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# Harbourne, Schenck and Secoleanu's Conjecture



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## ABSTRACT

In [2], Conjecture 5.5.2, Harbourne, Schenck and Secoleanu conjectured that, for  $r = 6$  and all  $r \geq 8$ , the artinian ideal  $I = (\ell_1^2, \dots, \ell_{r+1}^2) \subset K[x_1, \dots, x_r]$  generated by the square of  $r+1$  general linear forms  $\ell_i$  fails the Weak Lefschetz property. This paper is entirely devoted to prove this Conjecture. It is worthwhile to point out that half of the Conjecture – namely, the case when the number of variables  $r$  is even – was already proved in [5], Theorem 6.1.

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## 1. Introduction

Ideals generated by powers of linear forms have attracted great deal of attention recently. For instance, their Hilbert function has been the focus of the papers [1,9,2]; and the presence or failure of the Weak Lefschetz Property has been deeply studied in [2,5,7], among others.

Let  $A = R/I$  be a standard graded artinian algebra, where  $R = K[x_1, \dots, x_r]$  and  $K$  is an algebraically closed field of characteristic 0. If  $\ell$  is a linear form then multiplication by  $\ell$  induces a homomorphism from any graded component  $[A]_i$  to the next and the algebra  $A$  is said to have the *Weak Lefschetz Property (WLP)* if the “multiplication by a general linear form has maximal rank from each degree to the next.” There has been a long series of papers determining classes of algebras holding/failing the WLP but much more work remains to be done.

The first result in this direction is due to Stanley [8] and Watanabe [10] and it asserts that the WLP holds for an artinian complete intersection ideal generated by powers of linear forms. In [7], Schenck and Seceleanu gave the nice result that in three variables *any* ideal generated by powers of general linear forms has WLP. In contrast, in [5], we showed by examples that in 4 variables for  $d = 3, \dots, 12$  an ideal generated by the  $d$ -th power of five general linear forms does not have the WLP. Therefore, it is natural to ask when the WLP holds for artinian ideals  $I \subset K[x_1, \dots, x_r]$  generated by powers of  $\geq r+1$  general linear forms. The goal of this short note is to solve Conjecture 5.5.2 in [2] which says that for  $r = 6$  and  $r \geq 8$ , the artinian ideal  $I = (\ell_1^2, \dots, \ell_{r+1}^2) \subset K[x_1, \dots, x_r]$  generated by the square of  $r+1$  general linear forms  $\ell_i$  fails the Weak Lefschetz property. Half of the Conjecture – namely, the case when the number of variables  $r$  is even – was proved in [5], Theorem 6.1 where first we use the inverse system dictionary to relate an ideal  $I \subset K[x_1, \dots, x_r]$  generated by powers of linear forms to an ideal of fat points in  $\mathbb{P}^{r-1}$ , and then we show that the WLP problem of an ideal generated by powers of linear forms is closely connected to the geometry of the linear system of hypersurfaces in  $\mathbb{P}^{r-2}$  of fixed degree with preassigned multiple points. In this short note, we solve the remaining half – the case of an odd number of variables. The key point is to determine the degree where the WLP fails.

Next, we outline the structure of the paper. In Section 2, we fix notation and we briefly discuss general facts on Weak Lefschetz property needed later on. Section 3 is the heart of the paper and contains the proof of the conjecture stated by Harbourne, Schenck and Seceleanu in [2], Conjecture 5.5.2.

## 2. Background and preparatory results

In this section we fix notation, we recall the definition of Weak/Strong Lefschetz Property and we state some open conjectures related to the Weak Lefschetz property of ideals generated by powers of linear forms which are motivating current research in this vast topic of research which touch numerous and different areas like algebraic geometry, commutative algebra and combinatorics.

Throughout this work  $K$  will be an algebraically closed field of characteristic zero. Given a graded artinian  $K$ -algebra  $A = R/I$  where  $R = K[x_1, x_2, \dots, x_r]$  and  $I$  is a homogeneous ideal of  $R$ , we denote by  $H_A : \mathbb{Z} \rightarrow \mathbb{Z}$  with  $H_A(j) = \dim_K[A]_j$  its Hilbert function. Since  $A$  is artinian, its Hilbert function is captured in its  $h$ -vector  $h = (h_0, h_1, \dots, h_e)$  where  $h_i = H_A(i) > 0$  and  $e$  is the last index with this property. The integer  $e$  is called the *socle degree* of  $A$ .

