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Motivic Serre invariants and Weil restriction

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Abstract

We study the interactions between Weil restriction for formal schemes and rigid varieties, Greenberg schemes, and motivic Serre invariants, and their behavior with respect to finite extensions of the base ring R, which is a complete discrete valuation ring with perfect residue field.

Keywords: Formal geometry; Rigid geometry; Motivic integration; Weil restriction

1. Introduction

Let R be a complete discrete valuation ring, with quotient field K, and perfect residue field k.

We will study the interactions between Weil restriction for formal schemes and rigid varieties, Greenberg schemes, and motivic Serre invariants, and their behavior with respect to finite extensions of the base ring R. The basic definitions and properties are recalled in Section 2: in Section 2.1, we gather the basic results on Weil restriction of formal schemes and rigid varieties, and we establish some new elementary properties for later use. Section 2.2 contains the definition of the Greenberg schemes, and lists their main properties. Because of the confusion that seems to exist in literature, we spend some time on the study of the ring schemes $\mathscr{R}_n$ when R is absolutely ramified. In Section 2.3, we briefly recall the definition of the motivic Serre invariant.

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Next, in Section 3, we study the behavior of Greenberg schemes under ramification of the ring R. If X_∞ is a *stft* (separated, topologically of finite type) formal scheme over R, its Greenberg scheme $\mathrm{Gr}^R(X_\infty)$ is a k-scheme, generally of infinite type, that parametrizes the unramified sections of X_∞. When R' is a finite extension of R, with residue field k', we construct a canonical closed immersion $\iota_R^{R'}$ of $\mathrm{Gr}^R(X_\infty)\times_k k'$ into $\mathrm{Gr}^{R'}(X_\infty\times_R R')$ (Theorem 3.8). When R' is totally ramified over R, the Greenberg scheme $\mathrm{Gr}^{R'}(X_\infty\times_R R')$ carries a natural action of the Galois group $G(K'/K)$, where K' is the quotient field of R'. If K'/K is Galois and tame, $\iota_R^{R'}$ is an isomorphism from $\mathrm{Gr}^R(X_\infty)$ onto the fixed locus of $\mathrm{Gr}^{R'}(X_\infty\times_R R')$ w.r.t. the action of $G(K'/K)$ (Theorem 3.12).

Under the assumption that k is algebraically closed, we define the k-Ind-scheme of sections, and the k-Ind-scheme of tame sections, as direct limits of the inductive systems of k-schemes $\mathrm{Gr}^{R'}(X_\infty\times_R R')$, where R' runs over the finite (respectively finite and tame) extensions of R inside a fixed separable closure of K, and where the transition maps are the morphisms $\iota_{R'}^{R''}$, for $R'\subset R''$. The k-points on these Ind-schemes are in canonical bijective correspondence with $X_\eta(K^s)$, respectively $X_\eta(K^t)$, where K^s and K^t denote the separable, respectively tame closure of K, and where X_η is the generic fiber of X_∞.

Section 4 investigates the relation between Greenberg schemes and Weil restriction. If R'/R is totally ramified of degree n, and Y_∞ is a *stft* formal R'-scheme such that the Weil restriction of Y_∞ to R is representable by a *stft* formal R-scheme, we construct a canonical isomorphism between the special fiber of this Weil restriction, and the Greenberg scheme $\mathrm{Gr}_{n-1}^{R'}(Y_\infty)$ of length $n-1$ (Theorem 4.1).

Finally, in Section 5, we come to the main results of the paper: we show that the motivic Serre invariant (of a formal R'-scheme, respectively a rigid K'-variety) is stable under Weil restriction with respect to any finite totally ramified extension R'/R (Section 5.3).

In analogy with the p-adic case, the motivic Serre invariant of a smooth separated quasi-compact rigid variety over K was introduced in [16]. This notion was refined to define the motivic Serre invariant of a generically smooth *stft* formal R-scheme X_∞ in [19], assuming that the generic fiber X_η has pure dimension. Here we generalize the definition of the motivic Serre invariant of X_∞ to the non-equidimensional case (Definition 5.2).

Roughly speaking (see [18] for more details about this point of view), the motivic Serre invariant of a smooth rigid variety serves as a measure for the "number" of unramified points on the variety (which is infinite, in general). In this paper, we justify this point of view, by showing that the motivic Serre invariant is stable under Weil restriction w.r.t. a finite totally ramified extension (Theorem 5.16).

The idea behind the construction of the motivic Serre invariant is the following: let X_∞ be a *stft* formal R-scheme. Suppose that the generic fiber X_η of X_∞ is smooth over K. By a process of *Néron smoothening*, we can dominate X_∞ by another model Y_∞ of X_η, with the following property: if K' is any finite unramified extension of K, then all K'-points on X_η are contained in the generic fiber of the R-smooth locus $\mathrm{Sm}(Y_\infty)$ of Y_∞. Since $\mathrm{Sm}(Y_\infty)$ is smooth, its special fiber is a good measure for the number of unramified points on its generic fiber. The class $S(X_\infty)$ of this special fiber in an appropriate Grothendieck ring only depends on X_∞, and not on Y_∞ (Theorem 5.4). It is called the *motivic Serre invariant* of X_∞.

A certain specialization of $S(X_\infty)$ (forgetting the X_s-structure) only depends on X_η; this is the motivic Serre invariant $S(X_\eta)$ of the separated smooth quasi-compact rigid K-variety X_η. We will show that this construction makes sense for a broader class of rigid varieties: the *bounded* separated smooth rigid varieties over K (Section 5.2), whose unramified points are concentrated in a quasi-compact open subspace. This generalization is necessary because the category of smooth quasi-compact rigid varieties is not closed under Weil restriction; however, the Weil restriction of such a rigid variety w.r.t. a finite extension of the base field will be bounded and smooth.

Although all results in this paper are thematically intertwined, Section 3 is logically independent of the subsequent sections. The results of Section 4 are used in the proof of Theorem 5.16.

Notation

Throughout this note, R denotes a complete discrete valuation ring, with residue field k, and quotient field K. We suppose that k is perfect. We fix a generator π of the maximal ideal $\mathfrak{M}$ of R. For any integer $n \geqslant 0$, we denote by R_n the quotient $R/(\pi^{n+1})$. We denote by K^{sh} the strict henselization of K, and by R^{sh} the normalization of R in K^{sh}.

We say that a formal scheme over $\mathrm{Spf}\, R$ is *tft* if it is topologically of finite type over $\mathrm{Spf}\, R$, and *stft* if it is *tft* and separated over $\mathrm{Spf}\, R$. For any *stft* formal R-scheme X_∞, we denote by X_s its special fiber (which is a separated scheme of finite type over k), and by X_η its generic fiber (which is a separated quasi-compact rigid variety over the non-Archimedean field K). We say X_∞ is generically smooth if X_η is smooth over K. We denote by $\mathrm{Sm}(X_\infty)$ the smooth part of X_∞ over R. For any integer $n \geqslant 0$, we put $X_n := X_\infty \times_R R_n$; this is a separated R_n-scheme of finite type. In particular, $X_0 = X_s$.

If S is a scheme, a S-*variety* is a reduced separated S-scheme of finite type. We denote by S_{red} the underlying reduced scheme of S. We denote by (Sch/S) the category of schemes over S. One can view any object of (Sch/S) as a presheaf on (Sch/S): by Yoneda's lemma, the functor from (Sch/S) to the category of presheaves on (Sch/S), mapping a S-scheme X to the presheaf $Y \mapsto \mathrm{Hom}_S(Y, X)$, is a fully faithful embedding.

For any separated scheme X of finite type over a field F, we denote by $K_0(Var_X)$ the Grothendieck ring of varieties over X_{red} (see, for instance, [9, 2.3] for a definition). For any separated scheme Y of finite type over X, we denote by $[Y]$ the class of Y_{red} in $K_0(Var_X)$. We write $\mathbb{L}_X$ for the class of the affine line $[\mathbb{A}^1_X]$. Finally, when $X = \mathrm{Spec}\, k$, we will write $K_0(Var_k)$ and $\mathbb{L}$ instead of $K_0(Var_{\mathrm{Spec}\, k})$ and $\mathbb{L}_{\mathrm{Spec}\, k}$.

We denote by (For/R) the category of separated formal schemes, topologically of finite type over $\mathrm{Spf}\, R$, and by (Rig/K) the category of separated rigid varieties over K. If X is a separated K-scheme of finite type, we denote the associated rigid K-variety by X^{an}.

2. Preliminaries on Weil restriction, Greenberg transforms and motivic Serre invariants

In this section, we recall the definitions of these notions, and we establish some basic properties, which we need in the following sections.

2.1. Weil restriction

Let $h : S' \to S$ be a morphism of schemes, and let X be a presheaf on (Sch/S'). In [13, p. 13], the Weil restriction of X to S is defined as the presheaf

$$\prod_{S'/S} X : (Sch/S) \to (Sets) : T \mapsto X(T \times_S S').$$

This construction defines the Weil restriction functor $\prod_{S'/S}$ from the category of presheaves on (Sch/S') to the category of presheaves on (Sch/S).

If h is proper and flat, and X is a quasi-projective variety over S', the presheaf $\prod_{S'/S} X$ is representable, by [14, p. 20].

Now let X be any scheme over S', and suppose that for any point y on S and any finite set P of points on the fiber X_y of X over y, the set P is contained in an affine open subscheme of X. If h is finite and locally free, the Weil restriction of X to S is still representable (see [4, 7.6.4]). The following lemma is well known. A proof can be found in [17, 3.3.36(b)]. The second author would like to thank Qing Liu for this reference.

