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Finitely generated algebras with involution and multiplicities bounded by a constant

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ABSTRACT

Let A be an algebra with involution $*$ over a field F of characteristic zero, and let $\chi_n^*(A)$, $n = 1, 2, \cdots$, be the sequence of $*$-cocharacters of A. For every $n \geq 1$, let $l_n^*(A)$ denote the nth $*$-colength of A which is the sum of the multiplicities in $\chi_n^*(A)$. In this article, we classify in two different ways the finitely generated $*$-algebras satisfying an ordinary polynomial identity whose multiplicities of the $*$-cocharacters $\chi_n^*(A)$ are bounded by a constant. As a consequence this also yields a characterization of the $*$-varieties whose $*$-colength $l_n^*(A)$, $n = 1, 2, \ldots$, is bounded by a constant.

1. Introduction

Let F be a field of characteristic 0 and A an associative algebra with involution $*$ over F. In the last years, several authors have extensively studied the $*$-identities satisfied by A (see for instance [5,12,14]). In particular some of the results are about the asymptotic behavior of two special numerical sequences associated to A. The first one

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is the sequence of $*$-codimensions $c_n^*(A)$, given by the corresponding degrees measuring the growth of the ideal of $*$-identities of A, and the second is the sequence of $*$-colengths $l_n^*(A)$ given by the sum of the multiplicities in the decomposition of the $*$-cocharacter $\chi_n^*(A)$, for $n \geq 1$. Here we study a specific property that connects those two sequences.

In [8] it was noticed that, as in the ordinary (non-involution) case, if A satisfies a nontrivial $*$-identity, then the sequence of $*$-codimensions is exponentially bounded and while in the theory of associative algebras the space of multilinear identities of degree n is studied through the ordinary representation theory of the symmetric group S_n, in case of algebras with involution we exploit the representation theory of $H_n = \mathbb{Z}_2 \wr S_n$, the hyperoctahedral group of degree n (see [8]).

We denote by $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ the free algebra with involution freely generated by a countable set of indeterminates $X = \{x_1, x_2, \ldots\}$ over F. Recall that a polynomial $f(x_1, x_1^*, \ldots, x_n, x_n^*)$ is a $*$-identity of A if $f(a_1, a_1^*, \ldots, a_n, a_n^*) = 0$ for all $a_1, \ldots, a_n \in A$; it is well known that, in characteristic zero, the T-ideal $Id(A, *)$ of $*$-identities of A is completely determined by its multilinear polynomials. Let P_n^* be the space of all multilinear polynomials of degree n in $x_1, x_1^*, \ldots, x_n, x_n^*$. We shall consider an action of the group $H_n = \mathbb{Z}_2 \wr S_n$ on P_n^*.

We note that $\dim_F P_n^* = 2^n n!$. We can also observe that the space P_n^* modulo $Id(A, *) \cap P_n^*$ has a natural structure of left H_n-module and $c_n^*(A)$ is its dimension, known as the nth $*$-codimension of A, while $\chi_n^*(A)$ is its character, known as the nth $*$-cocharacter of A.

By complete reducibility we write such a character as a sum of irreducible characters as below

$$\chi_n^*(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} \chi_{\lambda,\mu}$$

where λ and μ are partitions of r and of $n-r$, respectively, $m_{\lambda,\mu} \geq 0$ is the corresponding multiplicity of the irreducible H_n-character $\chi_{\lambda,\mu}$ associated to the pair (λ, μ). To simplify the notation we shall use $|\lambda|+|\mu| = n$ to indicate $\lambda \vdash r$ and $\mu \vdash n-r$ for all $r = 0, 1, \ldots, n$ and so

$$\chi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} \chi_{\lambda,\mu}. \tag{1.1}$$

In [6] Giambruno and Mishchenko characterized the ideals of $*$-identities in case the corresponding multiplicities are bounded by one. In this paper, we characterize in two different ways the ideal of $*$-identities of a finitely generated algebra A with involution in case the multiplicities are bounded by a constant. In fact we shall prove that the multiplicities in $\chi_n^*(A)$ are bounded by a constant if and only if $Id(A, *)$ contains at least one $*$-polynomial which is not a $*$-identity of a specific subalgebra of the algebra of 4×4 upper triangular matrices endowed with the involution obtained by reflecting a matrix along its secondary diagonal.

Another characterization will be given in terms of H_n-characters and we prove that the multiplicities are bounded by a constant if and only if for any irreducible character $\chi_{\lambda,\mu}$ appearing in $\chi_n^*(A)$ with non-zero multiplicity, the number of boxes out of the first row of the Young diagram D_λ plus the number of boxes out of the first three rows of the Young diagram D_μ is not greater than a constant q.

As a corollary of our main theorem we obtain the characterization that $l_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}$ is bounded by a constant if and only if the $*$-codimensions $c_n^*(A)$ are polynomially bounded.

We remark that our characterizations are motivated by the results obtained by Mishchenko et al. in [11] concerning the ordinary case and also by the results obtained by Cirrito and Giambruno in [2] concerning the graded case.

2. Preliminaries

Throughout this paper, we will denote by F a field of characteristic zero. An anti-automorphism $*$ of the second order of an associative algebra A over F is called an involution. For an algebra A with involution $*$ we have $A = A^+ \oplus A^-$ where A^+ is the subspace formed by all symmetric elements, i.e. such that $a^* = a$, and A^- is the subspace of all skew elements, i.e. such that $a^* = -a$, with $a \in A$.

It is useful to regard the free algebra with involution $F\langle X, *\rangle$ as generated by symmetric and skew variables: if for $i = 1, \dots, n$ we let $s_i = x_i + x_i^*$ and $t_i = x_i - x_i^*$ then $F\langle X, *\rangle = F\langle s_1, t_1, s_2, t_2, \dots\rangle$ and P_n^* will be the space of multilinear polynomials of degree n in $s_1, t_1, \dots, s_n, t_n$. So given a monomial f in P_n^* either s_i or t_i appears in f at degree 1 (but not both), for any $i = 1, \dots, n$.

Recall that if $\mathbb{Z}_2 = \{1, *\}$ is the multiplicative group of order 2, then the hyperoctahedral group H_n is the wreath product $\mathbb{Z}_2 \wr S_n$ and the space P_n^* has a structure of left H_n-module induced by defining for $k = (a_1, \dots, a_n; \sigma) \in H_n$, a natural action: $ks_i = s_{\sigma(i)}$ and $kt_i = t_{\sigma(i)}$ or $-t_{\sigma(i)}$ according to whether $a_{\sigma(i)} = 1$ or $*$, respectively.