**Definition 2.1.** Let  $A = R/I$  be a graded artinian  $K$ -algebra. We say that  $A$  is *level of Cohen–Macaulay type  $t$*  if its socle is concentrated in one degree and has dimension  $t$ . E.g. a complete intersection is level of Cohen–Macaulay type 1 and a Gorenstein artinian  $K$ -algebra is also level of Cohen–Macaulay type 1.

For fixed  $e$  and  $t$ , a level graded artinian  $K$ -algebra of socle degree  $e$ , Cohen–Macaulay type  $t$  and of maximal Hilbert function among all level graded artinian  $K$ -algebras with that socle degree and Cohen–Macaulay type is said to be *compressed*. We can extend this notion as follows (see [6], Definition 1.1, for more details):

**Definition 2.2.** Let  $\mathfrak{a} \subset R$  be a homogeneous ideal. Then a level graded artinian  $K$ -algebra  $A$  of socle degree  $e$  and Cohen–Macaulay type  $t$  is said to be *relatively compressed with respect to  $\mathfrak{a}$*  if  $A$  has maximal length among all level graded artinian  $K$ -algebras  $R/I$  satisfying

- (i)  $\text{Soc } R/I \cong K(-e)^t$
- (ii)  $\mathfrak{a} \subset I$ .

Equivalently,  $A = R/I$  is relatively compressed with respect to  $\mathfrak{a}$  if it is a quotient of  $R/\mathfrak{a}$  having maximal length among all quotients of  $R/\mathfrak{a}$  with prescribed socle degree and Cohen–Macaulay type.

For level artinian algebras  $A$  relatively compressed with respect to a complete intersection  $\mathfrak{a} \subset R$  we have an upper bound for their Hilbert function. In fact, we have

**Lemma 2.3.** Let  $A = R/I$  be a graded level artinian  $K$ -algebra of socle degree  $e$ , Cohen–Macaulay type  $t$  and relatively compressed with respect to a complete intersection  $\mathfrak{a} \subset R$ . Then, we have

$$h_A(i) \leq \min\{\dim[R/\mathfrak{a}]_i, t \cdot \dim[R/\mathfrak{a}]_{e-i}\}.$$

**Remark 2.4.** For  $t = 1$ , we take  $F \in [\mathfrak{a}_e]^\perp$  and we consider the Gorenstein artinian graded  $K$ -algebra  $A = R/\text{Ann}(F)$ . Because  $R/\text{Ann}(F)$  is a quotient of  $R/\mathfrak{a}$ , which is Gorenstein, we clearly have

$$h_A(i) \leq \min\{\dim[R/\mathfrak{a}]_i, \dim[R/\mathfrak{a}]_{e-i}\}. \quad (2.1)$$

Note that by [4], Theorem 4.16, if  $\mathfrak{a}$  and  $F$  are both general (or if  $\mathfrak{a}$  is a monomial complete intersection and  $F$  is general) then we have equality in (2.1).

**Definition 2.5.** Let  $A = R/I$  be a graded artinian  $K$ -algebra. We say that  $A$  has the *Weak Lefschetz Property* (WLP) if there is a linear form  $L \in [A]_1$  such that, for all integers  $i \geq 0$ , the multiplication map

$$\times L : [A]_i \longrightarrow [A]_{i+1}$$

has maximal rank, i.e. it is injective or surjective. (We will often abuse notation and say that the ideal  $I$  has the WLP.) In this case, the linear form  $L$  is called a Lefschetz element of  $A$ . If for the general form  $L \in [A]_1$  and for an integer number  $j$  the map  $\times L : [A]_{j-1} \longrightarrow [A]_j$  does not have maximal rank, we will say that the ideal  $I$  fails the WLP in degree  $j$ .

$A$  has the *Strong Lefschetz Property* (SLP) if there is a linear form  $L \in [A]_1$  such that, for all integers  $i \geq 0$  and  $k \geq 1$ , the multiplication map

$$\times L^k : [A]_i \longrightarrow [A]_{i+k}$$

has maximal rank.

In this paper we will deal with artinian ideals  $I = (\ell_1^{a_1}, \dots, \ell_s^{a_s}) \subset K[x_1, \dots, x_r]$  generated by powers of  $s \geq r$  general linear forms  $\ell_i$  and we relate them to the presence or failure of the WLP. The first result in this topic is due to R. Stanley [8] and J. Watanabe [10] and it has motivated this entire area of research. It says:

**Proposition 2.6.** Let  $R = K[x_1, \dots, x_r]$  and let  $\mathfrak{a}$  be an artinian monomial complete intersection, i.e.  $\mathfrak{a} = (x_1^{a_1}, \dots, x_r^{a_r})$ . Then,  $R/I$  has both the WLP and the SLP.

