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A note on Itoh (e)-valuation rings of an ideal

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ABSTRACT

Let I be a regular proper ideal in a Noetherian ring R , let $e \geq 2$ be an integer, let $\mathbf{T}_e = R[u, tI, u^{\frac{1}{e}}]' \cap R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$ (where t is an indeterminate and $u = \frac{1}{t}$), and let $\mathbf{r}_e = u^{\frac{1}{e}} \mathbf{T}_e$. Then the Itoh (e)-valuation rings of I are the rings $(\mathbf{T}_e/z)_{(p/z)}$, where p varies over the (height one) associated prime ideals of \mathbf{r}_e and z is the (unique) minimal prime ideal in \mathbf{T}_e that is contained in p . We show, among other things:

(1) \mathbf{r}_e is a radical ideal if and only if e is a common multiple of the Rees integers of I .

(2) For each integer $k \geq 2$, there is a one-to-one correspondence between the Itoh (k)-valuation rings (V^*, N^*) of I and the Rees valuation rings (W, Q) of $uR[u, tI]$; namely, if $F(u)$ is the quotient field of W , then V^* is the integral closure of W in $F(u^{\frac{1}{k}})$.

(3) For each integer $k \geq 2$, if (V^*, N^*) and (W, Q) are corresponding valuation rings, as in (2), then V^* is a finite integral extension domain of W , and W and V^* satisfy the Fundamental Equality with no splitting. Also, if $uW = Q^e$, and if the greatest common divisor of e and k is d , and c is the integer such that $cd = k$, then $QV^* = N^{*c}$ and $[(V^*/N^*) : (W/Q)] = d$. Further, if $uW = Q^e$ and $k = ge$ is a multiple of e , then there exists a unit $\theta_e \in V^*$ such that $V^* = W[\theta_e, u^{\frac{1}{k}}]$ is a finite free integral extension domain of W , $QV^* = N^{*g}$, $N^* = u^{\frac{1}{k}} V^*$, and $[V^* : W] = k$.

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(4) If the Rees integers of I are all equal to e , then $V^* = W[\theta_e]$ is a simple free integral extension domain of W , $QV^* = N^* = u^{\frac{1}{e}}V^*$, and $[V^* : W] = e = [(V^*/N^*) : (W/Q)]$.

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1. Introduction

All rings in this paper are commutative and have an identity element $1 \neq 0$, and our terminology is mainly as in Nagata [7]. Thus a **basis** of an ideal is a generating set of the ideal, the term **altitude** refers to what is often called dimension or Krull dimension, and for a pair of local rings (R, M) and (S, N) , S **dominates** R in case $R \subseteq S$ and $N \cap R = M$, and we then write $\mathbf{R} \leq \mathbf{S}$ or $(\mathbf{S}, \mathbf{N}) \geq (\mathbf{R}, \mathbf{M})$.

In 1988, Shiroh Itoh proved the following interesting and useful theorem in [5, p. 392, lines 3–11] (the terminology is defined in Section 2):

Theorem 1.1. *Let I be a regular proper ideal in a Noetherian ring R , let $\mathbf{R} = R[u, tI]$ be the Rees ring of R with respect to I , let $(W_1, Q_1), \dots, (W_n, Q_n)$ be the Rees valuation rings of $u\mathbf{R}$, for $j = 1, \dots, n$, let $uW_j = Q_j^{e_j}$ (so e_1, \dots, e_n are the Rees integers of I), and let $e \geq 2$ be an arbitrary common multiple of e_1, \dots, e_n . Also, let $\mathbf{S} = \mathbf{R}[u^{\frac{1}{e}}]$, let $\mathbf{T} = \mathbf{S}' \cap R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$, and let $\mathbf{r} = u^{\frac{1}{e}}\mathbf{T}$. Then:*

(1.1.1) *\mathbf{r} is a radical ideal, so the Rees integers of \mathbf{r} and of $(u^{\frac{1}{e}}\mathbf{S})_a$ are all equal to one.*

(1.1.2) *There is a one-to-one correspondence between the Rees valuation rings (V^*, N^*) of \mathbf{r} and the Rees valuation rings (W, Q) of $u\mathbf{R}$; namely, if $F(u)$ is the quotient field of W , then V^* is the integral closure of W in $F(u^{\frac{1}{e}})$.*

(1.1.3) *Let (V^*, N^*) and (W, Q) be corresponding Rees valuation rings of \mathbf{r} and $u\mathbf{R}$, respectively, as in (1.1.2), so $W = W_j$ for some $j \in \{1, \dots, n\}$. Then $QV^* = N^* \frac{e}{e_j}$, so the ramification index of V^* relative to W is equal to $\frac{e}{e_j}$.*

Actually, the only part of this theorem that S. Itoh specifically stated in [5] was that \mathbf{r} is a radical ideal when e is the least common multiple of e_1, \dots, e_n . His proof of this essentially shows that (1.1.1)–(1.1.3) hold, but his goals in [5] were to prove several nice applications of the radicality of the ideal \mathbf{r} , not to find additional properties of the Rees valuation rings of this ideal.

However, it turns out that the Rees valuation rings of ideals like \mathbf{r} have some additional nice properties, and the goal of this present paper is to derive some of these properties. To facilitate discussing these valuation rings we make the following definition.

Definition 1.2. For an arbitrary integer $e \geq 2$, the **Itôh (e)-valuation rings of I** are the Rees valuation rings of $\mathbf{r}_e = u^{\frac{1}{e}}\mathbf{T}_e$, where $\mathbf{T}_e = R[u, tI, u^{\frac{1}{e}}]' \cap R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$.

2. Definitions and known results

In this section we recall the needed definitions and mention the needed known results concerning them.

Definition 2.1. Let I be an ideal in a ring R . Then:

(2.1.1) R' denotes the **integral closure** of R in its total quotient ring.

(2.1.2) I_a denotes the **integral closure** of I in R , so I_a is the ideal $\{x \in R \mid x \text{ is a root of an equation of the form } X^n + i_1 X^{n-1} + \cdots + i_n = 0\}$, where $i_j \in I^j$ for $j = 1, \dots, n$. The ideal I is **integrally closed** in case $I = I_a$.

(2.1.3) The **Rees ring of R with respect to I** is the graded subring $\mathbf{R}(R, I) = R[u, tI]$ of $R[u, t]$, where t is an indeterminate and $u = \frac{1}{t}$.

(2.1.4) Assume that R is Noetherian and that I is a regular proper ideal in R (that is, I contains a regular element of R and $I \neq R$), and let b_1, \dots, b_g be regular elements in R that generate I . For $i = 1, \dots, g$, let $C_i = R[I/b_i]$, let $p_{i,1}, \dots, p_{i,n_i}$ be the (height one) associated prime ideals of $b_i C_i'$ (see (2.1.1)), let $z_{i,j}$ be the (unique; see Remark 2.2.1 below) minimal prime ideal in C_i' that is contained in $p_{i,j}$ (possibly $z_{i,h} = z_{i,j}$ for some $h \neq j$ or $z_{i,j} = (0)$), and let $V_{i,j} = (C_i'/z_{i,j})_{(p_{i,j}/z_{i,j})}$, so $V_{i,j}$ is a DVR. Then the set $\mathbf{RV}(I)$ of all $V_{i,j}$ ($i = 1, \dots, g$ and $j = 1, \dots, n_i$) is the set of **Rees valuation rings** of I . (The Rees valuation rings of I are well defined by I ; they do not depend on the basis b_1, \dots, b_g of I .)

(2.1.5) If (V, N) is a Rees valuation ring of I (see (2.1.4)), then the **Rees integer** of I with respect to V is the positive integer e such that $IV = N^e$. If $\mathbf{RV}(I) = \{(V_{1,1}, N_{1,1}), \dots, (V_{g,n_g}, N_{g,n_g})\}$, and if $IV_{i,j} = N_{i,j}^{e_{i,j}}$ for $i = 1, \dots, g$ and $j = 1, \dots, n_i$, then $e_{1,1}, \dots, e_{g,n_g}$ are called the **Rees integers** of I .

(2.1.6) If $(W, Q) \leq (V^*, N^*)$ are DVRs such that V^* is a localization of an integral extension domain of W , then the **ramification index of V^* relative to W** is the positive integer k such that $QV^* = N^{*k}$.

Remark 2.2. Let I be a regular proper ideal in a Noetherian ring R . Then:

(2.2.1) Concerning (2.1.4), it is shown in [9, Definition p. 213 and Propositions 2.7 and 2.13] that if b is a regular nonunit in the integral closure A' of a Noetherian ring A , then bA' has a finite primary decomposition, each associated prime ideal p of bA' has height one, p contains exactly one associated prime ideal z of (0) , and $(A'/z)_{(p/z)}$ is a DVR.

(2.2.2) It is shown in [11, Proposition 10.2.3] that if V_1, \dots, V_n are the Rees valuation rings of I , then $(I^k)_a = \cap \{I^k V_j \cap R \mid j = 1, \dots, n\}$ for all $k \in \mathbb{N}_{>0}$. ($\mathbb{N}_{>0}$ denotes the set of positive integers.)

(2.2.3) It readily follows from Definition 2.1.4 that the set $\mathbf{RV}(I)$ of Rees valuation rings of I is the disjoint union of the sets $\mathbf{RV}((I+z)/z)$, where z runs over the minimal prime ideals w in R such that $I + w \neq R$. Also, if V is a Rees valuation ring of I and of $(I+z)/z$, then the Rees integer of I with respect to V is the Rees integer of $(I+z)/z$ with respect to V .

(2.2.4) It follows from Definition 2.1.4 (and is proved in [11, Example 10.3.2]) that if $I = bR$ is a regular proper principal ideal in R , then the Rees valuation rings of I are the rings $(R'/z)_{(p/z)}$, where p varies over the (height one) associated prime ideals of bR' and z is the (unique) minimal prime ideal in R' that is contained in p . (It is readily checked that if $A = R' \cap R[\frac{1}{b}]$, then there is a one-to-one correspondence between the associated prime ideals p of bR' and the associated prime ideals q of bA ; namely, $q = p \cap A$, and then $(R'/z)_{(p/z)} = (A/z')_{(q/z')}$, where $z' = z \cap A$.)

(2.2.5) It follows from Definitions 2.1.4 and 2.1.5 that if A is a ring such that $R \subseteq A \subseteq R'$, then I and IA have the same Rees valuation rings and the same Rees integers. And it also follows that, for all positive integers k , IA and $I^k A$ have the same Rees valuation rings.

(2.2.6) It is well known (and is readily proved, much as in the proof of [1, (2.5)]), that if R is a Noetherian integral domain and A is an integral extension domain of R , then the Rees valuation rings of IA are the extensions of the Rees valuation rings of I to the quotient field of A .

The following theorem is a special case of [13, Theorems 19 and 20, pp. 55 and 60–61]; the terminology “Fundamental Inequality” is due to O. Endler in [2, pp. 127–128].

