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On the Frattini subgroup of a finite group[☆]



Stefanos Aivazidis^{*}, Adolfo Ballester-Bolinches

*Departament d' Àlgebra, Universitat de València, C/ Dr. Moliner, 50,
46100-Burjassot, València, Spain*

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ABSTRACT

We study the class of finite groups G satisfying $\Phi(G/N) = \Phi(G)N/N$ for all normal subgroups N of G . As a consequence of our main results we extend and amplify a theorem of Doerk concerning this class from the soluble universe to all finite groups and answer in the affirmative a long-standing question of Christensen whether the class of finite groups which possess complements for each of their normal subgroups is subnormally closed.

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1. Introduction and statement of results

The only groups considered in this paper are finite. In the present article we shall examine certain questions concerning the behaviour of the Frattini subgroup in epimorphic images.

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^{*} Corresponding author.

E-mail addresses: s.aivazidis@qmul.ac.uk (S. Aivazidis), adolfo.ballester@uv.es (A. Ballester-Bolinches).

Following Gaschütz [10], we call a group Φ -free if its Frattini subgroup is trivial. We denote by \mathfrak{B} the class of groups G such that G/N is Φ -free for all normal subgroups N of G . It is clear that a group $G \in \mathfrak{B}$ if and only if G has no Frattini chief factors (recall that a chief factor K/L of a group G is said to be Frattini if $\Phi(G/L) \geq K/L$).

Our first main result is a reduction theorem of wide applicability that provides a sufficient condition for a Φ -free group to belong to \mathfrak{B} .

Theorem A. *Suppose that G is a Φ -free group. If G/N is Φ -free for all normal subgroups N of G containing the generalised Fitting subgroup $F^*(G)$ then G belongs to \mathfrak{B} .*

Theorem A is important in the study of groups that behave like nilpotent groups with respect to the Frattini subgroup, that is, $\Phi(G/N) = \Phi(G)N/N$ for all $N \triangleleft G$. In one of his last papers, Doerk [9] examined the soluble case and proved that the class $\mathfrak{F}_{\text{sol}}$ of soluble groups G where $\Phi(G/N) = \Phi(G)N/N$ for every normal subgroup N of G is a saturated formation, that is, a class of groups which is closed under epimorphic images, subdirect products and Frattini extensions. Further, he obtained several equivalent conditions for a soluble group to be in $\mathfrak{F}_{\text{sol}}$.

Theorem 1.1 ([9, Satz 2']). *Let G be a soluble group. Then the following statements are pairwise equivalent:*

- (1) $G \in \mathfrak{F}_{\text{sol}}$.
- (2) $G/\Phi(G)$ has no Frattini chief factors.
- (3) $G/F(G)$ has no Frattini chief factors.
- (4) If H/K is a chief factor of G then $G/C_G(H/K)$ has no Frattini chief factors.

One well-known feature of a saturated formation \mathfrak{F} is that in each group G , every chief factor of the form $G^{\mathfrak{F}}/K$ is supplemented in G , where $G^{\mathfrak{F}}$ is the \mathfrak{F} -residual of G , that is, the smallest normal subgroup of G with quotient in \mathfrak{F} . *Totally non-saturated formations* (or *tn-formations* for short) are studied in [8,5] in the soluble universe, and in [1,3] in the general case, as the formations \mathfrak{F} such that, in each group G , every chief factor of the form $G^{\mathfrak{F}}/K$ is Frattini.

Totally non-saturated formations are significant in many ways, one of which is the fact that every soluble group can be embedded in a multiprimitive group ([12]), and these groups belong to every totally non-saturated formation of full characteristic.

Our second main result is a generalisation of Doerk's theorem, valid for all groups. Its proof depends heavily on Theorem A.