It is worthwhile to point out that the above result is false in positive characteristic. Trying to extend the above proposition we can ask which ideals generated by powers of general linear forms define  $K$ -algebras that fail WLP. In two and three variables all such algebras satisfy the WLP. In fact, we have:

**Proposition 2.7.**

- (1) Any homogeneous artinian ideal in  $K[x, y]$  has WLP. In particular, every artinian ideal generated by powers of linear forms in 2 variables has the WLP.
- (2) Every artinian ideal generated by powers of general linear forms in 3 variables has the WLP.

**Proof.** (1) See [3], Proposition 4.4.

(2) See [7], Theorem 2.4.  $\square$

The above proposition is no longer true for ideals generated by powers of linear forms in  $r \geq 4$  variables. In fact, in [5] we prove that  $A := K[x_1, x_2, x_3, x_4]/(x_1^3, x_2^3, x_3^3, x_4^4, (x_1 + x_2 + x_3 + x_4)^3)$  does not have the WLP. The  $h$ -vector of  $A$  is  $(1, 4, 10, 15, 15, 6)$  and  $A_3 \rightarrow A_4$  has not maximal rank. The following questions naturally arise from this example:

**Question 2.8.**

- (1) (Almost complete intersections and uniform powers.) Let  $I = (\ell_1^t, \dots, \ell_{r+1}^t) \subset K[x_1, \dots, x_r]$  be an almost complete intersection ideal generated by uniform powers of general linear forms. For which values of  $r$  and  $t$  does  $K[x_1, \dots, x_r]/I$  fail WLP?
- (2) (Uniform powers.) Let  $I = (\ell_1^t, \dots, \ell_s^t) \subset K[x_1, \dots, x_r]$  be an artinian ideal generated by uniform powers of general linear forms. For which values of  $r$ ,  $s$  and  $t$  does  $K[x_1, \dots, x_r]/I$  fail WLP?
- (3) (Mixed powers.) Let  $I = (\ell_1^{a_1}, \dots, \ell_s^{a_s}) \subset K[x_1, \dots, x_r]$  be an artinian ideal generated by powers of general linear forms. For which values of  $r$ ,  $s$  and  $a_i$  does  $K[x_1, \dots, x_r]/I$  fail WLP?

Nice contributions to these problems are given in [7,5,2] but no complete answers to them are known and more work has to be done. Here there are three open conjectures from [5] and [2] related to these questions:

**Conjecture 2.9.** ([2], Conjecture 5.5.2) For  $r = 6$  and all  $r \geq 8$ ,  $K[x_1, \dots, x_r]/(\ell_1^2, \dots, \ell_{r+1}^2)$  where  $\ell_i$  are general linear forms, fails WLP.

**Conjecture 2.10.** ([2], Conjecture 1.2) For  $I = (\ell_1^t, \dots, \ell_n^t) \subset K[x_1, \dots, x_r]$  with  $\ell_i$  general linear forms and  $n \geq r + 1 \geq 5$ ,  $A = K[x_1, \dots, x_r]/I$  fails WLP for  $t \gg 0$ .

**Conjecture 2.11.** ([5], Conjecture 6.6) Let  $R = K[x_1, \dots, x_{2n+1}]$ , where  $n \geq 4$ . Let  $\ell \in R$  be a general linear form, and let  $I = \langle x_1^d, \dots, x_{2n+1}^d, \ell^d \rangle$ . Then the ring  $R/I$  fails the WLP if and only if  $d > 1$ . Furthermore, if  $n = 3$  then  $R/I$  fails the WLP when  $d = 3$ .

This paper will be devoted to proving Conjecture 2.9. More precisely, we will prove that for almost complete intersection ideals generated by squares of general linear forms we have the following full characterization which solves Conjecture 2.9.

**Theorem 2.12.** Set  $A = K[x_1, \dots, x_r]/(\ell_1^2, \dots, \ell_{r+1}^2)$  where  $\ell_i$  are general linear forms. We have:

- (1) If  $2 \leq r \leq 5$  or  $r = 7$  then  $A$  has the WLP.
- (2) If  $r = 6$  or  $r \geq 8$  then  $A$  fails WLP.

### 3. Proof of the main theorem

This section is entirely devoted to prove [Theorem 2.12](#). In order to make the proof self-contained we start with a series of technical lemmas and propositions which have as a goal to compute the Hilbert function of an almost complete intersection ideal generated by the quadratic powers of general linear forms.