Lemma 2.1. *Let X be a quasi-projective variety over k. Any finite set P of points on X is contained in some affine open subscheme of X.*

One can also define Weil restriction for the categories (For/R) and (Rig/K), as was done in [1] and [2]. We establish some elementary properties of the restriction functor, which will be of use later on.

Definition 2.2. We will say a formal *stft* scheme X_∞ over $\mathrm{Spf}\, R$ is nice, if any finite set of points on X_∞ is contained in an affine open formal subscheme of X_∞. We say that a separated quasi-compact rigid space X_η over K is nice, if it admits a nice model X_∞ over R.

In particular, the formal completion of a quasi-projective variety X/R is nice, by Lemma 2.1.

If X_∞ is a *stft* formal R-scheme, an *admissible blow-up* $Y_\infty \to X_\infty$ is the formal blow-up of an ideal sheaf $\mathcal{I}$ on X_∞ which contains a power of π.

Lemma 2.3. *If X_∞ is a nice stft formal R-scheme, and $Y_\infty \to X_\infty$ is any admissible blow-up, then Y_∞ is a nice stft formal R-scheme.*

Proof. We may assume that X_∞ is affine, i.e. $X_\infty = \mathrm{Spf}\, A$. Let $I = (f_1, \ldots, f_m, \pi^a)$ be an open ideal of A, and let Y_∞ be the admissible formal blow-up of X_∞ with center I. The special fiber of Y_∞ is isomorphic to the special fiber of the blow-up Y' of $\mathrm{Spec}\, A$ with center I. The blow-up scheme Y' is a closed subscheme of $\mathbb{P}^m_A$, so it suffices to prove that any finite set of points on the special fiber of $\mathbb{P}^m_A$ is contained in an affine open subscheme. This follows from Lemma 2.1. $\square$

The following proposition follows immediately from the results in [2]. Our definition of a nice formal scheme is there called condition (P_{For}) on p. 441, while our definition of a nice rigid variety is stronger than condition (P_{rig}) on p. 452, by the remark on p. 453.

Proposition 2.4. *Let K' be a finite extension of K, and let R' be the normalization of R in K'. Let X_∞ be a nice stft formal scheme over* $\mathrm{Spf}\, R'$.

(1) *The direct limit of locally ringed spaces*

$$\varinjlim_{n} \prod_{(R'/\pi^{n+1})/R_n} X_\infty \times_{R'} \left(R'/\pi^{n+1}\right)$$

is a stft formal R-scheme, and represents the functor

$$\prod_{R'/R} X_\infty : (For/R)^{op} \to (Sets) : Y_\infty \mapsto \operatorname{Hom}(Y_\infty \times_R R', X_\infty).$$

We call it the Weil restriction of X_∞ to $\operatorname{Spf} R$.

(2) *The functor*

$$\prod_{K'/K} X_\eta : (Rig/K)^{op} \to (Sets) : Y_\eta \mapsto \operatorname{Hom}(Y_\eta \times_K K', X_\eta)$$

is representable by a separated rigid space over K. We call it the Weil restriction of X_η to K.

(3) *There is a canonical open immersion*

$$\Big(\prod_{R'/R} X_\infty\Big)_\eta \hookrightarrow \prod_{K'/K} X_\eta.$$

Proof. Since R'/R is finite and free, these properties follow from [2, 1.3-4-5-9-16-22]. □

Remark. In general, taking generic fibers does *not* commute with Weil restriction, i.e. the open embedding

$$\Big(\prod_{R'/R} X_\infty\Big)_\eta \hookrightarrow \prod_{K'/K} X_\eta$$

is not an isomorphism. A counter-example is given in [2, p. 442]. Let us give a more elementary one: take $R = k[[t]]$, where k is any field of characteristic $\neq 2$, take $R' = k[[\sqrt{t}\,]]$, and consider $X_\infty := \operatorname{Spf} R'\{x\}/(x^2 - t)$. Let $Y_\infty \to X_\infty$ be the admissible blow-up of the ideal $(x, \sqrt{t}\,)$. Of course, this blow-up induces an isomorphism on the generic fibers. However, direct computation shows that $(\prod_{R'/R} Y_\infty)_\eta$ consists of two K-points, while $(\prod_{R'/R} X_\infty)_\eta$ consists of two K-points and a K'-point.

Now we prove some basic properties of the Weil restriction functor which are crucial for the applications in this article.

Proposition 2.5. *Let K' be a finite extension of K, and let R' be the normalization of R in K'. Let X_∞ be a nice stft formal scheme over* $\operatorname{Spf} R'$.

(1) *The functors $\prod_{R'/R}$ and $\prod_{K'/K}$ respect (open, closed) immersions.*
(2) *If X_∞ is smooth, then so is $\prod_{R'/R} X_\infty$. If X_η is smooth, then so are $\prod_{K'/K} X_\eta$ and $(\prod_{R'/R} X_\infty)_\eta$.*
(3) *If K'/K is separable, then $\prod_{K'/K} X_\eta$ is quasi-compact.*
(4) *The canonical open immersion $(\prod_{R'/R} X_\infty)_\eta \hookrightarrow \prod_{K'/K} X_\eta$ induces a bijection $(\prod_{R'/R} X_\infty)_\eta(L) = (\prod_{K'/K} X_\eta)(L)$ for any finite unramified extension L of K.*
(5) *If K'/K is unramified, then the canonical open immersion $(\prod_{R'/R} X_\infty)_\eta \hookrightarrow \prod_{K'/K} X_\eta$ is an isomorphism.*

(6) *If Y_∞ is a nice stft formal scheme over* $\operatorname{Spf} R'$*, and $Y_\infty \to X_\infty$ is a morphism of formal R'-schemes that induces an open embedding $Y_\eta \hookrightarrow X_\eta$ on the generic fibers, then the induced morphism $(\prod_{R'/R} Y_\infty)_\eta \to (\prod_{R'/R} X_\infty)_\eta$ is an open embedding.*

(7) *If X is a separated K'-scheme of finite type such that any finite set of points of X is contained in an open affine subscheme, then the functor $\prod_{K'/K} X^{an}$ is represented by the rigid K-variety $(\prod_{K'/K} X)^{an}$.*

Proof. (1) See [1, 1.3.3] and [1, 2.1].

(2) Since R'/R is finite and free, $\prod_{R'/R} X_\infty$ is smooth if X_∞ is, by [2, 1.5]. If X_η is smooth, then $\prod_{K'/K} X_\eta$ is smooth, by [1, 1.1.6], and so is $(\prod_{R'/R} X_\infty)_\eta$, by the canonical open embedding

$$\Bigl(\prod_{R'/R} X_\infty\Bigr)_\eta \hookrightarrow \prod_{K'/K} X_\eta.$$

(3) In the affinoid case, this follows from [2, 1.8.iii]. The general case is deduced by using [2, 1.13].

(4) Denote by R^L the normalization of R in L. Since R^L/R is unramified, the spectrum of $R' \otimes_R R^L$ is a disjoint union $\bigsqcup_i \operatorname{Spec} R_i$, with R_i a complete discrete valuation ring, finite and unramified over R' (see, for instance, [15, 6.1]). Moreover, $R^L \otimes_R K' = L \otimes_K K'$. Hence, $X_\infty(R' \otimes_R R^L) = X_\eta(K' \otimes_K L)$, and by the definition of the Weil restriction functor,

$$\Bigl(\prod_{R'/R} X_\infty\Bigr)_\eta(L) = \Bigl(\prod_{R'/R} X_\infty\Bigr)\bigl(R^L\bigr) = X_\infty\bigl(R' \otimes_R R^L\bigr) = X_\eta(K' \otimes_K L) = \Bigl(\prod_{K'/K} X_\eta\Bigr)(L).$$

(5) Since

$$\Bigl(\prod_{R'/R} X_\infty\Bigr)_\eta \hookrightarrow \prod_{K'/K} X_\eta$$

is an open embedding, it is sufficient to check that it induces a bijection on the underlying sets, i.e. on L-valued points with L a finite extension of K. Now one can copy the arguments from (4), interchanging the roles of K' and L.

(6) By (1), we get an open embedding $\prod_{K'/K} Y_\eta \hookrightarrow \prod_{K'/K} X_\eta$. Since the open embeddings

$$\Bigl(\prod_{R'/R} X_\infty\Bigr)_\eta \hookrightarrow \prod_{K'/K} X_\eta \quad \text{and} \quad \Bigl(\prod_{R'/R} Y_\infty\Bigr)_\eta \hookrightarrow \prod_{K'/K} Y_\eta$$

are canonical, we get a commutative diagram

$$\begin{tikzcd}
(\prod_{R'/R} Y_\infty)_\eta \arrow[r] \arrow[d] & (\prod_{R'/R} X_\infty)_\eta \arrow[d] \\
\prod_{K'/K} Y_\eta \arrow[r] & \prod_{K'/K} X_\eta.
\end{tikzcd}$$

This implies that the induced morphism $(\prod_{R'/R} Y_\infty)_\eta \to (\prod_{R'/R} X_\infty)_\eta$ is an open embedding.

(7) This is a special case of [2, 1.19]. □

2.2. Greenberg transforms

In this section, we recall the construction of the Greenberg schemes associated to a *stft* formal R-scheme, as defined in [11], and we will establish certain properties we will need in the remainder of this article.

First, we recall the construction of the ring schemes $\mathscr{R}_n$ associated to the Artin local rings R_n in [11]. If R has equal characteristic, then R is (non-canonically) an algebra over k, isomorphic to $k[[\pi]]$ (see [22, II, §4]), and $\mathscr{R}_n$ is simply the ring k-scheme representing the functor

$$\mathscr{R}_n : (k\text{-}algebras) \to (rings) : A \mapsto A \otimes_k R_n.$$

It depends on the chosen k-algebra structure on R.