Theorem B. *The following assertions are valid:*

- (1) The class \mathfrak{B} is a subnormally closed tn-formation.
- (2) The class \mathfrak{F} of all groups G such that $\Phi(G/N) = \Phi(G)N/N$ for all $N \triangleleft G$ satisfies $\mathfrak{F} = E_{\Phi} \mathfrak{B} = \mathfrak{N} \mathfrak{B}$, where \mathfrak{N} is the class of all nilpotent groups.

- (3) The class \mathfrak{F} is the smallest saturated formation containing \mathfrak{B} and it is locally defined by the formation function f given by $f(p) = \mathfrak{B}$ for all primes p . Moreover, \mathfrak{F} is closed under taking subnormal subgroups.

Before stating our next result, we recall some notation essentially due to Förster. For any group G , $F'(G)$ denotes the normal subgroup of G such that

$$F'(G)/\Phi(G) = \text{Soc}(G/\Phi(G)).$$

Note that $F(G)$ is a subgroup of $F'(G)$, and if G is soluble then $F(G) = F'(G)$.

Corollary 1.2. *Let G be a group. Then the following statements are pairwise equivalent:*

- (1) If $N \triangleleft G$ then $\Phi(G/N) = \Phi(G)N/N$.
- (2) $G/\Phi(G)$ has no Frattini chief factors.
- (3) $G/F(G)$ has no Frattini chief factors.
- (4) $G/F'(G)$ has no Frattini chief factors.
- (5) If H/K is a chief factor of G , then $G/C_G(H/K)$ has no Frattini chief factors.

Following Christensen [6], we say that G is an **nC**-group if every normal subgroup of G is complemented. The class of all **nC**-groups is a tn-formation which is contained in every tn-formation of full characteristic. In particular, every **nC**-group is a \mathfrak{B} -group. However, the containment of **nC** in \mathfrak{B} is proper. An example of a \mathfrak{B} -group which has a normal subgroup with no complement is $\text{Aut}(A_6)$.

Theorem 1.3. *Suppose that G is a \mathfrak{B} -group. Then G is an **nC**-group if and only if G splits over each member of its generalised Fitting series.*

The following known result is a direct consequence of Theorem 1.3.

Corollary 1.4 ([4]). *Every soluble group is a \mathfrak{B} -group if and only if it is an **nC**-group.*

Theorem 1.3 also serves to answer in the affirmative a long standing question whether a subnormal subgroup of an **nC**-group is also an **nC**-group (see [6]).

Corollary 1.5. *The class **nC** is subnormally closed.*

2. Notation and preliminaries

It is assumed that the reader is familiar with the notation presented in [2] and [7]. In order to make our paper reasonably self-contained, we collect in the following lemma some well-known facts about the Frattini subgroup of a group. Most of these will be used

without further explicit reference. For full proofs the reader should consult Gaschütz's early paper [10] or [7, pp. 30–32].

Lemma 2.1. *Let G be a group.*

- (i) *Let $N \triangleleft G$. Then N has a proper supplement in G if and only if N is not contained in $\Phi(G)$. If U is minimal in the set of supplements for N in G then $N \cap U = N \cap \Phi(U)$.*
- (ii) *If $G = H \times K$ then $\Phi(G) = \Phi(H) \times \Phi(K)$.*
- (iii) *If $N \triangleleft G$ then $\Phi(G)N/N \leq \Phi(G/N)$, and if $N \leq \Phi(G)$ then $\Phi(G)/N = \Phi(G/N)$.*
- (iv) *If $L \leq G$ and $U \leq \Phi(L)$ with $U \triangleleft G$ then $U \leq \Phi(G)$; thus if $L \triangleleft G$ then $\Phi(L) \leq \Phi(G)$.*
- (v) *If G is a p -group then $\Phi(G) = G'G^p$, thus $G/\Phi(G)$ is elementary abelian. Therefore, if G is nilpotent and $N \triangleleft G$ then $\Phi(G/N) = \Phi(G)N/N$.*
- (vi) $\Phi(G) \geq Z(G) \cap G'$.
- (vii) *If A is an abelian normal subgroup of G and $A \cap \Phi(G) = 1$ then A is complemented in G .*

Let us also fix some further notation and terminology. We say that a group G is *quasi-simple* if G is perfect and $G/Z(G)$ is simple. A subnormal quasi-simple subgroup of an arbitrary group G is called a *component* of G and the *layer* of G , denoted by $E(G)$, is the product of its components. It is known that if A and B are components of G , then either $A = B$ or $[A, B] = 1$ (see [2, Section 2.2]).