**Lemma 3.1.** *Let  $\mathfrak{a} = (x_1^2, x_2^2, \dots, x_r^2)$ . Then the Hilbert function of  $A = K[x_1, \dots, x_r]/\mathfrak{a}$  is*

$$H_A(j) = \begin{cases} \binom{r}{j} & \text{if } 0 \leq j \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** To compute the Hilbert function  $H_A(t)$  of  $A$ , it is enough to describe bases  $B_j$  of  $[A]_j$ . We choose the residue classes of the elements in the following sets

$$B_j = \{x_{i_1} \cdots x_{i_j} \mid 1 \leq i_1 < \cdots < i_j \leq r\} \text{ for } 1 \leq j \leq r,$$

and this concludes the proof.  $\square$

Given the polynomial ring  $R = K[x_1, \dots, x_r]$  we will consider its Macaulay–Matlis inverse system  $E = K[X_1, \dots, X_r]$  which means that  $E$  is a graded  $R$ -module and  $x_i$  acts as  $\frac{\partial}{\partial X_i}$ .

**Proposition 3.2.** *Set  $g := \sum_{1 \leq i_1 < \cdots < i_{r-2} \leq r} X_{i_1} X_{i_2} \cdots X_{i_{r-2}}$  and let  $G := K[x_1, \dots, x_r]/\text{Ann}(g)$  be the associated artinian Gorenstein  $K$ -algebra. The following hold:*

(1) *The Hilbert function of  $G$  is given by*

$$H_G(t) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t > r - 2, \\ \binom{r}{t} & \text{if } 0 \leq t \leq \frac{r-2}{2}, \\ \binom{r}{r-2-t} & \text{if } \frac{r-2}{2} < t \leq r - 2. \end{cases}$$

(2)  *$G$  has socle degree  $r - 2$  and it is relatively compressed with respect to the complete intersection monomial ideal  $\mathfrak{a} = (x_1^2, x_2^2, \dots, x_r^2)$ .*

(3) *For  $r \geq 3$  odd, we define*

$$S := \{x_1^2, x_2^2, \dots, x_r^2, (x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{r-2}} - x_{i_{r-1}})\}$$

*where  $1 \leq i_j \leq r$  and  $i_j \neq i_s$ . We have  $\langle S \rangle \subset \text{Ann}(g)$ .*

(4) For  $r \geq 4$  even, we define

$$S := \{x_1^2, x_2^2, \dots, x_r^2, (x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{r-3}} - x_{i_{r-2}})x_{i_{r-1}}\}$$

where  $1 \leq i_j \leq r$  and  $i_j \neq i_s$ . We have  $\langle S \rangle \subset \text{Ann}(g)$ .

**Proof.** (1) Since  $G$  is an artinian Gorenstein  $K$ -algebra, the Hilbert function of  $G$  is symmetric, i.e.  $H_G(i) = H_G(r-2-i)$  for  $0 \leq i \leq r-2$  and we only have to compute  $H_G(j)$  for  $\frac{r-2}{2} \leq j \leq r-2$ . By Macaulay–Matlis duality for any  $\frac{r-2}{2} \leq j \leq r-2$ , we have:

$$\begin{aligned} H_G(j) &= \dim_K \left\langle \frac{\partial^j g}{\partial X_1^{a_1} \cdots \partial X_r^{a_r}} \mid 0 \leq a_i \text{ and } a_1 + \cdots + a_r = j \right\rangle \\ &= \dim_K \left\langle \frac{\partial^j g}{\partial X_{i_1} \cdots \partial X_{i_j}} \mid 0 \leq i_1 < \cdots < i_j \leq r \right\rangle \\ &= \binom{r}{j}. \end{aligned}$$

(2) By definition  $A$  has socle degree  $r-2$ . Using (1) and [Lemma 3.1](#) we check that

$$H_G(i) = \min\{\dim[R/\mathfrak{a}]_i, \dim[R/\mathfrak{a}]_{r-2-i}\}$$

and we deduce from [Lemma 2.3](#) that  $G$  is relatively compressed with respect to the complete intersection monomial ideal  $\mathfrak{a}$ .

(3) Set  $C = K[x_1, \dots, x_r]/\mathfrak{a}$ . By item (2),  $H_G(j) = H_C(j)$  for all  $0 \leq j \leq \frac{r-2}{2}$  and we clearly have  $\mathfrak{a} \subset \text{Ann}(g)$  since  $x_i^2 \circ g = \frac{\partial^2 g}{\partial X_i^2} = 0$  for all  $0 \leq i \leq r$ . Therefore, for  $r \geq 7$ , the only generators of  $\text{Ann}(g)$  of degree  $\leq \frac{r-2}{2}$  are:  $x_1^2, x_2^2, \dots, x_r^2$ .

