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## Normal Sally modules of rank one



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### ABSTRACT

In this paper, we explore the structure of the normal Sally modules of rank one with respect to an  $\mathfrak{m}$ -primary ideal in a Nagata reduced local ring  $R$  which is not necessary Cohen–Macaulay. As an application of this result, when the base ring is Cohen–Macaulay analytically unramified, the extremal bound on the first normal Hilbert coefficient leads to the depth of the associated graded rings  $\overline{\mathcal{G}}$  with respect to a normal filtration is at least  $\dim R - 1$  and  $\overline{\mathcal{G}}$  turns in to Cohen–Macaulay when the third normal Hilbert coefficient is vanished.

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## 1. Introduction

Throughout this paper, let  $R$  be an analytically unramified Noetherian local ring with the maximal ideal  $\mathfrak{m}$  and  $d = \dim R > 0$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  and suppose that  $I$  contains a parameter ideal  $Q = (a_1, a_2, \dots, a_d)$  of  $R$  as a reduction. Let  $\ell_R(M)$  denote the length of an  $R$ -module  $M$  and  $\overline{I^{n+1}}$  denote the integral closure of  $I^{n+1}$  for each  $n \geq 0$ . Since  $R$  is an analytically unramified, there are integers  $\{\overline{e}_i(I)\}_{0 \leq i \leq d}$  such that the equality

$$\ell_R(R/\overline{I^{n+1}}) = \overline{e}_0(I) \binom{n+d}{d} - \overline{e}_1(I) \binom{n+d-1}{d-1} + \dots + (-1)^d \overline{e}_d(I)$$

holds true for all integers  $n \gg 0$ , which we call the normal Hilbert coefficients of  $R$  with respect to  $I$ . We will denote by  $\{e_i(I)\}_{0 \leq i \leq d}$  the ordinary Hilbert coefficients of  $R$  with respect to  $I$ . Let

$$\mathcal{R} = R(I) := R[It] \text{ and } T = R(Q) := R[Qt] \subseteq R[t]$$

denote, respectively, the Rees algebra of  $I$  and  $Q$ , where  $t$  stands for an indeterminate over  $R$ . Let

$$\mathcal{R}' = R'(I) := R[It, t^{-1}] \text{ and } \mathcal{G} = \mathcal{G}(I) := \mathcal{R}'/t^{-1}\mathcal{R}' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}$$

denote, respectively, the extended Rees algebra of  $I$  and the associated graded ring of  $R$  with respect to  $I$ . Let  $\overline{\mathcal{R}}$  denote the integral closure of  $\mathcal{R}$  in  $R[t]$  and  $\overline{\mathcal{G}} = \bigoplus_{n \geq 0} \overline{I^n}/\overline{I^{n+1}}$  denote the associated graded ring of the normal filtration  $\{\overline{I^n}\}_{n \in \mathbb{Z}}$ . Then  $\overline{\mathcal{R}} = \bigoplus_{n \geq 0} \overline{I^n}t^n$  and  $\overline{\mathcal{R}}$  is a module-finite extension of  $\mathcal{R}$  since  $R$  is analytically unramified (see [14, Corollary 9.2.1]). For the reduction  $Q$  of  $I$ , the reduction number of  $\{\overline{I^n}\}_{n \in \mathbb{Z}}$  with respect to  $Q$  is defined by

$$r_Q(\{\overline{I^n}\}_{n \in \mathbb{Z}}) = \min\{r \in \mathbb{Z} \mid \overline{I^{n+1}} = Q\overline{I^n}, \text{ for all } n \geq r\}.$$

The notion of Sally modules of normal filtrations was introduced by [1] in order to find the relationship between a bound on the first normal Hilbert coefficients  $\overline{e}_1(I)$  and the depth of  $\overline{\mathcal{G}}$  when  $R$  is an analytically unramified Cohen–Macaulay rings  $R$ . Following [1], we generalize the definition of normal Sally modules to the non-Cohen–Macaulay cases, and we define the normal Sally modules  $\overline{\mathcal{S}} = \overline{\mathcal{S}}_Q(I)$  of  $I$  with respect to a minimal reduction  $Q$  to be the cokernel of the following exact sequence

$$0 \longrightarrow \overline{IT} \longrightarrow \overline{\mathcal{R}}_+(1) \longrightarrow \overline{\mathcal{S}} \longrightarrow 0$$

of graded  $T$ -modules. Since  $\overline{\mathcal{R}}$  is a finitely generated  $T$ -module, so is  $\overline{\mathcal{S}}$  and we get

$$\overline{\mathcal{S}} = \bigoplus_{n \geq 1} \overline{I^{n+1}}/Q^n \overline{I}$$

by the following isomorphism

$$\overline{\mathcal{R}}_+(1) \xrightarrow{t^{-1}} \sum_{n \geq 0} \overline{I^{n+1}} t^n (\supseteq \sum_{n \geq 0} (Q^n \overline{I}) t^n = \overline{I} T)$$

of graded  $T$ -modules.

To state the results of this paper, let us consider the following four conditions:

- ( $C_0$ ) The sequence  $a_1, a_2, \dots, a_d$  is a  $d$ -sequence in  $R$  in the sense of [7].
- ( $C_1$ ) The sequence  $a_1, a_2, \dots, a_d$  is a  $d^+$ -sequence in  $R$ , that is for all integers  $n_1, n_2, \dots, n_d \geq 1$  the sequence  $a_1^{n_1}, a_2^{n_2}, \dots, a_d^{n_d}$  forms a  $d$ -sequence in any order.
- ( $C_2$ )  $(a_1, a_2, \dots, \check{a}_i, \dots, a_d) :_R a_i \subseteq I$  for all  $1 \leq i \leq d$ .
- ( $C_3$ )  $\text{depth} R > 0$  and  $\text{depth} R > 1$  if  $d \geq 2$ .

These conditions ( $C_0$ ), ( $C_1$ ), and ( $C_2$ ) are exactly the same as in [4]. The conditions ( $C_1$ ), ( $C_2$ ), and ( $C_3$ ) are automatically satisfied if  $R$  is Cohen–Macaulay. Conditions ( $C_1$ ) and ( $C_3$ ) imply the ring  $R$  satisfies Serre’s condition ( $S_2$ ) (see Remark 2.3 for an explanation of this fact).

Put  $B = T/\mathfrak{m}T$ . Then  $B \cong (R/\mathfrak{m})[X_1, \dots, X_d]$  a polynomial of  $d$  indeterminates over the field  $R/\mathfrak{m}$ . The main result of this research is as follows.

**Theorem 1.1.** *Let  $R$  be a Nagata and reduced local ring with the maximal ideal  $\mathfrak{m}$  and  $d = \dim R > 0$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  and suppose that  $I$  contains a parameter ideal  $Q$  of  $R$  as a reduction. Assume that conditions ( $C_1$ ), ( $C_2$ ), and ( $C_3$ ) are satisfied. Then the following are equivalent to each other.*

- (1)  $\overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I}) + 1$ .
- (2)  $\mathfrak{m}\overline{S} = (0)$  and  $\text{rank}_B \overline{S} = 1$ .
- (3)  $\overline{S} \cong B(-q)$  as graded  $T$ -modules for some integer  $q \geq 1$ .