If R has mixed characteristic and is absolutely unramified, then R is canonically isomorphic to the ring of Witt vectors $W(k)$, by [22, II§5, Theorem 4], and $\mathscr{R}_n$ is the ring k-scheme $\mathscr{W}_{n+1}$ of Witt vectors of length $n+1$.

Finally, we consider the case where R is of mixed characteristic and absolutely ramified with index e. Due to the confusion that seems to exist in literature, we do so with extra care. There exists a canonical injection of rings $W(k) \to R$ which makes R into a $W(k)$-algebra, free of rank e as a $W(k)$-module [22, II§5, Theorem 4]. In fact, R is isomorphic to $W(k)[\pi]/(p(\pi))$, where $p(\pi)$ is an Eisenstein polynomial

$$p(\pi) = \pi^e + a_1\pi^{e-1} + \cdots + a_e$$

over $W(k)$ of degree e. The ring R_n has characteristic p^m, with p the characteristic of k and $m = \lceil (n+1)/e \rceil$ the smallest integer bigger than or equal to $(n+1)/e$. Hence, R_n becomes a finite algebra over $W_m(k)$. As a $W_m(k)$-module, it can be written as an internal direct sum

$$R_n = W_m(k).e_1(\pi) \oplus \cdots \oplus W_m(k).e_r(\pi)$$

for some polynomials $e_i(\pi)$ over $W(k)$ of degree $< e$, and the multiplication in R_n is defined by the rules $p(\pi) = 0$ and $\pi^{n+1} = 0$. Each component $W_m(k) \cdot e_i(\pi)$ is isomorphic to $W_{n_i}(k)$, for some integer $n_i \leqslant m$, and $n+1 = n_1 + \cdots + n_r$. Considering the Witt coordinates on each of these rings $W_{n_i}(k)$, we obtain the affine space $\mathbb{A}_k^{n+1}$ endowed with the rules of addition and multiplication defined by this presentation; this is by definition the ring scheme $\mathscr{R}_n$ (see [11, Proposition 4]), which does not depend on the chosen presentation.

Let A be a perfect k-algebra, and consider the map

$$\Psi_A^{(n)} : \mathscr{R}_n(A) \to W_m(A) \otimes_{W_m(k)} R_n$$

mapping the point

$$(a_{1,0}, \ldots, a_{1,n_1-1}, \ldots, a_{r,0}, \ldots, a_{r,n_r-1}) \in A^{n_1+\cdots+n_r}$$

to

$$\sum_{i=1}^{r}\sum_{j=0}^{n_i}\tau\left(a_{i,j}^{p^{-j}}\right)p^j e_i(\pi)$$

where $\tau : A \to W_m(A)$ is the Teichmüller character. It is clear from the definition of the ring scheme $\mathscr{R}_n$ that this map is a morphism of rings, and that it does not depend on the chosen presentation for $\mathscr{R}_n$ (i.e. the choice of $e_1(\pi), \ldots, e_r(\pi)$).

Lemma 2.6. *If A is a perfect k-algebra, then the map*

$$\Psi_A^{(n)} : \mathscr{R}_n(A) \to W_m(A) \otimes_{W_m(k)} R_n$$

is an isomorphism, and $W_m(A) \otimes_{W_m(k)} R_n$ is canonically isomorphic to $W(A) \otimes_{W(k)} R_n$.

Proof. Since A is perfect, $W_a(A) \cong W(A)/(p^a)$, and $W(A) \otimes_{W(k)} W_a(k)$ is canonically isomorphic to $W_a(A)$ for each integer $a > 0$.

Therefore, $W_m(A) \otimes_{W_m(k)} R_n$ can be written as an internal direct sum

$$W_m(A) \cdot e_1(\pi) \oplus \cdots \oplus W_m(A) \cdot e_r(\pi)$$

and $W_m(A)e_i(\pi) \cong W_{n_i}(A)$ for each i. Hence, $\Psi_A^{(n)}$ is an isomorphism. $\square$

However, such an isomorphism $\mathscr{R}_n(A) \cong W(A) \otimes_{W(k)} R_n$ does not need to exist if A is not supposed to be perfect. Let us consider an easy example: take $n = 0$, i.e. $R_n = k$. In this case, $\mathscr{R}_n$ is simply the affine line $\mathbb{A}_k^1$, even as a ring scheme, and $\mathscr{R}_n(A) = A$ for any k-algebra A. On the other hand,

$$W(A) \otimes_{W(k)} R_n = W(A)/(p)$$

and this ring is not necessarily isomorphic to A if A is not perfect, since $p \cdot W(A)$ consists of the Witt vectors of the form $(0, a_1^p, a_2^p, \ldots)$ with $a_i \in A$ (such an isomorphism would yield an endomorphism

$$A \cong W(A)/p \cdot W(A) \to W(A)/W_1(A) \cong A$$

which is surjective but not injective, which is impossible if A is a field). This example contradicts [4, §9.6].

One might be tempted to expect that $\mathscr{R}_n(A) = W_m(A) \otimes_{W_m(k)} R_n$, with m as before. Let us give another example to show that this does not hold either. Put $R = W(k)[\pi]/(\pi^2 - p)$ and take $n = 2$. Then $m = 2$, and

$$R_n = W_2(k) \cdot 1 \oplus W_2(k) \cdot \pi$$

as a $W_2(k)$-module (internal direct sum). The relations $\pi^3 = 0$ and $\pi^2 - p = 0$ impose $\pi \cdot p = 0$, such that $W_2(k) \cdot \pi$ is isomorphic to $W_1(k)$. The ring scheme $\mathscr{R}_n$ is given by $\mathbb{A}_k^3$ with the following operations:

- Addition:

$$(x_0, x_1, x_2) + (y_0, y_1, y_2) = \big(x_0 + y_0, x_1 + y_1 - \big((x_0)^p + (y_0)^p - (x_0 + y_0)^p\big)/p, x_2 + y_2\big).$$

- Multiplication:

$$(x_0, x_1, x_2) \cdot (y_0, y_1, y_2) = \big(x_0 \cdot y_0, (x_0)^p \cdot y_1 + (y_0)^p \cdot x_1 + (x_2 \cdot y_2)^p, x_0 \cdot y_2 + x_2 \cdot y_0\big).$$

For any k-algebra A, we have $\mathscr{R}_n(A) = W_2(A) \oplus A$ with componentwise addition, and where multiplication is defined by the rule $(a, b) \cdot (a', b') = (a \cdot a' + pb \cdot b', ab' + a'b)$.

If A is not perfect, then the natural map

$$W_2(A) \otimes_{W_2(k)} R_2 \cong W_2(A)[\pi]/\big(\pi^2 - p, p \cdot \pi\big) \to \mathscr{R}_2(A)$$

is not an isomorphism in general, since $W_2(A)/(p) \neq W_1(A)$.

In any of the above cases, as a scheme, $\mathscr{R}_n$ is isomorphic to the affine space $\mathbb{A}_k^{n+1}$, but the ring structure can be fairly complicated (especially in the absolutely ramified case). The ring $\mathscr{R}_n(k)$ is canonically isomorphic to R_n. Hence, for any k-algebra A, $\mathscr{R}_n(A)$ is a R_n-algebra in a natural way.

The construction in [11] of the ring scheme $\mathscr{S}$ associated to an Artin local ring S is functorial in S: let $\varphi : S' \to S$ be a morphism of Artin local rings with the same residue field k. Assume that φ respects the chosen k-algebra structures if S, S' have equal characteristic, and that φ induces the identity on k if S, S' have mixed characteristic. Then φ induces a natural k-morphism of associated ring schemes $\mathscr{S}' \to \mathscr{S}$; see [12, p. 1]. Moreover, it is easy to see that $\mathscr{R}'_n \cong \mathscr{R}_n \times_k k'$ for any $n \geqslant 0$, if R' is an unramified extension of R with perfect residue field k'. Hence, if R' is a finite extension of R, of ramification index d, and with residue field k', then, for all $n \in \mathbf{N}$, the natural morphism of rings $R_{n-1} \to R'_{nd-1}$ induces canonically a k'-morphism of ring schemes:

$$\mathscr{R}_{n-1} \times_k k' \to \mathscr{R}'_{nd-1}$$

which is the composition of the canonical isomorphism $\mathscr{R}^o_{n-1} \cong \mathscr{R}_{n-1} \times_k k'$ (with R^o the maximal unramified extension of R in R') and the morphism $\mathscr{R}^o_{n-1} \to \mathscr{R}_{nd-1}$ induced by the morphism of Artin local rings $R^o_{n-1} \to R'_{nd-1}$.

