry there is a set of four logarithmic vector fields such that the corresponding determinant equals xf. The same is also true for yf, zf and tf. By Saito's criterion [21] the divisor $\{f=0\}$ is free outside the origin.

By [9] the equality $\mathrm{Ann}^{(1)}(\frac{1}{h})=\mathrm{Ann}(\frac{1}{h})$ holds for any free arrangement $h=0$ in $\mathbb{C}^n$. So, in our case, the surjective morphism

$$\widetilde{M}^{\log f}\to R_f\to 0$$

is an isomorphism when restricted to the neighborhood of any point $a\in\mathbb{C}^4$ different from the origin.

Notice that we have been using the field of complex numbers $\mathbb{C}$ to apply the results in [21] and [9]. Since Orlik–Terao's arrangement is in fact defined over the field of rational numbers $\mathbb{Q}$ we can extend this result to any field of characteristic zero by flat base change. $\square$

We applied the method explained in the previous example. The generic multiplicity of $T_0\mathbb{C}^4$ in $\mathrm{Ch}(\widetilde{M}^{\log f})$ is 104. The characteristic cycles of $\widetilde{M}^{\log f}$ and R_f coincide so we conclude that $\mathrm{Ann}_{A_n}(\frac{1}{f})$ is generated by operators of order 1 by Theorem 2.7.

5. Extending to larger examples

Let $f \in R = k[x_1, \dots, x_n]$ be the defining polynomial of an arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}^n_k$. Assume that $\mathrm{Ann}_{A_n}(\frac{1}{f})$ is generated by operators of order 1. Our aim in this section is to prove that the same property holds for the arrangement of hyperplanes $\mathcal{A}' \subseteq \mathbb{A}^{n+1}_k$ defined by the polynomial $f \cdot t \in R' := k[x_1, \dots, x_n, t]$.

Let A_{n+1} and A_n be the rings of linear differential operators with coefficients in R' and R, respectively. We will also denote the corresponding affine spaces $X' = \mathbb{A}^{n+1}_k$ and $X = \mathbb{A}^n_k$.

Let M be a finitely generated A_n-module. Its direct image corresponding to the injection $i : \mathbb{A}^n_k \to \mathbb{A}^{n+1}_k$ is the A_{n+1}-module $i_+(M)$ defined as

$$i_+(M) = k[\partial_t] \mathbin{\hat{\otimes}_k} M = M[\partial_t].$$

The characteristic variety of $i_+(M)$ can be computed from the characteristic variety of M. Namely, we have:

$$\mathrm{Ch}\big(i_+(M)\big) = \big\{(\mathbf{x}, 0, \xi, \tau) \in T^*X' \;\big|\; (\mathbf{x}, \xi) \in \mathrm{Ch}(M)\big\},$$

where we have considered $\mathrm{Ch}(M) \subseteq T^*X$.

Characteristic cycle of $R'_{f \cdot t}$

The characteristic cycle of the A_{n+1}-module $R'_{f \cdot t}$ can be deduced from the characteristic cycle of A_n-module R_f as follows

Proposition 5.1. *Assume that* $\mathrm{Ch}\, C(R_f) = \sum_i m_i \; T^*_{X_i} X$. *Then we have*

$$\mathrm{Ch}\, C\big(R'_{f \cdot t}\big) = \sum_i m_i \big(T^*_{X_i} X' + T^*_{X'_i} X'\big)$$

where $X'_i = X_i \times \mathbb{C} \subset X' = X \times \mathbb{C}$ *and we identify* X_i *with* $X_i \times \{0\}$.

Proof. Localizations have a good behavior with respect to flat base change so

$$i_+(R_f) = \big(A_t / A_t \cdot (t)\big) \mathbin{\hat{\otimes}_k} R_f = H^1_{(t)}\big(k[t]\big) \mathbin{\hat{\otimes}_k} R_f = H^1_{(t)}\big(R_f \mathbin{\hat{\otimes}_k} k[t]\big) = H^1_{(t)}\big(R'_f\big)$$

where A_t is the ring of linear differential operators with coefficient in the polynomial ring $k[t]$.