The set $Id(A, *)$ of all $*$-identities of an F-algebra with involution A is a T-ideal of $F\langle X, *\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle X, *\rangle$ commuting with the involution $*$.

It is well known that since $\operatorname{char} F = 0$, the ideal $Id(A, *)$ is determined by the multilinear $*$-polynomials it contains and since the space $Id(A, *) \cap P_n^*$ is invariant under the H_n action, the space

$$P_n(A, *) := P_n^* / \big(Id(A, *) \cap P_n^*\big)$$

has a structure of left H_n-module and its character $\chi_n^*(A)$ has the decomposition given in (1.1).

From now on, y_i and z_i will always denote independent symmetric and skew variables, respectively. For a fixed $r = 0, \dots, n$ we will denote $y_1 = s_1, \dots, y_r = s_r$ and $z_1 = t_{r+1}, \dots, z_{n-r} = t_n$ and the space $P_{r,n-r}$ of multilinear polynomials in

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$s_1, \dots, s_r, t_{r+1}, \dots, t_n$ can be seen as the space of multilinear polynomials in the symmetric variables $y_1, \dots, y_r$ and skew variables $z_1, \dots, z_{n-r}$. If we let S_r act on the symmetric variables $y_1, \dots, y_r$ and S_{n-r} act on the skew variables $z_1, \dots, z_{n-r}$, we obtain an action of $S_r \times S_{n-r}$ on $P_{r,n-r}$. Since T-ideals are invariant under permutations of symmetric (respectively skew) variables, we get that

$$P_{r,n-r}(A,*) := P_{r,n-r}/\big(Id(A,*) \cap P_{r,n-r}\big)$$

has an induced structure of left $S_r \times S_{n-r}$-module and we write $\psi_n^*(A)$ for its character. By complete reducibility we decompose

$$\psi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m'_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu) \tag{2.1}$$

where χ_λ (respectively χ_μ) denotes the usual S_r-character (respectively S_{n-r}-character), $\chi_\lambda \otimes \chi_\mu$ is the irreducible $S_r \times S_{n-r}$-character associated to the pair of partitions (λ, μ) and $m'_{\lambda,\mu}$ is the corresponding multiplicity.

In [4], Drensky and Giambruno proved that the relationship between the multiplicities of the characters in the decompositions (1.1) and (2.1) is given by

$$m_{\lambda,\mu} = m'_{\lambda,\mu}, \quad \text{for all } \lambda \vdash r \text{ and } \mu \vdash n-r.$$

In this way, we can use the action of $S_r \times S_{n-r}$ on the space $P_n(A,*)$ in order to obtain the multiplicities $m_{\lambda,\mu}$ in $\chi_n^*(A)$, for $\lambda \vdash r$ and $\mu \vdash n-r$.

A numerical sequence that can be attached to an algebra A with involution whose nth $*$-cocharacter has a decomposition as in (1.1) is given by the sequence of $*$-colengths:

$$l_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}.$$

It was proved in [1] that if A satisfies a nontrivial identity, the sequence $l_n^*(A)$ is polynomially bounded.

In the language of varieties of algebras (see [3]), if $\mathcal{V}$ is a variety of algebras generated by an algebra A with involution, that is, a $*$-variety $\mathcal{V} = var(A,*)$ then we have $\chi_n^*(\mathcal{V}) = \chi_n^*(A)$ and also $l_n^*(\mathcal{V}) = l_n^*(A)$.

Next we present a basic result about the sequences of $*$-cocharacters and $*$-colenghts. The proof is trivial and will be omitted (see [9]).

Lemma 2.1. *Let A and B be two algebras with involution whose $*$-cocharacters have the following decompositions*

$$\chi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu} \quad \text{and} \quad \chi_n^*(B) = \sum_{|\lambda|+|\mu|=n} m'_{\lambda,\mu}\chi_{\lambda,\mu}.$$

Then:

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(1) *If* $B \in var(A,*)$ *then* $m'_{\lambda,\mu} \le m_{\lambda,\mu}$ *for all* λ, μ *such that* $|\lambda| + |\mu| = n$. *As a consequence,* $l_n^*(B) \le l_n^*(A)$ *for all* n.
(2) *The direct sum* $A \oplus B$ *is also an algebra with involution induced by the involutions defined on* A *and* B *and if* $\chi_n^*(A \oplus B) = \sum_{|\lambda|+|\mu|=n} \bar{m}_{\lambda,\mu}\chi_{\lambda,\mu}$ *is the decomposition of its nth* $*$*-cocharacter then* $\bar{m}_{\lambda,\mu} \le m_{\lambda,\mu} + m'_{\lambda,\mu}$ *for all* λ, μ *such that* $|\lambda| + |\mu| = n$.

3. $*$-Varieties with polynomial growth

Given an algebra A with involution over F, there is another numerical sequence that can be attached to it:

$$c_n^*(A) := \dim_F P_n(A,*), \quad n = 1, 2, \dots$$

called the sequence of $*$-codimensions of A, giving a measure of the $*$-polynomial identities satisfied by A. If $\mathcal{V}$ is a $*$-variety generated by A then we write $c_n^*(\mathcal{V}) = c_n^*(A)$.

We say that a $*$-variety $\mathcal{V}$ has polynomial growth if there exist k, t such that $c_n^*(\mathcal{V}) \le kn^t$; finally we say that $\mathcal{V}$ has almost polynomial growth if $c_n^*(\mathcal{V})$ cannot be bounded by any polynomial function but any proper subvariety of $\mathcal{V}$ has polynomial growth.

Next we introduce two algebras with involution generating $*$-varieties with almost polynomial growth. The first example is the algebra $D = F \oplus F$ with exchange involution $\bar{*}$ given by $(a,b)^{\bar{*}} = (b,a)$ constructed by Giambruno and Mishchenko in [5]. The authors proved that D has almost polynomial growth and

$$\chi_n^*(D) = \sum_{j=0}^{n} \chi_{(n-j),(j)} \quad \text{and} \quad l_n^*(D) = n+1, \text{ for all } n \ge 1. \tag{3.1}$$

Now we consider the algebra $UT_k(F)$ of $k \times k$ upper triangular matrices endowed with an involution ρ defined by flipping a matrix along its secondary diagonal. The following subalgebra of $UT_4(F)$ with induced involution also generates a $*$-variety with almost polynomial growth:

$$M = F(e_{11} + e_{44}) \oplus Fe_{12} \oplus F(e_{22} + e_{33}) \oplus Fe_{34} \tag{3.2}$$

and we have

$$\begin{pmatrix} x & y & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & w \\ 0 & 0 & 0 & x \end{pmatrix}^{\rho} = \begin{pmatrix} x & w & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & y \\ 0 & 0 & 0 & x \end{pmatrix}$$

for some $x, y, z, w \in F$.