Lemma 2.7. *Assume that R has mixed characteristic. Suppose that R' is a finite, totally ramified extension of R, of degree d. Fix a couple of integers $n, q \geqslant 1$ such that either $q = 1$ and $n \leqslant d$, or $n = d$. For any k-algebra A, there is a canonical isomorphism of R'_{nq-1}-algebras*

$$\mathscr{R}'_{nq-1}(A) \cong \mathscr{R}_{q-1}(A) \otimes_{R_{q-1}} R'_{nq-1}.$$

In particular, if e is the absolute ramification index of R' and $0 < n \leqslant e$, we get a canonical isomorphism

$$\mathscr{R}'_{n-1}(A) \cong A \otimes_k R'_{n-1}.$$

Proof. We only have to prove the result when $n > 1$. Note that both R'_{nq-1} and R_{q-1} have characteristic p^m with p the characteristic of k and

$$m = \lceil nq/de \rceil = \lceil q/e \rceil.$$

Choose elements $e_1, \dots, e_r$ in R_{q-1} such that R_{q-1} can be written (as a $W_m(k)$-module) as an internal direct sum

$$R_{q-1} = W_m(k) \cdot e_1 \oplus \cdots \oplus W_m(k) \cdot e_r.$$

Now we fix a uniformizing element π' in R'; this choice yields an isomorphism $R' \cong R[\pi']/(p(\pi'))$ with $p(\pi')$ an Eisenstein polynomial of degree d over R. It follows from our assumptions that R'_{nq-1} is a free module over R_{q-1}, and that $1, \pi', \dots, (\pi')^{n-1}$ is a basis (this is obvious if $q = 1$, since then $R_{q-1} = k$; in the other case $n = d$, it follows from the fact that $1, \pi', \dots, (\pi')^{n-1}$ is a basis for R' over R and that $R'_{nq-1} \cong R' \otimes_R R_{q-1}$). Hence, R'_{qn-1} can be written (as a $W_m(k)$-module) as an internal direct sum

$$R'_{qn-1} = \bigoplus_{i=1,\dots,r;\, j=0,\dots,n-1} W_m(k) \cdot e_i \cdot (\pi')^j$$

with

$$W_m(k) \cdot e_i \cong W_m(k) \cdot e_i \cdot (\pi')^j$$

for all i, j. This shows that

$$\mathscr{R}'_{nq-1}(A) \cong \mathscr{R}_{q-1}(A)[\pi']/\big(p(\pi'), (\pi')^{nq}\big) \cong \mathscr{R}_{q-1}(A) \otimes_{R_{q-1}} R'_{nq-1}$$

for any k-algebra A. $\square$

Consider the functor h_n^R from the category of k-schemes to the category of $\mathrm{Spec}(R_n)$-locally ringed spaces, defined by:

$$T \mapsto \big(T, \mathcal{H}\mathrm{om}_k(T, \mathscr{R}_n)\big),$$

the locally ringed space of germs of k-morphisms from T to $\mathscr{R}_n$. Since $\mathscr{R}_n(k) = R_n$, we see that $h_n^R(T)$ is a locally ringed space in R_n-algebras in a natural way. If $T = \mathrm{Spec}\, A$ is affine, we have $h_n^R(T) = \mathrm{Spec}\, \mathscr{R}_n(A)$ by [12, Proposition 1], so $h_n^R(T)$ is a scheme for any T. If R has equal characteristic, then $h_n^R(T)$ is canonically isomorphic to $T \times_k R_n$. The following statement follows from the Theorem in [11, §4], and from [4, §7.6]:

Definition 2.8. Let Y be a separated R_n-scheme of finite type. The functor

$$(Sch/k)^{op} \to (Sets) : T \mapsto \mathrm{Hom}_{R_n}\big(h_n^R(T), Y\big),$$

is representable by a separated k-scheme of finite type $\mathrm{Gr}_n^R(Y)$. This construction defines a functor Gr_n^R from the category of separated R_n-schemes of finite type, to the category of separated

k-schemes of finite type. The functor Gr_n^R is called the Greenberg transform of length n, and the k-scheme $\mathrm{Gr}_n^R(Y)$ is called the Greenberg scheme (of length n) associated to Y.

If X_∞ is a formal R-scheme topologically of finite type, we put

$$\mathrm{Gr}_n^R(X_\infty) := \mathrm{Gr}_n^R(X_n).$$

Remark. When R is a ring of equal characteristic, the Greenberg transform Gr_n^R is nothing but the Weil restriction functor $\prod_{R_n/k}$.

For any pair of integers $m \geqslant n \geqslant 0$, and any *stft* formal R-scheme X_∞, the truncation morphism $\mathscr{R}_m \to \mathscr{R}_n$ induces a canonical truncation morphism

$$\theta_n^m : \mathrm{Gr}_m^R(X_\infty) \to \mathrm{Gr}_n^R(X_\infty).$$

These truncation morphisms are affine (since the construction of the Greenberg scheme is local on X_∞ and $\mathrm{Gr}_n^R(X_\infty)$ is affine if X_∞ is affine; see [21, 3.1.1]). Hence, we can take the projective limit

$$\mathrm{Gr}^R(X_\infty) := \varprojlim_n \mathrm{Gr}_n^R(X_\infty)$$

in the category of k-schemes. This k-scheme $\mathrm{Gr}^R(X_\infty)$ is called the Greenberg scheme associated to X_∞. It is not of finite type, in general. We obtain a functor Gr^R from the category of *stft* formal R-schemes to the category of separated k-schemes, called the Greenberg transform (of length ∞).

Remark. In this paper, contrary to [21], we do not endow the Greenberg schemes with their reduced structure. Observe that these constructions depend on R: if R' is a finite ramified extension of R, then $\mathrm{Gr}^R(\mathrm{Spf}\, R') = \emptyset$, while $\mathrm{Gr}^{R'}(\mathrm{Spf}\, R')$ is a point.

2.3. *Motivic Serre invariants*

Let X_η be a separated smooth rigid K-variety. A *weak Néron R-model for X_η* is a smooth *stft* formal R-scheme U_∞, endowed with an open embedding $U_\eta \to X_\eta$, such that the natural map $U_\infty(R^{sh}) \to X_\eta(K^{sh})$ is bijective [6, 1.3]. If X_η is quasi-compact, then such a weak Néron model always exists, by [6, 3.1].

Definition 2.9. Let X_∞ be a generically smooth, *stft* formal R-scheme. We say that a morphism of *stft* formal R-schemes $U_\infty \to X_\infty$ is a weak Néron R-smoothening, if it has the following properties:

(1) the induced morphism $U_\eta \to X_\eta$ is an open embedding,
(2) U_∞ is a weak Néron R-model for X_η (w.r.t. this open embedding).

For any generically smooth *stft* formal R-scheme, there exists an admissible blow-up $X'_\infty \to X_\infty$ such that $\mathrm{Sm}(X'_\infty) \to X_\infty$ is a weak Néron R-smoothening, by [6, 3.1].

Definition 2.10. If X_∞ is a generically smooth, *stft* formal R-scheme, of pure relative dimension, and $U_\infty \to X_\infty$ is a weak Néron R-smoothening, then we define the motivic Serre invariant $S(X_\infty)$ of X_∞ by

$$S(X_\infty) := [U_s] \in K_0(Var_{X_s})/\big(\mathbb{L}_{X_s} - [X_s]\big).$$

This definition does not depend on the choice of weak Néron smoothening, by [19, Proposition 6.11]. If $Y_\infty \to X_\infty$ is a morphism of generically smooth, *stft* formal R-schemes, inducing an isomorphism on the generic fibers, the forgetful morphism

$$K_0(Var_{Y_s})/\big(\mathbb{L}_{Y_s} - [Y_s]\big) \to K_0(Var_{X_s})/\big(\mathbb{L}_{X_s} - [X_s]\big)$$

maps $S(Y_\infty)$ to $S(X_\infty)$ (this applies in particular to the case where Y_∞ is the maximal flat closed formal subscheme of X_∞, i.e. the closed subscheme of X_∞ defined by the π-torsion ideal). Moreover, the image of $S(X_\infty)$ under the forgetful morphism

$$K_0(Var_{X_s})/\big(\mathbb{L}_{X_s} - [X_s]\big) \to K_0(Var_k)/(\mathbb{L} - 1)$$

is the motivic Serre invariant $S(X_\eta)$ introduced in [16]. It only depends on the generic fiber X_η of X_∞.

3. Greenberg schemes and ramification

In this section, we study the behavior of the Greenberg transform under extensions of the complete discrete valuation ring R.

3.1. A direct system of closed immersions

The main result of this section is the following statement:

Theorem 3.1. *Let $n \geqslant 1$ be an integer. Let R' be a finite extension of R, of ramification index e. We denote the residue field of R' by k'. Let X be a R_{n-1}-scheme of finite type and $X' := X \times_{R_{n-1}} R'_{ne-1}$ the R'_{ne-1}-scheme obtained by base change. Then the morphism $R_{n-1} \to R'_{ne-1}$ induces a canonical morphism of k'-schemes*

$$\big(\iota_R^{R'}\big)_n : \mathrm{Gr}^R_{n-1}(X) \times_k k' \to \mathrm{Gr}^{R'}_{ne-1}(X').$$

This morphism $(\iota_R^{R'})_n$ is a closed immersion.

The proof of this theorem follows after Lemma 3.6. We need several preliminary results. First, we construct the morphism $(\iota_R^{R'})_n$.

Lemma 3.2. *We keep the notations of Theorem* 3.1. *The morphism $R_{n-1} \to R'_{ne-1}$ induces a canonical morphism of k'-schemes*:

$$\left(\iota_R^{R'}\right)_n : \mathrm{Gr}_{n-1}^R(X) \times_k k' \to \mathrm{Gr}_{ne-1}^{R'}(X').$$

Proof. For simplicity, we will write ι_n instead of $(\iota_R^{R'})_n$. By Yoneda's lemma, it suffices to construct a natural map

$$\mathrm{Hom}_{k'}\left(T, \mathrm{Gr}_{n-1}^R(X) \times_k k'\right) \to \mathrm{Hom}_{k'}\left(T, \mathrm{Gr}_{ne-1}(X')\right)$$

for any k'-scheme T. So, by definition of the Greenberg schemes, it suffices to construct a natural map ι_n from $\mathrm{Hom}_{R_{n-1}}(h_{n-1}^R(T), X)$ to

$$\mathrm{Hom}_{R'_{ne-1}}\left(h_{ne-1}^{R'}(T), X \times_{R_{n-1}} R'_{ne-1}\right) \simeq \mathrm{Hom}_{R_{n-1}}\left(h_{ne-1}^{R'}(T), X\right)$$

for any k'-scheme T.