The *generalised Fitting subgroup* of G , denoted by $F^*(G)$, is the product of $E(G)$ and $F(G)$, the Fitting subgroup of G . The associated generalised Fitting series of G is defined by induction as $F_0^* := 1$, and $F_{i+1}^*/F_i^* := F^*(G/F_i^*)$ for $i > 0$.

The proof of [Theorem A](#) depends on the following property of the generalised Fitting subgroup.

Lemma 2.2. *Suppose that G is a Φ -free group. Then $F'(G) = F^*(G) = F(G) \times E(G)$. If $N \triangleleft G$ with $N \leq F^*(G)$ then $N = F(N) \times E(N)$ and $E(N)$ is the unique complement for $F(N)$ in N .*

Proof. First, we argue that $F^*(G) = F(G) \times E(G)$ follows from $F(G) \cap E(G) = F(E(G)) = 1$ and we justify the latter. If Q is a component of G , then Q is subnormal in G so $\Phi(Q) \leq \Phi(G) = 1$ and thus $\Phi(Q) = 1$. Moreover, we have $Z(Q) = Q' \cap Z(Q) \leq \Phi(Q)$ from [Lemma 2.1 \(vi\)](#) and the fact that Q is perfect. Therefore $Z(Q) = \Phi(Q) = 1$ and so Q is non-abelian simple. Thus $E(G)$ is a direct product of non-abelian simple groups and so $F(E(G)) = 1$, as wanted. By [7, A, [Lemma 4.14](#)], $E(G)$ is contained in $\text{Soc}(G) = F'(G)$. Therefore, $F'(G) = F^*(G)$.

Let N be a normal subgroup of G contained in $F^*(G)$. Then $N = F^*(N) = F(N) \times E(N)$ since $\Phi(N) = 1$. Hence $E(N)$ is a direct product of components of G . By

[7, A, Proposition 4.13(b)], we have that $E(N) = N \cap E(G)$. Thus $N = (N \cap F(G)) \times (N \cap E(G)) = F(N) \times E(N)$. Since $F(N)$ is abelian, it follows that $F(N) = Z(N)$.

Suppose that V is a complement for $F(N)$ in N . Then $N = F(N)V = F(N) \times V$. Thus $V = N' = E(N)$, as desired. \square

3. Good normal subgroups

We begin by identifying certain normal subgroups N of a group G which satisfy the equation $\Phi(G/N) = \Phi(G)N/N$, even if G as a whole does not have this property (an example that easily comes to mind is the Frobenius group of order 20). We refer to these subgroups as “good” normal subgroups.

Lemma 3.1. *Let N be a Φ -free central subgroup of the group G . Then N is a good normal subgroup of G .*

Proof. Let E be a not necessarily proper or nontrivial complement for $N \cap \Phi(G)$ in N so that $N = (N \cap \Phi(G)) \times E$. The existence of E is guaranteed since N , being central and Φ -free, is a direct product of elementary abelian subgroups and such a group is complemented [11]. Then $E \cap \Phi(G) = 1$, hence E has a complement in G , say L . So $G = E \times L$ and $\Phi(G) = \Phi(L)$. Since G/N is isomorphic to $L/N \cap \Phi(G)$, it follows that

$$\Phi(G/N) \cong \Phi(L/N \cap \Phi(L)) = \Phi(L)/N \cap \Phi(L) \cong \Phi(L)N/N = \Phi(G)N/N.$$

Consequently, $\Phi(G/N) = \Phi(G)N/N$ and the claim follows. \square

In particular, if $N \leq \text{Soc}(Z(G))$ then $\Phi(G/N) = \Phi(G)N/N$. In fact, we can use this information to get more good normal subgroups.

