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STAR-POLYNOMIAL IDENTITIES: COMPUTING THE EXPONENTIAL GROWTH OF THE CODIMENSIONS

A. GIAMBRUNO, C. POLCINO MILIES, AND A. VALENTI

Abstract. Can one compute the exponential rate of growth of the $*$-codimensions of a PI-algebra with involution $*$ over a field of characteristic zero? It was shown in [2] that any such algebra A has the same $*$-identities as the Grassmann envelope of a finite dimensional superalgebra with superinvolution B. Here, by exploiting this result we are able to provide an exact estimate of the exponential rate of growth $exp^*(A)$ of any PI-algebra A with involution. It turns out that $exp^*(A)$ is an integer and, in case the base field is algebraically closed, it coincides with the dimension of an admissible subalgebra of maximal dimension of B.

1. Introduction

Let A be an algebra over a field F of characteristic zero and suppose that A is a PI-algebra i.e., it satisfies a non trivial polynomial identity. A celebrated result of Kemer states that any such algebra A has the same polynomial identities as the Grassmann envelope of a suitable finite dimensional superalgebra [20]. Recall that if G is the (infinite dimensional) Grassmann algebra over F, one can consider its standard $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$. Then if $B = B_0 \oplus B_1$ is a superalgebra over F, the Grassmann envelope of B is the algebra $G(B) = G_0 \otimes B_0 \oplus G_1 \otimes B_1$. This result has been extended to algebras graded by a finite group H in [4] (see also [22]). In this case one considers the Grassmann envelope of a finite dimensional $\mathbb{Z}_2 \times H$-graded algebra.

In this paper we are concerned with algebras with involution. In [2] a suitable superinvolution on the Grassmann algebra was introduced having the following property: if B is any algebra endowed with a superinvolution, then its Grassmann envelope has an induced involution. More generally given a superalgebra B and its Grassmann envelope $G(B)$ there is a well-understood duality between graded involutions and superinvolutions of the two algebras. The main outcome of this correspondence is the following result proved in [2]: any PI-algebra with involution $*$ has the same $*$-identities as the Grassmann envelope of a suitable finite dimensional algebra with superinvolution.

Here we are interested in the growth of the identities of an algebra. Recall that if P_n is the space of multilinear polynomials in n variables and $Id(A)$ is the T-ideal of identities of the algebra A, then

$$c_n(A) = \dim P_n/(P_n \cap Id(A))$$

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is the n-th codimension of A. It is well known ([21]) that the sequence of codimensions of an associative PI-algebra is exponentially bounded and in [13], [14] it was shown that if A is any PI-algebra there exist constants $C_1 > 0, C_2, t_1, t_2$ such that $C_1 n^{t_1} d^n \leq c_n(A) \leq C_2 n^{t_2} d^n$ holds for a suitable integer d. In particular the limit $\lim_{n\to\infty} \sqrt[n]{c_n(A)} = d = exp(A)$ exists and is an integer called the PI-exponent of A. We refer the reader to [16] for an account of the theory developed around the exponent.

We have to mention that an actual asymptotic estimate of the codimensions was established in [7] and [8] for algebras with 1 and it turns out that $c_n(A) \simeq Cn^t exp(A)^n$ where $t \in \frac{1}{2}Z$. Later in [17] it was shown that even if A does not have a unit element, still $C_1 n^t exp(A)^n \leq c_n(A) \leq C_2 n^t exp(A)^n$ holds where C_1 and C_2 are positive constants.

When an algebra A has an additional structure, such as a group grading or an involution, one can consider the corresponding codimension sequence and ask if the analogue of the theorem on the existence of the exponent holds.

In this setting it was recently shown that if A is any PI-algebra graded by a finite group then the corresponding exponent exists and is an integer ([3], [11], [1]). Also it turns out that if the algebra is finite dimensional and is acted on by a finite group of automorphisms and antiautomorphisms or by a finite dimensional Lie algebra of derivations or more generally there is a so-called generalized Hopf algebra action, still the corresponding exponent exists and is an integer ([18]).

Here we shall prove that if A is any PI-algebra with involution $*$, and $c_n^*(A)$, $n = 1, 2, \dots$, is the corresponding sequence of $*$-codimensions, then

$$C_1 n^{t_1} d^n \leq c_n^*(G(A)) \leq C_2 n^{t_2} d^n,$$

holds for all n, where $C_1 > 0, C_2, t_1, t_2, d$ are constants and d is an integer. As a corollary we get that the $*$-exponent of A, i.e., $exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$ exists and is an integer. Since A has the same $*$-identities as the Grassmann envelope $G(B)$ of a finite dimensional algebra with superinvolution B, we shall actually exploit such relation.

As in the ordinary case (no involution) we also give an explicit way of computing the $*$-exponent; il turns out that when F is algebraically closed, $exp^*(A) = exp^*(G(B))$ is the dimension of a so-called admissible subalgebra of B of maximal dimension. It should be mentioned that the existence of the $*$-exponent was proved in [15] for finite dimensional algebras.

Finally but most important, the proof we present here is not a generalization of the original proof; the computation of the upper bound of $c_n^*(A)$ is based on an idea of Procesi and the proof of the lower bound is a generalization and a simplification of the original proof.

2. Preliminaries

Let A be a superalgebra over a field F and suppose that A is endowed with a superinvolution $*$. Recall that in this case A is a superalgebra $A = A_0 \oplus A_1$ and * is a linear map of A of order two such that $(ab)^* = (-1)^{|a||b|} b^* a^*$, for any homogeneous elements $a, b \in A$, where $\mid a \mid$ denotes the homogeneous degree of A. Now, for any subset $S = S^*$ of A, let $S^+ = \{a \in S \mid a = a^*\}$ and $S^- = \{a \in S \mid a = -a^*\}$ be the subsets of symmetric and skew elements of S, respectively. Then, since it is well-known that $A_0^* \subseteq A_0, A_1^* \subseteq A_1$ we can decompose A as $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$.

Now, throughout the paper the word algebra may mean superalgebra depending on the context, for instance we shall say that A is an algebra with superinvolution meaning that A is a superalgebra with superinvolution.

Throughout this paper F will be a field of characteristic zero, and $F\langle X, s\rangle$ the free algebra with superinvolution on a countable set X over F. $F\langle X, s\rangle$ is defined by a universal property and can be explicitly described as follows. Starting with the free associative algebra $F\langle X\rangle$ on a countable set X, write $X = Y \cup Z$, the disjoint union of two countable sets. Then $F\langle X\rangle$ becomes a superalgebra denoted $F\langle Y, Z\rangle$ (the free superalgebra of countable rank) by requiring that the variables of Y have homogeneous degree zero and those of Z have homogenous degree one.

Next we write $Y = Y^+ \cup Y^-$ and $Z = Z^+ \cup Z^-$, disjoint unions of countable sets. Then we define a superinvolution on $F\langle Y, Z\rangle$ by requiring that the variables of Y^+ (Y^-) are symmetric (skew, resp.) of homogenous degree zero and the variables of Z^+ (Z^-) are symmetric (skew, resp.) of homogenous degree one. The resulting algebra is $F\{X, s\}$, the free algebra with superinvolution on X.

In what follows we shall write $Y^+ = \{y_1^+, y_2^+, \ldots\}, Y^- = \{y_1^-, y_2^-, \ldots\}, Z^+ = \{z_1^+, z_2^+, \ldots\}, Z^- = \{z_1^-, z_2^-, \ldots\}$. Hence an element of $F\langle X, s\rangle$ will be written as

$$f = f(y_1^+, \ldots, y_n^+, y_1^-, \ldots, y_m^-, z_1^+, \ldots, z_p^+, z_1^-, \ldots, z_q^-).$$

Let A be a superalgebra with a superinvolution $*$. A polynomial $f \in F\langle X, s\rangle$ is a *-polynomial identity (or an identity with superinvolution) if it vanishes in A when the homogeneous symmetric and skew variables are evaluated in symmetric and skew elements of A of the corresponding homogeneous degree, respectively.

We shall denote by $Id^s(A)$ the ideal of $F\langle X, s\rangle$ of *-polynomial identities of A. Also we shall denote by $Id(A)$ the ideal of (ordinary) polynomial identities of A. As an extension of the ordinary case, it is clear that $Id^s(A)$ is invariant under all superinvolution endomorphisms of $F\langle X, s\rangle$.

Let us denote by P_n^s, the space of multilinear polynomials with superinvolution of $F\langle X, s\rangle$ in the first n variables, i.e.,

$$P_n^s = \mathrm{span}_F\{w_{\sigma(1)}, \cdots, w_{\sigma(n)} \,|\sigma \in S_n, w_i = y_i^+ \text{ or } y_i^- \text{ or } z_i^+ \text{ or } z_i^-, 1 \le i \le n\}.$$

Let also $c_n^s(A) = \dim \frac{P_n^s}{P_n^s \cap Id^s(A)}$ be the n-th superinvolution codimension of A. The sequence $c_n^s(A)$, $n = 1, 2, \ldots$, will be one our main object of study.