Using the additivity of the characteristic cycle with respect to the short exact sequence

$$0 \to R'_f \to R'_{f\cdot t} \to H^1_{(t)}\big(R'_f\big) \to 0$$

we get the desired result

$$\operatorname{Ch} C\big(R'_{f\cdot t}\big) = \operatorname{Ch} C\big(R'_f\big) + \operatorname{Ch} C\big(i_+(M)\big) = \sum_i m_i T^*_{X_i} X' + \sum_i m_i T^*_{X'_i} X',$$

where we are considering X_i as a subvariety of X' and $X'_i = X_i \cap \{t=0\}$. $\square$

Characteristic cycle of $\widetilde{M}'^{\log f\cdot t}$

The characteristic cycle of the A_{n+1}-module

$$\widetilde{M}'^{\log f\cdot t} := \frac{A_{n+1}}{\operatorname{Ann}^{(1)}_{A_{n+1}}(\frac{1}{f\cdot t})}$$

can be deduced from the characteristic cycle of A_n-module $\widetilde{M}^{\log f} := \frac{A_n}{\operatorname{Ann}^{(1)}_{A_n}(\frac{1}{f})}$ as follows

Proposition 5.2. *Assume that* $\operatorname{Ch} C(\widetilde{M}^{\log f}) = \sum_i m_i T^*_{X_i} X$. *Then we have*

$$\operatorname{Ch} C\big(\widetilde{M}'^{\log f\cdot t}\big) = \sum_i m_i \big(T^*_{X_i} X' + T^*_{X'_i} X'\big)$$

where we are considering X_i *as a subvariety of* X' *and* $X'_i = X_i \cap \{t=0\}$.

Proof. From the equality $\operatorname{Der}(-\log(f\cdot t)) = \operatorname{Der}(-\log f) \cap \operatorname{Der}(-\log t)$ we deduce the equality

$$\operatorname{Ann}^{(1)}_{A_{n+1}}\left(\frac{1}{f\cdot t}\right) = A_{n+1}\cdot \operatorname{Ann}^{(1)}_{A_n}\left(\frac{1}{f}\right) + A_{n+1}(t\partial_t + 1).$$

Since the variables t and ∂_t commute with x_i and ∂_i for $i=1,\dots,n$ we also have

$$\operatorname{gr}\left(\operatorname{Ann}^{(1)}_{A_{n+1}}\left(\frac{1}{f\cdot t}\right)\right) = \operatorname{gr}(A_{n+1})\cdot \operatorname{gr}\left(\operatorname{Ann}^{(1)}_{A_n}\left(\frac{1}{f}\right)\right) + \operatorname{gr}(A_{n+1})(t\tau).$$

Then, from the above equality on graded ideals we get the desired result. $\square$

From the results above we get

Theorem 5.3. *Let* $f \in k[x_1,\dots,x_n]$ *be the defining polynomial of an arrangement of hyperplanes* $\mathcal{A} \subseteq \mathbb{A}^n_k$ *such that* $\operatorname{Ann}_{A_n}(\frac{1}{f})$ *is generated by operators of order* 1. *Then* $\operatorname{Ann}_{A_{n+m}}(\frac{1}{f\cdot t_1\cdots t_m})$ *is also generated by operators of order* 1 *for all* $m\in\mathbb{N}$, *where* $f\cdot t_1\cdots t_m \in k[x_1,\dots,x_n, t_1,\dots,t_m]$.

Remark 5.4. Extending Orlik–Terao's example.

Any arrangement of hyperplanes $\mathcal{A} \subseteq \mathbb{A}_k^n$ that contains Orlik–Terao's example as subarrangement is non-tame by [27, p. 1655(T2)] so it is easy to find examples of non-tame arrangements in any dimension just adding new factors to the defining polynomial of Orlik–Terao's example. Even though it is not known whether LCT holds for this kind of examples in dimension greater than 4 (greater than 5 over $\mathbb{C}$) we are able to prove that the corresponding annihilator is generated by operators of order 1. By Theorem 5.3 we may extend the example given by Orlik–Terao's arrangement to higher dimension just adding new variables.

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