Theorem 2.3. (*Fundamental Inequality*): Let (V, N) be a DVR with quotient field F , let E be a finite algebraic extension field of F , let $[E : F] = e$, let $(V_1, N_1), \dots, (V_n, N_n)$ be all of the valuation rings of E that are extensions of V to E (so the integral closure V' of V in E has exactly n maximal ideals M_1, \dots, M_n and $V_j = V'_{M_j}$, $j = 1, \dots, n$), and for $j = 1, \dots, n$, let $NV_j = N_j^{e_j}$ and $[(V_j/N_j) : (V/N)] = f_j$. Then

$$(FI) \quad \sum_{j=1}^n e_j f_j \leq e,$$

and the equality holds if the integral closure V' of V in E is a finite V -module.

Terminology 2.4. If (V, N) and $(V_1, N_1), \dots, (V_n, N_n)$ are as in Theorem 2.3, if $n = 1$, and if the equality in (FI) holds, then it will be said that (V, N) and (V_1, N_1) **satisfy the Fundamental Equality with no splitting**.

Notation 2.5. If R is an integral domain, then $R_{(0)}$ denotes the quotient field of R . Therefore, if S is a finite algebraic extension domain of R , then $[S_{(0)} : R_{(0)}]$ denotes the dimension of the quotient field $S_{(0)}$ of S over the quotient field $R_{(0)}$ of R . If S is a finite free integral extension domain of R , and if it is clear that the rank of S is equal to $[S_{(0)} : R_{(0)}]$, then we often write $[S : R]$ in place of $[S_{(0)} : R_{(0)}]$.

The next three propositions are known, but we do not know specific references for them, so we sketch their proofs.

Proposition 2.6. *Let (W, Q) be a DVR, let $Q = \pi W$, let $f \geq 2$ be an integer, let $D = W[\pi^{\frac{1}{f}}]$, and let $P' = \pi^{\frac{1}{f}} D$. Then D is a DVR that is a simple free integral extension domain of W , P' is the maximal ideal in D , $QD = P'^f$, $[D_{(0)} : W_{(0)}] = f$, and $D/P' \cong W/Q$. Therefore W and D satisfy the Fundamental Equality with no splitting (see Terminology 2.4).*

Proof. $P' = (Q, \pi^{\frac{1}{f}})D$ is a maximal ideal in D , since $D/P' \cong W/Q$ is a field, and $P' = \pi^{\frac{1}{f}} D$ is a principal ideal, since Q is generated by π . Also, D is integral over W and Q is the unique maximal ideal in W , so every maximal ideal in D contains QD , so it follows that P' is the only maximal ideal in D , hence D is a DVR. It therefore follows that $QD = P'^f$. Therefore, since $[D_{(0)} : W_{(0)}] \leq f$, it follows from the Fundamental Inequality FI that $[D_{(0)} : W_{(0)}] = f$, hence D is a simple free integral extension domain of W . Thus, by (2.4), the last statement is clear. \square

Proposition 2.7. *Let M be a maximal ideal in a Noetherian ring R , and let $m(X)$ be a monic polynomial in $R[X]$. If the image $\overline{m}(X)$ of $m(X)$ in $(R/M)[X]$ is irreducible, then $m(X)$ is irreducible in $R[X]$, $MR[x]$ is a maximal ideal in $R[x] = R[X]/(m(X)R[X])$, and $R[x]$ is a simple free integral extension ring of R of rank equal to $\deg(m(X))$.*

Proof. By considering the maps $R[X] \rightarrow (R/M)[X] \rightarrow (R/M)[\chi]$, where χ is a root of the irreducible (by hypothesis) polynomial $\overline{m}(X)$ in $(R/M)[X]$, it follows that: $m(X)$ is irreducible in $R[X]$; $(M, m(X))R[X]$ is a maximal ideal; $MR[x] = ((M, m(X))R[X])/(m(X)R[X])$ is a maximal ideal in $R[x]$; and, $R[x]$ is a simple free integral extension ring of R of rank equal to $\deg(m(X))$. \square

Proposition 2.8. *Let I be a regular proper ideal in a Noetherian ring R , let b_1, \dots, b_g be regular elements in I that are a basis of I , and let $\mathbf{R} = R[u, tI]$ be the Rees ring of R with respect to I . Then there exists a one-to-one correspondence between the Rees valuation rings (V, N) of I and the Rees valuation rings (W, Q) of $uR[u, tI]$. Namely, if $V = (R[I/b_i]'/z)_{(p/z)}$ (where $i \in \{1, \dots, g\}$, p is an associated prime ideal of $b_i R[I/b_i]'$, and z is the minimal prime ideal in $R[I/b_i]'$ that is contained in p), then $W = V[\overline{tb_i}]_{NV[\overline{tb_i}]}$, where $\overline{tb_i} = t(b_i + z)$, and $Q = NW$. (Note that $\overline{tb_i} = tb_i + z'$ (where $z' = zR[u, t] \cap R[u, tI]$) corresponds to $t \cdot \overline{b_i}$ in the isomorphism between $R[u, tI]/z'$ and $(R/z)[u, t((I + z)/z)]$; see [10].)*

Proof. Since t is transcendental over R and $u = \frac{1}{t}$, there is a one-to-one correspondence between the minimal prime ideals z in R and the minimal prime ideals z' in \mathbf{R} ; namely, $z' = zR[u, t] \cap \mathbf{R}$, and then $z = z' \cap R$. Thus it follows from Remark 2.2.3 that it suffices to prove this proposition for the case when R is a Noetherian integral domain.

Therefore assume that R is a Noetherian domain, fix $b \in \{b_1, \dots, b_g\}$, let $C = R[I/b]$, and let $\mathbf{C}' = \mathbf{R}[\frac{1}{tb}]'$, so $\mathbf{R}'[\frac{1}{tb}] = \mathbf{C}' = C'[tb, \frac{1}{tb}]$. Also, $u\mathbf{C}' = b\mathbf{C}'$, so, since tb is transcendental over \mathbf{C}' , there exists a one-to-one correspondence between the (height one)

associated prime ideals p of bC' and the (height one) associated prime ideals p' of uC' ; namely, $p' = pC'$ (and $p = p' \cap C'$). Therefore there exists a one-to-one correspondence between the DVRs $V = C'_p$ and the DVRs $W = C'_{p'}$, and for corresponding V and W , if $V = R[I/b]'_p$ and $N = pV$, then $W = V[tb]_{NV[tb]}$ and $Q = NW = pW$.

It follows that, for each Rees valuation ring (V, N) of I of the form $V = R[I/b]'_p$, the ring $W = V[tb]_{pV[tb]}$ is a Rees valuation ring of $u\mathbf{R}$, and $W \cap R_{(0)} = V$.

Finally, let W be a Rees valuation ring of $u\mathbf{R}$, say $W = \mathbf{R}'_{p'}$, where p' is a (height one) associated prime ideal of $u\mathbf{R}'$ (see Remark 2.2.4). To complete the proof of the one-to-one correspondence, it suffices to show that there exists $b \in \{b_1, \dots, b_g\}$ and a Rees valuation ring (V, N) of I such that $V = R[I/b]'_p$ and $W = V[tb]_{NV[tb]}$.

For this, $tb \notin p'$ for some $b \in \{b_1, \dots, b_g\}$, by [8, Lemma 3.2] (the assumption in [8] that R is analytically unramified is not used in the proof of Lemma 3.2). Therefore $W = \mathbf{R}[\frac{1}{tb}]'_{p''}$, where $p'' = p'W \cap \mathbf{R}[\frac{1}{tb}]'$ (so p'' is a (height one) associated prime ideal of $u\mathbf{R}[\frac{1}{tb}]' = b\mathbf{R}[\frac{1}{tb}]'$, and $\mathbf{R}[\frac{1}{tb}]' = C'[tb, \frac{1}{tb}]$, where $C = R[I/b]$). Let $p = p'' \cap C'$. Then p is a (height one) associated prime ideal of bC' and $p'' = pC'[tb, \frac{1}{tb}]$, so it follows from the second preceding paragraph that $W = V[tb]_{NV[tb]}$, where $V = C'_p$ and $N = pV$. \square

3. Properties of Itoh (e)-valuation rings

In this section we show that Itoh (e)-valuation rings have several nice properties. For this, we need the following proposition, which is essentially a corollary of Proposition 2.7.

Proposition 3.1. *Let I be a regular proper ideal in a Noetherian ring R , let b_1, \dots, b_g be regular elements in I that generate I , and let $\mathbf{R} = R[u, tI]$ be the Rees ring of R with respect to I . Let (V, N) and (W, Q) be corresponding (as in Proposition 2.8) Rees valuation rings of I and $u\mathbf{R}$, respectively, say $V = (R[I/b]'/z)_{(p/z)}$, where $b \in \{b_1, \dots, b_g\}$, p is a (height one) associated prime ideal of $bR[I/b]'$, and z is the minimal prime ideal in $R[I/b]'$ that is contained in p , so $W = V[t\bar{b}]_{NV[t\bar{b}]}$, where $\bar{b} = b + z$. Let $e \geq 2$ be an integer, let $v \in V - N$, let $m_e(X) = X^e - \frac{v}{t\bar{b}}$, and let θ_e be a root of $m_e(X)$ in a fixed algebraic closure of the quotient field of W . Then $U = W[\theta_e]$ is a DVR that is a simple free integral extension domain of W , QU is its maximal ideal, and $[U : W] = e = [U/(QU) : W/Q]$. Therefore W and U satisfy the Fundamental Equality with no splitting (see Terminology 2.4).*

Proof. Since $W = V[tb]_{NV[tb]}$ (with tb transcendental over V) and $Q = NW$ is an extended ideal from V , $t\bar{b} + Q \in W/Q$ is transcendental over V/N , so it follows that the image $\overline{m_e(X)} \in (W/Q)[X]$ of $m_e(X)$ is irreducible, hence $[(W/Q)[y] : (W/Q)] = e$, where $y = X + \overline{m_e(X)}(W/Q)[X]$. Therefore it follows from Proposition 2.7 that: $m_e(X)$ is irreducible in the UFD $W[X]$; $M = QU$ is a (principal) maximal ideal, where $U = W[\theta_e]$; and, $[U : W] = e$. Further, U is integral over W and Q is the unique maximal ideal in W , so every maximal ideal in U contains the maximal ideal QU , hence U is a DVR that is

a simple free integral extension domain of W and M is its maximal ideal. Therefore, by (2.4), the last statement is clear. \square

Our first theorem, Theorem 3.2, is an expanded version of Itoh's Theorem (see Theorem 1.1; note that (1.1.3) is proved in (3.2.5)).

Theorem 3.2. *Let I be a regular proper ideal in a Noetherian ring R , let b_1, \dots, b_g be regular elements in I that are a basis of I , let $(V_1, N_1), \dots, (V_n, N_n)$ be the Rees valuation rings of I , let e_1, \dots, e_n be the Rees integers of I , let e be an arbitrary common multiple of e_1, \dots, e_n , and let $f_j = \frac{e}{e_j}$ ($j = 1, \dots, n$). Also, let $\mathbf{R} = R[u, tI]$ be the Rees ring of R with respect to I , let $\mathbf{S} = \mathbf{R}[u^{\frac{1}{e}}]$, let $\mathbf{T} = \mathbf{S}' \cap R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$, and let $\mathbf{r} = u^{\frac{1}{e}}\mathbf{T}$. Then:*

(3.2.1) \mathbf{r} is a radical ideal.