Now let B be an algebra with involution $*$. As above, let us denote by $B^+ = \{b \in B \mid b = b^*\}$ and $B^- = \{b \in B \mid b^* = -b\}$ the sets of symmetric and skew elements of B, respectively.

Though we are using the same symbol $*$ for an involution or a superinvolution, in what follows the role of $*$ will be clear from the context. Also in this case we let $F\langle X, *\rangle = F\langle X^+, X^-\rangle$ be the free associative algebra with involution * on the countable set X. We shall write $X^+ = \{x_1^+, x_2^+, \ldots\}$ and $X^- = \{x_1^-, x_2^-, \ldots\}$ for countable sets of symmetric and skew variables generating $F\langle X, *\rangle$, respectively. If $f \in F\langle X, *\rangle$, f will be a $*$-polynomial identity (or identity with involution) of the algebra B if f vanishes under all evaluations of the symmetric and skew variables into symmetric and skew elements of B, respectively. We let $Id^*(B)$ be the ideal of $*$-polynomial identities satisfied by B.

We shall denote by

$$P_n^* = \mathrm{span}\{w_{\sigma(1)}, \cdots, w_{\sigma(n)} |\sigma \in S_n, w_i = x_i^+ \text{ or } x_i^-, 1 \le i \le n\}$$

the space of multilinear polynomials in the first n variables and $c_n^*(B) = \dim \frac{P_n^*}{P_n^* \cap Id^*(B)}$ will be the n-th $*$-codimension of B. Our main result here will be to prove that $C_1 n^{t_1} d^n \leq c_n^*(B) \leq C_2 n^{t_2} d^n$, holds for all n, for some constants $C_1 > 0, C_2, t_1, t_2, d$ where d is an integer, for any PI-algebra with involution B.

Actually we shall first compute a lower bound for a different kind of codimensions that here we define. For any t, $0 \leq t \leq n$, define $P_{t,n-t}$ to be the span of multilinear monomials in $x_1^+, \dots, x_t^+, x_{t+1}^-, \dots, x_n^-$, i.e.,

$$P_{t,n-t} = \operatorname{span}\{w_{\sigma(1)} \cdots w_{\sigma(n)} | \sigma \in S_n, w_i = x_i^+, 1 \leq i \leq t,$$

$$\text{and } w_i = x_i^-, t+1 \leq i \leq n\}.$$

Then one defines $c_{t,n-t}(B) = \dim \frac{P_{t,n-t}}{P_{t,n-t} \cap Id^*(B)}$.

The connection between $c_n^*(B)$ and $c_{t,n-t}(B)$ is given by the formula (see [16, Corollary 10.6.2])

$$c_n^*(B) = \sum_{t=0}^{n} \binom{n}{t} c_{t,n-t}(B). \tag{1}$$

In order to compute the lower bound of $c_n^*(B)$, we shall make use of the representation theory of the symmetric group as follows. Given non negative integers $n_1, n_2, , n_3, n_4$ we shall consider the space $P_{n_1,\dots,n_4}$ of multilinear polynomials in the four sets of variables $y_1^+, \dots, y_{n_1}^+, y_1^-, \dots, y_{n_2}^-, z_1^+, \dots, z_{n_3}^+, z_1^-, \dots, z_{n_4}^-$ on which we act with $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$, where S_{n_i} acts on the corresponding set of variables by permuting them.

We also recall that if λ is a partition of the integer n and T_λ is a Young tableau of shape λ, then R_{T_λ} and C_{T_λ} are the subgroups of S_n stabilizing the rows and the columns of T_λ, respectively. Then $e_{T_\lambda} = R_{T_\lambda}^+ C_{T_\lambda}^-$ is an essential idempotent of the group algebra FS_n, where $R_{T_\lambda}^+ = \sum_{\sigma \in R_{T_\lambda}} \sigma$ and $C_{T_\lambda}^- = \sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau$. Moreover every irreducible S_n-module corresponding to λ is isomorphic to $FS_n e_{T_\lambda}$.

3. The basic setting

Let G be the Grassmann algebra over F, i.e., the algebra generated by the elements $e_1, e_2, \dots$ subject to the condition $e_i e_j = -e_j e_i$, for all $i, j \geq 1$. Recall that G has a natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$ where G_0 and G_1 are the spans of the monomials in the e_i's of even and odd length, respectively. If $A = A_0 \oplus A_1$ is a superalgebra then the Grassmann envelope of A is defined as $G(A) = G_0 \otimes A_0 \oplus G_1 \otimes A_1$.

The relevance of $G(A)$ relies in a result of Kemer ([20, Theorem 2.3]) stating that if B is any PI-algebra, then its T-ideal of polynomial identities coincides with the T-ideal of identities of the Grassmann envelope of a suitable finite dimensional superalgebra.

Regarding algebras with involution, one can perform a construction involving G as follows. First one defines a superinvolution $*$ on G by requiring that $e_i^* = -e_i$, for any $i \geq 1$. Then it is easily checked that $G_0 = G^+$ and $G_1 = G^-$. Now, if A is a superalgebra one can construct its Grassmann envelope $G(A)$ and in ([2]) it was shown that if A has a superinvolution $*$ (a graded involution), then $*$ induces a graded superinvolution (a graded involution, resp.) on $G(A)$ by setting $(g \otimes a)^* = g^* \otimes a^*$, for homogeneous elements $g \in G, a \in A$.

In what follows we shall assume that A has a superinvolution and we shall denote by $Id_2^*(G(A))$ the ideal of graded identities (superidentities) with involution of $G(A)$ and by $Id^*(G(A))$ the ideal of identities with involution of $G(A)$.

Next we briefly recall the main results of [2]. First one defines a superinvolution, denoted $\sharp$, on the Grassmann algebra $G = G_0 \oplus G_1$ by requiring that $e_i^\sharp = -e_i$, for $i \geq 1$. A basic property of this superinvolution is that $G^+ = G_0$ and $G^- = G_1$. It follows that one can bridge between graded involutions and superinvolutions of a superalgebra A and its Grassmann envelope. In fact we have (see [2, Lemma 1]).

Proposition 1. *Let A be a superalgebra endowed with a graded involution or a superinvolution $*$. Then the linear map $* : G(A) \to G(A)$ such that $(a \otimes g)^* = a^* \otimes g^\sharp$, for homogeneous elements $g \in G$ and $a \in A$, induces a superinvolution or an involution on the Grassmann envelope $G(A)$, respectively.*

In case A is a superalgebra with superinvolution and we regard $G(A)$ as an algebra with involution, the following result holds.

Theorem 1. ([2, Theorem 4]) *If B is a PI-algebra with involution over a field F of characteristic 0, then there exists a finite dimensional superalgebra with superinvolution A such that $Id^*(B) = Id^*(G(A))$.*

Hence, by making use of this theorem, we shall compute the exponential rate of growth of the sequence of $*$-codimensions of the Grassmann envelope of a finite dimensional algebra with superinvolution A. To this end we need a Wedderburn-Malcev theorem for finite dimensional algebras with superinvolution whose proof can be found in [10].

Theorem 2. *Let A be a finite dimensional superalgebra with superinvolution $*$ over an algebraically closed field F of characteristic 0. Then $A = B + J$ where B is a maximal semisimple superalgebra with induced superinvolution and $J^* = J$ is the Jacobson radical of A.*

Throughout this paper, unless otherwise stated, A will be a finite dimensional algebra with superinvolution over an algebraic closed field of characteristic 0. Then we write $A = \bar{A} + J$ as in Theorem 2. Also we can write

$$\bar{A} = A_1 \oplus \cdots \oplus A_k$$

where $A_1, \cdots, A_k$ are simple algebras with superinvolution.

We make a definition.

Definition 1. *Given $A = \bar{A} + J = A_1 \oplus \cdots \oplus A_k + J$, a finite dimensional algebra with superinvolution, we say that a subalgebra $A_{i_1} \oplus \cdots \oplus A_{i_t}$ where $A_{i_1}, \ldots, A_{i_t}$ are distinct simple superalgebras with induced superinvolution, is admisible if for some permutation $(l_1, \ldots, l_t)$ of $(i_1, \ldots, i_t)$ we have that $A_{l_1} J A_{l_2} J \cdots J A_{l_t} \neq 0$.*

Definition 2. *If $A_{i_1} \oplus \cdots \oplus A_{i_t}$ is an admissible subalgebra of A then $A' = A_{i_1} \oplus \cdots \oplus A_{i_t} + J$ is called a reduced subalgebra.*

In what follows we shall prove that $c_n^*(G(A))$ is bounded from above and from below, up to a polynomial factor, by d^n where d is the maximal dimension of an admissible subalgebra of A.