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Clearly M is a 4-dimensional algebra and if we set $a = e_{11} + e_{44}$, $b = e_{22} + e_{33}$, $c = e_{12}$, $c^\rho = e_{34}$ then $M = \text{span}\{a, b, c, c^\rho\}$ with the following multiplication table

$$\begin{array}{c|cccc} & a & b & c & c^\rho \\ \hline a & a & 0 & c & 0 \\ b & 0 & b & 0 & c^\rho \\ c & 0 & c & 0 & 0 \\ c^\rho & c^\rho & 0 & 0 & 0 \\ \hline \end{array} \tag{3.3}$$

In [12], Mishchenko and Valenti proved that M has almost polynomial growth and also that the multiplicity $m_{\lambda,\mu}$ of an irreducible character in the decomposition of the nth $*$-cocharacter $\chi_n^*(M)$ is equal to $q + 1$ if either

$$\begin{aligned} &(1) \quad \lambda = (p + q, p), \quad \mu = (1), \quad \text{for all } p \geq 0,\ q \geq 0 \text{ or} \\ &(2) \quad \lambda = (p + q, p), \quad \mu = \emptyset, \quad \text{for all } p \geq 1,\ q \geq 0 \text{ or} \\ &(3) \quad \lambda = (p + q, p, 1), \quad \mu = \emptyset, \quad \text{for all } p \geq 1,\ q \geq 0. \end{aligned} \tag{3.4}$$

In all other cases $m_{\lambda,\mu} = 0$, except the case $m_{\lambda,\emptyset} = 1$. Then, it follows that

$$l_n^*(M) = \begin{cases} \frac{3n^2+5}{4}, & n \text{ odd} \\ \frac{3n^2+4}{4}, & n \text{ even.} \end{cases} \tag{3.5}$$

In fact the algebras $F \oplus F$ and M with the involutions defined above are important in the characterization of $*$-varieties of polynomial growth given by Giambruno and Mishchenko in [7].

Theorem 3.1. *(See [7, Theorem 4.7].) Let $\mathcal{V}$ be a $*$-variety. Then $\mathcal{V}$ has polynomial growth if and only if $(F \oplus F, \bar{*}), (M, \rho) \notin \mathcal{V}$.*

We finish this section with the following remarks which will be useful in the future.

Remark 3.2. The $*$-identities of $M_2(F)$ with transpose involution are consequences of the $*$-identities of M with involution ρ (see [10] and [12]). Thus $(M, \rho) \in var(M_2(F), t)$.

Remark 3.3. The algebra $M_2(F)$ with symplectic involution s contains a subalgebra with induced involution isomorphic to $F \oplus F$ with exchange involution. In fact, it is enough to consider $C = Fe_{11} \oplus Fe_{22} \subseteq M_2(F)$ and it is not difficult to see that (C, s) is isomorphic to $(F \oplus F, \bar{*})$.

4. $*$-Identities and the Wedderburn–Malcev decomposition

It is well known that an analogue of the Wedderburn–Malcev decomposition holds for finite dimensional algebras with involution.

Theorem 4.1. *(See [9, Theorems 3.4.3 and 3.4.4].) Any finite dimensional F-algebra with involution $*$ has a decomposition of the form*

$$A = A_1 \oplus \cdots \oplus A_m + J$$

where the Jacobson radical $J = J(A)$ is a $*$-ideal and A_i is a $*$-simple subalgebra of A, for all $i = 1, \cdots, m$.

If we consider F an algebraically closed field, the Wedderburn–Malcev decomposition of a finite dimensional $*$-algebra given above can be described in more details. In fact, since F is algebraically closed by Theorem 3.4.4 in [9] we have that each A_i is a $*$-simple algebra of one of the following types:

(1) $(M_k(F), t)$ – the full matrix algebra with the transpose involution $* = t$,
(2) $(M_l(F), s)$ – the full matrix algebra with the symplectic involution $* = s$,
(3) $(M_p(F) \oplus M_p(F)^{op}, \bar{*})$ – the direct sum of the full matrix algebra and its opposite algebra with the exchange involution $(a, b)^{\bar{*}} = (b, a)$.

Remark 4.2. The condition for the base field F to be algebraically closed is necessary for the description of the $*$-simple components A_i, $1 \leq i \leq m$. On the other hand any finite dimensional F-algebra with involution A can be naturally embedded in the $*$-algebra $A \otimes_F \bar{F}$ which is finite dimensional over the algebraic closure $\bar{F} \supseteq F$. By this argument the study of identities with involution can be reduced to finite dimensional algebras A with Wedderburn–Malcev decomposition $A_1 \oplus \cdots \oplus A_m + J$ where A_i are $*$-simple algebras specified in (1), (2), (3) above.

We shall assume without lost of generality that a finite dimensional $*$-algebra A over a field F of characteristic zero has a Wedderburn–Malcev decomposition such that the $*$-simple components A_i satisfy the claims above the previous remark.

We will also evoke the following theorem recently proved by Sviridova in [14].

Theorem 4.3. *(See [14, Theorem 1].) Let F be a field of zero characteristic. Then the ideal of $*$-identities of a finitely generated F-algebra with involution coincides with the ideal of $*$-identities of some finite dimensional F-algebra with involution.*

Lemma 4.4. *Let A be an algebra with involution $*$ over a field F of characteristic zero and suppose that $var(A, *)$ does not contain any algebra isomorphic to $(M_2(F), t)$. Then $(M_2(F) \oplus M_2(F)^{op}, \bar{*}), (M_4(F), s) \notin var(A, *)$.*

Proof. First we suppose that $(M_4(F), s) \in var(A, *)$. In this case we can consider its subalgebra

$$S = \text{span}\{\underbrace{e_{11} + e_{33}}_{a}, \underbrace{e_{14} - e_{32}}_{b}, \underbrace{e_{41} - e_{23}}_{c}, \underbrace{e_{22} + e_{44}}_{d}\} \subseteq M_4(F)$$

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with induced involution, where the e_{ij}'s are the usual matrix units of $M_n(F)$, $1 \le i,j \le n$.