(3.2.2) There is a one-to-one correspondence between the Itoh (e) -valuation rings (V^*, N^*) of I and the Rees valuation rings (W, Q) of $u\mathbf{R}$; namely, given W , if $F(u)$ is the quotient field of W , then V^* is the integral closure of W in $F(u^{\frac{1}{e}})$.

(3.2.3) Let (W, Q) and (V^*, N^*) be corresponding (as in (3.2.2)) valuation rings, and let $(V, N = \pi V)$ be the Rees valuation ring of I that corresponds (as in Proposition 2.8) to W , so $V \leq W \leq V^*$ and $NW = Q$. Assume that $V = V_j$, so $IV = N^{e_j}$ ($= N_j^{e_j}$).

Then there exists a unit $\theta \in V^*$ such that $(U, P) = (W[\theta], N^* \cap W[\theta])$ is an Itoh (e_j) -valuation ring of I that is a simple free integral extension domain of W and $P = QU$ is the maximal ideal in U , so the ramification index of U relative to W is equal to one (see Definition 2.1.6).

Also, $QU = \pi U = yU$, where $y = u^{\frac{1}{e_j}}$, and $[U : W] = e_j = [(U/P) : (W/Q)]$. Therefore W and U satisfy the Fundamental Equality with no splitting.

(3.2.4) Let $(V, N) \leq (U, P) \leq (V^*, N^*)$ (with $V = V_j$) be as in (3.2.3). Then there exists a nonunit $x \in V^*$ such that $V^* = U[x]$ is a simple free integral extension domain of U , $N^* = xV^*$ (where $x = y^{\frac{1}{f_j}}$ with $y = u^{\frac{1}{e_j}}$ (as in (3.2.3)), so $x = u^{\frac{1}{e}}$), and $PV^* = N^{*f_j}$, so the ramification index of V^* relative to U is equal to f_j . Also, $[V^* : U] = f_j$, and $V^*/N^* \cong U/P$, so U and V^* satisfy the Fundamental Equality with no splitting.

(3.2.5) $V^* = W[\theta, x]$, $N^* = xV^* = u^{\frac{1}{e}}V^*$, where θ is as in (3.2.3) and x is as in (3.2.4), and V^* is a finite free integral extension domain of W . Also, the ramification index of V^* relative to W is equal to f_j , $[(V^*/N^*) : (W/Q)] = e_j$, and $[V^* : W] = e$, so W and V^* satisfy the Fundamental Equality with no splitting.

(3.2.6) Assume that $e_1 = \dots = e_n = e$, and let (W, Q) and $U = W[\theta]$ be as in (3.2.3). Then $V^* = U$ is a simple free integral extension domain of W , $P = QU = u^{\frac{1}{e}}U$ is the maximal ideal in U , and $[U : W] = e_j = [(U/P) : (W/Q)]$. Therefore W and V^* ($= U$) satisfy the Fundamental Equality with no splitting, and $u^{\frac{1}{e}}\mathbf{T}$ is a radical ideal, where $\mathbf{T} = R[u, tI, u^{\frac{1}{e}}]' \cap R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$.

Proof. We first prove (3.2.3)–(3.2.5). For this, fix an Itoh (e) -valuation ring (V^*, N^*) of I . Then, by Definitions 1.2 and 2.1.4, there exists a (height one) associated prime ideal p^* of $u^{\frac{1}{e}}\mathbf{T}$ and a minimal prime ideal $z^* \subset p^*$ such that $V^* = (\mathbf{T}/z^*)_{(p^*/z^*)}$.

Since $\mathbf{T} \subset R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$ and \mathbf{T} and $R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$ have the same total quotient ring, it follows that $z = z^*R[u^{\frac{1}{e}}, t^{\frac{1}{e}}] \cap R$ is a minimal prime ideal in R and $z^* = zR[u^{\frac{1}{e}}, t^{\frac{1}{e}}] \cap \mathbf{T}$. Also, $z' = zR[u, t] \cap \mathbf{R}$ is a minimal prime ideal in \mathbf{R} and $z^* \cap \mathbf{R} = z'$.

Let F be the quotient field of R/z , so $F(u)$ (resp., $F(u^{\frac{1}{e}})$) is the quotient field of \mathbf{R}/z' (resp., V^* and \mathbf{T}/z^*). Also, by Definition 1.2, V^* is a Rees valuation ring of $u^{\frac{1}{e}}\mathbf{T}$, so V^* is also a Rees valuation ring of $(u^{\frac{1}{e}}\mathbf{T} + z^*)/z^*$, by Remark 2.2.3, so V^* is also a Rees valuation ring of $(u\mathbf{T} + z^*)/z^*$, by Remark 2.2.5.

Therefore $W = V^* \cap F(u)$ is a Rees valuation ring of the ideal $(u\mathbf{R} + z)/z$, by Remark 2.2.6 (and so also of $u\mathbf{R}$, by Remark 2.2.3), hence V^* is an extension of W to $F(u^{\frac{1}{e}})$.

Let V be the Rees valuation ring of I that corresponds (as in Proposition 2.8) to W , and assume that $V = V_j = (C'/z)_{(p/z)}$, where C' is the integral closure of $C = R[I/b]$ in its total quotient ring, $b \in \{b_1, \dots, b_g\}$, p is a (height one) associated prime ideal of bC' , and z is the (unique) minimal prime ideal in C' that is contained in p . Let N and Q be the maximal ideals in V and W , respectively. Then $Q = NW$ by Proposition 2.8, so

$$(3.2.7) \quad t\bar{b} + Q \in W/Q \text{ is transcendental over } V/N, \text{ where } \bar{b} = b + z.$$

Let $N = \pi V$, so, by hypothesis, $\bar{b}V = IV = N^{e_j} = \pi^{e_j}V$. Therefore $uW = \bar{b}W = IW = Q^{e_j} = \pi^{e_j}W$, so

$$(3.2.8) \quad \text{there exist units } v \in V \text{ and } w \in W \text{ such that } \bar{b} = v\pi^{e_j} \text{ and } u = w\pi^{e_j}.$$

Since $\bar{b} = u(t\bar{b})$ (see the last part of Proposition 2.8), it follows from (3.2.8) that

$$(3.2.9) \quad w = \frac{v}{t\bar{b}}.$$

Let $\theta = w^{\frac{1}{e_j}}$ (in a fixed algebraic closure $(F(u))^*$ of $F(u)$), and let $U = W[\theta]$. Then it follows from Proposition 3.1 that: U is a DVR that is a simple free integral extension domain of W ; QU is its maximal ideal; $[U : W] = e_j = [(U/(QU)) : (W/Q)]$; and, W and U satisfy the Fundamental Equality with no splitting.

Continuing with the proof of (3.2.3), we next show that $U \leq V^*$, and we first show that $\theta \in V^*$. For this, $w\pi^{e_j} = u$, by (3.2.8), $\theta = w^{\frac{1}{e_j}}$, and $u^{\frac{1}{e}} \in V^*$, so $\theta\pi = (w\pi^{e_j})^{\frac{1}{e_j}} = u^{\frac{1}{e_j}} = (u^{\frac{1}{e}})^{f_j} \in V^*$, hence $\theta\pi = u^{\frac{1}{e_j}} \in V^*$. Further, $\pi \in Q \subset V^*$, so θ is in the quotient field $F(u^{\frac{1}{e}})$ of V^* . Moreover, $\theta = w^{\frac{1}{e_j}}$ is integral over W and is a unit, $W \leq V^*$, and V^* is integrally closed in $F(u^{\frac{1}{e}})$, so $\theta \in V^*$ and is a unit in V^* . Therefore $U = W[\theta] \leq V^*$, and $u^{\frac{1}{e_j}}U = \theta\pi U = \pi U = QU = N^* \cap U$ is the maximal ideal in U . Thus, to complete the proof of (3.2.3), it remains to show that U is an Itoh (e_j)-valuation ring of I .

For this, $U = W[\theta] \supseteq W[u^{\frac{1}{e_j}}] \supseteq \mathbf{R}[u^{\frac{1}{e_j}}] \supseteq \mathbf{T}_{e_j} + \mathbf{R}'$, where $\mathbf{T}_{e_j} = R[u, tI, u^{\frac{1}{e_j}}]' \cap R[u^{\frac{1}{e_j}}, t^{\frac{1}{e_j}}]$, and $P \cap \mathbf{R}' = (P \cap W) \cap \mathbf{R}' = Q \cap \mathbf{R}'$ is a height one associated prime

ideal of $u\mathbf{R}'$, so $P \cap \mathbf{T}_{e_j} = q$, say, is a height one associated prime ideal of $u^{\frac{1}{e_j}} \mathbf{T}_{e_j}$ (since $u^{\frac{1}{e_j}} \in P$).

Therefore $U \geq (\mathbf{T}_{e_j})_q$, $(\mathbf{T}_{e_j})_q$ is an Itoh (e_j) -valuation ring of I , and U and $(\mathbf{T}_{e_j})_q$ are DVRs with the same quotient field, so $U = (\mathbf{T}_{e_j})_q$ is an Itoh (e_j) -valuation ring of I . Thus (3.2.3) holds.

For (3.2.4), let $y = u^{\frac{1}{e_j}}$, let $x = y^{\frac{1}{f_j}}$ (so $x = u^{\frac{1}{e}}$, since $e_j \cdot f_j = e$), and let $D = U[x]$. By (3.2.3), $P = yU$ is the maximal ideal in U , so it follows from Proposition 2.6 that: D is a DVR that is a simple free integral extension domain of U ; $(QU)D = P'^{f_j}$, where $P' = \pi^{\frac{1}{f_j}} D$ is the maximal ideal in D (so $D/P' \cong U/P$); $[D : U] = f_j$; and, U and D satisfy the Fundamental Equality with no splitting.

Further, since $u^{\frac{1}{e}} = x \in D$, D and V^* have the same quotient field $F(u^{\frac{1}{e}})$. Thus, V^* and D are DVRs in $F(u^{\frac{1}{e}})$, and $D \leq V^*$, so it follows that $D = V^*$ and $P' = N^*$. Therefore (3.2.4) holds.

It follows from (3.2.3) and (3.2.4) that $V^* = W[\theta, x]$ ($= W[\theta, u^{\frac{1}{e}}]$) is a finite free integral extension domain of W , that $N^* = xV^* = u^{\frac{1}{e}}V^*$, and that $[V^* : W] = e_j \cdot f_j = e$. Also, since the ramification index of U over W is equal to one, by (3.2.3), and the ramification index of $V^* = D$ over U is equal to f_j , by (3.2.4), it follows that the ramification index of V^* over W is equal to f_j . Further, $[(U/P) : (W/Q)] = e_j$, by (3.2.3), and $V^*/N^* \cong U/P$, by (3.2.4), so $[(V^*/N^*) : (W/Q)] = e_j$. Thus it follows that W and V^* satisfy the Fundamental Equality with no splitting, hence (3.2.5) holds.