4. The map $\sim$

In this section we shall introduce a map relating the $*$-identities of a superalgebra with superinvolution and the $*$-identities of its Grassmann envelope (see [2, Section 2]). This generalizes the map introduced by Kemer in the ordinary case (see [20, Section 2]).

As we remarked above, since A has a superinvolution $*$, $G(A)$ has a graded involution $*$, i.e., the homogeneous components of $G(A)$ are invariant under $*$. Also in this case we decompose $G(A) = G(A)_0^+ \oplus G(A)_0^- \oplus G(A)_1^+ \oplus G(A)_1^-$. We recall that $Id_2^*(G(A))$ is the ideal of the free superalgebra with graded involution of graded $*$-identities of $G(A)$.

Let $f(y_1^+, \ldots, y_n^+, y_1^-, \ldots, y_m^-, z_1^+, \ldots, z_p^+, z_1^-, \ldots, z_q^-) \in F\langle X, s\rangle$ be a polynomial with superinvolution, and suppose that f is multilinear. We write f in the form

$$f = \sum_{w_1,\ldots,w_{p+q+1}} \sum_{\sigma\in S_{p+q}} \alpha_\sigma w_1 \zeta_{\sigma(1)} w_2 \zeta_{\sigma(2)} \ldots w_{p+q} \zeta_{\sigma(1)} w_{p+q+1}$$

where $\zeta_i = z_i^+$ for $1 \le i \le p$ and $\zeta_i = z_{p+i}^-$ for $1 \le i \le q$. Here $w_1, \ldots, w_{p+q+1}$ are eventually empty words in the variables of homogeneous degree zero. Then we define

$$\tilde{f} = \sum_{w_1,\ldots,w_{p+q+1}} \sum_{\sigma\in S_{p+q}} (\operatorname{sgn}\sigma) \alpha_\sigma w_1 \zeta_{\sigma(1)} w_2 \zeta_{\sigma(2)} \ldots w_{p+q} \zeta_{\sigma(1)} w_{p+q+1}.$$

The basic properties of the map $\sim$ are given in the following.

Lemma 1. 1) $\tilde{\tilde{f}} = f$;
2) *$f \in Id^s(A)$ if and only $\tilde{f} \in Id_2^*(G(A))$;*
3) *for any subset of variables Z' of $\{z_1^+, \ldots, z_p^+, z_1^-, \ldots, z_q^-\}$, f is alternating on Z' if and only if $\tilde{f}$ is symmetric on Z'.*

Proof. 1) is clear.

Recall that since $G^+ = G_0$ and $G^- = G_1$, we have that $G(A)_0{}^+ = A_0{}^+ \otimes G_0$, $G(A)_0{}^- = A_0{}^- \otimes G_0$, $G(A)_1{}^+ = A_1{}^- \otimes G_1$ and $G(A)_1{}^- = A_1{}^+ \otimes G_1$. Hence

$$\begin{aligned} & f(a_{1,0}^+ \otimes g_{1,0}, \ldots, a_{n,0}^+ \otimes g_{n,0}, a_{1,0}^- \otimes h_{1,0}, \ldots, a_{m,0}^- \otimes h_{m,0}, \\ & \qquad b_{1,1}^- \otimes g_{1,1}, \ldots, b_{p,1}^- \otimes g_{p,1}, b_{1,1}^+ \otimes h_{1,1}, \ldots, b_{q,1}^+ \otimes h_{q,1}) \\ &= \tilde{f}(a_{1,0}^+, \ldots, a_{n,0}^+, a_{1,0}^-, \ldots, a_{m,0}^-, b_{1,1}^-, \ldots, b_{p,1}^-, b_{1,1}^+, \ldots, b_{q,1}^+) \otimes g_{1,0} \\ & \qquad \cdots g_{n,0} h_{1,0} \cdots h_{m,0} g_{1,1} \cdots g_{n,1} h_{1,1} \cdots h_{m,1}, \end{aligned}$$

where $a_{i,0}^+ \otimes g_{i,0} \in A_0^+ \otimes G_0$, $a_{i,0}^- \otimes h_{i,0} \in A_0^- \otimes G_0$, $b_{i,1}^- \otimes g_{i,1} \in A_1^- \otimes G_1$, $b_{i,1}^+ \otimes h_{i,1} \in A_1^- \otimes G_1$. Thus $f \in Id_2^*(G(A))$ if and only if $\tilde{f} \in Id^s(G(A))$. $\square$

5. Multialternating polynomials

In what follows we shall denote by $Y_1^+, \ldots, Y_r^+$ finite disjoint sets of symmetric variables of homogeneous degree zero, by $Y_1^-, \ldots, Y_r^-$ finite disjoint sets of skew variables of homogeneous degree zero. The meaning of $Z_1^+, \ldots, Z_r^+$ and $Z_1^-, \ldots, Z_r^-$ is clear. Also, in order to simplify the notation, for $r \ge 1$, we shall write

$$Y_{(r)}^+ = Y_1^+ \cup \cdots \cup Y_r^+,$$

where $\mid Y_1^+ \mid = \cdots = \mid Y_r^+ \mid$; similarly $Y_{(r)}^- = Y_1^- \cup \cdots \cup Y_r^-$, and so on.

Also, if $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ are two partitions, we write $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$, for all i.

It is well-known that any finite dimensional algebra A has nilpotent Jacobson radical J. In the next lemmas we assume that $J \neq 0$ and s in any positive integer such that $J^s = 0$.

Lemma 2. *Let $A = \bar{A} + J$ be a finite dimensional algebra with superinvolution. Suppose that $\dim(\bar{A})_0^+ = p_0 > 0$ and let $s > 0$ be such that $J^s = 0$. Let*

$$f(Y_1^+, \dots, Y_r^+, X') \notin Id_2^*(G(A))$$

be a multilinear polynomial with $r \geq 1$, such that

1) *f is alternating on each set Y_i^+ with $|Y_i^+| = p_0$, $1 \leq i \leq r$;*
2) *X' contains a set Y^0 of s symmetric variables of homogeneous degree zero.*

Then there exists a partition $\lambda \geq (r^{p_0})$ with $|\lambda| - rp_0 = s$ and a tableau T_λ such that $e_{T_\lambda} f \notin Id_2^*(G(A))$ where e_{T_λ} acts on the set $Y_1^+ \cup \dots \cup Y_r^+ \cup Y^0$.

Proof. We let the symmetric group $S_{p_0 r+s}$ act on the variables of the set $Y_1^+ \cup \dots \cup Y_r^+ \cup Y^0$. Let $M = FS_{p_0 r+s} f$ be the left $S_{p_0 r+s}$-module generated by f. Then $M \not\subseteq Id_2^*(G(A))$ since $f \notin Id_2^*(G(A))$. Hence there exists an irreducible submodule M' of M not contained in $Id_2^*(G(A))$.

This says that there exists a partition $\lambda = (\lambda_1, \dots, \lambda_t) \vdash p_0 r + s$ and a tableau T_λ such that $e_{T_\lambda} \notin Id_2^*(G(A))$. We shall prove that $\lambda \geq (r^{p_0})$.

To this end, suppose that $\lambda_1 \geq r + s$. Then $e_{T_\lambda} f$ is symmetric on at least $r + s + 1$ variables of the set $Y_i^+ \cup \dots \cup Y_r^+ \cup Y^0$. Since f is alternating on each Y_i^+ by hypothesis, we obtain that f is symmetric and alternating on at least two variables. Thus by linearity $f = 0$ and, so $\lambda_1 \leq r + s$.

Next we prove that λ contains at most s boxes below the first p_0 rows, i.e. $\lambda_{p_0+1} + \dots + \lambda_t \leq s$.

Consider the polynomial $C_{T_\lambda}^- f$ which is alternating on disjoint sets of symmetric variables of homogeneous degree zero corresponding to the columns of λ. If $\sim$ is the map defined in Section 4, then $C_{T_\lambda}^- \tilde{f}$ is still alternating on the same sets of variables. Since $\dim(\bar{A})_0^+ = p$, in order to get a non-zero evaluation of $C_{T_\lambda}^- \tilde{f}$ in A, we must substitute at most p_0 elements of a basis of $(\bar{A})_0^+$ in each alternating set and the remaining variables in J_0^+. It follows that if there are at least s boxes outside the first p_0 rows, then $C_{T_\lambda}^- \tilde{f} \in Id^s(A)$, since $J^s = 0$. By applying the map $\sim$ again by Lemma 1 we get that $C_{T_\lambda}^- f \in Id_2^*(G(A))$, a contradiction.