As we observed right before Lemma 2.7, the natural morphism $R_{n-1} \to R'_{ne-1}$ induces a natural k'-morphism of ring schemes $\mathscr{R}_{n-1} \times_k k' \to \mathscr{R}'_{ne-1}$ and hence, by definition of the functors h_n^R, a natural R_{n-1}-morphism of locally ringed spaces

$$i_n : h_{ne-1}^{R'}(T) \to h_{n-1}^R(T).$$

Now the map ι_n is obtained by composition. $\square$

Proposition 3.3. *We keep the notation of Theorem* 3.1. *If R' is unramified over R, the k'-morphism $(\iota_R^{R'})_n$ is a k'-isomorphism of schemes.*

Proof. We noted in the paragraph preceding Lemma 2.7 that there exists a canonical isomorphism $\mathscr{R}'_n \cong \mathscr{R}_n \times_k k'$ for all $n \geqslant 0$, so the morphism $i_n : h_{n-1}^{R'}(T) \to h_{n-1}^R(T)$ constructed in the proof of Lemma 3.2 is an isomorphism. $\square$

Lemma 3.4. *We keep the notation of Theorem* 3.1. *The construction of the k'-morphism $(\iota_R^{R'})_n$ is functorial in X in the following sense*: *if $f : Y \to X$ is a morphism of R_{n-1}-schemes of finite type, and if we denote by Y' the base change $Y \times_{R_{n-1}} R'_{ne-1}$, then the square*

$$\begin{tikzcd}
\mathrm{Gr}_{n-1}^R(Y) \times_k k' \arrow[r, "(\iota_R^{R'})_n"] \arrow[d, "\mathrm{Gr}_{n-1}^R(f)"'] & \mathrm{Gr}_{ne-1}^{R'}(Y') \arrow[d, "\mathrm{Gr}_{ne-1}^{R'}(f)"] \\
\mathrm{Gr}_{n-1}^R(X) \times_k k' \arrow[r, "(\iota_R^{R'})_n"] & \mathrm{Gr}_{ne-1}^{R'}(X')
\end{tikzcd}$$

commutes.

Proof. This follows immediately from Yoneda's lemma and adjunction properties. $\square$

Lemma 3.5. *We keep the notations of Theorem* 3.1. *Suppose that R' is totally ramified over R. The k-morphism $(\iota_R^{R'})_n$ constructed in Lemma* 3.2 *is a closed immersion.*

Proof. Since being a closed immersion is a local property, we can suppose that X is an affine R_{n-1}-scheme of finite type. Consider a closed immersion $X \hookrightarrow \mathbf{A}^N_{R_{n-1}}$. By Lemma 3.4, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}^R_{n-1}(X) \arrow[r] \arrow[d] & \mathrm{Gr}^{R'}_{ne-1}(X') \arrow[d] \\
\mathrm{Gr}^R_{n-1}(\mathbf{A}^N_{R_{n-1}}) \arrow[r, "(\iota_R^{R'})_n"'] & \mathrm{Gr}^{R'}_{ne-1}(\mathbf{A}^N_{R'_{ne-1}})
\end{tikzcd}$$

where the vertical arrows are closed immersions, by [11, §4, Corollary 2]. Hence, we may assume that $X = \mathbf{A}^N_{R_{n-1}}$. In this case, we can give an explicit description of the morphism $(\iota_R^{R'})_n$. By the description of the ring scheme $\mathscr{R}_n$ at the beginning of Section 2.2 and the proof of Lemma 2.7, there exist isomorphisms

$$\mathrm{Gr}^R_{n-1}\big(\mathbf{A}^N_{R_{n-1}}\big) \cong \mathrm{Spec}\, k\big[(X_{i,j})_{1\leqslant i\leqslant N;\, j=0,\ldots,n-1}\big],$$
$$\mathrm{Gr}^R_{ne-1}\big(\mathbf{A}^N_{R'_{ne-1}}\big) \cong \mathrm{Spec}\, k\big[(Y_{i,j})_{1\leqslant i\leqslant N;\, j=0,\ldots,ne-1}\big]$$

such that the morphism $(\iota_R^{R'})_n$ is induced by the morphism of k-algebras

$$k\big[(Y_{i,j})_{1\leqslant i\leqslant N;\, j=0,\ldots,ne-1}\big] \to k\big[(X_{i,j})_{1\leqslant i\leqslant N;\, j=0,\ldots,n-1}\big]$$

mapping $Y_{i,j}$ to $X_{i,j/e}$ if e divides j, and to zero else. This morphism is clearly surjective. □

Lemma 3.6. *We keep the notations of Theorem* 3.1. *The construction of the k'-morphism $(\iota_R^{R'})_n$ is functorial in the extension R'/R in the following sense: if R'' is a finite extension of R' of ramification index e', and if we denote by k'' the residue field of R'', then*

$$\big(\iota_R^{R''}\big)_n = \big(\iota_{R'}^{R''}\big)_{ne} \circ \big(\big(\iota_R^{R'}\big)_n \times_{k'} k''\big).$$

Proof. By construction of the morphism $(\iota_R^{R'})_n$, it suffices to show that, for any k''-scheme T, the composition

$$h^{R''}_{nee'-1}(T) \to h^{R'}_{ne-1}(T) \to h^{R}_{n-1}(T)$$

coincides with the natural map

$$h^{R''}_{nee'-1}(T) \to h^{R}_{n-1}(T).$$

This follows at once from the fact that the composition $R_{n-1} \to R'_{ne-1} \to R''_{nee'-1}$ coincides with $R_{n-1} \to R''_{nee'-1}$. □

Proof of Theorem 3.1. Let R^o be the maximal unramified extension of R in R'. The result follows from Lemmas 3.3, 3.5, and 3.6, applied to the tower $R'/R^o/R$. $\square$

Lemma 3.7. *We keep the notations of Theorem* 3.1. *Consider a stft formal R-scheme X_∞, and denote by X'_∞ the base change $X_\infty \times_R R'$. For any pair of integers $m \geqslant n \geqslant 1$, the square*

$$\begin{tikzcd}
\mathrm{Gr}^R_{m-1}(X_\infty) \times_k k' \arrow[r, "(\iota^{R'}_R)_m"] \arrow[d, "\theta^{m-1}_{n-1}"'] & \mathrm{Gr}^{R'}_{me-1}(X'_\infty) \arrow[d, "\theta^{me-1}_{ne-1}"] \\
\mathrm{Gr}^R_{n-1}(X_\infty) \times_k k' \arrow[r, "(\iota^{R'}_R)_n"] & \mathrm{Gr}^{R'}_{ne-1}(X'_\infty)
\end{tikzcd}$$

commutes.

Proof. This follows from the construction of the morphism $(\iota^{R'}_R)_n$, and the commutativity of the square

$$\begin{tikzcd}
R_{m-1} \arrow[r] \arrow[d] & R'_{me-1} \arrow[d] \\
R_{n-1} \arrow[r] & R'_{ne-1}
\end{tikzcd} \quad \square$$

Theorem 3.8. *Let X_∞ be a stft formal R-scheme, let R' be finite extension of R, with residue field k', and denote by X'_∞ the base change $X_\infty \times_R R'$.*

(1) *The morphisms $(\iota^{R'}_R)_n$ induce a canonical morphism of k'-schemes*

$$\iota^{R'}_R : \mathrm{Gr}^R(X_\infty) \times_k k' \to \mathrm{Gr}^{R'}\bigl(X'_\infty\bigr).$$

(2) *The morphism $\iota^{R'}_R$ is a closed immersion.*
(3) *If R' is unramified over R, the morphism $\iota^{R'}_R$ is an isomorphism.*
(4) *The construction of the k'-morphism $\iota^{R'}_R$ is functorial in X_∞ in the following sense: if $f : Y_\infty \to X_\infty$ is a morphism of stft formal R-schemes, and if we denote by Y'_∞ the base change $Y_\infty \times_R R'$, then the square*

$$\begin{tikzcd}
\mathrm{Gr}^R(Y_\infty) \times_k k' \arrow[r, "\iota^{R'}_R"] \arrow[d, "\mathrm{Gr}^R(f)"'] & \mathrm{Gr}^{R'}(Y'_\infty) \arrow[d, "\mathrm{Gr}^{R'}(f)"] \\
\mathrm{Gr}^R(X_\infty) \times_k k' \arrow[r, "\iota^{R'}_R"] & \mathrm{Gr}^{R'}(X'_\infty)
\end{tikzcd}$$

commutes.

(5) *The construction of the k'-morphism $\iota_R^{R'}$ is functorial in the extension R'/R in the following sense*: *if R'' is a finite extension of R', and if we denote by k'' the residue field of R'', then*

$$\iota_R^{R''} = \iota_{R'}^{R''} \circ \left(\iota_R^{R'} \times_{k'} k''\right).$$

Proof. The existence of $\iota_R^{R'}$ follows from Lemma 3.7. Point (3), (4) and (5) follow immediately from the corresponding statements for the truncations $(\iota_R^{R'})_n$.

To prove that $\iota_R^{R'}$ is a closed immersion, we may assume that X_∞ is affine; then $\mathrm{Gr}_n^R(X_\infty)$ and $\mathrm{Gr}_n^{R'}(X_\infty)$ are affine for all n, and the result follows from the fact that the direct limit of an inductive system of surjective ring morphisms $A_n \to B_n$ is again surjective, by exactness of the direct limit functor. $\square$

3.2. Galois action on the Greenberg scheme

Let K' be a totally ramified extension of K, of degree e, and denote by R' the normalization of R in K'. Let X_∞ be a *stft* formal R-scheme, and denote by X'_∞ the base change $X_\infty \times_R R'$.

By functoriality, the action of the Galois group $G = G(K'/K)$ on R'_{ne-1} over R_{n-1} induces an action of G on the ring scheme $\mathscr{R}'_{ne-1}$ over $\mathscr{R}_{n-1}$, for any $n > 0$.