Moreover, since V^* is a finite free integral extension domain of W , and since V^* is integrally closed, it follows that V^* is the integral closure of W ($= W_j$) in the quotient field $F(u^{\frac{1}{e}})$ of V^* . Therefore it has been shown that if V^* is an Itoh (e) -valuation ring of I , then: $V^* = (\mathbf{T}/z^*)_{(p^*/z^*)}$ (for some (height one) associated prime ideal p^* of $u^{\frac{1}{e}}\mathbf{T}$, where z^* is the minimal prime ideal in \mathbf{T} that is contained in p^*); $z = z^* \cap R$ and $z' = z^* \cap \mathbf{R}$ are minimal prime ideals in R and \mathbf{R} , respectively; and, if F (resp., $F(u)$) is the quotient field of R/z (resp., \mathbf{R}/z'), then V^* is the integral closure of W ($= V^* \cap F(u)$) in $F(u^{\frac{1}{e}})$, and W is a Rees valuation ring of $u\mathbf{R}$ (and of $(u\mathbf{R} + z')/z'$).

It follows that each Itoh (e) -valuation ring V^* (with quotient field $F(u^{\frac{1}{e}})$) corresponds to the Rees valuation ring W ($= V^* \cap F(u)$) of $u\mathbf{R}$. On the other hand, if W is a Rees valuation ring of $u\mathbf{R}$, then: $W = (\mathbf{R}'/z')_{(p'/z')}$ for some (height one) associated prime ideal p' of $u\mathbf{R}$, where z' is the minimal prime ideal in \mathbf{R} that is contained in p' ; $z = z' \cap R$ and $z^* = zR[u^{\frac{1}{e}}, t^{\frac{1}{e}}] \cap \mathbf{T}$ are minimal prime ideals in R and \mathbf{T} , respectively; and, if F (resp., $F(u^{\frac{1}{e}})$) is the quotient field of R/z (resp., \mathbf{T}/z^*) and W'' is the integral closure of W in $F(u^{\frac{1}{e}})$, then for each maximal ideal M in W'' , W''_M is a Rees valuation ring of $(u\mathbf{T} + z^*)/z^*$ and of $(u^{\frac{1}{e}}\mathbf{T} + z^*)/z^*$, by Remarks 2.2.6 and 2.2.5, so each V''_M is a Rees valuation ring of $u\mathbf{T}$ and of $u^{\frac{1}{e}}\mathbf{T}$, by Remark 2.2.3, hence each such W''_M is an Itoh (e) -valuation ring of I , by Definition 1.2. It therefore follows from the first part of this paragraph that W'' has exactly one maximal ideal, so the one-to-one correspondence of (3.2.2) holds.

For (3.2.1), $u^{\frac{1}{e}}V^* = N^*$, by (3.2.4). Therefore, since V^* is an arbitrary Itoh (e)-valuation ring of I , it follows from Definitions 1.2 and 2.1.5 that the Rees integers of $u^{\frac{1}{e}}\mathbf{T}$ are all equal to one. Also, since $\mathbf{T} = \mathbf{R}[u^{\frac{1}{e}}]' \cap R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$, by [9, Remarks (ii) p. 215] it follows that $u^{\frac{1}{e}}\mathbf{T} = (u^{\frac{1}{e}}\mathbf{T})_a$. Therefore it follows from Remark 2.2.2 that $\mathbf{r} = u^{\frac{1}{e}}\mathbf{T} = (u^{\frac{1}{e}}\mathbf{T})_a = \cap\{u^{\frac{1}{e}}V_j^* \cap \mathbf{T} \mid j = 1, \dots, n\} = \cap\{N_j^* \cap \mathbf{T} \mid j = 1, \dots, n\}$ is a radical ideal. Thus (3.2.1) holds, so (3.2.1)–(3.2.5) hold.

Finally, for (3.2.6), let W be a Rees valuation ring of $u\mathbf{R}$, let V^* be the corresponding (as in (3.2.2)) Itoh (e)-valuation ring of I , and let U be as in (3.2.3), so $W \leq U \leq V^*$. Then there exists $j \in \{1, \dots, n\}$ such that $[U : W] = e_j$ (by (3.2.3)) = e (by the hypothesis in (3.2.6)) = $[V^* : W]$ (by (3.2.5)), so $V^* = U$ is a simple free integral extension domain of W . Also, $P = N^* = u^{\frac{1}{e}}V^*$ (by (3.2.5)) = $u^{\frac{1}{e}}U$. Since this holds for all Itoh (e)-valuation rings (U, P) of I , $u^{\frac{1}{e}}\mathbf{T}$ is a radical ideal, hence (3.2.6) holds. \square

Concerning Theorem 3.2.6, it is shown in Corollary 4.12 below that there always exists a finite integral extension ring A of R such that the Rees integers of IA are all equal to e , where e is an arbitrary common multiple of the Rees integers e_1, \dots, e_n of I , so Theorem 3.2.6 directly applies to IA in place of I .

Remark 3.3. Let (V, N) be a Rees valuation ring of I and assume that $IV = N^k$. Let (W, Q) be the Rees valuation ring of $uR[u, tI]$ that corresponds (as in Proposition 2.8) to V , and let h be an arbitrary positive integer. Then it follows from Theorem 3.2.3 and 3.2.4 that: $U = W[u^{\frac{1}{k}}]'$ and $D = W[u^{\frac{1}{hk}}]' = U[(u^{\frac{1}{k}})^{\frac{1}{h}}]'$ are DVRs that are finite free integral extension domains of W ; the maximal ideal of U (resp., D) is $P = u^{\frac{1}{k}}U$ (resp., $M = u^{\frac{1}{hk}}D$); the ramification index of D relative to U (resp., U relative to W) is equal to h (resp., 1); $[D : U] = h$ and $[U : W] = k$; and, $D/M \cong U/P$ and $[(U/P) : (W/Q)] = k$. It therefore follows that W and U (resp., W and D , U and D) satisfy the Fundamental Equality with no splitting. Also, it follows from Theorem 3.2.3 (resp., 3.2.4) that U (resp., D) is an Itoh (k)-valuation (resp., (hk)-valuation) ring of I .

Remark 3.4. The following result is called the Theorem of Independence of Valuations, and it is proved in [7, (11.11)]: Let $(V_1, N_1), \dots, (V_n, N_n)$ be valuation rings with the same quotient field F , and assume there are no containment relations among the V_j . Let $\mathbf{V} = V_1 \cap \dots \cap V_n$ and let $P_j = N_j \cap \mathbf{V}$ ($j = 1, \dots, n$). Then P_1, \dots, P_n are the maximal ideals in \mathbf{V} and $V_j = \mathbf{V}_{P_j}$ for $j = 1, \dots, n$.

Corollary 3.5. Let $I = (b_1, \dots, b_g)R$ be a nonzero proper ideal in a Noetherian integral domain R , let F be the quotient field of R , let $\mathbf{R} = R[u, tI]$, let $(W_1, Q_1), \dots, (W_n, Q_n)$ be the Rees valuation rings of $u\mathbf{R}$, let e be a positive common multiple of the Rees integers e_1, \dots, e_n of $u\mathbf{R}$, let $\mathbf{W} = W_1 \cap \dots \cap W_n$, and let $\mathbf{D} = \mathbf{W}[x]$, where $x = u^{\frac{1}{e}}$ in a fixed algebraic closure $F(u)^*$ of $F(u)$. Then:

- (1) \mathbf{W} is a semi-local Dedekind domain with exactly n maximal ideals $P_j = Q_j \cap \mathbf{W}$, $j = 1, \dots, n$.

- (2) \mathbf{D} is a simple free integral extension domain of \mathbf{W} of rank e , and \mathbf{D} has exactly n maximal ideals $M_j = (P_j, x)\mathbf{D}$.
- (3) There exist distinct elements $\theta_1, \dots, \theta_n$ in the quotient field $F(x)$ of \mathbf{D} such that:
- (3.1) \mathbf{D}' is the intersection of the Itoh (e)-valuation rings $(V_1^*, N_1^*), \dots, (V_n^*, N_n^*)$ of I , where $V_j^* = W_j[\theta_j, x] = \mathbf{D}'_{\mathbb{M}_j}$, where $\mathbb{M}_j = N_j^* \cap \mathbf{D}'$.
- (3.2) $\mathbf{D}' = \mathbf{D}[\theta_1, \dots, \theta_n]$ is a semi-local Dedekind domain that is a finite integral extension domain of \mathbf{D} , and \mathbf{D}' has exactly n maximal ideals $\mathbb{M}_j = N_j^* \cap \mathbf{D}' = (P_j, x, \theta_1, \dots, \theta_{j-1}, \theta_{j+1}, \dots, \theta_n)\mathbf{D}'$.
- (3.3) The Jacobson radical of \mathbf{D}' is $x\mathbf{D}'$.
- (4) Assume that $e_1 = \dots = e_n = e$ and that there exists $b \in I$ such that $bW_j = IW_j$ for $j = 1, \dots, n$. Then, with \mathbf{D}' , as in (3.1), and $Q_j = \pi_j W_j$ for $j = 1, \dots, n$, $\frac{x}{\pi_1 \dots \pi_n}$ is a unit in \mathbf{D}' , $\mathbf{D}' = \mathbf{D}[\frac{x}{\pi_1 \dots \pi_n}] = \mathbf{W}[\frac{x}{\pi_1 \dots \pi_n}]$, and $x\mathbf{D}'$ is the Jacobson radical of \mathbf{D}' .

Proof. (1) and part of (3.1) follow immediately from the Independence of Valuations Theorem (see Remark 3.4). The first part of (2) is clear. Also, each M_j is a maximal ideal in \mathbf{D} , since $\mathbf{D}/M_j \cong W_j/Q_j$. Further, u (resp., x) is in the Jacobson radical of \mathbf{W} (resp., \mathbf{D}), $x\mathbf{D} \cap \mathbf{W} = u\mathbf{W}$ (since \mathbf{D} is a free \mathbf{W} -algebra), and $\mathbf{D}/(x\mathbf{D}) \cong \mathbf{W}/(u\mathbf{W})$, so it follows that \mathbf{W} , $\mathbf{W}/(u\mathbf{W})$, \mathbf{D} , and $\mathbf{D}/(x\mathbf{D})$ each have exactly n maximal ideals, so (2) holds.

For (3), for $j = 1, \dots, n$, let (V_j, N_j) be the Rees valuation ring of I that corresponds (as in Proposition 2.8) to (W_j, Q_j) (so there exists $b_{\sigma(j)}$ (where $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, g\}$) in $\{b_1, \dots, b_g\}$ such that $V_j = (R[\frac{I}{b_{\sigma(j)}}])'_{p_j}$, for some (height one) prime ideal p_j in $R[\frac{I}{b_{\sigma(j)}}]'$), so $W_j = V_j[t b_{\sigma(j)}]_{N_j V_j[t b_{\sigma(j)}]}$, $W_j \cap F = V_j$, and $Q_j \cap F = N_j$.