Then by considering $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ the basis of matrix units of $M_2(F)$ we see that the application $\varphi : M_2(F) \to S$ given by

$$\varphi(e_{11}) = a, \qquad \varphi(e_{12}) = b, \qquad \varphi(e_{21}) = c \quad \text{and} \quad \varphi(e_{22}) = d$$

is an isomorphism such that $\varphi(X^t) = \varphi(X)^s$ for all $X \in M_2(F)$, that is, S is an algebra with involution isomorphic to $(M_2(F), t)$ and $S \in var(A, *)$, a contradiction.

Next we suppose that $B = (M_2(F) \oplus M_2(F)^{op}, \bar{*}) \in var(A, *)$ and consider the elements $U = (e_{11}, e_{11})$, $V = (e_{12}, e_{21})$, $V^{\bar{*}} = (e_{21}, e_{12})$ and $W = (e_{22}, e_{22})$ in B. Then by setting $H = \text{span}\{U, V, V^{\bar{*}}, W\} \subseteq B$ the association

$$e_{11} \mapsto U, \qquad e_{12} \mapsto V, \qquad e_{21} \mapsto V^{\bar{*}}, \qquad e_{22} \mapsto W$$

is clearly an isomorphism preserving the involutions t and $\bar{*}$ of $M_2(F)$ and H, respectively. This completes the proof. □

As a consequence we get the following result.

Corollary 4.5. *Let A be a finite dimensional F-algebra with involution $*$ and let $A = A_1 \oplus \cdots \oplus A_m + J$ be its Wedderburn–Malcev decomposition. If A does not contain any subalgebra isomorphic to $(M_2(F), t)$ then for each $i = 1, \cdots, m$, either $A_i \cong F$ with $F^* = F$ or $A_i \cong (F \oplus F, \bar{*})$ or $A_i \cong (M_2(F), s)$.*

Lemma 4.6. *Let A be a finite dimensional F-algebra with involution $*$ such that $var(A, *)$ does not contain any algebra isomorphic to (M, ρ), where (M, ρ) is given by (3.2). Then $var(A, *) = var(B_1 \oplus \cdots \oplus B_m, *)$ where for $1 \le i \le m$, either*

(i) *$B_i \cong F + J_i$ with $F^* = F$, or*
(ii) *$B_i \cong F \oplus F + J_i$ and $F \oplus F$ has exchange involution $\bar{*}$, or*
(iii) *$B_i \cong M_2(F) + J_i$ and $M_2(F)$ has symplectic involution s,*

and J_i is the Jacobson radical of B_i.

Proof. By Remark 3.2, A does not contain a subalgebra isomorphic to $(M_2(F), t)$ then by the previous corollary we can decompose

$$A = (A_1 \oplus \cdots \oplus A_m) + J$$

where $J = J(A)$ and for each $i = 1, \cdots, m$ we have

$$\text{either } A_i \cong F \text{ with } F^* = F \quad \text{or} \quad A_i \cong (F \oplus F, \bar{*}) \quad \text{or} \quad A_i \cong \big(M_2(F), s\big). \tag{4.1}$$

Suppose that there exist two $*$-simple components, say A_1 and A_2 such that $A_1JA_2 \neq 0$. If, for instance, $A_1 \cong (M_2(F), s)$ then according to Remark 3.3, A_1 contains a subalgebra $C \cong F \oplus F$ with exchange involution and we also have $CJA_2 \neq 0$. Thus we can reduce our analysis to the case $A_1JA_2 \neq 0$ where either $A_i \cong F$ with $F^* = F$ or $A_i \cong (F \oplus F, \bar{*})$, for $i = 1, 2$.

Let $u \in J$ be such that $e_1ue_2 \neq 0$ where e_1 and e_2 denote the unit elements of A_1 and A_2, respectively. Define $B = A_1 \oplus A_2 + J$, an algebra with induced involution and consider $k \geq 1$ such that $u \in J^k$ and $u \notin J^{k+1}$ and set $\bar{B} = B/J^{k+1}$.

We can write $\bar{B} = C_1 \oplus C_2 + \bar{J}$ where $C_i \cong A_i$, $i = 1, 2$, and $\bar{J}$ is the Jacobson radical of $\bar{B}$. Notice that since J^{k+1} is stable under $*$, $\bar{B}$ has induced involution. Write $\bar{a} = \bar{e}_1$ and $\bar{b} = \bar{e}_2$ for the images of e_1 and e_2, respectively. Then, if we let $\bar{c} = \bar{a}\bar{u}\bar{b}$ we get $\bar{c}^* = \bar{b}\bar{u}^*\bar{a}$.

We now define the algebra $R = \operatorname{span}\{\bar{a}, \bar{b}, \bar{c}, \bar{c}^*\}$. Then $\dim R = 4$ and it is easy to check that R has the same multiplication table given in (3.3).

Hence the algebra $R \cong (M, \rho)$ and since $R \in var(A, *)$ we reach a contradiction. Thus we must have that

$$\text{for all } i \neq k, \quad A_iJA_k = 0 \quad \text{and} \quad A_iA_k = 0. \tag{4.2}$$

Define now $B_i = A_i + J$, $i = 1, \ldots, m$. Then $A = (A_1 \oplus \cdots \oplus A_m) + J = (A_1 + J) + \ldots + (A_m + J) = B_1 + \ldots + B_m$. Furthermore for each $i = 1, \ldots, m$, $J = J_i \subseteq B_i$ is the Jacobson radical of B_i and $B_i/J_i \cong A_i$. So according to (4.1), each B_i satisfies (i), (ii) or (iii) of this lemma.

Next we shall prove that

$$Id(B_1 + \ldots + B_m, *) = Id(B_1, *) \cap \ldots \cap Id(B_m, *) \tag{4.3}$$

and since $A = B_1 + \ldots + B_m$ and $Id(B_1, *) \cap \ldots \cap Id(B_m, *) = Id(B_1 \oplus \cdots \oplus B_m, *)$ this implies that $Id(A, *) = Id(B_1 \oplus \cdots \oplus B_m, *)$, where each B_i is under one of the conditions (i), (ii) or (iii) and in conclusion $var(A, *) = var(B_1 \oplus \cdots \oplus B_m, *)$.