**Claim.**  $(x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{r-2}} - x_{i_{r-1}}) \in \text{Ann}(g)$ .

**Proof of the Claim.** Obviously, it is enough to check one case, namely  $(x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \circ g = 0$ . We set  $r = 2q + 1$  and we proceed by induction on  $q$ .

For  $q = 1$  and  $2$  the claim is true since we have  $(x_1 - x_2) \circ (X_1 + X_2 + X_3) = 0$  and  $(x_1 - x_2)(x_3 - x_4) \circ (X_1 X_2 X_3 + X_1 X_2 X_4 + X_1 X_2 X_5 + X_1 X_3 X_4 + X_1 X_3 X_5 + X_1 X_4 X_5 + X_2 X_3 X_4 + X_2 X_3 X_5 + X_2 X_4 X_5 + X_3 X_4 X_5) = 0$ .

Assume  $q \geq 3$ . We define

$$\begin{aligned} g_1 &:= \sum_{\substack{1 \leq j_1 < \cdots < j_{r-3} \leq r \\ j_i \neq r-2}} X_{j_1} \cdots X_{j_{r-3}} \\ &= X_r \sum_{\substack{1 \leq j_1 < \cdots < j_{r-4} \leq r-1 \\ j_i \neq r-2}} X_{j_1} \cdots X_{j_{r-4}} + \sum_{\substack{1 \leq j_1 < \cdots < j_{r-3} \leq r-1 \\ j_i \neq r-2}} X_{j_1} \cdots X_{j_{r-3}} \\ &=: X_r g_1^1 + g_1^2 \end{aligned}$$

and, analogously,

$$\begin{aligned} g_2 &:= \sum_{\substack{1 \leq j_1 < \cdots < j_{r-3} \leq r \\ j_i \neq r-1}} X_{j_1} \cdots X_{j_{r-3}} \\ &= X_r \sum_{\substack{1 \leq j_1 < \cdots < j_{r-4} \leq r-1 \\ j_i \neq r-1}} X_{j_1} \cdots X_{j_{r-4}} + \sum_{\substack{1 \leq j_1 < \cdots < j_{r-3} \leq r-1 \\ j_i \neq r-1}} X_{j_1} \cdots X_{j_{r-3}} \\ &=: X_r g_2^1 + g_2^2. \end{aligned}$$

We have:

$$\begin{aligned} (x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \circ g &= \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3})x_{r-2} \circ g - (x_1 - x_2) \cdots (x_{r-4} - x_{r-3})x_{r-1} \circ g &= \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ g_1 - (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ g_2 &= \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ (X_r g_1^1 + g_1^2) - (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ (X_r g_2^1 + g_2^2). \end{aligned}$$

Using the induction hypothesis we get

$$(x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ X_r g_1^1 = 0$$

and

$$(x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ X_r g_2^1 = 0.$$

Therefore, we have

$$\begin{aligned} (x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \circ g &= \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ g_1^2 - (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ g_2^2 &= \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ (X_1 \cdots X_{r-3}) + X_{r-1}h - & \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ (X_1 \cdots X_{r-3}) + X_{r-2}h \end{aligned}$$

where

$$h = \sum_{1 \leq s_1 < \cdots < s_{r-4} \leq r-3} X_{s_1} \cdots X_{s_{r-4}}$$

Applying again hypothesis of induction we obtain

$$\begin{aligned} (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ X_{r-1}h &= 0 \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ X_{r-2}h &= 0 \end{aligned}$$

and, hence, we get

$$\begin{aligned} (x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \circ g &= \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ (X_1 \cdots X_{r-3}) + X_{r-1}h &= \\ - (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ (X_1 \cdots X_{r-3}) + X_{r-2}h &= \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ X_1 \cdots X_{r-3} - & \\ (x_1 - x_2) \cdots (x_{r-4} - x_{r-3}) \circ X_1 \cdots X_{r-3} &= \\ 0. \end{aligned}$$

(4) It is analogous to (3) and we leave it to the reader.  $\square$

**Remark 3.3.** In this paper we do not need an explicit description of a (minimal) system of generators of  $\text{Ann}(g)$ . Computer evidences, using Macaulay2, suggest that  $S$  is a full



system of generators of  $\text{Ann}(g)$  and, hence  $\text{Ann}(g)$  is generated by  $r$  quadrics and  $\binom{r-1}{q-1}$  forms of degree  $q$  if  $r = 2q + 1 \geq 7$  and by  $r$  quadrics and  $\binom{r}{q} - \binom{r}{q-2}$  forms of degree  $q$  if  $r = 2q \geq 6$ .