We have proved that $\lambda_1 \leq r+s$ and $\lambda_{p_0+1} + \dots + \lambda_t \leq s$. It follows that $\lambda \geq (r^{p_0})$ and $e_{T_\lambda} f \notin Id_2^*(G(A))$, for some tableau T_λ. □

The skew analogue of Lemma 2 is the following.

Lemma 3. *Let $A = \bar{A} + J$ be a finite dimensional algebra with superinvolution. Suppose that $\dim(\bar{A})_1^+ = p_1 > 0$ and let $s > 0$ be such that $J^s = 0$. Let*

$$f(Z_1^+, \dots, Z_r^+, X') \notin Id_2^*(G(A))$$

be a multilinear polynomial with $r \geq 1$, such that

1) *f is symmetric on each set Z_i^+ with $|Z_i^+| = p_1, 1 \leq i \leq r$;*
2) *X' contains a set Z^0 of s symmetric variables of homogeneous degree one.*

Then there exists a partition $\mu \geq ({p_1}^r)$ with $|\mu| - rp_1 = s$ and a tableau T_μ such that $e_{T_\mu} f \notin Id_2^*(G(A))$ where e_{T_μ} acts on the set $Z_1^+ \cup \dots \cup Z_r^+ \cup Z^0$.

Proof. We consider the action of S_{p_1r+s} on the variables of the set $Z_1^+\cup\cdots\cup Z_r^+\cup Z^0$. and the left S_{p_1r+s}-module $M = FS_{p_1r+s}f$. As in the previous lemma there exists a partition $\mu = (\mu_1,\dots,\mu_t) \vdash p_1r+s$ and a tableau T_μ such that $e_{T_\mu} \notin Id_2^*(G(A))$.

Let $\mu' = (\mu_1',\dots,\mu_u')$ be the conjugate partition of μ, and suppose that $\mu' > r+s$. Then the polynomial $C_{T_\mu}^- f'$ is alternating on at least $r+s+1$ variables of the set $Z_1^+\cup\cdots\cup Z_r^+\cup Z^0$ corresponding to the first column of μ. Since f is symmetric on each of the r sets Z_i^+, we get that $C_{T_\mu}^- f$ is symmetric and alternating on at least two variables. Hence $C_{T_\mu}^- f = 0$ and, so, $e_{T_\mu} f = 0$, a contradiction. Thus $\mu' \le r+s$.

Suppose now that $\mu_{p_1+1}' + \cdots, \mu_u' \ge s$. Since the polynomial $e_{T_\mu} f$ is symmetric on disjoint sets of variables corresponding to the rows of μ, the polynomial $e_{T_\mu}\tilde{f}$ is alternating on the same sets of variables. Since $e_{T_\mu}\tilde{f} \notin Id^s(A)$ and $p_1 = \dim(\bar{A})_1^+$, in order to get a non-zero evaluation, we must substitute at most p_1 elements from a basis of $(\bar{A})_1^+$ in each alternating set, and the remaining variables from J_1^+.

It follows that if there are at least s boxes below the first p_1 rows of μ', then $e_{T_\mu}\tilde{f} \in Id^s(A)$, since $J^s = 0$. But then by Lemma 1 $e_{T_\mu} f = e_{T_\mu}\tilde{\tilde{f}} \in Id_2^*(G(A))$, a contradiction.

We have proved that $\mu_1' \le r+s$ and $\mu_{p_1+1}' + \ldots + \mu_u < s$. Thus $\mu \ge ({p_1}^r)$ and $e_{T_\mu} f \notin Id_2^*(G(A))$, for some partition μ. □

Remark 1. It is clear that if $\dim(\bar{A})_0^- = q_0 > 0$, then Lemma 2 has an analogue: if $f(Z_1^+,\dots,Z_r^+,X'') \notin Id_2^*(G(A))$, then $e_{T_\mu} f \notin Id_2^*(G(A))$ where f has the analogous alternating properties as in that lemma and $\lambda \ge (r^{q_0})$. Similarly, if $\dim(\bar{A})_1^- = q_1 > 0$, Lemma 3 has a corresponding analogue: if $f(Z_1^-,\dots,Z_r^-,X''') \notin Id_2^*(G(A))$, then $e_{T_\mu} f \notin Id_2^*(G(A))$ where f has the analogous symmetric properties as in that Lemma 3 and $\lambda \ge (q_1^r)$.

The previous two lemmas and Remark 1 imply the following.

Remark 2. Since the symmetric groups of the previous two lemmas and Remark 1 act on disjoint sets of variables, we can put together the above constructions. Hence, for instance, if p_0, q_0, p_1, q_1 are all non-zero we have the following: let

$$f(Y_{(r_1)}^+, Y_{(r_2)}^+, Z_{(r_3)}^+, Z_{(r_4)}^-, X') \notin Id_2^*(G(A))$$

be a multilinear polynomial with $r_i \ge 1, 1 \le i \le 4$, and suppose that

1) f is alternating on each of the sets Y_i^+ and Y_j^-, $1 \le i \le r_1, 1 \le j \le r_2$,
2) f is symmetric on each of the set Z_i^+ and Z_j^-, $1 \le i \le r_3$, $1 \le j \le r_4$,
3) $|Y_i^+| = \dim(\bar{A}_0)^+ = p_0$, $|Y_i^-| = \dim(\bar{A}_0)^- = q_0$, $|Z_i^+| = \dim(\bar{A}_1)^+ = p_1$, $|Z_i^-| = \dim(\bar{A}_1)^- = q_1$,
3) X' contains four disjoint sets of order s, and of these sets two are of symmetric variables of different homogeneous degree and two are of skew variables of different homogeneous degree.

Then there exist partitions, $\lambda(1) \ge (r_1^{p_0}), \lambda(2) \ge (r_2^{q_0}), \lambda(3) \ge (p_1^{r_3}), \lambda(4) \ge (q_1^{r_4})$ with $|\lambda(1)| - r_1p_0 = |\lambda(2)| - r_2q_0 = |\lambda(3)| - r_3p_0 = |\lambda(4)| - r_4q_1 = s$ such that $\prod_{i=1}^4 e_{T_{\lambda(i)}} f \notin Id_2^*(G(A))$, for suitable tableaux $T_{\lambda(i)}, 1 \le i \le 4$.

6. Simple algebras with superinvolution

We recall the classification of the finite dimensional simple algebras with superinvolution over an algebraically closed field F of charateristic 0 (see [5]).

First, if A is a finite dimensional simple superalgebra over F, then it is well known (see [20, Section 3]) that either $A \cong M_n(F), n \geq 1$ or $A \cong M_{n,m}(F)$ or $A \cong M_n(F) \oplus cM_n(F)$, with $c^2 = 1$.

Here $M_{n,m}(F)$ is the algebra of $(n+m) \times (n+m)$ matrices with $\mathbb{Z}_2$-grading

$$M_{n,m}(F)_0 = \{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M_n(F), B \in M_m(F)\},$$

$$M_{n,m}(F)_1 = \{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C = n \times m \text{ matrix}, D = m \times n \text{ matrix}\},$$

and $M_n(F) \oplus cM_n(F) = A$ has a $\mathbb{Z}_2$-grading such that $A_0 = M_n(F), A_1 = cM_n(F)$.

Now let A be a finite dimensional algebra with superinvolution. Then, according to [5], either

1) $A \cong M_{n,m}(F)$ with orthosymplectic or transpose involution, or
2) $M_{n,m}(F) \oplus cM_{n,m}(F)^{osp}$ with exchange involution, or
3) $A = Q(n) \oplus Q(n)^{osp}$ with exchange involution, where $Q(n) = M_n(F) \oplus cM_n(F)$.

Recall that $M_{n,m}(F)$ has an orthosymplectic involution osp if and only if $m = 2s$ is even and osp in defined as follows

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix}$$

where $X \in M_n(F), T \in M_{2s}(F), I_n$ is the identity $n \times n$ matrix, Y and Z are rectangular matrices of suitable size, t is the transpose involution and $P = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} \in M_{2s}(F)$.

Also, $M_{n,m}(F)$ has a transpose superinvolution trp if $n = m$ and trp is defined as follows

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix}$$

where $X, Y, Z, T \in M_n(F)$.

Finally we remind the reader that if $A = A_0 \oplus A_1$ is a superalgebra then the superopposite algebra A^{sop} of A is the algebra A with the same underlying vector space and a new multiplication $\circ$ defined on homogeneous elements as follows: $a \circ b = (-1)^{|a||b|}ba$; here $|a|$ is the homogeneous degree of a.