The inclusion $Id(B_1 + \ldots + B_m, *) \subseteq Id(B_1, *) \cap \ldots \cap Id(B_m, *)$ is obvious. Conversely, we shall prove that if $f = f(y_1, \ldots, y_r, z_1, \ldots, z_{n-r}) \in P_{r,n-r}$ is a $*$-polynomial in $Id(B_1, *) \cap \ldots \cap Id(B_m, *)$ then f is a $*$-identity of $A = B_1 + \ldots + B_m$. It suffices to check substitutions in $B_1 \cup \ldots \cup B_m$, that is, substitutions of the type

$$y_i \to \bar{y}_i \in B_1^+ \cup \ldots \cup B_m^+ \quad \text{and} \quad z_j \to \bar{z}_j \in B_1^- \cup \ldots \cup B_m^-.$$

If $\bar{y}_1, \ldots, \bar{y}_r, \bar{z}_1, \ldots, \bar{z}_{n-r} \in B_k$ for a single k, we get a zero value for f because $f \in Id(B_k, *)$. Otherwise, by observing that $B_i = A_i + J$ for all i, there exist k, l with $k \neq l$ such that one of the following occurs: either $\bar{y}_k \in A_k^+$ and $\bar{y}_l \in A_l^+$, or $\bar{z}_k \in A_k^-$ and $\bar{z}_l \in A_l^-$, or $\bar{y}_k \in A_k^+$ and $\bar{z}_l \in A_l^-$. In all cases, by (4.2) we get $\bar{w}_{\sigma(1)} \ldots \bar{w}_{\sigma(n)} = 0$, for any monomial $w_{\sigma(1)} \ldots w_{\sigma(n)}$ in f, $\sigma \in S_n$, under the substitution $w_j \to \bar{y}_j$, for $1 \leq j \leq r$ and $w_t \to \bar{z}_{n-t+1}$ for $r + 1 \leq t \leq n$.

Thus we have proved the equality (4.3) and we are done. □

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5. Bounded multiplicities of the $*$-cocharacters

In this section we shall characterize T-ideals of $*$-identities with multiplicities bounded by a constant.

Recall that if A is a finite dimensional algebra with involution $*$ such that $\dim A^+ = u$ and $\dim A^- = v$ then according to [4, Lemma 1.2] we can refine the decomposition of its nth $*$-cocharacter as follows

$$\chi_n^*(A) = \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda)\leq u \\ h(\mu)\leq v}} m_{\lambda,\mu}\chi_{\lambda,\mu} \tag{5.1}$$

where $h(\lambda)$ (respectively $h(\mu)$) is the height of the Young diagram D_λ (respectively D_μ).

In the next result we shall use the considerations above.

Lemma 5.1. *Let $A = C + J$ be a finite dimensional F-algebra with involution $*$ where $J = J(A)$ is its Jacobson radical and C is a $*$-simple subalgebra of A which is isomorphic to $(M_2(F), s)$. If the $*$-cocharacter of A has a decomposition as in (1.1) then there exists a constant $\tilde{N}_0$ such that $m_{\lambda,\mu} \leq \tilde{N}_0$, for all $n \geq 1$ and $|\lambda| + |\mu| = n$.*

Proof. Let $\dim A = d$. By hypothesis we can consider $\{a_0, a_1, \ldots, a_{u-1}\}$ a basis of A^+ and $\{b_0, b_1, \ldots, b_{v-1}\}$ a basis of A^- such that $a_0 \in C^+$, $a_1, \ldots, a_{u-1} \in J^+$, $b_0, b_1, b_2 \in C^-$ and $b_3, \ldots, b_{v-1} \in J^-$. Since $C \cong (M_2(F), s)$ we also admit that for $0 \leq i, j \leq 2$, we have

$$b_ib_j = 0 \quad \text{if and only if} \quad i = j = 1 \quad \text{or} \quad i = j = 2. \tag{5.2}$$

In fact we note that it is true for $b_0 = e_{11} - e_{22}$, $b_1 = e_{12}$ and $b_2 = e_{21}$ in $M_2(F)$.

Let q be the least positive integer such that $J^q = 0$ and note that we can assume $q \geq 2$. In fact if $q = 1$ then $A \cong (M_2(F), s)$ and in this case the multiplicities are bounded by a constant (see [4, Theorem 4.1]). We shall prove that any multiplicity $m_{\lambda,\mu}$ in (5.1) is not greater than $\tilde{N}_0 = dN_0$, where $N_0 = (q^d)^{uv}$.

We start by considering partitions $\lambda \vdash r$ and $\mu \vdash n - r$ such that $h(\lambda) \leq u$ and $h(\mu) \leq v$. Let (T_λ, T_μ) be a pair of Young tableaux corresponding to (λ, μ); also R_{T_λ} and C_{T_λ} are the subgroups of row and column stabilizers of T_λ, respectively and analogously for T_μ. Let

$$R^+_{T_\lambda} = \sum_{\sigma\in R_{T_\lambda}} \sigma \quad \text{and} \quad C^-_{T_\lambda} = \sum_{\tau\in C_{T_\lambda}} (-1)^\tau\tau \quad (\text{analogously for } T_\mu).$$

It is known that (see [3]) one can construct the quasi-idempotents

$$e_{T_\lambda} = R^+_{T_\lambda}C^-_{T_\lambda} \quad \text{and} \quad e_{T_\mu} = R^+_{T_\mu}C^-_{T_\mu} \tag{5.3}$$

in the group algebras $F[S_r]$ and $F[S_{n-r}]$, respectively. Clearly $e = e_{T_\lambda} e_{T_\mu} = e_{T_\mu} e_{T_\lambda}$ is a quasi-idempotent in the group algebra $F[S_r \times S_{n-r}]$.

For each $j = 1, \ldots, u$ let Y_j^λ be the set of symmetric variables whose indices lie in the jth row of T_λ and for each $k = 1, \ldots, v$ let Z_k^μ be the set of skew variables whose indices lie in the kth row of T_μ.