Corollary 3.2. *Let G be a group and define by induction $S_0 := 1$, and $S_{i+1}/S_i := \text{Soc}(Z(G/S_i))$ for $i > 0$. Then every normal subgroup of G lying in an interval $[[S_j, S_{j+1}]]$, $j \geq 0$ is good.*

Proof. We induct on j . For $j = 0$ the claim is true from Lemma 3.1. Now, let $H \triangleleft G$ with $H \in [[S_j, S_{j+1}]]$, $j > 0$: Write $\overline{G} = G/S_j$, and use the bar convention. Then \overline{H} is a subgroup of $\text{Soc}(Z(\overline{G}))$, so by Lemma 3.1 we have $\Phi(\overline{G}/\overline{H}) = \Phi(\overline{G})\overline{H}/\overline{H}$. Note that $\Phi(\overline{G}/\overline{H}) \cong \Phi(G/H)$, so it suffices to show that $\Phi(\overline{G})\overline{H}/\overline{H} \cong \Phi(G)H/H$ to complete the proof. By induction $\Phi(\overline{G}) = \Phi(G)S_j/S_j$, thus $\Phi(\overline{G})\overline{H} = (\Phi(G)S_j/S_j)H/S_j = \Phi(G)H/S_j$. Then clearly $\Phi(\overline{G})\overline{H}/\overline{H} \cong \Phi(G)H/H$ and the induction is complete. \square

Note that if M, N are good normal subgroups of G , then MN is not necessarily good.

Example 3.3. Let E be elementary abelian of order 5^2 and assume that $C = \langle c \rangle$ is cyclic of order 4, acting on E by $x^c = x^2$, $x \in E$. Let $G = E \rtimes C$, the semidirect product,

and write $E = M \times N$, where each of M and N has order 5. Note that M and N are normal in G . Also $\Phi(G) = 1$. To see this, note that MC and NC are maximal in G , so $\Phi(G) \leq MC \cap NC = C$. Then $\Phi(G) \cap E = 1$ so $\Phi(G)$ centralises E . Since E is self-centralising in G , this forces $\Phi(G) = 1$. Now $K \triangleleft G$ is good if and only if $\Phi(G/K)$ is trivial. Then M and N are good because G/M and G/N are each isomorphic to the Frobenius group of order 20 and so have trivial Frattini subgroups. But $MN = E$ is a normal subgroup of G which is not good.

Remark 3.4. There is an obvious exception to the general rule that if M, N are normal good subgroups of a group G , then the product MN is not good. If M, N are normal in G , both good, and both lie in the same interval $\llbracket S_j, S_{j+1} \rrbracket$ for some index j , where S_j, S_{j+1} are successive terms of the S -series of [Corollary 3.2](#), then their product is normal in G and lies in $\llbracket S_j, S_{j+1} \rrbracket$. Therefore, in this particular case, MN is good.

4. Proofs of the main results

Proof of Theorem A. Suppose that G/N is Φ -free for all normal subgroups N of G containing $F^*(G)$. Assume, arguing by contradiction, that the result is not true and let G be a counterexample of minimal order. Then G has a normal subgroup E such that G/E is not Φ -free. Let us choose E of minimal order. If $1 < N < E$ with $N \triangleleft G$, we will show that G/N satisfies the hypotheses of the theorem. First, $\Phi(G/N) = 1$ by the choice of E . Next, if $F^*(G/N) \leq M/N$ with $M \triangleleft G$ then $F^*(G) \leq M$, so by hypothesis $\Phi(G/M) = 1$, thus $\Phi(G/N/M/N) = 1$. The minimal choice of G implies that $\Phi(G/N/E/N) = 1$, so $\Phi(G/E) = 1$, contrary to supposition. We may thus assume that E is minimal normal in G . Write $U/E = \Phi(G/E)$. Any maximal subgroup of G which contains $F^*(G)$ contains E and thus contains U and so contains $UF^*(G)$. Therefore $UF^*(G)/U$ is a subgroup of $\Phi(G/F^*(G)) = 1$, thus $U \leq F^*(G)$. By [Lemma 2.2](#) we deduce that $U = F(U) \times E(U)$.