When this is the case

- (a)  $\overline{S}$  is a Cohen–Macaulay  $T$ -module.
- (b) Put  $t = \text{depth} R$ . Then

$$\text{depth} \overline{\mathcal{G}} \geq \begin{cases} d-1 & \text{if } t \geq d-1, \\ t & \text{if } t < d-1. \end{cases}$$

- (c) For all  $n \geq 0$ ,

$$\begin{aligned} \ell_R(R/\overline{I^{n+1}}) &= e_0(I) \binom{n+d}{d} - \{e_0(I) + e_1(Q) - \ell_R(R/\overline{I})\} \binom{n+d-1}{d-1} \\ &\quad + \sum_{i=2}^d (-1)^i \{e_{i-1}(Q) + e_i(Q)\} \binom{n+d-i}{d-i} \end{aligned}$$

if  $n < q$ , and

$$\begin{aligned} \ell_R(R/\overline{I}^{n+1}) &= e_0(I) \binom{n+d}{d} - \{e_0(I) + e_1(Q) - \ell_R(R/\overline{I}) + 1\} \binom{n+d-1}{d-1} \\ &\quad + \sum_{i=2}^d (-1)^i \{e_{i-1}(Q) + e_i(Q) + \binom{q}{i-1}\} \binom{n+d-i}{d-i} \end{aligned}$$

if  $n \geq q$ . Hence  $\overline{e}_i(I) = e_{i-1}(Q) + e_i(Q) + \binom{q}{i-1}$  for all  $2 \leq i \leq d$ .

The relationship between the equality  $\overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1$  and the depth of  $\overline{\mathcal{G}}$  in an analytically unramified Cohen–Macaulay local ring was examined in [1]. In their paper, they proved that if  $R$  is an analytically unramified Cohen–Macaulay ring possessing a canonical module  $\omega_R$ , then  $\overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1$  makes  $\text{depth} \overline{\mathcal{G}} \geq d-1$  ([1, Theorem 2.6]). Moreover if  $d \geq 3$  and  $\overline{e}_3(I) = 0$ , then this equality  $\overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1$  leads to the Cohen–Macaulayness of  $\overline{\mathcal{G}}$  ([1, Proposition 3.4]). The assumption  $R$  has a canonical module assures that  $\overline{\mathcal{R}}$  satisfies the Serre condition  $(S_2)$  as a ring, which is essential for the proofs of their results. In this paper, as an application of Theorem 1.1, we will prove that the above results [1, Theorem 2.6 and Proposition 3.4] still hold true even when we delete the assumption that the base ring possessing a canonical module  $\omega_R$ , as stated in the following.

**Theorem 1.2.** *Let  $R$  be a analytically unramified Cohen–Macaulay local ring with the maximal ideal  $\mathfrak{m}$ , dimension  $d = \dim R > 0$ , and  $I$  an  $\mathfrak{m}$ -primary ideal of  $R$  containing a parameter ideal  $Q$  of  $R$  as a reduction. Assume that  $\overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1$ . Then the following assertions hold true.*

- (1)  $\text{depth} \overline{\mathcal{G}} \geq d-1$ .
- (2) If  $d \geq 3$  and  $\overline{e}_3(I) = 0$  then  $\overline{\mathcal{G}}$  is Cohen–Macaulay,  $\overline{e}_2(I) = 1$ , and the normal filtration has reduction number two.

Now it is a position to explain how this paper is organized. This paper contains of 3 sections. The introduction part is this present section. In Section 2 we will collect some auxiliary results on normal Sally modules and normal Hilbert functions. We will prove Theorem 1.1, Theorem 1.2, and explore a consequence of Theorem 1.1 in the Cohen–Macaulay case in Section 3.

## 2. Auxiliaries

In this section we will collect properties of the normal Sally modules and the normal Hilbert coefficients which are essential for the proof of our main results. Throughout this section, let  $R$  be an analytically unramified Noetherian local ring with the maximal

ideal  $\mathfrak{m}$  and  $\dim R = d \geq 1$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal and assume that  $I$  contains a minimal reduction  $Q = (a_1, \dots, a_d)$ .

Let us begin with the following lemma which play an important role on computing the normal Hilbert functions and on examining the structure of  $\overline{S}$ .

**Lemma 2.1.** [4, Lemma 2.1] *Suppose that conditions  $(C_0)$  and  $(C_2)$  are satisfied. Then*

$$T/\overline{I}T \cong (R/\overline{I})[X_1, X_2, \dots, X_d]$$

*as graded  $R$ -algebras, where  $(R/\overline{I})[X_1, X_2, \dots, X_d]$  denotes the polynomial ring with  $d$  indeterminates over Artinian local ring  $R/\overline{I}$ . Hence  $T/\overline{I}T$  is a Cohen–Macaulay ring with  $\dim T/\overline{I}T = d$ .*

Under our generalized assumption, the results [1, Proposition 2.2] on the set of associated prime ideals and the dimension of  $\overline{S}$  do not change, and moreover we obtain a formula on  $\text{depth} \overline{G}$  as follows.

**Lemma 2.2.** *The following assertions hold true.*

- (1) [4, Lemma 2.1]  $\mathfrak{m}^\ell \overline{S} = 0$  for integer  $\ell \gg 0$ . Hence  $\dim_T \overline{S} \leq d$ .
- (2) [4, Lemma 2.3] *Suppose that conditions  $(C_0)$ ,  $(C_2)$  and  $(C_3)$  are satisfied and  $T$  is a  $(S_2)$  ring. Then  $\text{Ass}_T(\overline{S}) \subseteq \{\mathfrak{m}T\}$ . Hence  $\dim_T \overline{S} = d$  provided  $\overline{S} \neq (0)$ .*
- (3) *Suppose that conditions  $(C_0)$ ,  $(C_2)$  and  $(C_3)$  are satisfied. Then  $\text{depth} \overline{G} \geq \text{depth} R$  if  $\overline{S} = (0)$  and*

$$\text{depth} \overline{G} \geq \begin{cases} \text{depth} R & (\text{depth}_T \overline{S} \geq \text{depth} R + 1), \\ \text{depth}_T \overline{S} - 1 & (\text{depth}_T \overline{S} \leq \text{depth} R \text{ or } \text{depth} R = d - 1) \end{cases}$$

*if  $\overline{S} \neq (0)$ .*

**Proof.** The proof of (1) is the same as that of [3, Lemma 2.1]. The proof of (2) is almost the same as that of [4, Lemma 2.3], let us include a proof for the sake of completeness. We may assume that  $\overline{S} \neq (0)$ . Let  $P \in \text{Ass}_T \overline{S}$ . Then  $\mathfrak{m}T \subseteq P$ . Assume that  $P \neq \mathfrak{m}T$ . Then  $\text{ht}_T P \geq 2$  since  $\text{ht}_T \mathfrak{m}T = 1$ . Therefore  $\text{depth}_T P \geq 2$  by condition  $(S_2)$ . Now we consider the following exact sequences

$$0 \longrightarrow (\overline{I}T)_P \longrightarrow (\overline{\mathcal{R}}_+(1))_P \longrightarrow \overline{S}_P \longrightarrow 0$$

and

$$0 \longrightarrow (\overline{I}T)_P \longrightarrow T_P \longrightarrow T_P/(\overline{I}T)_P \longrightarrow 0$$

of graded  $T_P$ -modules. Since  $\text{depth}_{T_P}(\overline{\mathcal{R}}_+(1))_P \geq 1$  and  $\text{depth}_{T_P} \overline{S}_P = 0$ ,  $\text{depth}_{T_P}(\overline{I}T)_P = 1$ . Therefore  $\text{depth}_{T_P} T_P/(\overline{I}T)_P = 0$  by the second exact sequence. Moreover since

conditions  $(C_0)$  and  $(C_2)$  are satisfied,  $T/\bar{I}T$  is a Cohen–Macaulay ring by Lemma 2.1, so is  $T_P/(\bar{I}T)_P$ . Therefore  $P \in \text{Min}_T T_P/(\bar{I}T)_P = \{\mathfrak{m}T\}$ , which is a contradiction. Thus  $P = \mathfrak{m}T$  as desired.