Then, for every $*$-polynomial $f \in P_{r,n-r}$ we have that the polynomial $ef = e_{T_\lambda} e_{T_\mu} f$ is symmetric on each of the sets $Y_1^\lambda, \ldots, Y_u^\lambda$ and $Z_1^\mu, \ldots, Z_v^\mu$, respectively. Thus, the variables of ef are partitioned into $u + v$ disjoint subsets

$$Y_1^\lambda \cup \cdots \cup Y_u^\lambda \cup Z_1^\mu \cup \cdots \cup Z_v^\mu \tag{5.4}$$

and ef is symmetric on each set as described above.

Note that for $j = 1, \ldots, u$, the set Y_j^λ may be empty if $h(\lambda) < j \leq u$ i.e. if the height of the Young diagram D_λ is less than u. A similar condition holds for the sets Z_k^μ, for $k = 1, \ldots, v$.

Notice that for any $\sigma_1 \in S_r$ and any $\sigma_2 \in S_{n-r}$ we have $\sigma_1 e_{T_\lambda} \neq 0$ and $\sigma_2 e_{T_\mu} \neq 0$ and so for $\eta = (\sigma_1, \sigma_2) \in S_r \times S_{n-r}$ we have $\eta e \neq 0$. It follows that if $f \in P_{r,n-r}$ is a $*$-polynomial such that $ef \neq 0$ then the polynomials ef and ηef generate the same irreducible $S_r \times S_{n-r}$-module.

Now we consider multilinear $*$-polynomials $f_1, \ldots, f_L$ such that f_i and f_j generate different but isomorphic irreducible $S_r \times S_{n-r}$-modules corresponding to the same pair of partitions (λ, μ) for $i \neq j$. Then, by the above, we can choose $\eta_1, \ldots, \eta_L \in S_r \times S_{n-r}$ and a decomposition as in (5.4) such that $\eta_1 f_1, \ldots, \eta_L f_L$ are simultaneously symmetric on Y_j^λ and on Z_k^μ, for all $j = 1, \ldots, u$ and $k = 1, \ldots, v$. Therefore, we shall assume that $f_1, \ldots, f_L$ satisfy the above condition.

Assume by contradiction that $m_{\lambda,\mu} = L > \tilde{N}_0 = d(q^d)^{uv}$. We shall prove that A satisfies a $*$-identity of the type

$$f = t_1 f_1 + \ldots + t_L f_L \tag{5.5}$$

where $t_1, \ldots, t_L \in F$ are not all zero. This will say that the $*$-polynomials $f_1, \ldots, f_L$ are linearly dependent modulo the $*$-identities of A and this is a contradiction. Since f is a multilinear $*$-polynomial it is sufficient to verify that f takes zero value on the given basis $\{a_0, \ldots, a_{u-1}, b_0, \ldots, b_{v-1}\}$.

We start by considering substitutions of a special kind and so let

$$0 \leq \alpha_{j0}^\lambda, \alpha_{j1}^\lambda, \ldots, \alpha_{j(u-1)}^\lambda, \beta_{k0}^\mu, \beta_{k1}^\mu, \ldots, \beta_{k(v-1)}^\mu$$

be integers satisfying the following equalities:

$$\sum_{i=0}^{u-1} \alpha_{ji}^\lambda = \left|Y_j^\lambda\right| \quad \text{and} \quad \sum_{i=0}^{v-1} \beta_{ki}^\mu = \left|Z_k^\mu\right|,$$

for $1 \leq j \leq u$ and $1 \leq k \leq v$.

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Now we set $X_{jk}^{\lambda,\mu} = Y_j^\lambda \cup Z_k^\mu$, $j = 1, \dots, u$, $k = 1, \dots, v$. We say that a substitution γ has type

$$\left(\alpha_{j0}^\lambda, \alpha_{j1}^\lambda, \dots, \alpha_{j(u-1)}^\lambda, \beta_{k0}^\mu, \beta_{k1}^\mu, \dots, \beta_{k(v-1)}^\mu\right), \quad 1 \le j \le u,\ 1 \le k \le v,$$

if we replace the variables in the following way: for fixed j and k, we replace the first α_{j0}^λ symmetric variables from $X_{jk}^{\lambda,\mu}$ by a_0, the next α_{j1}^λ symmetric variables by a_1, and so on up to the last $\alpha_{j(u-1)}^\lambda$ symmetric variables from $X_{jk}^{\lambda,\mu}$ by a_{u-1}; also we replace the first β_{k0}^μ skew variables from $X_{jk}^{\lambda,\mu}$ by b_0, the next β_{k1}^μ skew variables by b_1, and so on up to the last $\beta_{k(v-1)}^\mu$ skew variables from $X_{jk}^{\lambda,\mu}$ by b_{v-1}.

In order to get a non-zero evaluation of f given in (5.5), we should also take into account (5.2) and the exponent q of the Jacobson radical. Thus any substitution from variables in $X_{jk}^{\lambda,\mu}$ should also satisfy the restrictions below

(1) $\beta_{k1}, \beta_{k2} \le 1$ and $\sum_{i=3}^{v-1} \beta_{ki} \le q - 1$,
(2) $\alpha_{j1}^\lambda + \ldots + \alpha_{j(u-1)}^\lambda \le q - 1$, $1 \le j \le u$,
(3) $\alpha_{j0}^\lambda = |Y_j^\lambda| - (\alpha_{j1}^\lambda + \ldots + \alpha_{j(u-1)}^\lambda)$ and $\beta_{k0}^\mu = |Z_k^\mu| - (\beta_{k1}^\mu + \ldots + \beta_{k(v-1)}^\mu)$.

Note that the total number of special substitutions is the total number of distinct types $(\alpha_{j0}^\lambda, \dots, \alpha_{j(u-1)}^\lambda, \beta_{k0}^\mu, \dots, \beta_{k(v-1)}^\mu)$, $1 \le j \le u$, $1 \le k \le v$. Now, from the restrictions (1), (2), (3) above we get that for each $j = 1, \dots, u$, $k = 1, \dots, v$, the number of distinct v-tuples $(\beta_{k0}^\mu, \dots, \beta_{k(v-1)}^\mu)$ is at most $2^2 q^{v-2}$ and the number of distinct u-tuples $(\alpha_{j0}^\lambda, \dots, \alpha_{j(u-1)}^\lambda)$ is at most q^u. Since $2^2 q^{v-2} \le q^v$ we have that the number of distinct $(u+v)$-tuples $(\alpha_{j0}^\lambda, \dots, \alpha_{j(u-1)}^\lambda, \beta_{k0}^\mu, \dots, \beta_{k(v-1)}^\mu)$ is at most $q^u q^v = q^{u+v}$.