Lemma 4. *Let A be a finite dimensional simple algebra with superinvolution over an algebraically closed field. Then, for every $t \geq 1$ there exists a multilinear polynomial*

$$f(Y_1^+, \ldots, Y_{2t}^+, Y_1^-, \ldots, Y_{2t}^-, Z_1^+, \ldots, Z_{2t}^+, Z_1^-, \ldots, Z_{2t}^-)$$

which is alternating on each of the sets $Y_i^+, Y_i^+, Z_i^+, Z_i^-$, where $|Y_i^+| = \dim A_0^+, |Y_i^-| = \dim A_0^-, |Z_i^+| = \dim A_1^+, |Z_i^-| = \dim A_1^-, 1 \leq i \leq 2t$. Moreover f is a central polynomial that takes an invertible value in A.

Proof. Let $g_n(x_1, \ldots, x_{n^2}, y_1, \ldots, y_{n^2})$ be the central polynomial of Regev for $M_n(F)$ (see [16, Theorem 5.7.4] or [9]). Recall that g_n is alternating on each of the two sets of variables $\{x_1, \ldots, x_{n^2}\}$ and $\{y_1, \ldots, y_{n^2}\}$. Let $f_n^{(t)}$ be the product of t copies of g_n on disjoint sets of variables. Then $f_n^{(t)}$ is alternating on $2t$ sets of variables each of order n^2 and $f_n^{(t)}$ takes a non-zero central value in $M_n(F)$.

Now let A be a finite dimensional simple algebra with superinvolution and suppose first that $A \cong M_{n,m}(F)$ with orthosymplectic or transpose superinvolution. Then the polynomial $f^{(t)}_{n+m}$ is alternating on $2t$ sets each of cardinality $(n+m)^2$; since $\dim A_0^+ + \dim A_0^- + \dim A_1^+ + \dim A_1^- = (n+m)^2$, each set can be partitioned into four sets of the type $Y_i^+ \cup Y_i^- \cup Z_i^+ \cup Z_i^-$, and $f^{(t)}_{n+m}$ is alternating on each of the four sets separately. Hence $f^{(t)}_{n+m}$ is the desired polynomial.

Next suppose that $A \cong M_{n,m}(F) \oplus cM_{n,m}(F)^{sop}$ with exchange involution. For such algebra $A^+ = \{(a,a) \mid a \in M_{n,m}(F)\}$ and $A^- = \{(a,-a) \mid a \in M_{n,m}(F)\}$. Then we take $g_1^{(t)}$ to be $f^{(t)}_{n+m}$ on symmetric variables and $g_2^{(t)}$ to be $f^{(t)}_{n+m}$ on skew variables. Since $A^+ = A_0^+ \oplus A_1^+$ and $A^- = A_0^- \oplus A_1^-$, it turns out that $g_1^{(t)} g_2^{(t)}$ is the desired polynomial.

Now let $A = Q(n) \oplus Q(n)^{sop}$ with exchange involution, where $Q(n) = M_n(F) \oplus cM_n(F)$. Since $A^+ = \{(a,a) \mid a \in Q(n)\}$ and $A^- = \{(a,-a) \mid a \in Q(n)\}$, we consider $g^{(t)} = f_n^{(t)} \cdot c f_n^{(t)}$, which is a central polynomial for $Q(n)$. Then we consider $g_1^{(t)} = g^{(t)}$ on symmetric variables of A and $g_2^{(t)} = g^{(t)}$ on skew variables of A. It follows that $g_1^{(t)} g_2^{(t)}$ is the desired polynomial. □

7. Reduced algebras

The aim of this section is to construct multilinear polynomials corresponding to hook shaped diagrams of suitable size, which are not $*$-identities of $G(A)$, in case A is a reduced algebra (see Definition 2).

We start by recalling two basic facts about the degrees of S_n-characters.

Il $\lambda \vdash n$, we shall denote by $d_\lambda = \chi_\lambda(1)$. the degree of the corresponding S_n-character. χ_λ. Then we have the following easy fact.

Remark 3. *([16, Lemma 6.2.4]) Let $\lambda \vdash n$ and $\mu \leq \lambda$. If $|\lambda| - |\mu| = k$, then $d_\lambda \geq n^{-2k} d_\mu$.*

Next we recall the following definition.

Definition 3. *For any integers $k, l, t \geq 0$, we let $h(k,l,t)$ be the hook shaped diagram $h(k,l,t) = ((l+t)^k, l^t)$.*

The asymptotics of $d_h(k,l,t)$ are given in the following ([16, Lemma 6.2.5]).

Lemma 5. *Let $k, l \geq 0$ be fixed integers. If $h(k,l,t) \vdash n$, then*

$$d_{h(k,l,t)} \simeq_{n\to\infty} Cn^r (k+l)^n,$$

for some constants C, r.

Lemma 6. *Let $A = \bar{A} + J$ be a finite dimensional reduced algebra with superinvolution over an algebraically closed field F. Let $\dim(\bar{A})_0^+ = p_0$, $\dim(\bar{A})_0^- = q_0$, $\dim(\bar{A})_1^+ = p_1$, $\dim(\bar{A})_1^- = q_1$. Then, for every $t \geq 1$ there exists a multilinear polynomial*

$$f = f(Y_1^+, \dots, Y_{2t}^+, Y_1^-, \dots, Y_{2t}^-, Z_1^+, \dots, Z_{2t}^+, Z_1^-, \dots, Z_{2t}^-, \bar{X})$$

where $|Y_i^+| = p_0, |Y_i^-| = q_0, |Z_i^+| = p_1, |Y_i^-| = q_1$ such that

1) *f is alternating on each of the sets $Y_i^+, Y_i^-, Z_i^+, Z_i^-, 1 \leq i \leq 2t$;*
2) *$\bar{X}$ is a set of $k-1$ of homogeneous symmetric variables, where k is the number of simple summands of $\bar{A}$;*

3) $f \notin Id^s(A)$.

Proof. Let $\bar{A} = A_1 \oplus \cdots \oplus A_k$ where each A_i is a simple algebra with superinvolution. For each i, $1 \le i \le k$, write $A_i = (A_i)_0^- \oplus (A_i)_0^+ \oplus (A_i)_1^+ \oplus (A_i)_1^-$ and let $p_{0i} = \dim(A_i)_0^+$, $q_{0i} = \dim(A_i)_0^+$, $p_{1i} = \dim(A_i)_1^+$, $q_{1i} = \dim(A_i)_1^-$. Let $f_i = f_i(Y_{i,1}^+, \dots, Y_{i,2t}^+, Y_{i,1}^-, \dots, Y_{i,2t}^-, Z_{i,1}^+, \dots, Z_{i,2t}^+, Z_{i,1}^-, \dots, Z_{i,2t}^-)$ be the polynomial constructed in Lemma 4 which is alternating on each of the sets $Y_{i,j}^+, Y_{i,j}^+, Z_{i,j}^+, Z_{i,j}^-$ and takes a central invertible value on A_i. Recall that $|Y_{i,j}^+| = p_{0i}$, $|Y_{i,j}^-| = q_{0i}$, $|Z_{i,j}^+| = p_{1i}$, $|Z_{i,j}^-| = q_{1i}$, for all $1 \le j \le 2t$. Now since A is reduced, $A_1JA_2J \cdots JA_k \neq 0$, and let $a_i \in A_i, 1 \le i \le k, j_u \in J, 1 \le u \le k-1$ such that

$$a_1 j_1 a_2 j_2 \cdots j_{k-1} a_k \neq 0.$$

We may clearly assume that each element a_i and j_i is homogeneous and either symmetric or skew. Define a new polynomial

$$f = f_1 x_1 f_2 x_2 \cdots x_{k-1} f_k,$$

where x_i is a symmetric variable of the same homogeneous degree as $a_i j_i a_{i+1}, 1 \le i \le k-1$.

Let φ be an evaluation of f such that $\varphi(f_i) = e_i$, where e_i is the unit element of $A_i, 1 \le i \le k-1$; also $\varphi(x_i) = a_i j_i a_{i+1} \pm a_{i+1} j_i a_i$, for $1 \le i \le k-1$, where we take the plus or minus sign so that the resulting element is symmetric. Then we get

$$\begin{aligned}\varphi(f) &= e_1(a_1 j_1 a_2 \pm a_2 j_1 a_1)e_2 \cdots (a_{k-1} j_{k-1} a_k \pm a_k j_{k-1} a_{k-1})e_k \\ &= e_1 a_1 j_1 a_2 e_2 \cdots a_{k-1} j_{k-1} a_k e_k = a_1 j_1 a_2 \cdots a_{k-1} j_{k-1} a_k \neq 0.\end{aligned}$$

Notice that here we are using the fact that $e_i a_{i+1} \in A_i A_{i+1} = 0$.