The statement (3) follows by comparing depths of  $T$ -modules in the following exact sequences

$$\begin{aligned} (1) \quad & 0 \longrightarrow \bar{I}T \longrightarrow T \longrightarrow T/\bar{I}T \longrightarrow 0, \\ (2) \quad & 0 \longrightarrow \bar{I}T \longrightarrow \bar{\mathcal{R}}_+(1) \longrightarrow \bar{\mathcal{S}} \longrightarrow 0, \\ (3) \quad & 0 \longrightarrow \bar{\mathcal{R}}_+(1) \longrightarrow \bar{\mathcal{R}} \longrightarrow \bar{\mathcal{G}} \longrightarrow 0, \\ (4) \quad & 0 \longrightarrow \bar{\mathcal{R}}_+ \longrightarrow \bar{\mathcal{R}} \longrightarrow R \longrightarrow 0 \end{aligned}$$

of graded  $T$ -modules.  $\square$

Applying Lemma 2.2 to the case where the base ring  $R$  is analytically unramified Cohen–Macaulay, we get  $\text{depth} \bar{\mathcal{G}} \geq \text{depth}_T \bar{\mathcal{S}} - 1$  ([1, Proposition 2.4](b)) and  $\bar{\mathcal{G}}$  is Cohen–Macaulay if  $\mathcal{S} = (0)$  ([1, Proposition 2.4](a)).

**Remark 2.3.** Suppose that conditions  $(C_1)$  and  $(C_3)$  are satisfied. Then  $T$  is a  $(S_3)$ -ring by using [16, Theorem 6.2]. Therefore if we assume that conditions  $(C_1)$ ,  $(C_2)$  and  $(C_3)$  are satisfied, then by Lemma 2.2  $\text{Ass}_T(\bar{\mathcal{S}}) \subseteq \{\mathfrak{m}T\}$ . Hence  $\dim_T \bar{\mathcal{S}} = d$  provided  $\bar{\mathcal{S}} \neq (0)$ .

The following lemma play a crucial role on computing the normal Hilbert polynomial in Theorem 1.1.

**Lemma 2.4.** Suppose that conditions  $(C_0)$  and  $(C_2)$  are satisfied. Then the following assertions hold true.

(1) [4, Proposition 2.4] For every  $n \geq 0$

$$\begin{aligned} \ell_R(R/\overline{I^{n+1}}) &= e_0(I) \binom{n+d}{d} - \{e_0(I) + e_1(Q) - \ell_R(R/\bar{I})\} \binom{n+d-1}{d-1} \\ &\quad + \sum_{i=2}^d (-1)^i \{e_{i-1}(Q) + e_i(Q) \binom{n+d-i}{d-i}\} - \ell_R(\bar{\mathcal{S}}_n). \end{aligned}$$

(2) [4, Proposition 2.5]  $\overline{e_1}(I) = e_0(I) + e_1(Q) - \ell_R(R/\bar{I}) + \ell_{T_{\mathfrak{m}T}}(\bar{\mathcal{S}}_{\mathfrak{m}T})$ , whence  $\overline{e_1}(I) \geq e_0(I) + e_1(Q) - \ell_R(R/\bar{I})$ .

We omit the proof of the above lemma because they are the same as in [4]. Here we notice that under the condition  $(C_0)$ ,  $\ell_R(R/Q^{n+1}) = \sum_{i=0}^d (-1)^i e_i(Q) \binom{n+d-i}{d-i}$  for every  $n \geq 0$  by [15, Theorem 4.1].

The following lemma shows that the equality  $\overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I})$  corresponds to the case where either  $\overline{S}$  vanishes or the reduction number of the normal Hilbert filtration is at most one. And the equality  $\overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I}) + 1$  corresponds to the normal Sally module of rank one.

**Lemma 2.5.** *Assume that conditions  $(C_1)$ ,  $(C_2)$ , and  $(C_3)$  are satisfied. Then the following assertions hold true.*

(1) *The following are equivalent to each other*

(a)  $\overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I})$ .

(b)  $\overline{S} = (0)$ .

(c)  $r_Q(\{\overline{I}^n\}_{n \in \mathbb{Z}}) \leq 1$ .

*When this is the case we get the following.*

(i)  $\text{depth} \overline{\mathcal{G}} \geq \text{depth} R$ .

(ii) *For all  $n \geq 0$*

$$\begin{aligned} \ell_R(R/\overline{I}^{n+1}) &= e_0(I) \binom{n+d}{d} - \{e_0(I) + e_1(Q) - \ell_R(R/\overline{I})\} \binom{n+d-1}{d-1} \\ &\quad + \sum_{i=2}^d (-1)^i \{e_{i-1}(Q) + e_i(Q)\} \binom{n+d-i}{d-i}. \end{aligned}$$

(2) [4, Theorem 2.9]  $\overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I}) + 1$  if and only if  $\mathfrak{m}\overline{S} = 0$  and  $\text{rank}_B \overline{S} = 1$ .

**Proof.** The statement (2) is by [4, Theorem 2.9]. Now we will prove (1). Since conditions  $(C_1)$ ,  $(C_2)$ , and  $(C_3)$  are satisfied,  $\text{Ass}_T \overline{S} \subseteq \{\mathfrak{m}T\}$  by Lemma 2.1 and we get  $\overline{S}_{\mathfrak{m}T} = (0)$  if and only if  $\overline{S} = (0)$ . Moreover by Lemma 2.4(2)  $\overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I}) + \ell_{T_{\mathfrak{m}T}}(\overline{S}_{\mathfrak{m}T})$ , therefore  $\overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I})$  if and only if  $\overline{S} = (0)$ . The equivalence of (b) and (c) is clear. Now assume that  $\overline{S} = (0)$ . The statement (i) follows by Lemma 2.2(3). The last assertion (ii) follows by Lemma 2.4(1).  $\square$

### 3. Proof of Theorem 1.1 and Theorem 1.2

This section is devoted for presenting the proofs of Theorem 1.1 and Theorem 1.2. In order to do this we need the result that the integral closure of  $R(I)$  in  $R[t]$  is a  $(S_2)$ -ring. Here we notice that a Noetherian ring  $R$  is called Nagata if for every  $P \in \text{Spec} R$ , for any finite extension  $L$  of  $\mathcal{Q}(R/P)$ , the integral closure of  $R/P$  in  $L$  is a finite  $R/P$ -module, where  $\mathcal{Q}(R/P)$  denotes the quotient field of  $R/P$  (see [12, 31.A DEFINITIONS]). Let  $\mathcal{I} = \{I_n\}_{n \in \mathbb{Z}}$  be a filtrations of ideals in  $R$ , that is  $I_n$  is an ideal of  $R$  for every  $n \in \mathbb{Z}$ ,  $I_0 = R$ ,  $I_n \supseteq I_{n+1}$  for every  $n \in \mathbb{Z}$ , and  $I_m I_n \subseteq I_{mn}$  for all  $m, n \in \mathbb{Z}$ . Let  $R(\mathcal{I}) := \sum_{n \geq 0} I_n t^n$  denote the Rees algebra of the filtration  $\mathcal{I}$ . Then we get the following result which is belong to Shiro Goto.