Thus, for given $1 \le j \le u$, $1 \le k \le v$ the total number of different special substitutions is less than $q^{u+v} = q^d$. Since the number of pairs (j, k) is uv it follows that the total number N of distinct types of substitutions is less than $N_0 = (q^d)^{uv}$.

So, let us consider all these N distinct special substitutions $\gamma_1, \dots, \gamma_N$ and set

$$\gamma_s(f_i) = \omega_{is} \in A, \quad 1 \le i \le L,\ 1 \le s \le N. \tag{5.6}$$

The matrix (ω_{is}) has L rows and N columns of elements from A, and since we assume $L > dN_0$ we have that the rows of (ω_{is}) are linearly dependent.

Then there exist $t_1, \dots, t_L \in F$, not all equal zero, such that

$$\sum_{i=1}^{L} t_i \omega_{is} = 0, \quad 1 \le s \le N.$$

This, together with (5.6), implies that $\gamma_s(\sum_{i=1}^L t_i f_i) = 0$, $1 \le s \le N$, that is, the $*$-polynomial $f = \sum_{i=1}^L t_i f_i$ takes zero value under all special substitutions γ_s, $1 \le s \le N$. We claim that this implies that $f \in Id(A, *)$. In fact by multilinearity it is enough to check only substitutions φ where the variables are evaluated into elements in the basis $\{a_0, \ldots, a_{u-1}, b_0, \ldots, b_{v-1}\}$.

Let l^λ_{j0} be the number of variables in Y^λ_j mapped by φ in a_0; let l^λ_{j1} be the number of variables in Y^λ_j mapped by φ in a_1 and so on. Let r^μ_{k0} be the number of variables in Z^μ_k mapped by φ in b_0; let r^μ_{k1} be the number of variables in Z^μ_k mapped by φ in b_1 and so on, up to $r^\mu_{k(v-1)}$ which is the number of variables in Z^μ_k mapped by φ in b_v.

Since f is simultaneously symmetric on $Y^\lambda_1, \ldots, Y^\lambda_u, Z^\mu_1, \cdots, Z^\mu_v$ it follows that, for any $\eta \in S_r \times S_{n-r}$ such that $\eta(Y^\lambda_j) = Y^\lambda_j$ and $\eta(Z^\mu_k) = Z^\mu_k$, for all $j = 1, \ldots, u$ and $k = 1, \ldots, v$ we have

$$\varphi(f) = \varphi(\eta f) = (\varphi\eta)f.$$

In particular, we can choose $\eta \in S_r \times S_{n-r}$ such that $\varphi\eta$ is the special substitution φ' corresponding to the type

$$\left(l^\lambda_{j0}, l^\lambda_{j1}, \ldots, l^\lambda_{j(u-1)}, r^\mu_{k0}, r^\mu_{k1}, \ldots, r^\mu_{k(v-1)}\right).$$

Thus $\varphi(f) = \varphi'(f) = 0$ and f is a $*$-identity for A. We conclude that $m_{\lambda,\mu} \leq \tilde{N}_0$ for any pair (λ, μ) and the proof of the lemma is complete. $\square$

The next result is essentially Lemma 7 in [13] for the involution case. The proof uses the same argument as in the previous lemma and will be omitted.

Lemma 5.2. *Let $A = C' + J$ be a finite dimensional F-algebra with involution $*$ where $J = J(A)$ is its Jacobson radical and C' is a $*$-simple subalgebra of A which is isomorphic to either F with $F^* = F$ or $(F \oplus F, \bar{*})$. If the $*$-cocharacter of A has a decomposition as in (1.1) then there exists a constant N'_0 such that $m_{\lambda,\mu} \leq N'_0$, for all $n \geq 1$ and $|\lambda| + |\mu| = n$.*

Next we prove a property of the mutiplicities of the $*$-cocharacters of a finite dimensional algebra A with involution when (M, ρ) is excluded from $var(A, *)$.

Lemma 5.3. *Let A be a finite dimensional F-algebra with involution $*$ such that $(M, \rho) \notin var(A, *)$. Then there exists a constant q such that in (1.1) we have $m_{\lambda,\mu} = 0$ whenever $((|\lambda| - \lambda_1) + (|\mu| - (\mu_1 + \mu_2 + \mu_3)) \geq q$.*

Proof. Let q be the index of nilpotence of the Jacobson radical J of A. Consider (λ, μ) a pair of partitions with $|\lambda| + |\mu| = n$ such that $(|\lambda| - \lambda_1) + (|\mu| - \mu_1 + \mu_2 + \mu_3) \geq q$ and suppose, by contradiction, that $m_{\lambda,\mu} \neq 0$.

Then there exist a pair of Young tableaux (T_λ, T_μ) and a $*$-polynomial $f \in P_{r,n-r}$ such that $ef \notin Id(A, *)$ where $e = e_{T_\lambda} e_{T_\mu}$ as in (5.3) and $F[S_r \times S_{n-r}]ef$ is a minimal left ideal.

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Now let $e' = C^{-}_{T_\lambda} e_{T_\lambda} C^{-}_{T_\mu} e_{T_\mu}$. We have

$$0 \neq F[S_r \times S_{n-r}]e'f \subseteq F[S_r \times S_{n-r}]ef$$

and since $F[S_r \times S_{n-r}]ef$ is a minimal left ideal, it follows that $F[S_r \times S_{n-r}]e'f = F[S_r \times S_{n-r}]ef$. This means that $e'f$ is not a $*$-identity of A and note that $h = e'f$ is alternating on each of the λ_1 disjoint sets of symmetric variables corresponding to the columns of T_λ, and on each of the μ_1 disjoint sets of skew variables corresponding to the columns of T_μ.

We shall reach a contradiction by proving that h vanishes in A. Since f is multilinear, clearly it is sufficient to prove that h vanishes for any substitution by elements of a basis of A.

By hypothesis and Remark 3.2 we have $(M_2(F), t) \notin var(A, *)$. So by Corollary 4.5, A has Wedderburn–Malcev decomposition $A_1 \oplus \cdots \oplus A_m + J$ where for each $i = 1, \cdots, m$ either $A_i \cong F$ with $F^* = F$ or $A_i \cong (F \oplus F, \bar{*})$ or $A_i \cong (M_2(F), s)$. We also have (4.2) as in Lemma 4.6.