Assume that E is abelian. Then U is soluble, $E(U) = 1$ and $U = F(U)$ is nilpotent. Also $U \triangleleft G$ so $\Phi(U) \leq \Phi(G) = 1$, hence $\Phi(U) = 1$ and thus U is abelian. Then [Lemma 2.1 \(vii\)](#) guarantees the existence of a complement for U in G . Call this complement X . Then XE/E is a complement for U/E in G/E and since $U/E = \Phi(G/E)$ we have $G = XE$. By Dedekind’s lemma $U = U \cap XE = (U \cap X)E = E$, thus $\Phi(G/E) = 1$, contrary to our assumption.

Assume that E is non-abelian. Then $E \leq E(U)$. In fact, E must be the whole of $E(U)$ for if not then $U/F(U)E$, which is a homomorphic image of the nilpotent $U/E = \Phi(G/E)$ thus nilpotent itself, would also be a direct product of non-abelian simple groups. Therefore $U = F(U) \times E$. By [Lemma 2.1 \(vii\)](#), and since $F(U)$ is abelian, there is a complement X for $F(U)$ in G . Let $V = U \cap X$. Then Dedekind’s lemma yields $U = U \cap XF(U) = VF(U)$ with $V \cap F(U) = X \cap F(U) = 1$. From [Lemma 2.2](#) we know that $V = E$ thus $X \geq E$. Therefore, X/E complements $U/E = \Phi(G/E)$ in G/E thus

$X = G$ and so $F(U) = 1$. Finally, this implies that $U = E$ so $\Phi(G/E)$ is trivial. This final contradiction completes the proof of the theorem. \square

Let us now demonstrate how **Theorem A** can be used effectively to establish the generalisation of Doerk’s theorem asserted in the introduction.

Proof of Theorem B. (1) First, we prove that \mathfrak{B} is a formation. Let G be a group and suppose that $G/M, G/N \in \mathfrak{B}$ for some normal subgroups M, N of G with $M \cap N = 1$. Note that $\Phi(G)$ is contained in the full preimage of $\Phi(G/M)$, which is M , and in the full preimage of $\Phi(G/N)$, which is N . Thus $\Phi(G) \leq M \cap N = 1$, and so $\Phi(G) = 1$. Now, let G have least possible order among groups which have two trivially intersecting normal subgroups, each defining a quotient which is in \mathfrak{B} , but the whole group is not in \mathfrak{B} . We assume, as we may, that M, N are minimal subgroups of G with respect to $G/M, G/N \in \mathfrak{B}$. We claim that M, N are both minimal normal subgroups of G . If not, then $1 < U < M$ with $U \triangleleft G$, for instance. By the minimality of M , $G/U \notin \mathfrak{B}$. Consider the group $\overline{G} = G/U$ and its normal subgroups $\overline{M} = M/U, \overline{N} = UN/U$. Notice that $\overline{M} \cap \overline{N} = 1$ and both $\overline{G}/\overline{M}, \overline{G}/\overline{N}$ are in \mathfrak{B} since $G/M, G/N$ are in \mathfrak{B} . However, this violates the minimality of G since $|\overline{G}| < |G|$ and $\overline{M}, \overline{N}$ satisfy the initial hypotheses. Then $M \leq \text{Soc}(G) \leq F^*(G)$. Since $G/M \in \mathfrak{B}$, we have $\Phi(G/K) = 1$ for all $M \leq K$, thus $\Phi(G/K) = 1$ for all $F^*(G) \leq K$. Now **Theorem A** yields $G \in \mathfrak{B}$, a contradiction. Therefore, there exists no such G .