**Proposition 3.1.** Assume that  $R$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$ ,  $d = \dim R > 0$  such that  $R$  is a reduced, Nagata, and  $(S_2)$ -ring. Let  $\mathcal{I} = \{I_n\}_{n \in \mathbb{Z}}$  be a filtrations of ideals in  $R$  such that  $I_1 \neq R$ . Suppose that  $\text{ht}_R I_1 \geq 2$ , and  $R(\mathcal{I}) \subseteq R[t]$  is Noetherian. Then the integral closure of  $R(\mathcal{I})$  in  $R[t]$  is a  $(S_2)$ -ring.

**Proof.** We denote  $\mathcal{Q}(-)$  the quotient field of  $(-)$ . Put  $\mathcal{R} := R(\mathcal{I})$  and  $\mathcal{F} = \mathcal{Q}(R[t])$ . Let  $\mathcal{S}$  and  $\mathcal{T}$  denote the integral closure of  $\mathcal{R}$  in  $R[t]$  and  $\mathcal{F}$ , respectively. Since  $\text{ht}_R I_1 \geq 2$ , there exists an  $R$ -regular element  $a \in I_1$ . Put  $f = at$ . Then  $\mathcal{Q}(R[f]) = \mathcal{Q}(R[t]) \supseteq \mathcal{Q}(R)$ . Let  $\overline{R}$  be the integral closure of  $R$  in  $\mathcal{Q}(R)$ . Then  $\overline{R}$  is a finitely generated graded  $R$ -module. Since  $\overline{R}[t]$  is integrally closed in  $\mathcal{F}$ ,  $\mathcal{T} \subseteq \overline{R}[t]$ . Therefore  $\mathcal{S} \subseteq \mathcal{T}$  and  $\mathcal{S} = \mathcal{T} \cap R[t]$ . Since  $R$  is Nagata and  $\mathcal{R}$  is Noetherian,  $\mathcal{T}$  is a finitely generated graded  $\mathcal{R}$ -module and hence  $\mathcal{S}$  is Noetherian. Assume on the contrary that  $\mathcal{S}$  is not a  $(S_2)$ -ring. Then there is a prime ideal  $P$  of  $\mathcal{S}$  such that

$$\text{depth} \mathcal{S}_P < \inf\{2, \dim \mathcal{S}_P\}.$$

If  $\dim \mathcal{S}_P \leq 1$ , then  $\text{depth} \mathcal{S}_P = 0$ ,  $\dim \mathcal{S}_P = 1$ , which contradicts to the fact that  $\mathcal{S}$  is reduced. Therefore  $\dim \mathcal{S}_P \geq 2$  and  $\text{depth} \mathcal{S}_P = 1$ . Moreover this ideal  $P$  is graded. In fact, assume on the contrary that  $P$  is not graded. Then  $P \neq P^*$ , where  $P^*$  denotes the graded ideal generated by all homogeneous elements of  $P$ . Therefore  $\text{depth} \mathcal{S}_{P^*} = 0$ . Furthermore since  $\mathcal{S}$  is reduced,  $\text{depth} \mathcal{S}_{P^*} = \dim \mathcal{S}_{P^*}$  and from this we get  $\dim \mathcal{S}_P = 1$ , which is a contradiction. Now we put  $p = P \cap R$ .

**Claim 3.2.**  $p \supseteq I_1$ .

**Proof of Claim 3.2.** Assume on the contrary that  $p \not\supseteq I_1$ . Then  $\mathcal{R}_p = \mathcal{S}_p = R_p[t]$ . Thanks to the embedding  $R_p \hookrightarrow \mathcal{S}_p$  and the fact that  $\text{depth} \mathcal{S}_P = 1$  we have  $\text{depth} R_p \leq 1$ . Since  $R$  is  $(S_2)$ ,  $\text{depth} R_p \geq \inf\{2, \dim R_p\}$  and we have  $R_p$  is Cohen–Macaulay and so is  $\mathcal{S}_p$ . Since  $\mathcal{S}_P = (\mathcal{S}_p)_{P \cap \mathcal{S}_p}$ ,  $\mathcal{S}_P$  is Cohen–Macaulay, which is impossible. Thus  $p \supseteq I_1$  as wanted.  $\square$

**Claim 3.3.** For all graded prime ideal  $\mathfrak{p}$  of  $\mathcal{S}$  such that  $\text{ht} \mathfrak{p} \geq 2$ , we have  $\text{ht}_{\mathcal{T}} \mathfrak{P} \geq 2$  for all prime ideal  $\mathfrak{P}$  of  $\mathcal{T}$  such that  $\mathfrak{P} \cap \mathcal{S} = \mathfrak{p}$ .

**Proof of Claim 3.3.** Take  $\mathfrak{P}_0 \in \text{Min} \mathcal{T}$  such that  $\mathfrak{P}_0 \subseteq \mathfrak{P}$  and  $\dim \mathcal{T}_{\mathfrak{P}} = \dim \mathcal{T}_{\mathfrak{P}_0} / \mathfrak{P}_0 \mathcal{T}_{\mathfrak{P}_0}$ . Since  $\mathfrak{P}_0 \in \text{Ass}_{\mathcal{T}} \overline{R}[t]$ , there is  $U \in \text{Ass}_{\overline{R}[t]} \overline{R}[t]$  such that  $U \cap \mathcal{T} = \mathfrak{P}_0$ . Put  $W = \mathfrak{P}_0 \cap \overline{R}$ . Then  $W \in \text{Ass}_{\overline{R}} \overline{R}$  and  $U = W \overline{R}[t]$ . Put  $p_0 = W \cap R$ . Then  $p_0 \in \text{Ass} R$  and therefore  $\text{ht}_{R p_0} = 0$  as  $R$  is  $(S_1)$ . Since

$$\mathfrak{P}_0 \cap \mathcal{S} = U \cap \mathcal{S} = W \overline{R}[t] \cap \mathcal{S} = (W \overline{R}[t] \cap R[t]) \cap \mathcal{S} = p_0 R[t] \cap \mathcal{S},$$

we get  $(\mathfrak{P}_0 \cap \mathcal{S}) \cap R = p_0$  and  $\mathcal{S}/(\mathfrak{P}_0 \cap \mathcal{S}) \cong R/(\{\overline{I_n} + p_0\}_{n \in \mathbb{Z}})$ . Therefore  $\dim \mathcal{S}/(\mathfrak{P}_0 \cap \mathcal{S}) = d + 1$ , where  $d = \dim R \geq 2$ . Now let  $M$  be the graded maximal



of  $\mathcal{S}$ . Then  $\mathfrak{p} \subseteq M$ . Since the extension  $\mathcal{S} \subseteq \mathcal{T}$  is finite, by Going Up theorem, there exists a graded maximal ideal  $N$  of  $\mathcal{T}$  such that  $\dim \mathcal{T}_N/\mathfrak{P}\mathcal{T}_N = \dim \mathcal{S}_M/\mathfrak{p}\mathcal{S}_M =: \alpha$ .