We shall fix a basis $\beta = \beta^+ \cup \beta^-$ of A where β^+ (respectively β^-) is the union of the base of symmetric elements (respectively skew elements) of $A_1, \ldots, A_m$ and J.

In order to get a nonzero value of h, by using (4.2) we must replace all the variables by elements of J and by elements of only one $*$-simple component, say A_i.

Since $\dim A_i^+ = 1$ we can substitute at most one element of A_i^+ in each alternating set of symmetric variables.

On the other hand, either $\dim A_i^- \leq 1$ or $\dim A_i^- = 3$ in case $A_i \cong (M_2(F), s)$. In this last case, we can substitute at most three elements of A_i^- in each alternating set of skew variables. Hence, in order to get a nonzero value, we can substitute at most λ_1 elements from A_i^+ and at most $\mu_1 + \mu_2 + \mu_3$ elements from A_i^-. This means that we must substitute at least $(|\lambda| - \lambda_1) + (|\mu| - (\mu_1 + \mu_2 + \mu_3))$ elements from J. Since $J^q = 0$ and by hypothesis, $(|\lambda| - \lambda_1) + (|\mu| - (\mu_1 + \mu_2 + \mu_3)) \geq q$, we obtain that h vanishes under all substitutions.

In case $\dim A_i^- \leq 1$ we can substitute at most one element of A_i^- in each alternating set of skew variables. So in order to get a nonzero value, we can substitute at most λ_1 elements from A_i^+ and at most μ_1 elements from A_i^-. In this way we must substitute at least $(|\lambda| - \lambda_1) + (|\mu| - \mu_1)$ elements from J. Since

$$\big(|\lambda| - \lambda_1\big) + \big(|\mu| - \mu_1\big) \geq \big(|\lambda| - \lambda_1\big) + \big(|\mu| - (\mu_1 + \mu_2 + \mu_3)\big) \geq q,$$

again h vanishes under all substitutions, i.e., h is a $*$-identity of A. With this contradiction the proof is finished. $\square$

Finally we are in a position to prove our main result.

Theorem 5.4. *Let A be a finitely generated F-algebra with involution $*$ satisfying a nontrivial identity. If the $*$-cocharacter of A has a decomposition as in (1.1) then the following conditions are equivalent:*

(1) *There exists a constant N_0 such that for all $n \geq 1$ and $|\lambda| + |\mu| = n$, the inequality*

$$m_{\lambda,\mu} \leq N_0$$

holds.

(2) $(M, \rho) \notin var(A, *)$.

(3) *There exists a constant q such that for all $n \geq 1$ and $|\lambda| + |\mu| = n$ the inequality*

$$\big(|\lambda| - \lambda_1\big) + \big(|\mu| - (\mu_1 + \mu_2 + \mu_3)\big) < q$$

holds whenever $m_{\lambda,\mu} \neq 0$.

Proof. Suppose that A satisfies condition (1) and assume that $(M, \rho) \in var(A, *)$. By (3.4) and Lemma 2.1 we get a contradiction. Therefore, we have that condition (1) implies condition (2). In order to prove the converse, we use Theorem 4.3 and so we may assume that A is a finite dimensional algebra with involution. Now the proof follows from Lemmas 2.1, 4.6, 5.1 and 5.2.

Finally, we prove the equivalence of conditions (2) and (3). As above we may assume that A is a finite dimensional algebra and so by Lemma 5.3, condition (2) implies condition (3). Conversely, we suppose $(M, \rho) \in var(A, *)$. In this case, if $\chi_n^*(M) = \sum_{|\lambda|+|\mu|=n} m'_{\lambda,\mu}\chi_{\lambda,\mu}$ then by (3.4) for $\lambda = (\lambda_1, \lambda_2, 1)$ and $\mu = \emptyset$ we have $m'_{\lambda,\mu} = \lambda_1 - \lambda_2 + 1 > 0$. It turns out that $m_{\lambda,\mu} \neq 0$ for any pair of partitions (λ, μ) with $\mu = \emptyset$ and $|\lambda| - \lambda_1$ arbitrary large and so A does not satisfy condition (3). In this way the proof of the theorem is complete. □

As a consequence, we have the following result about algebras with involution whose $*$-colengths are bounded by a constant.

Corollary 5.5. *Let A be a finite dimensional algebra with involution $*$ over a field of characteristic zero. Then $c_n^*(A)$ is polynomially bounded if and only if $l_n^*(A) \leq K$, for some constant K and for all $n \geq 1$.*

Proof. First we assume that $c_n^*(A)$ is polynomially bounded. Then by Theorem 3.1 it follows that $M, F \oplus F \notin var(A, *)$. Consider the nth $*$-cocharacter

$$\chi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}.$$

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Since $M \notin var(A,*)$, by Theorem 5.4 all the multiplicities $m_{\lambda,\mu}$ are bounded by some constant N_0. On the other hand, by Lemma 5.3 there is a constant q such that

$$m_{\lambda,\mu} = 0 \quad \text{whenever } \big(|\lambda| - \lambda_1\big) + \big(|\mu| - (\mu_1 + \mu_2 + \mu_3)\big) \geq q. \tag{5.7}$$

Furthermore since $F \oplus F \notin var(A,*)$, by [5, Theorem 2] we have $z^m \in Id(A,*)$ for some $m \geq 1$. So by [7, Theorem 2.5] there exists $s \geq 1$ such that

$$z_1 w_1 z_2 w_2 \dots z_s w_s \equiv 0 \quad \text{on } A \tag{5.8}$$

where $w_i's$ are eventually empty words in symmetric and skew variables.

From (5.7) and (5.8) we can conclude that $m_{\lambda,\mu} = 0$ for pairs of partitions (λ,μ) such that $|\lambda| - \lambda_1 \geq q$ or $|\mu| \geq s$. Thus $m_{\lambda,\mu} = 0$ if n is large enough.

Independently of n, only finite number of pairs of partitions (λ,μ) satisfy the conditions $|\lambda| - \lambda_1 < q$ and $|\mu| < s$ and since the multiplicities are bounded by a constant it follows that for all n

$$\sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} = l_n^*(A) \leq K, \quad \text{for some constant } K.$$

Conversely, assume that $l_n^*(A)$ is bounded by some constant. In this case we use Lemma 2.1 and also (3.1) and (3.5) to get that $M, F \oplus F \notin var(A,*)$. But by Theorem 3.1 this implies that A has a polynomially bounded growth which completes the proof of the corollary. $\square$

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