Lemma 3.9. *Suppose that K'/K is tame and Galois, with group G. For any integer $n \geqslant 1$, and any k-algebra A, there is a canonical isomorphism*

$$\mathscr{R}_{n-1}(A) \cong \left(\mathscr{R}'_{ne-1}(A)\right)^G.$$

Proof. Since K'/K is tame, K' is of the form $K' \cong K[T]/(T^e - \rho)$, with ρ a uniformizing element in K, and $G = \mu_e(R) = \mu_e(k)$ acts by multiplication on T. Let ξ be a primitive eth root of unity in R.

We have

$$\mathscr{R}'_{ne-1}(A) \cong \mathscr{R}_{n-1}(A) \otimes_{R_{n-1}} R'_{ne-1}$$

by Lemma 2.7. Hence, $\mathscr{R}'_{ne-1}(A)$ is free over $\mathscr{R}_{n-1}(A)$, with basis $1, T, \ldots, T^{e-1}$, and any element a of $\mathscr{R}'_{ne-1}(A)$ can be written in a unique way as $a = \sum_{i=0}^{e-1} a_i T^i$ with $a_i \in \mathscr{R}_{n-1}(A)$. The image of a under the action of ξ is given by $\xi \cdot a = \sum_{i=0}^{e-1} \xi^i a_i T^i$, so it suffices to show that $(1 - \xi^i)$ is invertible in R for $0 < i < e$. However, reduction modulo ρ yields an isomorphism between $\mu_e(R)$ and $\mu_e(k)$, so in particular, $(1 - \xi^i) \notin (\rho)$ for $0 < i < e$. $\square$

Remark. Lemma 3.9 is false without the tameness condition. Consider, for example, the extension $R' = \mathbb{Z}_2[T]/(T^2 - 2)$ of the ring of 2-adic integers $\mathbb{Z}_2$. The extension K'/K is Galois, and its Galois group $G = \mathbb{Z}/2\mathbb{Z}$ acts by $T \mapsto -T$. Putting $n = 1$, we get $R_{n-1} = \mathbb{F}_2$ and $R'_{ne-1} = \mathbb{F}_2[T]/(T^2)$, and we see that G acts trivially on R'_{ne-1}.

The main result of this subsection is the following:

Theorem 3.10. *For any integer $n \geqslant 0$, the Greenberg scheme $\mathrm{Gr}_n^{R'}(X'_\infty)$ carries a natural action of the Galois group $G = G(K'/K)$. The functor of fixed points*

$$(Sch/k)^{op} \to (Sets) : T \to \left(\mathrm{Gr}_n^{R'}\left(X'_\infty\right)(T)\right)^G$$

is representable by a closed subscheme $\mathrm{Gr}_n^{R'}(X'_\infty)^G$ *of* $\mathrm{Gr}_n^{R'}(X'_\infty)$. *Moreover, for* $n \geqslant 1$, *the morphism* $(\iota_R^{R'})_n : \mathrm{Gr}_{n-1}^R(X_\infty) \to \mathrm{Gr}_{ne-1}^{R'}(X'_\infty)$ *factors through a closed immersion*

$$\mathrm{Gr}_{n-1}^R(X_\infty) \to \left(\mathrm{Gr}_{ne-1}^{R'}\left(X'_\infty\right)\right)^G$$

and this is an isomorphism if K'/K *is tame and Galois.*

Proof. First, we construct the natural action of G on $\mathrm{Gr}_n^{R'}(X'_\infty)$. By Yoneda's lemma, it suffices to construct a natural G-action on $\mathrm{Hom}_k(T, \mathrm{Gr}_n^{R'}(X'_\infty))$, for any k-scheme T. By definition,

$$\mathrm{Hom}_k\left(T, \mathrm{Gr}_n^{R'}\left(X'_\infty\right)\right) = \mathrm{Hom}_{R'_n}\left(h_n^{R'}(T), X'_n\right) = \mathrm{Hom}_{R_m}\left(h_n^{R'}(T), X_m\right)$$

for any integer $m \geqslant 0$ with $(m+1)e \geqslant n+1$. Hence, it suffices to construct a canonical action of G on the locally ringed space in R_m-algebras $h_n^{R'}(T)$. This action is induced from the G-action on R'_n over R_m, via the resulting action of G on $\mathscr{R}'_n$ over $\mathscr{R}_m$.

The functor of fixed points is representable by a closed subscheme $\mathrm{Gr}_n^{R'}(X'_\infty)^G$ of $\mathrm{Gr}_n^{R'}(X'_\infty)$, by [10, 3.1].

For any k-algebra A, and any integer $n \geqslant 1$, the closed embedding

$$\left(\iota_R^{R'}\right)_n : \mathrm{Gr}_{n-1}^R(X_\infty) \to \mathrm{Gr}_{ne-1}^{R'}\left(X'_\infty\right)$$

induces a map

$$X_{n-1}\left(\mathscr{R}_{n-1}(A)\right) = \mathrm{Gr}_{n-1}^R(X_\infty)(A) \to \mathrm{Gr}_{ne-1}^{R'}\left(X'_\infty\right)(A) = X'_{ne-1}\left(\mathscr{R}'_{ne-1}(A)\right)$$

whose image is invariant under the action of G by Lemma 2.7. Hence, $(\iota_R^{R'})_n$ factors through a closed immersion

$$\left(\iota_R^{R'}\right)_n : \mathrm{Gr}_{n-1}^R(X_\infty) \to \mathrm{Gr}_{ne-1}^{R'}\left(X'_\infty\right)^G.$$

If K'/K is tame and Galois, then this map is an isomorphism by Lemma 3.9. $\square$

Proposition 3.11. *For any pair of integers* $m \geqslant n \geqslant 0$, *the* $G = G(K'/K)$*-action on* $\mathrm{Gr}_n^{R'}(X'_\infty)$ *is compatible with the truncations* θ_n^m, *i.e. for any element* g *of* G, *and any* k*-scheme* T, *the square*

$$\begin{tikzcd}
\mathrm{Gr}_m^{R'}(X'_\infty)(T) \arrow[r, "g"] \arrow[d, "\theta_n^m"'] & \mathrm{Gr}_m^{R'}(X'_\infty)(T) \arrow[d, "\theta_n^m"] \\
\mathrm{Gr}_n^{R'}(X'_\infty)(T) \arrow[r, "g"] & \mathrm{Gr}_n^{R'}(X'_\infty)(T)
\end{tikzcd}$$

commutes.

Proof. This follows from the fact that the square

$$\begin{tikzcd} R'_m \arrow[r, "g"] \arrow[d] & R'_m \arrow[d] \\ R'_n \arrow[r, "g"] & R'_n \end{tikzcd}$$

commutes. □

Hence, our constructions pass to the limit $n = \infty$, and we get the following result.

Theorem 3.12. *The $G = G(K'/K)$-action on the schemes $\mathrm{Gr}_n^{R'}(X'_\infty)$ induces a natural G-action on $\mathrm{Gr}^{R'}(X'_\infty)$. The functor of fixed points*

$$(Sch/k)^{op} \to (Sets) : T \to \left(\mathrm{Gr}^{R'}\left(X'_\infty\right)(T)\right)^G$$

is representable by a closed subscheme $\mathrm{Gr}^{R'}(X'_\infty)^G$ of $\mathrm{Gr}^{R'}(X'_\infty)$. Moreover, if $n \geqslant 1$ and K'/K is tame and Galois, then $\iota_R^{R'}$ is an isomorphism from $\mathrm{Gr}^R(X_\infty)$ onto $(\mathrm{Gr}^{R'}(X_\infty))^G$.

Proof. We only have to prove that the natural map

$$\mathrm{Gr}^{R'}\left(X'_\infty\right)^G \to \varprojlim\left(\mathrm{Gr}_n^{R'}\left(X'_\infty\right)^G\right)$$

is an isomorphism. This follows from Lemma 3.13 below. □

Lemma 3.13. *Let $Y_\bullet = (Y_i)_{i \in \mathbb{N}}$ be an inverse system of k-schemes, and suppose that the projective limit Y of the system $Y_\bullet$ exists in the category of k-schemes. Let G be a finite group acting on Y_i for each i, such that the transition morphisms of $Y_\bullet$ are equivariant. If we consider the induced G-action on Y, then there exists a canonical isomorphism of Y-schemes*

$$\left(Y^G \to Y\right) \cong \left(\varprojlim_i\left(Y_i^G\right) \to \varprojlim_i Y_i = Y\right).$$

Proof. Since the functor $(.)^G$ has a left adjoint (endowing a scheme with the trivial G-action), it commutes with projective limits. □

3.3. The Ind-scheme of sections

For simplicity, we suppose that k is algebraically closed. We denote by K^s a separable closure of K, and by K^t the maximal tamely ramified extension of K in K^s.

We denote by $\mathfrak{R}$ the directed set of finite extensions of R in K^s, ordered by inclusion. Likewise, we denote by $\mathfrak{R}^t$ the directed set of finite, tamely ramified extensions of R in K^t. Recall that a k-Ind-scheme is an object of the category of presheaves on (Sch/k), isomorphic to a direct limit of schemes.