Next we define new sets, for $1 \le i \le 2t$:

$$\begin{aligned} Y_i^+ &= Y_{1,i}^+ \cup \cdots \cup Y_{k,i}^+, \\ Y_i^- &= Y_{1,i}^- \cup \cdots \cup Y_{k,i}^-, \\ Z_i^+ &= Z_{1,i}^+ \cup \cdots \cup Z_{k,i}^+, \\ Z_i^+ &= Z_{1,i}^+ \cup \cdots \cup Z_{k,i}^+. \end{aligned}$$

Let Alt_{X_i} be the operator of alternation on the set X_i. Recall that Alt is defined on a multilinear polynomial $f(x_1, \dots, x_n)$ as follows:

$$Alt_{\{x_1,\dots,x_r\}} f(x_1, \dots, x_r, x_{r+1}, \dots, x_n) = \sum_{\sigma \in S_r} (\operatorname{sgn} \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(r)}, x_{r+1}, \dots, x_n).$$

Define a new polynomial

$$\begin{aligned} f' = \prod_{i=1}^{2t} Alt_{Y_i^+} Alt_{Y_i^-} Alt_{Z_i^+} Alt_{Z_i^-} f(&Y_1^+, \dots, Y_{2t}^+, Y_1^-, \\ &\dots, Y_{2t}^-, Z_1^+, \dots, Z_{2t}^+, Z_1^-, \dots, Z_{2t}^-). \end{aligned}$$

Notice that for $1 \le i \le 2t$, $|Y_i^+| = |Y_{1,i}^+| + \cdots + |Y_{k,i}^+| = p_{01} + \cdots + p_{0k} = p_0$ and similarly $|Y_i^-| = q_0$, $|Z_i^+| = p_1$, $|Z_i^-| = q_1$. Now, since each polynomial f_i is already alternating in each of the sets $Y_{i,j}^+, Y_{i,j}^-, Z_{i,j}^+, Z_{i,j}^-$, $1 \le j \le 2t$, and $A_i A_j = 0$ for $i \neq j$, it follows that under the evaluation φ we have that

$$\varphi(f') = \prod_{1=i}^{k} p_{0i}! q_{0i}! p_{1i}! q_{1i}! \varphi(f) \neq 0,$$

and hence f' does not vanish on A. Thus f' satisfies the conclusion of the lemma. □

Corollary 1. *Let f be the polynomial constructed in the previous lemma, $f \notin Id^s(A)$. Then there exists a polynomial $f' \notin Id^s(A)$ which is multialternating on the same sets of variables as f and f' has 4 extra sets of variables $X^{(p_0)} \subseteq Y^+, X^{(p_1)} \subseteq Y^-, X^{(q_0)} \subseteq Z^+, X^{(q_1)} \subseteq Z^-$ such that $|X^{(i)}| = 2m$, if $i \neq 0$ and $|X^{(i)}| = \emptyset$ otherwise.*

Proof. We start with the polynomial f constructed in the previous lemma and we recall that there is an evaluation φ on A such that $\varphi(f) = \alpha e_1 j_1 e_2 j_2 \cdots j_{k-1} e_k \neq 0$, where $\alpha \in F, e_i$ is the identity element of A_i and $j_i \in J$ is a homogeneous symmetric element, $1 \leq i \leq k$. Such evaluation coincides up to a scalar with the evaluation of the polynomial $f_1 x_1 f_2 x_2 \cdots x_{k-1} f_k$ where f_i is a central polynomial for A_i constructed in Lemma 4.

We shall show how to insert in f, $2m$ skew variables of homogeneous degree zero, provided $q_0 = \dim A_0^- > 0$. The insertion of the other sets of variables is similar.

Hence suppose that $q_0 = \dim A_0^- > 0$. Then there exists a simple algebra with superinvolution, say A_l such that $\dim A_l^- > 0$. Now, $e_1 j_1 \cdots j_{k-1} e_k \neq 0$ says that there exists a matrix unit $a \in A_l$ such that $e_1 j_1 \cdots j_{l-1} a j_l e_{l+1} \cdots j_{k-1} e_k \neq 0$.

Now recalling the classification of the finite dimensional simple algebras with involution, one can choose $2m$ elements of homogeneous degree zero $a_1, \ldots, a_{2m}$, taken from a standard basis of A_l, such that $a a_1 \cdots a_{2m} = a$. But then the polynomial

$$f_1 x_1 \cdots f_{l-1} x_{l-1} f_l x_{k+1} \cdots x_{2m+k} f_{l+1} x_{l+1} \cdots x_{k-1} f_k$$

has a nonzero evaluation in A and has the required $2m$ extra skew variables of homogeneous degree zero. □

Lemma 7. *Let $A = \bar{A} + J$ be a reduced algebra with superinvolution over an algebraically closed field F. Let $\dim A = m, \dim(\bar{A})_0^- = q_0, \dim(\bar{A})_1^+ = p_1, \dim(\bar{A})_1^- = q_1$. Then, for any $t \geq 1$, there exist partitions*

$$\lambda \vdash 2t(p_0 + p_1) + 4m, \ \ \mu \vdash 2t(q_0 + q_1) + 4m$$

such that

$$h(p_0, p_1, 2t - 2m) \leq \lambda \leq h(p_0 + 2m, p_1 + 2m, 2t + 2m)$$

$$h(q_0, q_1, 2t - 2m) \leq \mu \leq h(q_0 + 2m, q_1 + 2m, 2t + 2m),$$

and tableaux T_λ, T_μ with the following properties. If we let e_{T_λ} act on symmetric variables and e_{T_μ} on skew variables, then $e_{T_\lambda} e_{T_\mu} f \notin Id^*(G(A))$, for some multilinear polynomial f with $\deg f = 2t(\dim \bar{A}) + 8m$.

Proof. Let f be the polynomial constructed in the previous corollary. Recall that

$$f = f(Y^+_{(2t)}, Y^-_{(2t)}, Z^+_{(2t)}, Z^-_{(2t)}, X^{(p_0)}, X^{(p_1)}, X^{(q_0)}, X^{(q_1)})$$

is a multilinear polynomial alternating on each of the sets of $Y^+_{(2t)} \cup Y^-_{(2t)} \cup Z^+_{(2t)} \cup Z^-_{(2t)}$ and $|X^{(i)}| = 2m$ if $i \neq 0$ and $X^{(i)} = \emptyset$ otherwise. Moreover $f \notin Id^s(A)$.

By the property of the map $\sim$, the polynomial $\tilde{f}$ is alternating on each of the sets of $Y^+_{(2t)} \cup Y^-_{(2t)}$ and symmetric on each of the sets of $Z^+_{(2t)} \cup Z^-_{(2t)}$; moreover

$\tilde{f} \notin Id_2^*(G(A))$. The polynomial $\tilde{f}$ satisfies the hypothesis of Remark 2. Hence there exist partitions

$$\lambda(1) \geq ((2t)^{p_0}),\ \lambda(2) \geq ((2t)^{q_0}),\ \lambda(3) \geq (p_1^{2t}),\ \lambda(4) \geq (q_1^{2t})$$

with

$$|\lambda(1)| - 2tp_0 = |\lambda(2)| - 2tq_0 = |\lambda(3)| - 2tp_1 = |\lambda(4)| - 2tq_1 = 2m,$$

and tableaux $T_{\lambda(i)}, 1 \leq i \leq 4$, such that $\prod_{i=1}^4 e_{T_{\lambda(i)}} \tilde{f} \notin Id_2^*(G(A))$.

Let the symmetric groups $S_{2tp_0+2m}, S_{2tq_0+2m}, S_{2tp_1+2m}, S_{2tq_1+2m}$ act on the polynomial $g = \prod_{i=1}^4 e_{T_{\lambda(i)}} \tilde{f}$ by permuting the variables corresponding to the tableaux $T_{\lambda(i)}, 1 \leq i \leq 4$, respectively. Let M be the $S_{2tp_0+2m} \times S_{2tq_0+2m} \times S_{2tp_1+2m} \times S_{2tq_1+2m}$-module generated by the polynomial g. If $\bar{M}$ is the induced module

$$\bar{M} = M \uparrow^{S_{2t(p_0+p_1)+4m} \times S_{2t(q_0+q_1)+4m}},$$

then, since $M \not\subseteq Id_2^*(G(A))$, then also $\bar{M} \notin Id_2^*(G(A))$. We decompose $\bar{M}$ into irreducible modules $\bar{M} = \bar{M}_1 \oplus \cdots \oplus \bar{M}_r$ and since $\bar{M} \not\subseteq Id_2^*(G(A))$, we have that $\bar{M}_i \not\subseteq Id_2^*(G(A))$, for some i. Now by the Littlewood-Richardson rule (see [19]), $\bar{M}_i$ is associated to a pair of partitions $\lambda \vdash 2t(p_0+p_1)+4m,\ \mu \vdash 2t(q_0+q_1)+4m$ such that

$$h(p_0, p_1, 2t-r) \leq \lambda \leq h(p_0+2m, p_1+2m, 2t+2m),$$

and

$$h(q_0, q_1, 2t-s) \leq \mu \leq h(q_0+2m, q_1+2m, 2t+2m),$$

where $r = \max\{p_0, p_1\}$ and $s = \max\{q_0, q_1\}$. Clearly $\lambda \geq h(p_0, p_1, 2t-2m)$ and $\mu \geq h(q_0, q_1, 2t-2m)$. Now, since $\bar{M}_i \not\subseteq Id_2^*(G(A))$, there exists a multilinear polynomial g' and tableaux T_λ and T_μ such that $e_{T_\lambda} e_{T_\mu} g' \notin Id_2^*(G(A))$.