**Proposition 3.4.** *Fix an integer  $r \geq 3$ . Set  $g := \sum_{1 \leq i_1 < \dots < i_{r-2} \leq r} X_{i_1} X_{i_2} \dots X_{i_{r-2}}$ . Let  $G := K[x_1, \dots, x_r] / \text{Ann}(g)$  be the associated artinian Gorenstein  $K$ -algebra relatively compressed with respect to the monomial complete intersection  $\mathfrak{a} = (x_1^2, \dots, x_r^2)$ . We define  $J = (\mathfrak{a} : \text{Ann}(g))$ . We have:*

- (1)  *$J$  is an almost complete intersection artinian ideal generated by  $r + 1$  forms of degree 2 and with socle degree*

$$e(J) = \begin{cases} q + 1 & \text{if } r = 2q + 1, \\ q & \text{if } r = 2q. \end{cases}$$

- (2) *If  $r = 2q + 1$ , the Hilbert function of  $A := K[x_1, \dots, x_r] / J$  is given by*

$$H_A(t) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t > q + 1, \\ \binom{r}{t} - \binom{r}{t-2} & \text{if } 0 \leq t \leq q, \\ \binom{r}{q} - \binom{r}{q-1} & \text{if } t = q + 1. \end{cases}$$

*If  $r = 2q$ , we have*

$$H_A(t) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t > q, \\ \binom{r}{t} - \binom{r}{t-2} & \text{if } 0 \leq t \leq q. \end{cases}$$

- (3)  $J = (x_1^2, x_2^2, \dots, x_r^2, (x_1 + \dots + x_r)^2)$ .

**Proof.** (1) Since  $G$  is an artinian Gorenstein  $K$ -algebra relatively compressed with respect to  $\mathfrak{a}$  and with socle degree  $r - 2$  and  $J$  is linked to  $\text{Ann}(g)$  by means of  $\mathfrak{a}$ , we have that  $J$  is an almost complete intersection generated by  $x_1^2, \dots, x_r^2, f$  where  $f$  is a form of degree 2.

According to the proof of [Proposition 3.2](#) (3) and (4),  $\text{Ann}(g)$  has  $r$  minimal generators of degree 2, at least one minimal generator of degree  $\lfloor \frac{r}{2} \rfloor$  and all the others (if any) in degree  $\geq \lfloor \frac{r}{2} \rfloor$ . Therefore, the socle degree of  $J$  is

$$e(J) = -\lfloor \frac{r}{2} \rfloor + r = \begin{cases} q + 1 & \text{if } r = 2q + 1, \\ q & \text{if } r = 2q \end{cases}$$

which proves what we want.

(2) Since the Hilbert function of  $K[x_1, \dots, x_r]/\mathfrak{a}$  is known (Lemma 3.1) and of  $K[x_1, \dots, x_r]/\text{Ann}(g)$  is also known (Proposition 3.2), we can compute the Hilbert function of  $J = (\mathfrak{a} : \text{Ann}(g))$  and we have

$$H_{K[x_1, \dots, x_r]/J}(t) = H_{K[x_1, \dots, x_r]/\mathfrak{a}}(t) - H_G(r - t)$$

and we get what we want.

(3) By the proof of (1) we know that  $J = (x_1^2, x_2^2, \dots, x_r^2, f)$  where  $f$  is an homogeneous form of degree 2. Let us determine it. More precisely, let us check that we can take  $f = (x_1 + \dots + x_r)^2$ . We assume that  $r$  is odd, say,  $r = 2q + 1$ . Analogous argument works for  $r$  even. Set

$$S := \{x_1^2, x_2^2, \dots, x_r^2, (x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{r-2}} - x_{i_{r-1}})\}$$

where  $1 \leq i_j \leq r$  and  $i_j \neq i_s$ . By Proposition 3.2,  $S$  is part of a system (not necessarily minimal) of generators of  $\text{Ann}(g)$ . Therefore, it is enough to prove:

- (i) For all  $h \in S$ ,  $h \cdot (x_1 + \dots + x_r)^2 \in \mathfrak{a}$ .
- (ii) If  $f \in K[x_1, \dots, x_r]_2$  and  $h \cdot f \in \mathfrak{a}$  for all  $h \in S$  then  $f \in (x_1^2, x_2^2, \dots, x_r^2, (x_1 + \dots + x_r)^2)$ .