Let $\mathbf{V} = V_1 \cap \dots \cap V_n$, and for $j = 1, \dots, n$, let $q_j = N_j \cap \mathbf{V}$. Then $\mathbf{V} = \mathbf{W} \cap F$ and $\mathbf{W} = \mathbf{V}[u]_S$, where $S = \mathbf{V}[u] - (q_1 \mathbf{V}[u] \cup \dots \cup q_n \mathbf{V}[u])$, and $q_j = (Q_j \cap F) \cap \mathbf{V} = (Q_j \cap \mathbf{W}) \cap \mathbf{V} = P_j \cap \mathbf{V}$, for $j = 1, \dots, n$. Therefore it follows from the Independence of Valuations Theorem (see Remark 3.4) that \mathbf{V} is a semi-local Dedekind domain with exactly n maximal ideals q_j ($j = 1, \dots, n$). Thus \mathbf{V} is a Principal Ideal Domain, by [12, Theorem 16, p. 278], so for $j = 1, \dots, n$, there exists $\pi_j \in q_j$ such that $q_j = \pi_j \mathbf{V}$, so: $\pi_j V_j = (N_j \cap \mathbf{V}) \mathbf{V}_{q_j} = N_j$; $\pi_j \mathbf{W} = q_j \mathbf{W} = (P_j \cap \mathbf{V}) \mathbf{W} = P_j$; and, $\pi_j W_j = q_j \mathbf{W}_{Q_j \cap \mathbf{W}} = P_j W_j = Q_j$. Then, since the Rees integers of $u\mathbf{R}$ are e_1, \dots, e_n (by hypothesis), $u\mathbf{W} = P_1^{e_1} \cap \dots \cap P_n^{e_n} = P_1^{e_1} \dots P_n^{e_n} = \pi_1^{e_1} \dots \pi_n^{e_n} \mathbf{W}$, so there exists a unit $w \in \mathbf{W}$ such that $u = w\pi_1^{e_1} \dots \pi_n^{e_n}$, hence

- (*1) there exists a unit $w_j \in W_j$ such that $u = w_j \pi_j^{e_j}$ (in $W_j = \mathbf{W}_{P_j}$), for $j = 1, \dots, n$,

where

$$(*2) \quad w_j = \frac{u}{\pi_j^{e_j}} = w \prod_{i \neq j} \pi_i^{e_i} \in \left(\bigcap_{i \neq j} P_i \right) - P_j, \text{ for } j = 1, \dots, n.$$

Also, since e_j is the Rees integer of I with respect to V_j and $IV_j = b_{\sigma(j)}V_j$, there exists a unit $v_j \in V_j$ such that

$$(*3) \quad b_{\sigma(j)} = v_j \pi_j^{e_j} \text{ in } V_j, \text{ for } j = 1, \dots, n.$$

Therefore, since $b_{\sigma(j)} = u(tb_{\sigma(j)})$ (in $\mathbf{R} \subseteq W_j$), it follows from (*3) and (*1) that $v_j \pi_j^{e_j} = b_{\sigma(j)} = u(tb_{\sigma(j)}) = w_j \pi_j^{e_j} (tb_{\sigma(j)})$, hence

$$(*4) \quad w_j = \frac{v_j}{tb_{\sigma(j)}} \text{ and } w_j + P_j \text{ is transcendental over } \mathbf{V}/(P_j \cap \mathbf{V}) \text{ for } j = 1, \dots, n.$$

Therefore

$$(*5) \quad \overline{m}_{e_j}(X) = X^{e_j} - (w_j + P_j) \text{ is irreducible in } (\mathbf{W}/P_j)[X] \text{ for } j = 1, \dots, n.$$

For $j = 1, \dots, n$, let $\theta_j = w_j \frac{1}{e_j}$ in the (fixed) algebraic closure $F(u)^*$ of $F(u)$. Then it is shown in [Theorem 3.2.3–3.2.5](#) that, for the Itoh (e)-valuation ring V_j^* of I ,

$$(*6) \quad V_j^* = W_j[\theta_j, x] \text{ and } \theta_j \text{ is a unit in } V_j^*, \text{ for } j = 1, \dots, n.$$

Also, $V_1^* \cap \dots \cap V_n^*$ is the integral closure \mathbf{W}^* of \mathbf{W} in $F(x)$, by [Theorem 3.2.2](#), and $x = u \frac{1}{e} \in \mathbf{W}^*$ (since $u \in \mathbf{W}$), so it follows that $\mathbf{W}^* = \mathbf{W}[x]' = \mathbf{D}'$.

Further, for $j = 1, \dots, n$, $\theta_j = w_j \frac{1}{e_j} \in \mathbf{W}^*$, by (*2) and (*6), and by (*2)

$$\theta_1, \dots, \theta_{j-1}, \theta_{j+1}, \dots, \theta_n \in N_j^* \cap \mathbf{W}^* = N_j^* \cap \mathbf{D}' = \mathbb{M}_j, \text{ for } j = 1, \dots, n.$$

Hence by (*2) and (*6) we see that

$$(*7) \quad \text{for } i = 1, \dots, n, \theta_i \in \cap \{N_j^* \cap \mathbf{D}' \mid j = 1, \dots, i-1, i+1, \dots, n\} - (N_i^* \cap \mathbf{D}').$$

Let $\mathbf{E} = \mathbf{D}[\theta_1, \dots, \theta_n]$. Then, for $j = 1, \dots, n$, $x \in \mathbf{D} \subseteq \mathbf{E} \subseteq \mathbf{W}^*$ (by (*2) and (*6)) $\subseteq V_j^* = W_j[\theta_j, x]$. Therefore $W_j = \mathbf{W}_{N_j^* \cap \mathbf{W}} \subseteq \mathbf{D}_{N_j^* \cap \mathbf{D}} \subseteq \mathbf{E}_{N_j^* \cap \mathbf{E}} \subseteq \mathbf{W}^*_{N_j^* \cap \mathbf{W}^*} = V_j^* = (W_j[x])[\theta_j] \subseteq \mathbf{E}_{N_j^* \cap \mathbf{E}}$, so $V_j^* = \mathbf{E}_{N_j^* \cap \mathbf{E}}$, for $j = 1, \dots, n$. Also, \mathbf{D} and \mathbf{W}^* have exactly n maximal ideals, so it follows from integral dependence that \mathbf{E} has exactly n maximal ideals. Further, for each integral domain A , $A = \cap \{A_M \mid M \text{ is a maximal ideal in } A\}$, by [\[7, \(33.9\)\]](#). Therefore it follows that $\mathbf{E} = \mathbf{D}[\theta_1, \dots, \theta_n] = V_1^* \cap \dots \cap V_n^* = \mathbf{D}'$. Thus (3.1) and (3.2) hold.

(3.3) follows immediately from [Theorem 3.2.4](#).

Finally, for (4), let (V_j, N_j) be the Rees valuation ring of I that corresponds (as in [Proposition 2.8](#)) to (W_j, Q_j) . Then it follows from the hypothesis on b that, for $j = 1, \dots, n$, $W_j = V_j[tb]_{N_j V_j[tb]}$ and $Q_j = N_j W_j$.

Let $\mathbf{V} = V_1 \cap \dots \cap V_n$, so by [Remark 3.4](#) \mathbf{V} is a semi-local Dedekind domain with exactly n maximal ideals $N_j \cap \mathbf{V}$ ($= P_j \cap \mathbf{V}$). Thus \mathbf{V} is a Principal Ideal Domain, by [\[12, Theorem 16, p. 278\]](#), so for $j = 1, \dots, n$ there exists $\pi_j \in N_j \cap \mathbf{V}$ such that $\pi_j \mathbf{V} = N_j \cap \mathbf{V}$.

Let $\alpha = \pi_1 \cdots \pi_n$, let $\mathbf{N} = N_1 \cap \cdots \cap N_n$, and let $\mathbf{Q} = Q_1 \cap \cdots \cap Q_n$. Then, since the Rees integers of I are all equal to e , it follows that

$$(3.5.4.1) \quad I\mathbf{V} = b\mathbf{V} = \mathbf{N}^e = \alpha^e \mathbf{V} \text{ and } I\mathbf{W} = u\mathbf{W} = \mathbf{Q}^e = \alpha^e \mathbf{W}.$$

It follows from (3.5.4.1) that

$$(3.5.4.2) \quad \text{there exist units } v \in \mathbf{V} \text{ and } w \in \mathbf{W} \text{ such that } b = v\alpha^e \in \mathbf{V} \text{ and } u = w\alpha^e \in \mathbf{W}.$$

Since $b = u(tb)$ in $\mathbf{R} \subseteq \mathbf{W}$, and since $Q_j = N_j V_j [tb]_{N_j V_j [tb]}$ for $j = 1, \dots, n$, it follows from (3.5.4.2) that

$$(3.5.4.3) \quad w = \frac{v}{tb} \text{ and } w + P_j \text{ is transcendental over } \mathbf{V}/(P_j \cap \mathbf{V}) \text{ for } j = 1, \dots, n.$$

Therefore

$$(3.5.4.4) \quad \overline{m}_e(X) = X^e - (w + P_j) \text{ is irreducible in } (\mathbf{W}/P_j)[X] \text{ for } j = 1, \dots, n.$$

Let $\theta = w^{\frac{1}{e}}$ in the (fixed) algebraic closure $F(u)^*$ of $F(u)$. Then it follows from Proposition 2.7, together with (3.5.4.4), that: $m_e(X)$ is irreducible in the UFD $\mathbf{W}[X]$; for $j = 1, \dots, n$, $\mathbb{M}_j = P_j \mathbf{E}$ is a (principal) maximal ideal, where $\mathbf{E} = \mathbf{W}[\theta]$; and, $[\mathbf{E} : \mathbf{W}] = e$.

Also, \mathbf{E} is integral over \mathbf{W} , so it follows that $\mathbb{M}_1, \dots, \mathbb{M}_n$ are the only nonzero prime ideals in \mathbf{E} , hence \mathbf{E} is a semi-local Dedekind domain that is a simple free integral extension domain of \mathbf{W} and $\mathbf{Q}\mathbf{E} = (Q_1 \cap \cdots \cap Q_n)\mathbf{E} = (\mathbb{M}_1 \cap \cdots \cap \mathbb{M}_n)\mathbf{E} = \pi_1 \cdots \pi_n \mathbf{E} = \alpha \mathbf{E}$ is its Jacobson radical J . Therefore $\mathbf{D}' = \mathbf{E} = \mathbf{W}[\theta] = \mathbf{D}[\theta]$, and since $w = \frac{u}{\alpha^e}$ is a unit in \mathbf{W} , it follows that $\theta = w^{\frac{1}{e}} = \frac{x}{\alpha}$ is a unit in \mathbf{D}' and that $J = \alpha \mathbf{E} = x \mathbf{E}$. \square

The next remark lists several well known facts concerning finite field extensions and ramification.

Remark 3.6. Let $(U_1, P_1) \leq (U_2, P_2) \leq (U_3, P_3)$ be DVRs such that U_3 is a finite integral extension of U_1 . Then:

- (1) $[(U_3)_{(0)} : (U_1)_{(0)}] = [(U_3)_{(0)} : (U_2)_{(0)}] \cdot [(U_2)_{(0)} : (U_1)_{(0)}]$.
- (2) $[(U_3/P_3) : (U_1/P_1)] = [(U_3/P_3) : (U_2/P_2)] \cdot [(U_2/P_2) : (U_1/P_1)]$.

Also, if we let $r_{3,1}$ (resp., $r_{3,2}$, $r_{2,1}$) denote the ramification index of U_3 relative to U_1 (resp., U_3 relative to U_2 , U_2 relative to U_1), then:

- (3) $r_{3,1} = r_{3,2} \cdot r_{2,1}$.