If we consider the variables appearing in $e_{T_\lambda} e_{T_\mu} g'$ as symmetric ungraded or skew ungraded variables, we get that $e_{T_\lambda} e_{T_\mu} g'$ in these new variables is not a $*$-identity for $G(A))$, i.e., $e_{T_\lambda} e_{T_\mu} g' \notin Id^*(G(A))$. □

8. The lower bound

Lemma 8. *Let A be a finite dimensional algebra with superinvolution over an algebraically closed field F. Then there exist constants $C > 0, r$ such that*

$$c_n^*(G(A)) \leq Cn^r d^n,$$

where d is the maximal dimension of an admissible subalgebra of A.

Proof. Let $A = \bar{A} + J$ where $\bar{A} = A_1 \oplus \cdots \oplus A_r$ is a maximal semisimple subalgebra with superinvolution and the A_i's are simple algebras with superinvolution. Recall that by Definition 2, if C is an admissible subalgebra of A, then $B = C + J$ is a reduced algebra. Hence, since $c_n^*(G(A)) \geq c_n^*(G(B))$, for all $n \geq 1$, in order to prove the lemma we may clearly assume that A is a reduced algebra with $\dim \bar{A} = d$.

Let $m = \dim A$ and set $p_0 = \dim(\bar{A}_0)^+, q_0 = \dim(\bar{A}_0)^-, p_1 = \dim(\bar{A}_1)^+, q_1 = \dim(\bar{A}_1)^-$. Hence $d = p_0 + q_0 + p_1 + q_1$.

Take $n \geq 8m$ any integer, and divide $n - 8m$ by d. So we write $n = 2td + 8m + r$, with $0 \leq r \leq 2d$. Let f be the polynomial constructed in Lemma 7. Hence $e_{T_\lambda} e_{T_\mu} f \notin Id^*(G(A))$, for suitable tableaux T_λ, T_μ where

$$\lambda \geq h(p_0, p_1, 2t-2m), \quad \mu \geq h(q_0, q_1, 2t-2m)).$$

Moreover $\deg f = 2td + 8m$. We compute $n - \deg f = 2td + 8m + r - 2td - 8m = r \leq 2d$. Thus $n - \deg f = s$ is a constant that does not depend on t.

Let $g' = e_{T_\lambda} e_{T_\mu} f x_{s+1} \cdots x_n$, where $x_{s+1} \cdots x_n$ are symmetric variables distinct from the ones appearing in f. Recalling the construction of f, it is readily seen that still $g' \notin Id^*(G(A))$. Now, by the branching rule we add $n-s$ boxes to the diagrams of λ and μ In this way we obtain partitions

$$\hat{\lambda} \vdash n_1 \geq |\lambda| \geq |h(p_0, p_1, 2t-2m)|,$$

$$\hat{\mu} \vdash n_2 \geq |\mu| \geq |h(q_0, q_1, 2t-2m)|$$

and a polynomial g'' such that $e_{T_{\hat{\lambda}}} e_{T_{\hat{\mu}}} g'' \notin Id^*(G(A))$. Here n_1 and n_2 count the number of symmetric and skew variables appearing in the polynomial g'.

Notice that

$$n - |h(p_0, p_1, 2t-2m)| - |h(q_0, q_1, 2t-2m)|$$

$$= 2td + 8m + r - (p_0+p_1)(2t-2m) - p_0p_1 - (q_0+q_1)(2t-2m) - q_0q_1$$

$$\leq 8m + 2d - 2dm - p_0p_1 - q_0q_1,$$

a constant that does not depend on t.

It follows that $n_1 - |h(p_0, p_1, 2t-2m)|$ and $n_2 - |h(q_0, q_1, 2t-2m)|$ are also constant. Thus by Remark 3, $d_{\hat{\lambda}} \geq n_1^{-2k} d_{h(p_0,p_1,2t-2m)}$ and $d_{\hat{\mu}} \geq n_2^{-2k} d_{h(q_0,q_1,2t-2m)}$, for some constant k. It follows that

$$c_{n_1,n_2}(G(A)) \geq c^*_{n_1,n_2}(G(B)) \geq d_{\hat{\lambda}} d_{\hat{\mu}}$$

$$\geq (n_1n_2)^{-2k} d_{h(p_0,p_1,2t-2m)} d_{h(q_0,q_1,2t-2m)}$$

$$\geq (n_1n_2)^r (p_0+p_1)^{2t(p_0+p_1)} (q_0+q_1)^{2t(q_0+q_1)} = (n_1n_2)^r p^{2tp} q^{2tq},$$

where

$$|h(p_0, p_1, 2t-2m)| = (p_0+p_1)^{2t} - (p_0+p_1)2m + p_0p_1,$$

and similarly for $h(q_0, q_1, 2t-2m)$. In the above inequalities we have applied Lemma 6 which gives the asymptotics for the hooks.

Next we compute $c^*_n(G(A))$ (recall the connection between c^*_n and c_{n_1,n_2} given at the end of Section 2). Recalling Stirling formula $n! \simeq \sqrt{2\pi n}(\frac{n}{e})^n$, we get

$$\frac{n!}{n_1!n_2!} \geq \frac{(2tp+2tq)!}{(2tp)!(2tq)!} \simeq \frac{\sqrt{2\pi(2tp+2tq)}}{\sqrt{16\pi^2t^2pq}} \frac{(2tp+2tq)^{2tp+2tq}}{(2tp)^{2tp}(2tq)^{2tq}}$$

$$= \frac{C(p+q)^{2t(p+q)}}{p^{2tp}q^{2tq}},$$

for some constant $C > 0$. Hence

$$c^*_n(G(A)) \geq \binom{n}{n_1, n_2} c_{n_1,n_2}(G(A)) \geq C(n_1n_1)^r p^{2tp} q^{2tq} \frac{(p+q)^{2t(p+q)}}{p^{2tp}q^{2tq}}$$

$$\geq Cn^r (p+q)^{2t(p+q)} \geq Cn^r d^{2td} \geq C_1 n^{r'} d^n,$$

since $n - 2td$ is constant. $\square$

9. The upper bound

Let $n \geq 1$ and let S be the free supercommutative algebra over F on the countable sets T_1 and T_2 where $T_1 = \{\xi_{i,j} | 1 \leq i \leq n, j \geq 1\}$ and $T_2 = \{\eta_{i,j} | 1 \leq i \leq n, j \geq 1\}$ (see [6]). Recall that S is the algebra with 1 generated by $T_1 \cup T_2$ over F, subject to the conditions that the elements of T_1 are central and the elements of T_2 anticommute.

The algebra S has a natural $\mathbb{Z}_2$-grading $S = S_0 \oplus S_1$ if we require that the $\xi_{i,j}$'s are of homogeneous degree zero and the $\eta_{i,j}$'s are of homogeneous degree one. Hence S_0 (S_1) is the span of all monomials in the elements of $T_1 \cup T_2$ having an even (odd, resp.) number of $\eta_{i,j}$'s.

The Grassmann algebra G embeds into S with induced $\mathbb{Z}_2$-grading if one identifies the generating elements $e_1, e_2, \dots$ of G with the elements of T_2. In this embedding G_0 is spanned by the monomials in the $\eta_{i,j}$'s of even length and G_1 by the monomials in the $\eta_{i,j}$'s of odd length. Hence $S \cong F[\xi_{i,j}] \otimes_F G$. Notice that the superinvolution of G extends naturally to a superinvolution of S by requiring that $\xi_{i,j}^* = \xi_{i,j}$ for all i, j. And we have that $S_0 = S^+$ and $S_1 = S^-$.

Now let $A = A_0 \oplus A_1$ be a finite dimensional algebra with superinvolution $*$ and consider $G(A)$, its Grassmann envelope.

We fix a basis $\mathcal{B}$ of A which is the union $\mathcal{B} = \mathcal{B}_0^+ \cup \mathcal{B}_0^- \cup \mathcal{B}_1^+ \cup \mathcal{B}_1^-$ of bases of $A_0^+, A_0^-, A_1^+, A_1^-$, respectively.