$$\begin{array}{ccccc} \mathfrak{P}_0 & \xrightarrow{\quad} & \mathfrak{P} & \xrightarrow{\quad \alpha \quad} & \exists N \subseteq \mathcal{T} \\ | & & | & & | \\ \mathfrak{P}_0 \cap \mathcal{S} & \xrightarrow{\quad} & \mathfrak{p} & \xrightarrow{\quad \alpha \quad} & M \subseteq \mathcal{S} \end{array}$$

Since the extension  $(\mathcal{S}/(\mathfrak{P}_0 \cap \mathcal{S}))_M \subseteq (\mathcal{T}/\mathfrak{P}_0)_N$  is finite and  $R$  is universal catenary,  $(\mathcal{S}/(\mathfrak{P}_0 \cap \mathcal{S}))_M$  is universal catenary local domain. Therefore  $\text{ht} N/\mathfrak{P}_0 = \text{ht} M/(\mathfrak{P}_0 \cap \mathcal{S})$  and hence  $\dim \mathcal{T}/\mathfrak{P}_0 = d + 1$ . Now we assume on the contrary that  $\dim \mathcal{T}_{\mathfrak{P}} \leq 1$ . Then  $\alpha \geq d$ . On the other hand, since  $\text{ht} \mathfrak{p} \geq 2$ ,  $d + 1 = \alpha + \text{ht} \mathfrak{p} \geq \alpha + 2$ . Hence  $\alpha \leq d - 1$  which yields a contradiction. Thus  $\dim \mathcal{T}_{\mathfrak{P}} \geq 2$ .  $\square$

Therefore  $\text{depth}_{\mathcal{S}_P} \mathcal{T}_P \geq 2$  because of the following fact which we omit the proof.

**Claim 3.4.** *Let  $A$  be a Noetherian local ring with the maximal ideal  $\mathfrak{a}$  and  $B$  is a finite extension of  $A$  with  $(S_2)$  property. If for every maximal ideal  $\mathfrak{b}$  of  $B$ ,  $\dim B_{\mathfrak{b}} \geq 2$ , then  $\text{depth}_A B \geq 2$ .*

Next we consider the exact sequence

$$0 \longrightarrow \mathcal{S}_P \longrightarrow \mathcal{T}_P \longrightarrow (\mathcal{T}/\mathcal{S})_P \longrightarrow 0$$

of graded  $\mathcal{S}_P$ -modules. Here we notice that  $\mathcal{T}/\mathcal{S} \neq (0)$  since  $\text{depth}_{\mathcal{S}_P} \mathcal{T}_P \geq 2$  but  $\text{depth}_{\mathcal{S}_P} \mathcal{S}_P = 1$ . Applying the Depth lemma to the above exact sequence we get  $\text{depth}_{\mathcal{S}_P} (\mathcal{T}/\mathcal{S})_P = 0$ . Therefore  $P \in \text{Ass}_{\mathcal{S}} \mathcal{T}/\mathcal{S}$  and then  $P \in \text{Ass}_{\mathcal{S}} \overline{R}[t]/R[t]$  since  $\mathcal{T} \cap R[t] = \mathcal{S}$ . Hence  $P = \mathfrak{q} \cap \mathcal{S}$  for some  $\mathfrak{q} \in \text{Ass}_{R[t]} \overline{R}[t]/R[t]$ . Moreover since  $\overline{R}[t]/R[t] \cong (\overline{R}/R) \otimes_R R[t]$  and  $\text{Ass}_{R[t]} (\overline{R}/R) \otimes_R R[t] = \bigcup_{p \in \text{Ass}_R \overline{R}/R} \text{Ass}_{R[t]} (R/p) \otimes_R R[t]$ , there is  $p_1 \in \text{Ass}_R \overline{R}/R$  such that  $\mathfrak{q} \in \text{Ass}_{R[t]} R[t]/p_1 R[t]$ . Since  $\mathfrak{q} = p_1 R[t]$ ,  $p_1 = \mathfrak{q} \cap R = P \cap R = p$ . Therefore  $p \in \text{Ass}_R \overline{R}/R$ . Furthermore since  $p \supseteq I_1$ ,  $\dim R_p \geq 2$  and hence  $\text{depth} R_p \geq 2$  as  $R$  is  $(S_2)$ . On the other hand  $\text{depth}_{R_p} (\overline{R})_p > 0$  as there is an  $R$ -regular element in  $R$  which is also  $\overline{R}$ -regular. Now applying Depth lemma to the following exact sequence

$$0 \longrightarrow R_p \longrightarrow (\overline{R})_p \longrightarrow (\overline{R}/R)_p \longrightarrow 0$$

of  $R_p$ -modules we get a contradiction. Thus  $\mathcal{S}$  is an  $(S_2)$ -ring.  $\square$

Now it is a position to prove [Theorem 1.1](#).

**Proof of Theorem 1.1.** Since conditions  $(C_1)$ ,  $(C_2)$ , and  $(C_3)$  are satisfied, we get  $\overline{e_1}(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I}) + \ell_{T_{\mathfrak{m}T}}(\overline{S}_{\mathfrak{m}T})$ , by [Lemma 2.4\(2\)](#) and  $\text{Ass}_T(\overline{S}) \subseteq \{\mathfrak{m}T\}$ , by [Lemma 2.2\(2\)](#).

(3)  $\Rightarrow$  (2) This is obvious.

(2)  $\Leftrightarrow$  (1) This is by Lemma 2.5(2).

(1)  $\Rightarrow$  (3) Assume that  $\overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/\overline{I}) + 1$ . Then  $\overline{S} \neq (0)$  by Lemma 2.5 and hence  $\text{Ass}_T \overline{S} = \{\mathfrak{m}T\}$ . Therefore  $\overline{S}$  is a torsion free  $B$ -module. If  $d = 1$ , then  $B$  is a PID. Hence  $\overline{S}$  is  $B$ -free because every torsion free modules over a PID are free. Now we consider the case where  $d \geq 2$ . We will show that  $\overline{S}$  is a  $(S_2)$  module over  $B$ . When this is the case, since  $\text{rank}_B \overline{S} = 1$  and  $B$  is an UFD,  $\overline{S}$  is a reflexive  $B$ -module and hence a free  $B$ -module. Therefore  $\overline{S}/B_+ \overline{S} = (R/\mathfrak{m})\overline{\varphi}$  for some homogeneous element  $\varphi \in (\overline{S})_q$  of degree  $q \geq 1$ . Hence  $\overline{S} = B\varphi + B_+ \overline{S}$  and  $(\overline{S})_{B_+} = B_{B_+} \frac{\varphi}{1}$  by the graded Nakayama lemma. So  $(\overline{S}/B\varphi)_{B_+} = 0$  and  $\overline{S}/B\varphi = 0$ . Thus  $\overline{S} \cong B(-q)$  as desired.