Further, if U_1 and U_3 satisfy the Fundamental Equality with no splitting, then

- (4) $r_{3,1} \cdot [(U_3/P_3) : (U_1/P_1)] = [(U_3)_{(0)} : (U_1)_{(0)}]$.

It then follows from (1)–(4) and the Fundamental Inequality, (2.3), that:

- (5) $r_{3,2} \cdot [(U_3/P_3) : (U_2/P_2)] = [(U_3)_{(0)} : (U_2)_{(0)}]$.
- (6) $r_{2,1} \cdot [(U_2/P_2) : (U_1/P_1)] = [(U_2)_{(0)} : (U_1)_{(0)}]$.

(7) Hence, if U_1 and U_3 satisfy the Fundamental Equality with no splitting, then both $(U_1$ and $U_2)$ and $(U_2$ and $U_3)$ satisfy the Fundamental Equality with no splitting.

The next proposition shows that [Theorem 3.2.2](#) holds for all integers greater than one.

Proposition 3.7. *With the notation of [Theorem 3.2](#), let $k \geq 2$ be an arbitrary integer, let $\mathbf{S}_k = \mathbf{R}[u^{\frac{1}{k}}]$, and let $\mathbf{T}_k = \mathbf{S}_k' \cap R[u^{\frac{1}{k}}, t^{\frac{1}{k}}]$. Then:*

- (1) *There exists a one-to-one correspondence between the Itoh (k) -valuation rings (U, P) of I and the Rees valuation rings (W, Q) of $u\mathbf{R}$; namely, given W , if $F(u)$ is the quotient field of W , then U is the integral closure of W in $F(u^{\frac{1}{k}})$.*
- (2) *Let W and U be corresponding (as in (1)). Then U is a finite integral extension domain of W , and W and U satisfy the Fundamental Equality with no splitting,*

Proof. Let m be the least common multiple of e_1, \dots, e_n , and let $e = k \cdot m$. Then it is clear that $\mathbf{R} \subseteq \mathbf{T}_k \subseteq \mathbf{T}_e = \mathbf{R}[u^{\frac{1}{e}}]' \cap R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$ and that \mathbf{T}_e is integral over \mathbf{R} . Also, by [Theorem 3.2.2](#) there exists a one-to-one correspondence between the Rees valuation rings (W, Q) of $u\mathbf{R}$ and the Itoh (e) -valuation rings (V^*, N^*) of I , and V^* is the integral closure of W in the quotient field of V^* . It follows from this, and integral dependence, that (1) holds.

For (2), it follows from the proof of (1) that if W and V^* are as in (1), then there exists an Itoh (k) -valuation ring U of I such that $W \leq U \leq V^*$. Also, V^* is a finite free integral extension domain of W , by [Theorem 3.2.5](#), hence U is a finite integral extension domain of W . Further, W and V^* satisfy the Fundamental Equality with no splitting, by [Theorem 3.2.5](#), so it follows from [Remark 3.6.7](#) that (2) holds. \square

Terminology 3.8. If (W, Q) is a Rees valuation ring of $uR[u, tI]$ and (U, P) is the corresponding (as in [Proposition 3.7.1](#)) Itoh (k) -valuation ring of I (so $U \supseteq W$ and U is the integral closure of W in the quotient field $F(u^{\frac{1}{k}})$ of U), then we say that (U, P) is the Itoh (k) -valuation ring of I that corresponds to (W, Q) . Also, if (V, N) is the Rees valuation ring of I that corresponds (as in [Proposition 2.8](#)) to (W, Q) , then we say that (U, P) is the Itoh (k) -valuation ring of I that corresponds to (V, N) .

Proposition 3.9. *Let I be a regular proper ideal in a Noetherian ring R , let (V, N) be a Rees valuation ring of I , let (W, Q) be the Rees valuation ring of $uR[u, tI]$ that corresponds (as in [Proposition 2.8](#)) to (V, N) , and let (U_k, P_k) be the Itoh (k) -valuation ring of I that corresponds to (V, N) (see (3.8)), where $k \geq 2$ is an arbitrary integer. Assume that $IV = N^e$. Then the following hold:*

- (1) *If e is a multiple of k , then $P_k = NU_k$ and $[(U_k)_{(0)} : W_{(0)}] = k = [(U_k/P_k) : (W/Q)]$.*
- (2) *If e and k are relatively prime, then $NU_k = P_k^k$, $[(U_k)_{(0)} : W_{(0)}] = k$, and $U_k/P_k \cong W/Q$.*

- (3) If the greatest common divisor of e and k is d , and if $c \in \mathbb{N}_{>0}$ is such that $cd = k$, then $NU_k = P_k^c$, $[(U_k)_{(0)} : W_{(0)}] = k$, and $[(U_k/P_k) : (W/Q)] = d$.

Proof. For (1), let (U_e, P_e) be the Itoh (e)-valuation ring of I that corresponds to (V, N) . Then it follows from Remark 3.3 that $P_e = NU_e$ and $[(U_e)_{(0)} : W_{(0)}] = e = [(U_e/P_e) : (W/Q)]$. Also, it is clear that $W \leq U_k \leq U_e$ and that $[(U_k)_{(0)} : W_{(0)}] = k$. Item (1) readily follows from this, together with Remark 3.6.1–3.6.3.

For (2), let (U_{ke}, P_{ke}) be the Itoh (ke)-valuation ring of I that corresponds to (V, N) . Then it follows from Remark 3.3 that $P_{ke}^k = NU_{ke}$, $[(U_{ke})_{(0)} : W_{(0)}] = ke$, and $[(U_{ke}/P_{ke}) : (W/Q)] = e$. Also, it is clear that $W \leq U_k \leq U_{ke}$ and that $[(U_k)_{(0)} : W_{(0)}] = k$. Since e and k are relatively prime, (2) readily follows from this, together with Remark 3.6.2, 3.6.3, and 3.6.7.

For (3), let (U_c, P_c) (resp., (U_d, P_d)) be the Itoh (c)-valuation (resp., (d)-valuation) ring of I that corresponds to (V, N) . Since $cd = k$, it follows that $W \leq U_c \leq U_k$ and $W \leq U_d \leq U_k$. Since e is a multiple of d , it follows from (1) that

$$(a) \quad QU_d = P_d \text{ and } [(U_d)_{(0)} : W_{(0)}] = d = [(U_d/P_d) : (W/Q)].$$

Since c and e are relatively prime, it follows from (2) that

$$(b) \quad QU_c = P_c^c \text{ and } U_c/P_c \cong W/Q.$$

It follows from (a), (b), and Remark 3.6.2, 3.6.3, and 3.6.7 that (3) holds. \square

Proposition 3.10 gives several equivalences to Property 3.2.1 of Theorem 3.2 and also shows that the hypothesis of Theorem 3.2 is necessary for 3.2.1.

Proposition 3.10. Let I be a regular proper ideal in a Noetherian ring R , let $\mathbf{R} = R[u, tI]$ be the Rees ring of R with respect to I , let $k \geq 2$ be an arbitrary integer, let $\mathbf{S}_k = \mathbf{R}[u^{\frac{1}{k}}]$, and let $\mathbf{T}_k = \mathbf{S}_k' \cap R[u^{\frac{1}{k}}, t^{\frac{1}{k}}]$. Then the following statements are equivalent:

- (1) $u^{\frac{1}{k}}\mathbf{T}_k$ is a radical ideal.
- (2) The Rees integers of $u^{\frac{1}{k}}\mathbf{T}_k$ are all equal to one.
- (3) The Rees integers of $(u^{\frac{1}{k}}\mathbf{S}_k)_a$ are all equal to one.
- (4) k is a common multiple of the Rees integers of I .

Proof. A primary decomposition of the regular proper principal radical ideal $u^{\frac{1}{k}}\mathbf{T}_k$, together with Remark 2.2.4, shows that (1) \Rightarrow (2), and the next to last paragraph of the proof of Theorem 3.2 shows that (2) \Rightarrow (1).

Also, (2) \Leftrightarrow (3), by Remark 2.2.5, since $\mathbf{S}_k \subseteq \mathbf{T}_k \subseteq \mathbf{S}_k'$.

Assume that (2) holds and let $(V_1^*, N_1^*), \dots, (V_n^*, N_n^*)$ be the Itoh (k)-valuation rings of I , so $u^{\frac{1}{k}}V_j^* = N_j^*$ for $j = 1, \dots, n$. For $j = 1, \dots, n$, let $uW_j = Q_j^{e_j}$, where $\mathbf{RV}(uR[u, tI]) = \{(W_j, Q_j) \mid j = 1, \dots, n\}$. Suppose that k is not a multiple of e_j for

some j , let d be the greatest common divisor of k and e_j , and let $c \geq 1$ and $h > 1$ be integers such that $cd = k$ and $hd = e_j$. Then it follows from Proposition 3.9(3) (with (W_j, Q_j) (resp., (V_j^*, N_j^*)) in place of (W, Q) (resp., (U_k, P_k))) that $Q_j V_j^* = (N_j^*)^c$. Since $uW_j = Q_j^{e_j}$, it follows that $u^{\frac{1}{k}} V_j^* = (N_j^*)^{\frac{ce_j}{k}} = (N_j^*)^h$. However, $h > 1$, and this contradicts (2). Therefore the supposition that k is not a multiple of e_j leads to a contradiction, hence (2) \Rightarrow (4).

Finally, (4) \Rightarrow (1) by Theorem 3.2.1. \square

Corollary 3.11. *Let I be a regular proper ideal in a Noetherian ring R , let $\mathbf{R} = R[u, tI]$ be the Rees ring of R with respect to I , and for each integer $k \geq 2$ let $\mathbf{S}_k = \mathbf{R}[u^{\frac{1}{k}}]$ and let $\mathbf{T}_k = \mathbf{S}_k' \cap R[u^{\frac{1}{k}}, t^{\frac{1}{k}}]$. Then the following statements are equivalent:*

- (1) *The Rees integers of I are all equal to one.*
- (2) *For all integers $k \geq 2$, the ideal $u^{\frac{1}{k}} \mathbf{T}_k$ is a radical ideal.*

Proof. This follows immediately from Proposition 3.10(1) \Leftrightarrow (4). \square

4. A related theorem

In this section, we first prove an expanded version of Corollary 3.5 and Theorem 3.2.6, and we then prove a closely related and more general theorem.

The next theorem is an expanded version of Corollary 3.5 and Theorem 3.2.6. One of the main reasons for including this theorem is to show that it displays, except for separability, a realization (see Definition 4.3 below) of a powerful classical method. This is explained somewhat more fully in Remark 4.4 below.

Theorem 4.1. *Let I be a nonzero proper ideal in a Noetherian integral domain R , let F be the quotient field of R , let $\mathbf{R} = R[u, tI]$, let $(W_1, Q_1), \dots, (W_n, Q_n)$ be the Rees valuation rings of $u\mathbf{R}$, let e be a positive common multiple of the Rees integers e_1, \dots, e_n of $u\mathbf{R}$, let $\mathbf{W} = W_1 \cap \dots \cap W_n$, let $\mathbf{Q} = Q_1 \cap \dots \cap Q_n$, and for $j = 1, \dots, n$, let $P_j = Q_j \cap \mathbf{W}$, so \mathbf{W} is a semi-local Dedekind domain, \mathbf{Q} is its Jacobson radical, and the ideals P_1, \dots, P_n are the maximal ideals in \mathbf{W} . Then there exists an integral domain \mathbf{E} with an ideal J such that:*

(4.1.1) $\mathbf{E} = \mathbf{W}[x]'$ is a semi-local Dedekind domain that is a finite integral extension domain of \mathbf{W} , where $x = u^{\frac{1}{e}}$ in a fixed algebraic closure $F(u)^*$ of $F(u)$.