Definition 3.14. For any *stft* formal R-scheme X_∞, we define the k-Ind-scheme of sections $\mathrm{Sec}(X_\infty)$ of X_∞ by

$$\mathrm{Sec}(X_\infty) := \varinjlim_{R' \in \mathfrak{R}} \mathrm{Gr}^{R'}(X_\infty \times_R R').$$

Likewise, we define the k-Ind-scheme of tame sections $\mathrm{Sec}^t(X_\infty)$ of X_∞ by

$$\mathrm{Sec}^t(X_\infty) := \varinjlim_{R' \in \mathfrak{R}^t} \mathrm{Gr}^{R'}(X_\infty \times_R R').$$

Proposition 3.15. *For any stft formal R-scheme X_∞, the k-Ind-schemes $\mathrm{Sec}(X_\infty)$ and $\mathrm{Sec}^t(X_\infty)$ carry a canonical action of the absolute Galois group $G(K^s/K)$, respectively the tame Galois group $G(K^t/K)$.*

Proof. These actions are induced by the canonical Galois action on $\mathrm{Gr}^{R'}(X_\infty \times_R R')$, for any finite extension R' of R. □

Proposition 3.16. *There exist canonical bijections*

$$\mathrm{Sec}(X_\infty)(k) \cong X_\eta\big(K^s\big) \quad \textit{and} \quad \mathrm{Sec}^t(X_\infty)(k) \cong X_\eta\big(K^t\big)$$

which respect the actions of $G(K^s/K)$ and $G(K^t/K)$.

Proof. If K' is a finite extension of K, and R' denotes the normalization of R in K', then $X_\eta(K') = X_\infty(R') = \mathrm{Gr}^{R'}(X_\infty)(k)$ by definition. The result now follows from the fact that

$$X_\eta\big(K^s\big) = \varinjlim_{K'} X_\eta(K')$$

where K' runs over the finite extensions of K, and the analogous assertion for $X_\eta(K^t)$. □

4. Greenberg schemes and Weil restrictions

In this section, we will establish the following result:

Theorem 4.1. *Let R' be a finite, totally ramified extension of R, of degree n. Let X_∞ be a nice stft formal R'-scheme. There exists a canonical isomorphism of k-schemes*

$$\theta(X_\infty) : \Big(\prod_{R'/R} X_\infty\Big)_s \to \mathrm{Gr}^{R'}_{n-1}(X_\infty).$$

Proof. By Yoneda's lemma, it suffices to construct a natural bijection

$$\mathrm{Hom}_k\Big(T, \Big(\prod_{R'/R} X_\infty\Big)_s\Big) \to \mathrm{Hom}_k\big(T, \mathrm{Gr}^{R'}_{n-1}(X_\infty)\big)$$

for any k-scheme T. By definition of the Weil restriction functor and the Greenberg transform, it suffices to construct a natural bijection

$$j_n : \mathrm{Hom}_{R'_{n-1}}\bigl(T \times_k (R'/\mathfrak{M}R'), X_{n-1}\bigr) \to \mathrm{Hom}_{R'_{n-1}}\bigl(h^{R'}_{n-1}(T), X_{n-1}\bigr)$$

for any k-scheme T (recall that $\mathfrak{M}$ denotes the maximal ideal of R). Note that $R'/\mathfrak{M}R' = R'_{n-1}$.

By Lemma 2.7, there exists a natural R'_{n-1}-isomorphism of locally ringed spaces

$$h^{R'}_{n-1}(T) \to T \times_k R'_{n-1}$$

which induces the natural bijection j_n by composition. □

By composition with the truncation morphism $\theta_0^{n-1} : \mathrm{Gr}^{R'}_{n-1}(X_\infty) \to X_s$, we obtain a k-morphism

$$\widetilde{\theta}(X_\infty) : \Bigl(\prod_{R'/R} X_\infty\Bigr)_s \to X_s.$$

Proposition 4.2. *Let R' be totally ramified extension of R of degree n. Let X_∞ be a nice stft formal R'-scheme. If X_∞ is smooth over R', then*

$$\widetilde{\theta}(X_\infty) : \Bigl(\prod_{R'/R} X_\infty\Bigr)_s \to X_s$$

is a Zariski-locally trivial fibration, with fiber $\mathbf{A}_k^{n-1}$.

Proof. By [21, 3.4.2], $\theta_0^{n-1} : \mathrm{Gr}^{R'}_{n-1}(X_\infty) \to X_s$ is a Zariski-locally trivial fibration, with fiber $\mathbf{A}_k^{n-1}$. Now apply Theorem 4.1. □

Corollary 4.3. *Let R' be a totally ramified extension of R of degree n. If X_∞ is a nice smooth stft formal R'-scheme, then*

$$\Bigl[\Bigl(\prod_{R'/R} X_\infty\Bigr)_s\Bigr] = [\mathbb{A}^{n-1}_{X_s}]$$

in $K_0(Var_{X_s})$.

5. Motivic Serre invariants and Weil restriction

In this final section, we study the behavior of the motivic Serre invariant under Weil restriction.

5.1. Motivic Serre invariants of mixed dimension

First, we will generalize the definition of the motivic Serre invariant to formal schemes and rigid varieties of *mixed dimension*.

Let X_∞ be a flat generically smooth *stft* formal R-scheme, and denote by $\widetilde{X}_\infty \to X_\infty$ its normalization. Let $\widetilde{X}_{\infty,i}$, $i = 1, \dots, \ell$, be the connected components of $\widetilde{X}_\infty$. Since X_η is smooth over K, and normalization commutes with taking generic fibers, $\widetilde{X}_\eta = X_\eta$ (see [7, 2.1.3]). By [7, 2.3.1], the generic fibers $\widetilde{X}_{\eta,i}$ of the formal schemes $\widetilde{X}_{\infty,i}$, are exactly the connected components of X_η. In particular, each $\widetilde{X}_{\infty,i}$ is a normal *stft* formal R-scheme, whose generic fiber is smooth over K and connected, and hence, whose special fiber has pure dimension by [20, §1, Lemma 1].

Lemma 5.1. *Let X_∞ be a generically smooth stft formal R-scheme, with equidimensional generic fiber X_η. Let $\widetilde{X}_\infty \to X_\infty$ be the normalization of X_∞, and denote by $\widetilde{X}_{\infty,i}$ $(1 \leqslant i \leqslant \ell)$ the connected components of $\widetilde{X}_\infty$. Then*

$$S(X_\infty) = \sum_{i=1}^{\ell} S(\widetilde{X}_{\infty,i})$$

in $K_0(Var_{X_s})/(\mathbb{L}_{X_s} - [X_s])$, via the forgetful morphisms

$$K_0(Var_{\widetilde{X}_{s,i}})/\big(\mathbb{L}_{\widetilde{X}_{s,i}} - [\widetilde{X}_{s,i}]\big) \to K_0(Var_{X_s})/\big(\mathbb{L}_{X_s} - [X_s]\big).$$

Proof. Since $\widetilde{X}_\infty \to X_\infty$ induces an isomorphism on the generic fibers,

$$S(\widetilde{X}_\infty) = S(X_\infty)$$

in $K_0(Var_{X_s})/(\mathbb{L}_{X_s} - [X_s])$. We conclude by additivity of the motivic Serre invariant. □

Definition 5.2. Let X_∞ be a generically smooth, *stft* formal R-scheme. Let $\widetilde{X}_\infty \to X_\infty$ be the normalization of X_∞, and denote by $\widetilde{X}_{\infty,i}$, $i = 1, \dots, \ell$, the connected components of $\widetilde{X}_\infty$. We define the motivic Serre invariant of X_∞ by

$$S(X_\infty) := \sum_{i=1}^{\ell} S(\widetilde{X}_{\infty,i})$$

in $K_0(Var_{X_s})/(\mathbb{L}_{X_s} - [X_s])$.

We will need the following technical result.

Proposition 5.3. *Let $h : V_\infty \to X_\infty$ be a morphism of stft formal R-schemes, with V_∞ smooth over R and X_η smooth over K. Then h factors through a unique morphism $V_\infty \to \widetilde{X}_\infty$.*

Proof. The assertion is local on V_∞ and X_∞, so we may assume that both are affine formal R-schemes. In this case, the existence of a morphism of formal X_∞-schemes $V_\infty \to \widetilde{X}_\infty$ follows from [6, 2.2]. The unicity of this morphism is automatic, since any two morphisms of flat *stft* formal R-schemes that induce the same morphism on the generic fibers, are equal (see, for example, [5, proof of Theorem 4.1]). □

Theorem 5.4. *Let X_∞ be a generically smooth stft formal R-scheme, and let $U_\infty \to X_\infty$ be a weak Néron R-smoothening. Then*

$$S(X_\infty) = [U_s] \in K_0(Var_{X_s})/\big(\mathbb{L}_{X_s} - [X_s]\big).$$

Proof. Consider the normalization morphism $\widetilde{X}_\infty \to X_\infty$, and denote by $X_{\infty,i}$, $i = 1, \dots, \ell$, the connected components of $\widetilde{X}_\infty$. We have to show that

$$[U_s] = \sum_{i=1}^{\ell} S(\widetilde{X}_{\infty,i})$$

in $K_0(Var_{X_s})/(\mathbb{L}_{X_s} - [X_s])$.

Proposition 5.3 implies that the morphism $U_\infty \to X_\infty$ factors through a unique morphism $h : U_\infty \to \widetilde{X}_\infty$. It is clear that $h : U_\infty \to \widetilde{X}_\infty$ is a weak Néron R-smoothening for $\widetilde{X}_\infty$, and, for each $i = 1, \dots, \ell$, $h^{-1}(\widetilde{X}_{\infty,i}) \to \widetilde{X}_{\infty,i}$ is a weak Néron R-smoothening for $\widetilde{X}_{\infty,i}$. By Definition 2.10, we have

$$\big[\big(h^{-1}(\widetilde{X}_{\infty,i})\big)_s\big] = S(\widetilde{X}_{\infty,i})$$

in $K_0(Var_{X_s})/(\mathbb{L}_{X_s} - [X_s])$. Now we can apply the additivity of $[.]$. □

Definition 5.5. For any separated quasi-compact rigid variety X_η, smooth over K, the motivic Serre invariant $S(X_\eta)$ is defined as

$$S(X_\eta) := \sum_{i=1}^{\ell} S(X_{\eta,i}) \in K_0(Var_k)/(\mathbb{L} - 1)$$

where $X_{\eta,1}, \dots, X_{\eta,\ell}$ are the connected components of X_η, and $S(X_{\eta,i})$ is the motivic Serre invariant defined in [16, 4.5].