Next, we show that \mathfrak{B} is subnormally closed. For that it clearly suffices to prove that \mathfrak{B} is closed under taking normal subgroups. A straightforward induction on the subnormal defect then yields that \mathfrak{B} is subnormally closed. Let G have least possible order among groups which are in \mathfrak{B} , but at least one of their normal subgroups is not. Let N be one of them, note that $\Phi(N) = 1$, and consider $F^*(N)$, which is normal in G and nontrivial. Then $G/F^*(N)$ is in \mathfrak{B} , since G is in \mathfrak{B} , but $N/F^*(N)$ is not in \mathfrak{B} (if it were then **Theorem A** would yield $N \in \mathfrak{B}$). This, however, contradicts the minimality of G , so there is no such G to begin with.

Finally, we demonstrate that \mathfrak{B} is totally non-saturated. Let G be a group and consider the \mathfrak{B} -residual T of G . Let T/H be a minimal normal subgroup of G/H , where $H < T$ and $H \triangleleft G$. If T/H is not contained in $\Phi(G/H)$ then $T/H \cap \Phi(G/H) = 1$ by minimality of T/H . But $\Phi(G/H/T/H)$ is trivial since $\Phi(G/T)$ is trivial, so $\Phi(G/H)$ is contained in T/H , the full preimage of $\Phi(G/H/T/H)$, and thus $\Phi(G/H) = 1$. Therefore, $T/H \leq \text{Soc}(G/H) = F^*(G/H)$ and so $G/H/F^*(G/H) \in \mathfrak{B}$. Then, by **Theorem A**, we have that $G/H \in \mathfrak{B}$, which contradicts the fact that T is the \mathfrak{B} -residual of G , so T/H is contained in $\Phi(G/H)$, as wanted.

(2) Let \mathfrak{F} be the class of groups G such that $\Phi(G/N) = \Phi(G)N/N$ for all $N \triangleleft G$. Note that every Φ -free group in \mathfrak{F} is a \mathfrak{B} -group. Hence \mathfrak{F} is contained in $E_\Phi \mathfrak{B}$.

Assume that $X \in E_\Phi \mathfrak{B}$. Then $X/\Phi(X)$ belongs to \mathfrak{B} . Let Z be a normal subgroup of X . Then the full preimages of $\Phi(X/Z\Phi(X))$ and $\Phi(G/Z)$ are equal since the intersection of those maximal subgroups of X that contain Z is precisely the intersection

of the maximal subgroups that contain $Z\Phi(X)$. Since $X/Z\Phi(X) \in \mathfrak{B}$, it follows that $Z\Phi(X)/Z = \Phi(G/Z)$ and $X \in \mathfrak{F}$. Therefore $\mathfrak{F} = E_\Phi \mathfrak{B}$. In particular, \mathfrak{F} is contained in $\mathfrak{N}\mathfrak{B}$. Suppose that $\mathfrak{N}\mathfrak{B}$ is not contained in \mathfrak{F} and let $G \in \mathfrak{N}\mathfrak{B} \setminus \mathfrak{F}$ a group of minimal order. Then $G/F(G) \in \mathfrak{B}$. If G were Φ -free, then $\text{Soc}(G) = F'(G) = F^*(G)$ and so $G/F^*(G)$ would belong to \mathfrak{B} . By [Theorem A](#), G would be a \mathfrak{B} -group, contrary to our supposition. Therefore, $\Phi(G) \neq 1$. The minimal choice of G implies that $G/\Phi(G) \in \mathfrak{F}$ and so $G/\Phi(G) \in \mathfrak{B}$. This contradiction shows that $\mathfrak{N}\mathfrak{B}$ is contained in \mathfrak{F} , and so $\mathfrak{F} = E_\Phi \mathfrak{B} = \mathfrak{N}\mathfrak{B}$.