Now we assume on the contrary that  $\overline{S}$  is not a  $(S_2)$  module over  $B$ . Then

$$\text{depth}_{B_P} \overline{S}_P < \inf\{2, \dim_{B_P} \overline{S}_P\}$$

for some prime ideal  $P \in \text{Supp}_B \overline{S}$ . Therefore  $\dim_{B_P} \overline{S}_P \geq 2$  and  $\text{depth}_{B_P} \overline{S}_P = 1$ . Here we notice that  $\dim_{B_P} \overline{S}_P = \dim B_P$ . This ideal  $P$  is a graded ideal of  $B$ . In fact, assume on the contrary that  $P$  is not graded. Then  $1 = \text{depth}_{B_P} \overline{S}_P = \text{depth}_{B_{P^*}} \overline{S}_{P^*} + 1$ , where the graded ideal  $P^*$  denotes the ideal generated by all homogeneous elements in  $P$ . Therefore we have  $\text{depth}_{B_{P^*}} \overline{S}_{P^*} = 0$  and hence  $P^* \in \text{Ass}_B(\overline{S})$ . Thus  $\text{ht} P^* = 0$ . On the other hand, we have  $2 \leq \dim B_P = \dim_{B_P} \overline{S}_P = \dim_{B_{P^*}} \overline{S}_{P^*} + 1 = \dim B_{P^*} + 1$ . This means  $\text{ht} P^* \geq 1$  which is a contradiction. Thus  $P$  is graded. Now let  $p \in \text{Spec} T$  such that  $P = p + \mathfrak{m}T$ . Then  $p$  is also graded as  $\mathfrak{m}T$  is graded. Moreover  $\text{ht}_T p \geq 3$  because  $\text{ht}_B P \geq 2$ ,  $\mathfrak{m}T \subseteq p$  and  $\text{ht}_T \mathfrak{m}T = 1$ . We will prove that  $\text{depth}_{T_p}(\overline{\mathcal{R}})_p \geq 2$ . In order to prove this, it is enough to show the following.

**Claim 3.5.** For all graded prime ideal  $\mathfrak{p}$  of  $T$  such that  $\text{ht} \mathfrak{p} \geq 3$ , we have  $\text{ht}_{\overline{\mathcal{R}}} \mathfrak{P} \geq 2$  for all prime ideal  $\mathfrak{P}$  in  $\overline{\mathcal{R}}$  with  $\mathfrak{P} \cap T = \mathfrak{p}$ .

When this Claim 3.5 holds true, since  $\overline{\mathcal{R}}$  is a  $(S_2)$ -ring by Proposition 3.1, we get  $\text{depth}_{T_p}(\overline{\mathcal{R}})_p \geq 2$  by applying Claim 3.4.

**Proof of Claim 3.5.** Assume on the contrary that there exists a prime ideal  $\mathfrak{P}$  of  $\overline{\mathcal{R}}$  such that  $\mathfrak{P} \cap T = \mathfrak{p}$  but  $\text{ht}_{\overline{\mathcal{R}}} \mathfrak{P} \leq 1$ . Take  $\mathfrak{P}_0 \in \text{Min} \overline{\mathcal{R}}$  such that  $\mathfrak{P}_0 \subseteq \mathfrak{P}$  and  $\dim \overline{\mathcal{R}}_{\mathfrak{P}} = \dim \overline{\mathcal{R}}_{\mathfrak{P}_0}/\mathfrak{P}_0 \overline{\mathcal{R}}_{\mathfrak{P}_0}$ . Then  $\mathfrak{P}_0 = \mathfrak{q}R[t] \cap \overline{\mathcal{R}}$ , for some  $\mathfrak{q} \in \text{Min} R$ , and  $\mathfrak{P}_0 \cap T = \mathfrak{q}R[t] \cap T$ . Therefore  $(\mathfrak{P}_0 \cap T) \cap R = \mathfrak{q}$  and hence  $T/(\mathfrak{P}_0 \cap T) \cong R((Q + \mathfrak{q})/\mathfrak{q})$ . Since  $Q \notin \mathfrak{q}$ ,  $\dim T/(\mathfrak{P}_0 \cap T) = \dim R((Q + \mathfrak{q})/\mathfrak{q}) = \dim R/\mathfrak{q} + 1 = d + 1$  and we get  $\mathfrak{P}_0 \cap T \in \text{Min} T$ . Let  $\mathcal{M}$  be the unique graded maximal ideals of  $T$ . Since the extension  $T \subseteq \overline{\mathcal{R}}$  is finite, there is a graded maximal ideals  $\mathcal{N}$  of  $\overline{\mathcal{R}}$  such that  $\dim \overline{\mathcal{R}}_{\mathcal{N}}/\mathfrak{P} \overline{\mathcal{R}}_{\mathcal{N}} = \dim T_{\mathcal{M}}/\mathfrak{p} T_{\mathcal{M}} = \alpha$ .

$$\begin{array}{ccccc} \mathfrak{P}_0 & \xrightarrow{\quad} & \mathfrak{P} & \xrightarrow{\quad \alpha \quad} & \exists \mathcal{N} \subseteq \overline{\mathcal{R}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{P}_0 \cap T & \xrightarrow{\quad} & \mathfrak{P} \cap T = \mathfrak{p} & \xrightarrow{\quad \alpha \quad} & \mathcal{M} \subseteq T \end{array}$$

We notice that  $\mathfrak{p} \subseteq \mathcal{N}$ , as  $\mathfrak{p}$  is graded, and  $\text{ht}\mathcal{N}/\mathfrak{P}_0 = \text{ht}\mathcal{M}/(\mathfrak{P}_0 \cap T) = d + 1$ , because  $R$  is universal catenary and the extension  $T/(\mathfrak{P}_0 \cap T) \hookrightarrow \overline{\mathcal{R}}/\mathfrak{P}_0$  is finite. Therefore  $d + 1 = \text{ht}\mathcal{N}/\mathfrak{P}_0 \leq 1 + \alpha$  and  $d + 1 = \text{ht}\mathcal{M}/(\mathfrak{P}_0 \cap T) \geq 3 + \alpha$ , a contradiction.  $\square$

**Claim 3.6.**  $\text{depth}_{T_p}(\overline{\mathcal{R}}_+)_p \geq 2$ .

**Proof of Claim 3.6.** We consider the following exact sequence of  $T_p$ -modules.

$$0 \longrightarrow (\overline{\mathcal{R}}_+)_p \longrightarrow (\overline{\mathcal{R}})_p \longrightarrow R_p \longrightarrow 0.$$

If  $R_p = (0)$ , then  $\text{depth}_{T_p}(\overline{\mathcal{R}}_+)_p = \text{depth}_{T_p}(\overline{\mathcal{R}})_p \geq 2$  by Claim 3.5. If  $R_p \neq (0)$ , then  $p = \mathcal{M}$ , the unique graded maximal ideal of  $T$ . In fact, since  $R_p \neq (0)$  and  $R = \overline{\mathcal{R}}/\overline{\mathcal{R}}_+$ , we have  $p \supseteq \overline{\mathcal{R}}_+ \supseteq T_+$  and hence  $p \supseteq mT + T_+ = \mathcal{M}$ , which leads to the fact that  $p = \mathcal{M}$  as  $p$  is graded. Hence  $\text{depth}_{T_p}R_p = \text{depth}_{T_{\mathcal{M}}}R_{\mathcal{M}} = \text{depth}R \geq 2$  by the condition  $(C_3)$ . Now by using Claim 3.5 and applying Depth lemma to the above exact sequence we get the desired result.  $\square$

By Claim 3.5, Claim 3.6, and comparing depths of  $T_p$ -modules in the following exact sequences

$$0 \longrightarrow (\overline{IT})_p \longrightarrow (\overline{\mathcal{R}}_+(1))_p \longrightarrow (\overline{S})_p \longrightarrow 0$$

and

$$0 \longrightarrow (\overline{IT})_p \longrightarrow T_p \longrightarrow T_p/(\overline{IT})_p \longrightarrow 0$$

of  $T_p$ -modules, we get  $\text{depth}T_p = 2$ . On the other hand since  $T$  is an  $(S_3)$ -ring, by Remark 2.3, and  $\text{ht}T_p \geq 3$ , we get  $2 = \text{depth}T_p \geq \inf\{3, \dim T_p\} = 3$  which is a contradiction. Thus  $\overline{S}$  is a  $(S_2)$  module over  $B$  as desired.