Let $\mathcal{B}_0^+ = \{a_{0,1}, \dots, a_{0,r_0}\}, \mathcal{B}_0^- = \{b_{0,1}, \dots, b_{0,s_0}\}, \mathcal{B}_1^+ = \{a_{1,1}, \dots, a_{1,r_1}\}$ and $\mathcal{B}_1^- = \{b_{1,1}, \dots, b_{1,s_1}\}$. Next we define for $i = 1, \dots, n$,

$$\xi_i^+ = \sum_{j=1}^{r_0} \xi_{i,j} a_{0,j} + \sum_{l=1}^{s_1} \eta_{k,l} b_{1,l}, , \tag{2}$$

$$\xi_i^- = \sum_{j=1}^{s_0} \xi_{i,r_0+j} b_{0,j} + \sum_{l=1}^{r_1} \eta_{k,s_1+l} a_{1,l}, . \tag{3}$$

Let $\mathcal{H} = F\langle \xi_1^+, \dots, \xi_n^+, \xi_1^-, \dots, \xi_n^- \rangle$ be the algebra generated by the ξ_i^+ and ξ_i^-, $1 \leq i \leq n$, over F. Clearly $\mathcal{H} \subseteq F[\xi_{i,j}] \otimes G \otimes A \simeq S \otimes A$. Moreover, $\mathcal{H}$ has an involution $*$ such that $\xi_1^+, \dots, \xi_n^+$ are symmetric elements and $\xi_1^-, \dots, \xi_n^-$ are skew elements (this is easily checked by recalling that the $\eta_{k,l}$ are anticommuting variables). Now recall that the involution on the Grassmann envelope $G(A)$ is such that

$$G(A)^+ = G_0 \otimes A_0^+ \oplus G_1 \otimes A_1^-,$$
$$G(A)^- = G_0 \otimes A_0^- \oplus G_1 \otimes A_1^+.$$

Hence any element of $G(A)^+$ ($G(A)^-$) can be thought as obtained from ξ_i^+ (ξ_i^-, resp.) by evaluating the variables $\xi_{i,j}$ into G_0 and the variables $\eta_{i,j}$ into G_1. It follows that $f(x_1^+ \dots, x_n^+, x_1^- \dots, x_n^-) \in F\langle X, * \rangle$ is a $*$-identity of $G(A)$ if and only if $f(\xi_1^+ \dots, \xi_n^+, \xi_1^- \dots, \xi_n^-) = 0$. Hence $\mathcal{H}$ is the relatively free algebra with involution in n symmetric generators and n skew generators of the variety of algebras with involution generated by $G(A)$.

Lemma 9. *Let A be a finite dimensional algebra with superinvolution over an algebraically closed field F. Then there exist constants C, t such that*

$$c_n^*(G(A)) \leq C n^t d^n,$$

where d is the maximal dimension of an admissible subalgebra of A.

Proof. Let $\mathcal{H} = F\langle\xi_1^+,\dots,\xi_n^+,\xi_1^-,\dots,\xi_n^-\rangle$, be the relatively free algebra defined above. Then the n-th $*$-codimension of $G(A)$ is

$$c_n^*(G(A)) = \dim_F \operatorname{span}\{\eta_{\sigma(1)}\cdots\eta_{\sigma(n)} \mid \sigma\in S_n,\ \eta_i=\xi_i^+ \text{ or } \eta_i=\xi_i^-,\ 1\le i\le n\}.$$

For a fixed $t\geq 0$, let

$$\mathcal{P}_{t,m-t} = \operatorname{span}\{\eta_{\sigma(1)}\cdots\eta_{\sigma(n)} \mid \sigma\in S_n,\ \eta_i=\xi_i^+ \text{ if } 1\le i\le t,$$
$$\text{and } \eta_i=\xi_i^- \text{ if } t+1\le i\le n\}.$$

Our first aim is to compute an upper bound of $\dim_F \mathcal{P}_{t,n-t}$. To this end, take a monomial $\eta_{\sigma(1)}\cdots\eta_{\sigma(n)}\in\mathcal{P}_{t,n-t}$ and, by mean of the definition given in (2) and (3), write such monomial as a linear combination of products of n elements of the basis of A with coefficients polynomials in the $\xi_{i,j}$'s and the $\eta_{k,l}$'s.

Write $A = B + J$ where B is a maximal semisimple subalgebra with superinvolution and $J = J(A)$. Write $B = B_1\oplus\cdots\oplus B_k$, a direct sum of simple algebras with superinvolution. Then we choose our basis $\mathcal{B} = \mathcal{B}_0^+\cup\mathcal{B}_0^-\cup\mathcal{B}_1^+\cup\mathcal{B}_1^-$ in such a way that each of the four components is made of elements of J and of the simple algebras B_i. But then, by abuse of notation, since each variable $\xi_{i,j}$ or $\eta_{k,l}$ in (1) and (2) is attached to a basis element, we shall say that $\xi_{i,j}$ or $\eta_{k,l}$ is a radical variable or a semisimple variable.

Notice that if $J^{u+1}=0$, then clearly each non-zero monomial contains at most u radical variables and $n-i$ semisimple variables with $i\le u$.

Let us fix a distribution of the radical variables in a non-zero monomial of $\mathcal{P}_{t,n-t}$. For this fixed distribution the semisimple variables must come either from a simple component or from distinct simple components of $(i_1,\dots,i_v)$. This means that $D = B_{i_1}\oplus\cdots\oplus B_{i_v}\subseteq B$ is an admissible subalgebra of B.

Now let D be an admissible subalgebra of B and let $d_1 = |\mathcal{B}_0^+\cap D|, d_2 = |\mathcal{B}_1^+\cap D|, d_3 = |\mathcal{B}_0^-\cap D|$ and $d_4 = |\mathcal{B}_1^-\cap D|$. Hence $d_1+d_2 = \dim D^+$ and $d_3+d_4 = \dim D^-$. Now, each monomial has t coefficients taken in the set $\{\xi_{i,j}|1\le i\le t,\ 1\le j\le r_0\}\cup\{\eta_{k,l}|1\le k\le t,\ 1\le l\le s_1\}$ and $n-t$ taken in the set $\{\xi_{i,j}|\,t+1\le i\le n,\ r_0+1\le j\le r_0+s\}\cup\{\eta_{k,l}|\,t+1\le k\le n,\ s_1+1\le l\le s_1+r_1\}$. Moreover the number of possible distributions of i radical variables for $i=1,\dots,u$, is bounded by C_1n^u, for some constant C_1. Therefore in all we get $\le C_2n^u(d_1+d_2)^t(d_3+d_4)^{n-t}$ possible monomials, with C_2 a constant. Thus an upper bound for the monomials coming from an admissible subalgebra D is

$$C_2n^u(\dim D^+)^t(\dim D^-)^{n-t}.$$

Now, if M is the number of admissible subalgebras of B and E is an admissible subalgebra of maximal dimension, then an upper bound for the number of possible non-zero monomials is $MC_2n^u(\dim E^+)^t(\dim E^-)^{n-t}$. Taking into account that we rewrote any product of n basis elements of A as a linear combination of basis elements, we get that

$$\dim_F \mathcal{P}_{t,n-t}\le C_3n^u(\dim E^+)^t(\dim E^-)^{n-t},$$

where C_3 is a constant. The connection between $c_n^*(G(A))$ and $\dim\mathcal{P}_{t,n-t}$, $0\le t\le n$, is given in (1), and we have

$$c_n^*(G(A)) = \sum_{t=0}^{n}\binom{n}{t}\dim\mathcal{P}_{t,n-t}$$

$$\leq C_3 n^u \sum_{t=0}^{n} \binom{n}{t} (\dim E^+)^t (\dim E^-)^{n-t} = C_3 n^u d^n,$$

where $d = \dim E^+ + \dim E^- = \dim E$. □

10. The main theorem

Putting together the results of the previous sections we get.

Theorem 3. *Let A be a finite dimensional superalgebra with superinvolution over an algebraically closed field of characteristic zero. If $*$ is the induced involution on the Grassmann envelope $G(A)$, then there exist constants $C_1 > 0, C_2, t_1, t_2$ such that*

$$C_1 n^{t_1} d^n \leq c_n^*(G(A)) \leq C_2 n^{t_2} d^n.$$

where d is the maximal dimension of an admissible subalgebra of A.

Recalling that codimensions do not change by extending the base field, by Theorem 1 we have the following.

Theorem 4. *Let A be a PI-algebra with involution $*$ over a field of characteristic zero. Then there exist constants $C_1 > 0, C_2, t_1, t_2$ such that*

$$C_1 n^{t_1} d^n \leq c_n^*(A) \leq C_2 n^{t_2} d^n.$$

Hence $\lim_{n\to\infty} \sqrt[n]{c_n^*(A)} = exp^*(A)$, the $*$-exponent of A, exists and is an integer.

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