(4.1.2) $[\mathbf{E}_{(0)} : \mathbf{W}_{(0)}] = e$.

(4.1.3) $J = x\mathbf{E}$ is the Jacobson radical of \mathbf{E} , \mathbf{E} has exactly n maximal ideals $\mathbb{M}_1, \dots, \mathbb{M}_n$, and $\mathbb{M}_j \cap \mathbf{W} = P_j$ for $j = 1, \dots, n$.

(4.1.4) $[(\mathbf{E}/\mathbb{M}_j) : (\mathbf{W}/P_j)] = e_j$ for $j = 1, \dots, n$.

(4.1.5) $u\mathbf{E} = J^e$.

(4.1.6) The Rees integers of $I\mathbf{E} = u\mathbf{E}$ are all equal to e .

(4.1.7) If $e_1 = \cdots = e_n = e$ and if there exists $b \in I$ such that $bV = IV$ for each Rees valuation ring (V, N) of I , then there exists a unit $\theta \in \mathbf{E}$ such that $\mathbf{E} = \mathbf{W}[\theta]$, so \mathbf{E} is a simple free integral extension domain of \mathbf{W} of rank e .

(4.1.8) $\mathbf{E} = \mathbf{T}_{S'}$, where $\mathbf{T} = R[u, tI, u^{\frac{1}{e}}]' \cap R[u^{\frac{1}{e}}, t^{\frac{1}{e}}]$ and $S' = \mathbf{T} - \cup\{q \mid q \text{ is a (height one) associated prime ideal of } u^{\frac{1}{e}}\mathbf{T}\}$.

Proof. It is shown in Corollary 3.5(1) that: \mathbf{W} is a semi-local Dedekind domain; \mathbf{Q} is its Jacobson radical, and, the ideals P_1, \dots, P_n are the maximal ideals in \mathbf{W} .

(4.1.1) follows from Corollary 3.5(1) and Corollary 3.5(3.2).

(Note: \mathbf{E} is frequently denoted by \mathbf{D}' in Corollary 3.5 and its proof.)

(4.1.2) follows from Corollary 3.5(2).

(4.1.3) is proved in Corollary 3.5(3.3) and Corollary 3.5(3.2).

For (4.1.4), it is shown in Corollary 3.5(3) and 3.5(3.2) that there exist $\theta_1, \dots, \theta_n \in \mathbf{E}$ such that $\mathbf{E} = \mathbf{W}[x, \theta_1, \dots, \theta_n]$ and $\mathbb{M}_j = (P_j, x, \theta_1, \dots, \theta_{j-1}, \theta_{j+1}, \dots, \theta_n)\mathbf{E}$.

Therefore $\mathbf{E}/\mathbb{M}_j \cong \mathbf{W}[\theta_j]/P_j \cong (W_j/Q_j)[\theta_j + P_j]$, and $V_j^*/N_j^* = W_j[x, \theta_j]/N_j^* \cong (W_j/Q_j)[\theta_j + P_j]$, so $\mathbf{E}/\mathbb{M}_j \cong V_j^*/N_j^*$ for $j = 1, \dots, n$.

Also, $\mathbf{W}/P_j \cong W_j/Q_j$ for $j = 1, \dots, n$. Therefore (4.1.4) follows from Theorem 3.2.5.

Since $I\mathbf{E} = u\mathbf{E}$, (4.1.5) follows from Corollary 3.5(3.3), and then (4.1.6) follows from (4.1.5) and Definition 2.1.5. (4.1.7) is proved in Corollary 3.5(4).

Finally, for (4.1.8), $\mathbf{E} = \mathbf{W}[x]'$ is the intersection of the Itoh (e)-valuation rings of I , by Corollary 3.5(3.1). Also, $\mathbf{T}_{S'}$ is the intersection of the Itoh (e)-valuation rings of I , by Definition 1.2, so $\mathbf{E} = \mathbf{T}_{S'}$, so (4.1.8) holds. \square

We next consider a powerful classical theorem of Krull, and to state the theorem, we use the following terminology of Gilmer in [3].

Definition 4.2. Let $(V_1, N_1), \dots, (V_n, N_n)$ be distinct DVRs of a field F and for $j = 1, \dots, n$, let $K_j = V_j/N_j$ denote the residue field of V_j . Let m be a positive integer. By an **m-consistent system for** $\{V_1, \dots, V_n\}$, we mean a collection of sets $\mathbf{S} = \{S_1, \dots, S_n\}$ satisfying the following conditions:

- (1) For each j , $S_j = \{(K_{j,i}, f_{j,i}, e_{j,i}) \mid i = 1, \dots, s_j\}$, where $K_{j,i}$ is a simple algebraic field extension of K_j with $f_{j,i} = [K_{j,i} : K_j]$, and $s_j, e_{j,i} \in \mathbb{N}_{>0}$.
- (2) For each j , the sum $\sum_{i=1}^{s_j} e_{j,i} f_{j,i} = m$.

Definition 4.3. The m -consistent system \mathbf{S} as in Definition 4.2 is said to be **realizable** if there exists a separable algebraic extension field L of F such that:

- (1) $[L : F] = m$.
- (2) For $j = 1, \dots, n$, V_j has exactly s_j extensions $V_{j,1}, \dots, V_{j,s_j}$ to L .
- (3) For $j = 1, \dots, n$, the residue field of $V_{j,i}$ is $K_{j,i}$ -isomorphic to $K_{j,i}$, and the ramification index of $V_{j,i}$ relative to V_j is equal to $e_{j,i}$ (so $N_j V_{j,i} = N_{j,i}^{e_{j,i}}$).

If \mathbf{S} and L are as above, we say the field L **realizes** \mathbf{S} or that L is a **realization** of \mathbf{S} .

Remark 4.4. (4.4.1) With Definition 4.3 in mind, and with the notation of Theorem 4.1, the semi-local Dedekind domain $\mathbf{E} = \mathbf{W}[x]'$ (or its quotient field $F(x)$) in Theorem 4.1 is, except for separability, a realization of the e -consistent system $\mathbf{S} = \{S_1, \dots, S_n\}$ for the Rees valuation rings $(W_1, Q_1), \dots, (W_n, Q_n)$ of $uR[u, tI]$. Here, e is an arbitrary positive common multiple of the Rees integers e_1, \dots, e_n of $uR[u, tI]$ (and of I), and for $j = 1, \dots, n$, $S_j = \{(W_j/Q_j)[\overline{\theta_j}], e_j, \frac{e}{e_j}\}$, where θ_j is a root of $X^{e_j} - w_j$ (with w_j playing the role of w in (3.5.4.3) in the proof of Corollary 3.5), and $\overline{\theta_j} = \theta_j + Q_j W_j[\theta_j]$.

(4.4.2) More generally, let I be a nonzero proper ideal in a Noetherian integral domain R , let $\mathbf{R} = R[u, tI]$ be the Rees ring of R with respect to I , let $(W_1, Q_1), \dots, (W_n, Q_n)$ be the Rees valuation rings of $u\mathbf{R}$, let $\mathbf{W} = W_1 \cap \dots \cap W_n$, and let $k \geq 2$ be an arbitrary integer. For $j = 1, \dots, n$, let e_j be the Rees integer of $u\mathbf{R}$ with respect to W_j , and let d_j be the greatest common divisor of k and e_j . Also, let F be the quotient field of R , let $F(u)^*$ be an algebraic closure of $F(u)$, and let \mathbf{E} be the integral closure of $\mathbf{W}[x_k]$, where $x_k = u^{\frac{1}{k}} \in F(u)^*$. Then \mathbf{E} is a realization of the k -consistent system $\mathbf{S}' = \{S_1', \dots, S_n'\}$ for the valuation rings $(W_1, Q_1), \dots, (W_n, Q_n)$. Here, for $j = 1, \dots, n$, $S_j' = \{(W_j/Q_j)[\overline{\theta_{j,k}}], d_j, \frac{k}{d_j}\}$, where $\theta_{j,k}$ is a root of $X^{e_{j,k}} - w_{j,k}$ (with $w_{j,k}$ playing the role of w in (3.5.4.3) in the proof of Corollary 3.5), and $\overline{\theta_{j,k}} = \theta_{j,k} + Q_j W_j[\theta_{j,k}]$.

Proof. Item (4.4.1) follows immediately from Theorem 4.1 (it follows from Proposition 3.9(1) that $Q_j W_j[\theta_j]$ is a prime ideal), and Item (4.4.2) follows from Proposition 3.9(3). \square

Theorem 4.5. (Krull [6]): Let $(V_1, N_1), \dots, (V_n, N_n)$ be distinct DVRs of a field F with $K_j = V_j/N_j$ for $j = 1, \dots, n$, let m be a positive integer, and let $\mathbf{S} = \{S_1, \dots, S_n\}$ be an m -consistent system for $\{V_1, \dots, V_n\}$ with $S_j = \{(K_{j,i}, f_{j,i}, e_{j,i}) \mid i = 1, \dots, s_j\}$ for $j = 1, \dots, n$. Then \mathbf{S} is realizable if one of the following conditions is satisfied:

- (1) $s_j = 1$ for at least one j .
- (2) F has at least one DVR V distinct from V_1, \dots, V_n .
- (3) For each monic polynomial $X^t + a_1 X^{t-1} + \dots + a_t$ with $a_i \in \bigcap_{j=1}^n V_j = \mathbf{D}$, and for each $h \in \mathbb{N}_{>0}$ there exists an irreducible separable polynomial $X^t + b_1 X^{t-1} + \dots + b_t \in \mathbf{D}[X]$ with $b_l - a_l \in N_j^h$ for each $l = 1, \dots, t$ and $j = 1, \dots, n$.

Observation.

- (a) Condition (1) of Theorem 4.5 is a property of the m -consistent system $\mathbf{S} = \{S_1, \dots, S_n\}$.
- (b) Condition (2) of Theorem 4.5 is a property of the family of DVRs of the field F .
- (c) Condition (3) of Theorem 4.5 is a property of the family $(V_1, N_1), \dots, (V_n, N_n)$.

Remark 4.6. Let R be a Noetherian integral domain with quotient field F , and assume that $\text{altitude}(R) \geq 2$. Then there exist infinitely many height one prime ideals in R , so

R' has infinitely many height one prime ideals, by the Lying-Over Theorem ([7, (10.8)]). Therefore, since R' is a Krull domain, by ([7, (33.10)]), there exist infinitely many distinct DVRs with quotient field F , hence Theorem 4.5(2) is always satisfied for such fields F .

We can now state and prove the first new result in this section. It is closely related to Theorem 4.1, and it is also considerably more general.