The statement (b) is by Lemma 2.2(3). Lastly we will prove the statement (c). Since  $\overline{S} \cong B(-q)$  for some integer number  $q \geq 1$ ,  $\ell_R(\overline{S}_n) = \ell_R(B_{n-q})$  for all  $n \geq 0$ . If  $n < q$  then  $\ell_R(\overline{S}_n) = \ell_R(B_{n-q}) = (0)$  and hence

$$\begin{aligned} \ell_R(R/\overline{I}^{n+1}) &= e_0(I) \binom{n+d}{d} - \{e_0(I) + e_1(Q) - \ell_R(R/\overline{I})\} \binom{n+d-1}{d-1} \\ &\quad + \sum_{i=2}^d (-1)^i \{e_{i-1}(Q) + e_i(Q)\} \binom{n+d-i}{d-i} \end{aligned}$$

by Lemma 2.4(1). If  $n \geq q$  then

$$\ell_R(B_{n-q}) = \binom{n-q+d-1}{d-1} = \sum_{i=0}^q (-1)^i \binom{q}{i} \binom{n+d-1-i}{d-1-i}.$$

Therefore

$$\begin{aligned}
 \ell_R(R/\overline{I^{n+1}}) &= e_0(I) \binom{n+d}{d} - \{e_0(I) + e_1(Q) - \ell_R(R/\overline{I})\} \binom{n+d-1}{d-1} \\
 &\quad + \sum_{i=2}^d (-1)^i \{e_{i-1}(Q) + e_i(Q)\} \binom{n+d-i}{d-i} \\
 &\quad - \sum_{i=0}^q (-1)^i \binom{q}{i} \binom{n+d-1-i}{d-1-i} \\
 &= e_0(I) \binom{n+d}{d} - \{e_0(I) + e_1(Q) - \ell_R(R/\overline{I}) + 1\} \binom{n+d-1}{d-1} \\
 &\quad + \sum_{i=2}^d (-1)^i \{e_{i-1}(Q) + e_i(Q) + \binom{q}{i-1}\} \binom{n+d-i}{d-i}
 \end{aligned}$$

also by Lemma 2.4(1). Thus the last statement (c) follows.  $\square$

Now we will prove Theorem 1.2 and examine some corollaries of Theorem 1.1 in the Cohen–Macaulay case.

**Proof of Theorem 1.2.** Since  $\widehat{\overline{\mathfrak{a}R}} = \overline{\widehat{\mathfrak{a}R}}$  for every  $\mathfrak{m}$ -primary ideal  $\mathfrak{a}$  in  $R$ , by passing to the  $\mathfrak{m}$ -adic completion  $\widehat{R}$  of  $R$ , without loss of generality we may assume that  $R$  is complete. Therefore  $R$  is a Nagata reduced Cohen–Macaulay ring. Since  $\overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1$ , we have  $\text{depth} \overline{\mathcal{G}} \geq d-1$  by Theorem 1.1(b). Therefore we get the assertion (1). Now we will prove the assertion (2). Since  $\text{depth} \overline{\mathcal{G}} \geq d-1$ , by [9, Proposition 4.6] the normal Hilbert coefficients have the following forms

$$\overline{e}_i(I) = \sum_{n \geq i} \binom{n-1}{i-1} \ell_R(\overline{I^n}/Q\overline{I^{n-1}})$$

for  $1 \leq i \leq d$ . Now, since  $\overline{e}_3(I) = 0$ , we get  $\overline{I^n} = Q\overline{I^{n-1}}$  for all  $n \geq 3$  and therefore  $\overline{\mathcal{G}}$  is Cohen–Macaulay, by [8, Theorem 4.6(ii)], because  $\overline{I^2} \cap Q = Q\overline{I}$  by [10, THEOREM 1]. Since  $\overline{e}_1(I) = \sum_{n \geq 1} \ell_R(\overline{I^n}/Q\overline{I^{n-1}})$  and  $\overline{e}_2(I) = \sum_{n \geq 2} (n-1) \ell_R(\overline{I^n}/Q\overline{I^{n-1}})$ , we have  $\overline{e}_1(I) = \ell_R(\overline{I}/Q) + \ell_R(\overline{I^2}/Q\overline{I})$  and  $\overline{e}_2(I) = \ell_R(\overline{I^2}/Q\overline{I})$ . Therefore  $\overline{e}_2(I) = \ell_R(\overline{I^2}/Q\overline{I}) = 1$ . The last statement follows from the results that  $\ell_R(\overline{I^2}/Q\overline{I}) = 1$  and  $\overline{I^n} = Q\overline{I^{n-1}}$  for all  $n \geq 3$ .  $\square$

When  $R$  is a Nagata reduced Cohen–Macaulay ring, with a further assumption that  $\overline{e}_3(I) = 0$ , the number  $q$  in Theorem 1.1 turns into exactly 1 and  $\overline{\mathcal{G}}$  and  $\overline{\mathcal{R}}$  are Cohen–Macaulay as follows.

**Corollary 3.7.** Assume that  $R$  be a Nagata reduced Cohen–Macaulay local ring with the maximal ideal  $\mathfrak{m}$  and  $d = \dim R \geq 3$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  and suppose that

$I$  contains a parameter ideal  $Q$  of  $R$  as a reduction. Assume that  $\overline{e}_3(I) = 0$ . Then the following are equivalent to each other.

- (1)  $\overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1$ .
- (2)  $\mathfrak{m}\overline{S} = (0)$  and  $\text{rank}_B \overline{S} = 1$ , where  $B = T/\mathfrak{m}T$ .
- (3)  $\overline{S} \cong B(-1)$  as graded  $T$ -modules.
- (4)  $\overline{e}_2(I) = 1$  and  $\ell_R(\overline{I}^2/Q\overline{I}) = 1$ .