For any *stft* formal R-model X_∞ of X_η, the Serre invariant $S(X_\eta)$, is the image of $S(X_\infty)$ under the forgetful morphism

$$K_0(Var_{X_s})/\big(\mathbb{L}_{X_s} - [X_s]\big) \to K_0(Var_k)/(\mathbb{L} - 1).$$

5.2. Bounded rigid varieties

If K'/K is a finite extension, and X_η is a smooth quasi-compact rigid variety over K', then $\prod_{K'/K} X_\eta$ is not necessarily quasi-compact, so in order to associate a motivic Serre invariant to this Weil restriction, we have to extend the construction.

Definition 5.6. We say a rigid variety X over K is bounded, if there exists a quasi-compact open subspace V of X, such that $V(K^{sh}) = X(K^{sh})$.

There exists an important natural class of bounded rigid varieties.

Definition 5.7. A special formal R-scheme is a separated Noetherian adic formal scheme $\mathfrak{X}$ over R, such that $\mathfrak{X}/J$ is a scheme of finite type over R, for any ideal of definition J on $\mathfrak{X}$.

Berthelot constructs in [3, 0.2.6] the generic fiber $\mathfrak{X}_\eta$ of a special formal R-scheme $\mathfrak{X}$ (this is carefully explained in [8, §7]). This generic fiber $\mathfrak{X}_\eta$ is a separated rigid variety over K, not quasi-compact in general.

Proposition 5.8. *If $\mathfrak{X}$ is a special formal R-scheme, then its generic fiber $\mathfrak{X}_\eta$ is a bounded rigid variety over K.*

Proof. We may assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Let J be the biggest ideal of definition on $\mathfrak{X}$. Following [8, 7.1], we denote by $A[J/\pi]$ the subalgebra of $A \otimes_R K$ generated by A and elements of the form j/π with j in J, and we denote by B_1 the π-adic completion of $A[J/\pi]$. This is an algebra topologically of finite type over R, and if we put $C_1 = B_1 \otimes_R K$, then $\operatorname{Sp} C_1$ is a quasi-compact open subspace of $\mathfrak{X}_\eta$ in a natural way.

By [8, 7.1.9], the points of $\mathfrak{X}_\eta$ correspond canonically and bijectively to the maximal ideals of $A \otimes_R K$. Hence, it suffices to prove that any morphism of K-algebras $A \otimes_R K \to K^{sh}$ factors through the natural map $A \otimes_R K \to C_1$, or, equivalently, that any continuous morphism of R-algebras $A \to R^{sh}$ factors through $A \to A[J/\pi]$. This, however, is clear, since continuity of $A \to R^{sh}$ guarantees that the image of any element j of J belongs to the maximal ideal of R^{sh}. $\square$

Now we define motivic integrals on bounded rigid varieties, using appropriate quasi-compact models. We refer to [16] for the theory of motivic integrals of differential forms on smooth rigid varieties. We recall that $\mathcal{M}_k$ denotes the localized Grothendieck ring $K_0(Var_k)[\mathbb{L}^{-1}]$, and that $\widehat{\mathcal{M}_k}$ denotes its dimensional completion (see for instance [9, 4.3]).

Proposition 5.9. *Let X be a separated smooth rigid variety over K of pure dimension m, and let ω be a section of $\Omega^m_{X/K}(X)$. If V_1, V_2 are quasi-compact open subspaces of X such that $V_1(K^{sh}) = V_2(K^{sh}) \subset X(K^{sh})$, then*

$$\int_{V_1} |\omega| = \int_{V_2} |\omega| \quad \text{in } \widehat{\mathcal{M}_k}.$$

If ω is a gauge form on X, then this equality holds already in $\mathcal{M}_k$.

Proof. Passing to the union $V_1 \cup V_2$, we may as well assume that $V_1 \subset V_2$. By [5, 4.4], we can find a formal R-model X_∞ for V_2, and an open formal subscheme U_∞ of X_∞ such that $V_1 = U_\eta$. By the assumption on V_1 and V_2, the induced morphism $\mathrm{Gr}^R(U_\infty) \to \mathrm{Gr}^R(X_\infty)$ is an isomorphism (it is an open immersion by [21, 3.3.2], and it is bijective by [16, 2.7.3]). This concludes the proof. $\square$

Definition 5.10. If X is a bounded separated smooth rigid variety over K, and ω is a differential form on X which is of maximal degree on each connected component of X, then we choose a quasi-compact open subspace V of X with $V(K^{sh}) = X(K^{sh})$, and we put

$$\int_X |\omega| := \int_V |\omega| \quad \text{in } \widehat{\mathcal{M}}_k.$$

If ω is a gauge form on X, we put

$$\int_X |\omega| := \int_V |\omega| \quad \text{in } \mathcal{M}_k.$$

This definition does not depend on the choice of V, by Proposition 5.9.

Definition 5.11. If X is a bounded separated smooth rigid variety over K, then we choose a quasi-compact open subspace V of X with $V(K^{sh}) = X(K^{sh})$, and we define the motivic Serre invariant of X by

$$S(X) := S(V) \quad \text{in } K_0(Var_k)/(\mathbb{L}-1).$$

This definition does not depend on the choice of V, by Proposition 5.9. In fact, we have the following result, which follows immediately from the definitions.

Proposition 5.12. *If ω is any gauge form on X, then*

$$S(X) = \int_X |\omega| \quad \text{in } \mathcal{M}_k/(\mathbb{L}-1) \cong K_0(Var_k)/(\mathbb{L}-1).$$

5.3. Motivic Serre invariants and Weil restriction

Now we check how the motivic Serre invariant behaves under Weil restriction. If X_∞ is a nice *stft* formal R-scheme, $\prod_{K'/K} X_\eta$ might not be quasi-compact (it is, though, if K'/K is separable, by Proposition 2.5(3)). However:

Lemma 5.13. *Let K' be a finite extension of K, and denote by R' the normalization of R in K'. If X_∞ is a nice stft formal R'-scheme, then $\prod_{K'/K} X_\eta$ is a bounded rigid variety over K. In fact, the canonical open embedding*

$$\Bigl(\prod_{R'/R} X_\infty\Bigr)_\eta \to \prod_{K'/K} X_\eta$$

induces a bijection on the K^{sh}-valued points.

Proof. This follows immediately from Proposition 2.5(4). □

Corollary 5.14. *If, moreover, X_∞ is generically smooth, then*

$$S\Bigl(\prod_{K'/K} X_\eta\Bigr) = S\Bigl(\Bigl(\prod_{R'/R} X_\infty\Bigr)_\eta\Bigr)$$

in $K_0(Var_k)/(\mathbb{L}-1)$.

Lemma 5.15. *Let K' be a finite extension of K, and let R' be the normalization of R in K'. Let X_∞ be a nice, generically smooth stft formal scheme over R', and let $Y_\infty \to X_\infty$ be a weak Néron R'-smoothening, with Y_∞ nice. The Weil restriction $\prod_{R'/R} Y_\infty \to \prod_{R'/R} X_\infty$ is a weak Néron R-smoothening.*

Proof. Since Y_∞ is smooth over $\mathrm{Spf}\, R'$, the formal scheme $\prod_{R'/R} Y_\infty$ is smooth over $\mathrm{Spf}\, R$, by Proposition 2.5(2). By Proposition 2.5(6), the morphism $(\prod_{R'/R} Y_\infty)_\eta \to (\prod_{R'/R} X_\infty)_\eta$ is an open embedding. If L is any finite unramified extension of K, Proposition 2.5(4) implies

$$\Big(\prod_{R'/R} Y_\infty\Big)_\eta(L) = \Big(\prod_{R'/R} X_\infty\Big)_\eta(L). \qquad \square$$

Remark. If X_∞ is nice, we can always find a Néron R'-smoothening $Y_\infty \to X_\infty$ such that Y_∞ is nice, by [6, 3.1] and Lemma 2.3.

Theorem 5.16. *Let R' be a finite totally ramified extension of R. Let X_∞ be a generically smooth stft-formal scheme over R'. Suppose that X_∞ is nice. Then*

$$S\Big(\prod_{R'/R} X_\infty\Big) = S(X_\infty) \in K_0(Var_{X_s})/\big(\mathbb{L}_{X_s} - [X_s]\big)$$

where the left-hand side carries a X_s-structure via the morphism of k-schemes $\widetilde{\theta}(X_\infty): (\prod_{R'/R} X_\infty)_s \to X_s$ from Section 4.

Proof. Let $Y_\infty \to X_\infty$ be a weak Néron R'-smoothening for X_∞, with Y_∞ nice over R'. The Weil restriction $\prod_{R'/R} Y_\infty \to \prod_{R'/R} X_\infty$ is a weak Néron R-smoothening for $\prod_{R'/R} X_\infty$, by Proposition 5.15. By Theorem 5.4, $[(\prod_{R'/R} Y_\infty)_s] = S(\prod_{R'/R} X_\infty)$ in $K_0(Var_{X_s})/(\mathbb{L}_{X_s} - [X_s])$. Applying Corollary 4.3 concludes the proof. $\square$

Corollary 5.17. *With the notation and hypotheses from Theorem* 5.16*,*

$$S\Big(\prod_{K'/K} X_\eta\Big) = S(X_\eta)$$

in $K_0(Var_k)/(\mathbb{L} - 1)$.

Proof. This follows from Theorem 5.16 and Corollary 5.14. $\square$

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