(3) According to [[7, IV, Example 3.4 b](#))], \mathfrak{F} is a saturated formation which is locally defined by the formation function f given by $f(p) = \mathfrak{B}$ for all primes p . Suppose that \mathfrak{H} is a saturated formation containing \mathfrak{B} . Then $\mathfrak{F} = E_\Phi \mathfrak{B} \subseteq E_\Phi \mathfrak{H} = \mathfrak{H}$. Consequently, \mathfrak{F} is the smallest saturated formation containing \mathfrak{B} . By [[7, IV, Proposition 3.14](#)], \mathfrak{F} is closed under taking subnormal subgroups. \square

Proof of [Theorem 1.3](#). If G is an \mathbf{nC} -group, then certainly G splits over each member of its generalised Fitting series, since each said member is normal. For the other direction, we induct on the generalised Fitting length k of the Φ -free group G , that is, the smallest integer k such that $F_k^* = G$. If $k = 1$ then $G = F^*(G) = F(G) \times E(G)$ and if N is a normal subgroup of G , it follows that $N = F^*(N) = F(N) \times E(N)$ by [Lemma 2.2](#). Since $E(N)$ is a normal subgroup of G which is a direct product of non-abelian simple groups, it follows that $E(N)$ is complemented in $E(G)$. Moreover $F(N)$ is a normal subgroup of G which is complemented in $F(G)$ by a normal subgroup of G since $F(G)$ is a direct product of semisimple modules. Therefore G splits over N .

Assume now that $k > 1$. Let $G \in \mathfrak{B}$ have length k , and suppose that G splits over each member of its generalised Fitting series. Let H be a complement for $F^*(G)$ in G . Then $H \in \mathfrak{B}$ has generalised Fitting length $k - 1$, since $G/F^*(G) \cong H$, and is therefore an \mathbf{nC} -group by the inductive hypothesis. Let $N \triangleleft G$ and write $X = F^*(G)N$. Then $X \triangleleft G$ and $X \cap H$ has a complement in H , say C . Let V be a complement for $F^*(N) = F^*(G) \cap N$ in $F^*(G)$. By our earlier remark $V \triangleleft G$ and we claim that VC is a complement for N in G . First, $X = NV F^*(N) = NV$, thus $NVC = XC = (X \cap H)C F^*(G) = H F^*(G) = G$. On the other hand $|C| = |H : X \cap H| = |XH : X| = |G : X|$ and $|V| = |F^*(G) : F^*(G) \cap N| = |X : N|$. Thus $|C||V| = |G : N|$ and $C \cap V \leq H \cap F^*(G) = 1$. We conclude that $|VC| = |G : N|$, thus VC is a complement for N in G , as claimed. The induction is now complete. \square

Proof of [Corollary 1.5](#). First, we show that if $G \in \mathbf{nC}$ and $N \triangleleft G$ then $N \in \mathbf{nC}$. Since $G \in \mathbf{nC} \subseteq \mathfrak{B}$ and \mathfrak{B} is subnormally closed it follows that $N \in \mathfrak{B}$ and so by [Theorem 1.3](#) $N \in \mathbf{nC}$ if and only if N splits over $F_i^*(N)$ for all i . But $F_i^*(N)$ is characteristic in N thus normal in G , so if C is a complement for $F_i^*(N)$ in G then $N \cap C$ is a complement for $F_i^*(N)$ in N by a standard application of Dedekind’s lemma. Since N splits over $F_i^*(N)$ for all i and $N \in \mathfrak{B}$ we deduce that $N \in \mathbf{nC}$.

For the general case of subnormal subgroups of an \mathbf{nC} -group G we argue by induction on its order, the base case being vacuously true. Let $G \in \mathbf{nC}$ and $H \triangleleft\triangleleft G$ and proper. Then $H \triangleleft\triangleleft N$ for some proper normal subgroup N of G which, by the previous paragraph, is an \mathbf{nC} -group. Applying the inductive hypothesis to $N < G$ we conclude that $H \in \mathbf{nC}$. Since H was arbitrary our induction is complete. \square

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