Let us prove (i). By symmetry we only have to check that  $(x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \cdot (x_1 + \dots + x_r)^2 \in \mathfrak{a}$  or, equivalently,  $(x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \cdot (\sum_{1 \leq i_1 < i_2 \leq r} x_{i_1} x_{i_2}) \in \mathfrak{a}$ . We proceed by induction on  $q$ . For  $q = 1$  we have  $(x_1 - x_2)(x_1 x_2 + x_1 x_3 + x_2 x_3) = x_1^2(x_2 + x_3) - x_2^2(x_1 + x_3) \in \mathfrak{a}$ . Assume  $q \geq 2$ . By hypothesis of induction, it holds

$$(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \cdot \left( \sum_{3 \leq i_1 < i_2 \leq r} x_{i_1} x_{i_2} \right) \in (x_3^2, \dots, x_r^2).$$

Therefore, we have

$$\begin{aligned} (x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \cdot (\sum_{1 \leq i_1 < i_2 \leq r} x_{i_1} x_{i_2}) &= (\text{mod. } \mathfrak{a}) \\ (x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \cdot (x_1(x_2 + \dots + x_r) + x_2(x_3 + \dots + x_r)) &= (\text{mod. } \mathfrak{a}) \\ x_1(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \cdot (x_2(x_3 + \dots + x_r)) & \\ -x_2(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \cdot (x_1(x_2 + \dots + x_r)) &= (\text{mod. } \mathfrak{a}) \\ 0 &= (\text{mod. } \mathfrak{a}). \end{aligned}$$

To prove (ii) we will check that if  $f \notin (x_1^2, \dots, x_r^2, (x_1 + \dots + x_r)^2) = (x_1^2, \dots, x_r^2, \sum_{1 \leq i_1 < i_2 \leq r} x_{i_1} x_{i_2})$  then  $\exists h \in S$  such that  $f \cdot h \notin \mathfrak{a}$ . Reordering the variables, if necessary, any form of degree two  $f \notin (x_1^2, \dots, x_r^2, \sum_{1 \leq i_1 < i_2 \leq r} x_{i_1} x_{i_2})$  can be written as

$$f = a_1 x_1^2 + \dots + a_r x_r^2 + a \left( \sum_{1 \leq i_1 < i_2 \leq r} x_{i_1} x_{i_2} \right) + \sum_{1 \leq i_1 < i_2 \leq r} b_{i_1 i_2} x_{i_1} x_{i_2}$$

with  $b_{12} = 0$  and  $b_{13} \neq 0$  or  $b_{12} = \dots = b_{1r} = 0$  and  $b_{23} \neq 0$ . If  $b_{12} = 0$  and  $b_{13} \neq 0$ , then

$$\begin{aligned} & (x_2 - x_3)(x_4 - x_5) \cdots (x_{r-1} - x_r) \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} b_{i_1 i_2} x_{i_1} x_{i_2} \right) \\ &= b_{13} x_1 x_2 x_3 \prod_{j=2}^q x_{2j} + (\text{other monomials}) \notin \mathfrak{a}, \end{aligned}$$

and, if  $b_{12} = \dots = b_{1r} = 0$  and  $b_{23} \neq 0$ , then

$$\begin{aligned} & (x_1 - x_2)(x_3 - x_4) \cdots (x_{r-2} - x_{r-1}) \cdot \left( \sum_{1 \leq i_1 < i_2 \leq r} b_{i_1 i_2} x_{i_1} x_{i_2} \right) \\ &= b_{13} x_1 x_2 x_3 \prod_{j=2}^q x_{2j} + (\text{other monomials}) \notin \mathfrak{a}. \quad \square \end{aligned}$$

**Lemma 3.5.** Set  $A = K[x_1, \dots, x_r]/(x_1^2, \dots, x_r^2, \ell^2)$ ,  $B = K[x_1, \dots, x_r]/(\ell_1^2, \dots, \ell_r^2, \ell_{r+1}^2)$  and  $C = K[x_1, \dots, x_r]/(q_1, \dots, q_r, q_{r+1})$  where  $\ell$  and  $\ell_i$  are general linear forms and  $q_j$  are general forms of degree 2. Then

$$H_A(t) = H_B(t) = H_C(t) \text{ for all } t \in \mathbb{Z}.$$

**Proof.** By semicontinuity we have

$$H_A(t) \leq H_B(t) \leq H_C(t) \text{ for all } t \in \mathbb{Z}. \quad (3.1)$$

By Proposition 3.4(2), if  $r = 2q + 1$  we have

$$H_A(t) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t > q + 1, \\ \binom{r}{t} - \binom{r}{t-2} & \text{if } 0 \leq t \leq q, \\ \binom{r}{q} - \binom{r}{q-1} & \text{if } t = q + 1 \end{cases}$$

and, if  $r = 2q$ , we have

$$H_A(t) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t > q \\ \binom{r}{t} - \binom{r}{t-2} & \text{if } 0 \leq t \leq q. \end{cases}$$

By Proposition 2.6,  $J = (q_1, \dots, q_r)$  has the SLP. Hence, we have

$$H_C(t) = \max\{H_{K[x_1, \dots, x_r]/J}(t) - H_{K[x_1, \dots, x_r]/J}(t-2), 0\} = H_A(t)$$

and we conclude  $H_C(t) = H_A(t)$  for all  $t \in \mathbb{Z}$ . Therefore, using the inequalities (3.1), we get  $H_A(t) = H_B(t) = H_C(t)$  for all  $t \in \mathbb{Z}$  which proves what we want.  $\square$

We are now ready to prove the main result of this paper.