When this is the case

- (a)  $\overline{S}$  is a Cohen–Macaulay  $T$ -module.
- (b)  $\overline{\mathcal{G}}$  is Cohen–Macaulay.
- (c)  $\overline{\mathcal{R}}$  is Cohen–Macaulay.
- (d)  $r_Q(\{\overline{I}^n\}_{n \in \mathbb{Z}}) = 2$ .
- (e) For all  $n \geq 0$

$$\ell_R(R/\overline{I}^{n+1}) = e_0(I) \binom{n+d}{d} - \overline{e}_1(I) \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}.$$

**Proof.** The equivalent statements (1)  $\Leftrightarrow$  (2) is exactly as in [Theorem 1.1](#). By [\[13, Proposition 4.9\]](#) we have  $\overline{e}_2(I) = 1$  implies  $\overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1$  and then we get the implication (4)  $\Rightarrow$  (1). The implication (1)  $\Rightarrow$  (4) is followed by [Theorem 1.2](#) and its proof. Again by [Theorem 1.1](#) we get the implication (3)  $\Rightarrow$  (1). Assume that we have condition (1), then by [Theorem 1.1\(c\)](#) we have  $\overline{S} \cong B(-q)$  and  $\overline{e}_2(I) = q$  for some  $q \geq 1$ . Moreover since  $\overline{e}_2(I) = 1$  by the implication (1)  $\Rightarrow$  (4), we get the implication (1)  $\Rightarrow$  (3).

Now we assume that  $\overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1$  and  $\overline{e}_3(I) = 0$ . Then (a) follows by [Theorem 1.1 \(a\)](#). The statements (e) follows by [Theorem 1.1 \(c\)](#) and the fact that  $\overline{e}_2(I) = 1$ . (b) and (d) are by [Theorem 1.2 \(2\)](#). For a proof of (c), we use [\[9, Proposition 4.10\]](#) which said that  $\overline{\mathcal{R}}$  is Cohen–Macaulay if and only if  $\overline{e}_1(I) = \sum_{n=1}^{d-1} \ell_R(\overline{I}^n + Q/Q)$ . Since  $\overline{I}^n = Q\overline{I}^{n-1}$  for all  $n \geq 3$ , by (d) and  $\ell_R(\overline{I}^2 + Q/Q) = \ell_R(\overline{I}^2/Q\overline{I}) = 1$  we get  $\overline{\mathcal{R}}$  is Cohen–Macaulay as desired.  $\square$

We end this research by giving some remarks of [Theorem 1.2](#) on the Cohen–Macaulayness of  $\overline{\mathcal{G}}$  and  $\overline{\mathcal{R}}$  in the case where  $d \leq 2$ . In the case where  $d = 2$ , we do not have any information about the Cohen–Macaulayness of  $\overline{\mathcal{G}}$ . However  $\overline{\mathcal{R}}$  may not be Cohen–Macaulay as showing in the following example.

**Example 3.8.** Let  $S = k[x, y]$  be the polynomial ring over a field  $k$  and  $A = k[x^2, xy, xy^2]$  ( $\subseteq S$ ). We set  $R = A_M$  where  $M = (x^2, xy, xy^2)A$  and let  $\mathfrak{m}$  denote the maximal ideal of  $R$ . We then have the following.

- (1)  $R$  is a Gorenstein local integral domain such that  $\dim R = 2$ ,  $e_0(\mathfrak{m}) = 3$ , and  $\mathfrak{m}^3 = Q\mathfrak{m}^2$  where  $Q = (x^2 - xy^2, xy)$ , but  $\ell_R(\mathfrak{m}^2/Q\mathfrak{m}) = 1$ .

- (2)  $R$  is not a normal ring but  $\mathfrak{m}$  is a normal ideal in  $R$ .
- (3)  $\ell_R(R/\mathfrak{m}^{n+1}) = 3\binom{n+2}{2} - 3\binom{n+1}{1} + 1$  for all  $n \geq 0$ .
- (4)  $\overline{S}_Q(\mathfrak{m}) = S_Q(\mathfrak{m}) \cong B(-1)$  as a graded  $T$ -module.
- (5)  $\mathcal{G}(\mathfrak{m})$  is a Cohen–Macaulay ring with  $\mathfrak{a}(\mathcal{G}(\mathfrak{m})) = 0$ , so that  $\mathcal{R}(\mathfrak{m})$  is not a Cohen–Macaulay ring.

**Proof.** Since  $A \cong k[x, y, z]/(z^4 - xy^2)$  where  $k[x, y, z]$  denotes the polynomial ring, we have  $\dim R = 2$  and  $e_0(\mathfrak{m}) = 3$ . It is direct to check the rest of Assertion (1). The ring  $A$  is clearly not normal, whence so is  $R$ . To check that  $\mathfrak{m}$  is normal, one needs some computation which we leave to readers. Assertions (3) and (4) now follow from [3, Theorem 1.1]. As  $\mathcal{G}(\mathfrak{m}) \cong k[x, y, z]/(xy^2)$ ,  $\mathcal{G}(\mathfrak{m})$  is a Cohen–Macaulay ring with  $\mathfrak{a}(\mathcal{G}(\mathfrak{m})) = 0$ . Therefore  $\mathcal{R}(\mathfrak{m})$  is not a Cohen–Macaulay ring (see, e.g., [5]).  $\square$

When  $d = 1$ , the following remark gives a note of the Cohen–Macaulayness of  $\overline{\mathcal{G}}$  and  $\overline{\mathcal{R}}$ .

**Remark 3.9.** When  $(R, \mathfrak{m})$  is a one-dimensional analytically unramified Cohen–Macaulay local ring, with the same notations for as above,  $\overline{\mathcal{G}}$  is a Cohen–Macaulay ring and  $\overline{\mathcal{R}}$  is a Cohen–Macaulay ring if and only if  $R$  is a discrete valuation ring. In fact, the fact that  $\overline{\mathcal{G}}$  is a Cohen–Macaulay ring is by [11, Proposition 3.25]. For a proof of the second fact, if  $R$  is a discrete valuation ring, then  $\overline{I}$  is a parameter ideal and hence  $\overline{\mathcal{R}}$  is Cohen–Macaulay. Conversely let  $a \in I$  such that  $(a) \subseteq I$  as a reduction. Notice that by the module version of [5, Theorem 1.1],  $\overline{\mathcal{R}}$  is Cohen–Macaulay if and only if  $\overline{\mathcal{G}}$  is Cohen–Macaulay and  $\mathfrak{a}(\overline{\mathcal{G}}) < 0$ , where  $\mathfrak{a}(\overline{\mathcal{G}})$  denotes the  $\mathfrak{a}$ -invariant of  $\overline{\mathcal{G}}$  ([6, DEFINITION (3.1.4)]). Since

$$\overline{\mathcal{G}}/at\overline{\mathcal{G}} \cong \mathcal{G}(\{\overline{I}^n + (a)/(a)\}_{n \in \mathbb{Z}}),$$

the associated graded ring of the filtration  $\{\overline{I}^n + (a)/(a)\}_{n \in \mathbb{Z}}$ ,  $\mathfrak{a}(\overline{\mathcal{G}}/at\overline{\mathcal{G}}) = \mathfrak{a}(\overline{\mathcal{G}}) + 1 \leq 0$ . Therefore for all  $n \geq 1$ ,  $\overline{I}^n \subseteq (a) + \overline{I}^{n+1}$ . Moreover since  $\overline{I}^\ell = a\overline{I}^{\ell-1}$  for  $\ell \gg 0$ ,  $\overline{I}^n \subseteq (a)$  for all  $n \geq 1$  and hence  $\overline{I}^n = a\overline{I}^{n-1}$  for all  $n \geq 1$ . In particular  $\overline{I} = (a)$ , whence  $R$  is a discrete valuation ring by [2, Corollary 2.5].

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