**Proof of Theorem 2.12.** (1) For  $2 \leq r \leq 5$  or  $r = 7$  we can use Macaulay2 to check that  $A$  has the WLP.

(2) For  $r$  even the reader can see [5], Theorem 6.1. Let us assume  $r = 2q + 1$ ,  $q \geq 4$ . Set  $I = (\ell_1^2, \dots, \ell_{r+1}^2) \subset R := K[x_1, \dots, x_r]$  where  $\ell_i$  are general linear forms. We consider the exact sequence:

$$0 \longrightarrow [I : \ell]/I \longrightarrow R/I \xrightarrow{\times \ell} R/I(1) \longrightarrow R/(I, \ell)(1) \longrightarrow 0$$

where  $\ell$  is a general linear form. The multiplication by  $\ell$  will fail to have maximal rank from degree  $i - 1$  to degree  $i$  exactly when

$$\dim_K[R/(I, \ell)]_i \neq \max\{\dim_K[R/I]_i - \dim_K[R/I]_{i-1}, 0\}. \quad (3.2)$$

By Lemma 3.5 and Proposition 3.4 we have

$$\dim_K(R/I)_q = \binom{2q+1}{q} - \binom{2q+1}{q-2} \text{ and } \dim_K(R/I)_{q-1} = \binom{2q+1}{q-1} - \binom{2q+1}{q-3}.$$

Since  $\dim_K[R/I]_q > \dim_K[R/I]_{q-1}$ , to prove that  $R/I$  fails WLP it will be enough to check that

$$\dim_K[R/I]_q < \dim_K[R/(I, \ell)]_q + \dim_K[R/I]_{q-1}.$$

By [9], Corollaries 7.3 and 7.4, we have  $\dim_K[R/(I, \ell)]_q = 2^q$ . Therefore, we have to prove

$$\dim_K[R/I]_q < 2^q + \dim_K[R/I]_{q-1}.$$

Let us prove it. Since  $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n} \binom{n}{k}$  and  $\binom{n}{k} = \frac{n+1-k}{k} \binom{n}{k-1}$ , we obtain

$$\begin{aligned} \dim_K[(R/I)]_q &= \binom{2q+1}{q} - \binom{2q+1}{q-2} < 2^q + \dim_K[(R/I)]_{q-1} \\ &= 2^q + \binom{2q+1}{q-1} - \binom{2q+1}{q-3} \end{aligned}$$

if and only if

$$\binom{2q+1}{q} - \binom{2q+1}{q-1} < 2^q + \binom{2q+1}{q-2} - \binom{2q+1}{q-3}$$

if and only if

$$\binom{2q+2}{q} \frac{1}{q+1} < 2^q + \binom{2q+2}{q-2} \frac{3}{q+1}$$

if and only if

$$2^q > \frac{1}{q+1} \left[ \binom{2q+2}{q} - 3 \binom{2q+2}{q-2} \right] = \frac{1}{q+1} \binom{2q+2}{q} \left[ 1 - 3 \frac{(q-1)q}{(q+4)(q+3)} \right].$$

We easily check that the last equality is true for  $q = 4$  or  $5$  and it is obviously true for  $q \geq 6$  since we have

$$1 - 3 \frac{(q-1)q}{(q+4)(q+3)} = \frac{-2q^2 + 10q + 12}{(q+4)(q+3)} = \frac{-2(q-6)(q+1)}{(q+4)(q+3)} \leq 0. \quad \square$$

It is worthwhile to point out that if instead of considering an almost complete intersection ideal  $I = (\ell_1^2, \dots, \ell_8^2) \subset R := K[x_1, \dots, x_7]$  generated by the square of general linear forms  $\ell_i$ , we take an almost complete intersection ideal  $J = (q_1, \dots, q_8)$  generated by general quadrics then  $J$  *does* have the WLP. We observe that  $I$  and  $J$  have the same Hilbert function but the WLP behaviour is very different. This example illustrates how subtle the WLP is since a minuscule change in the ideal can affect the WLP.

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