Theorem 4.7. *Let R be a Noetherian integral domain with quotient field F , and assume that $\text{altitude}(R) \geq 2$. Let I be a nonzero proper ideal in R , let $(V_1, N_1), \dots, (V_n, N_n)$ ($n \geq 2$) be the Rees valuation rings of I , let $\mathbf{D} = V_1 \cap \dots \cap V_n$ (so \mathbf{D} is a Dedekind domain with exactly n maximal ideals $M_j = N_j \cap \mathbf{D}$ ($j = 1, \dots, n$)), and let $M_1^{e_1} \dots M_n^{e_n}$ ($= M_1^{e_1} \cap \dots \cap M_n^{e_n}$) be an irredundant primary decomposition of $I\mathbf{D}$ (so e_1, \dots, e_n are the Rees integers of I). Let m be the least common multiple of e_1, \dots, e_n , let $d_j = \frac{m}{e_j}$ ($j = 1, \dots, n$), and let $e = km$ be a positive multiple of m . Then there exists an integral domain \mathbf{E} with an ideal J such that:*

(4.7.1) $\mathbf{E} = \mathbf{D}[\theta]$ is a semi-local Dedekind domain that is a simple free separable integral extension domain of \mathbf{D} .

(4.7.2) $[\mathbf{E} : \mathbf{D}] = e$.

(4.7.3) J is the Jacobson radical of \mathbf{E} , and \mathbf{E} has exactly $k(e_1 + \dots + e_n)$ maximal ideals.

(4.7.4) $[(\mathbf{E}/Q_{j,i}) : (\mathbf{D}/M_j)] = 1$ for $j = 1, \dots, n$ and for the ke_j associated prime ideals $Q_{j,i}$ of $M_j\mathbf{E}$.

(4.7.5) $I\mathbf{E} = J^m$.

(4.7.6) The Rees integers of $I\mathbf{E}$ are all equal to m .

Proof. Let $\mathbf{S} = \{S_1, \dots, S_n\}$ with $S_j = \{(K_{j,i}, 1, d_j) \mid i = 1, \dots, ke_j\}$ ($j = 1, \dots, n$). Observe that $d_j \cdot ke_j = km = e$ for $j = 1, \dots, n$, so \mathbf{S} is an e -consistent system for $\{\mathbf{D}_{M_1}, \dots, \mathbf{D}_{M_n}\}$, so \mathbf{S} is a realizable e -consistent system for $\{\mathbf{D}_{M_1}, \dots, \mathbf{D}_{M_n}\}$, by Theorem 4.5(2) and Remark 4.6. Therefore the integral closure \mathbf{E} of \mathbf{D} in a realization L of \mathbf{S} for $\{\mathbf{D}_{M_1}, \dots, \mathbf{D}_{M_n}\}$ is a simple free separable integral extension domain of \mathbf{D} such that $[\mathbf{E} : \mathbf{D}] = e$ and \mathbf{E} is a Dedekind domain (by [12, Theorem 19, p. 281]). Also, since $S_j = \{(K_{j,i}, 1, d_j) \mid i = 1, \dots, ke_j\}$ ($j = 1, \dots, n$): each V_j has exactly ke_j extensions $(V_{j,i}, N_{j,i})$ to L ; $\mathbf{E} = V_{1,1} \cap \dots \cap V_{n,ke_n}$ and $Q_{j,i} = N_{j,i} \cap \mathbf{E}$; $\mathbf{E}/Q_{j,i} \cong V_{j,i}/N_{j,i} \cong V_j/N_j \cong \mathbf{D}/M_j$ ($j = 1, \dots, n$ and $i = 1, \dots, ke_j$); and, $N_j V_{j,i} = N_{j,i}^{d_j}$ ($j = 1, \dots, n$ and $i = 1, \dots, ke_j$).

Therefore (4.7.1)–(4.7.4) hold, and

$$I\mathbf{E} = \prod_{j=1}^n (M_j^{e_j} \mathbf{E}) = \prod_{j=1}^n \left(\prod_{i=1}^{ke_j} (N_{j,i}^{d_j}) \right)^{e_j} = \left(\prod_{j=1}^n \prod_{i=1}^{ke_j} N_{j,i} \right)^m = J^m,$$

where $J = \prod_{j=1}^n \prod_{i=1}^{ke_j} N_{j,i}$ is the Jacobson radical J of \mathbf{E} . Thus $I\mathbf{E} = J^m$, so (4.7.5) holds, and since \mathbf{E} is a semi-local domain with Jacobson radical J , (4.7.6) follows immediately from (4.7.5), Definition 2.1.5, and Remark 2.2.4. \square

Remark 4.8. In the proof of Proposition 4.7, there are many cases when the simpler e -consistent system $\mathbf{T} = \{T_1, \dots, T_n\}$ with $T_j = \{(K_{j,1}, ke_j, d_j)\}$ ($j = 1, \dots, n$) can be used in place of $S_j = \{(K_{j,i}, 1, d_j) \mid i = 1, \dots, ke_j\}$ ($j = 1, \dots, n$), and then the resulting realization \mathbf{E} of \mathbf{T} for $\{\mathbf{D}_{M_1}, \dots, \mathbf{D}_{M_n}\}$ has the same number of maximal ideals as \mathbf{D} . However, \mathbf{T} cannot always be used, since, for example, if \mathbf{D}/M_j is algebraically closed and $ke_j \geq 2$, then there are no extension fields $K_{j,1}$ of $K_j = \mathbf{D}/M_j$ such that $[K_{j,1} : K_j] = ke_j$.

Remark 4.9. In the proof of Proposition 4.7, if $S_j = \{(K_{j,i}, 1, d_j) \mid i = 1, \dots, ke_j\}$ is replaced with $U_j = \{(K_{j,i}, 1, kd_j) \mid i = 1, \dots, e_j\}$ ($j = 1, \dots, n$), then the same conclusions hold, but replace:

- (a) “each V_j has exactly ke_j extensions $(V_{j,i}, N_{j,i})$ to L ” with “each V_j has exactly e_j extensions $(V_{j,i}, N_{j,i})$ to L ”;
- (b) “the Rees integers of $I\mathbf{E}$ are all equal to m ,” with “the Rees integers of $I\mathbf{E}$ are all equal to e ”;
- (c) “ $I\mathbf{E} = J^m$ ” with “ $I\mathbf{E} = J^e$ ”;
- (d) “ $N_j V_{j,i} = N_{j,i}^{d_j}$ ($j = 1, \dots, n$ and $i = 1, \dots, ke_j$)” with “ $N_j V_{j,i} = N_{j,i}^{kd_j}$ ($j = 1, \dots, n$ and $i = 1, \dots, e_j$)”;
- (e) “ $I\mathbf{E} = \prod_{j=1}^n (M_j^{e_j} \mathbf{E}) = \prod_{j=1}^n (\prod_{i=1}^{ke_j} (N_{j,i}^{d_j}))^{e_j} = (\prod_{j=1}^n \prod_{i=1}^{ke_j} N_{j,i})^m$ ” with “ $I\mathbf{E} = \prod_{j=1}^n (M_j^{e_j} \mathbf{E}) = \prod_{j=1}^n (\prod_{i=1}^{e_j} (N_{j,i}^{kd_j}))^{e_j} = (\prod_{j=1}^n \prod_{i=1}^{e_j} N_{j,i})^e$ ”.

Our first corollary of Theorem 4.7 is a more complete and detailed version of [4, Theorem 2.8(1)].

Corollary 4.10. *Let I be a nonzero proper ideal in a Noetherian domain R , let $(V_1, N_1), \dots, (V_n, N_n)$ ($n \geq 2$) be the Rees valuation rings of I , let e_j be the Rees integer of I with respect to V_j ($j = 1, \dots, n$), and let $e = km$ be a positive multiple of the least common multiple m of e_1, \dots, e_n . Then there exists an integral domain B_e such that:*

- (1) B_e is a semi-local Dedekind domain that is a simple free separable integral extension domain of R .
- (2) $[B_e : R] = e$.
- (3) The Rees integers of IB_e are all equal to m .

Proof. Let \mathbf{D} be the intersection of the n Rees valuation rings (V_j, N_j) of I , and let $M_j = N_j \cap \mathbf{D}$ ($j = 1, \dots, n$), so \mathbf{D} is a semi-local Dedekind domain with exactly n maximal ideals M_j and $I\mathbf{D} = M_1^{e_1} \cap \dots \cap M_n^{e_n} = M_1^{e_1} \dots M_n^{e_n}$. Therefore by Proposition 4.7 there exists a simple free separable extension field L_e of the quotient field F of R such that: $[L_e : F] = e$; the integral closure \mathbf{E}_e of \mathbf{D} in L_e is a finite (by free separability) integral extension domain of \mathbf{D} and is a semi-local Dedekind domain with exactly ke_j maximal ideals $Q_{j,i}$ lying over each M_j ($j = 1, \dots, n$); and, $(I\mathbf{D})\mathbf{E}_e = J_e^m$, where J_e is the Jacobson radical of \mathbf{E}_e . By separability $L_e = F[\theta]$, so there exists $r \in R$ such that $r \cdot \theta$ is integral over R , so let $B_e = R[r \cdot \theta]$. Then B_e is a simple free separable integral

extension domain of R , $[B_e : R] = e$, and \mathbf{E}_e is the intersection of the Rees valuation rings of IB_e by Remark 2.2.6. Since $(IB_e)\mathbf{E}_e = (ID)\mathbf{E}_e = J_e^m$ and since J_e is the Jacobson radical of \mathbf{E}_e , it follows that the Rees integers of IB_e are all equal to m . \square

Remark 4.11. (4.11.1) With the notation of Corollary 4.10, there exists an integral domain C_e such that:

(1) C_e is a semi-local Dedekind domain that is a simple free separable integral extension domain of R .

(2) $[C_e : R] = e$.

(3) The Rees integers of IC_e are all equal to e .

(4.11.2) Let I be a regular proper ideal in a Noetherian ring R , let $(V_1, N_1), \dots, (V_n, N_n)$ ($n \geq 2$) be the Rees valuation rings of I , let e_j be the Rees integer of I with respect to V_j ($j = 1, \dots, n$), and let $e = km$ be a positive multiple of the least common multiple m of e_1, \dots, e_n . Assume that $\text{Rad}(0_R)$ is prime, say $\text{Rad}(0_R) = z$. Then there exists a ring B_e such that:

(1) B_e is a simple free integral extension ring of R .

(2) $[B_e : R] = e$.

(3) The Rees integers of IB_e are all equal to m .

(4) $zB_e = \text{Rad}(0_{B_e})$, so zB_e is the only minimal prime ideal in B_e .

(4.11.3) With the notation of (4.11.2), there exists a ring C_e such that:

(1) C_e is a simple free integral extension ring of R .

(2) $[C_e : R] = e$.

(3) The Rees integers of IC_e are all equal to e .

(4) $zC_e = \text{Rad}(0_{C_e})$, so zC_e is the only minimal prime ideal in C_e .

Proof. The proof of (4.11.1) is the same as the proof of Corollary 4.10, but use Remark 4.9 in place of Proposition 4.7.

For (4.11.2), since $\text{Rad}(0_R) = z$ is the only minimal prime ideal in R , the Rees valuation rings of I are the Rees valuation rings of $\bar{I} = (I + z)/z$ (see Remark 2.2.3). Also, by Corollary 4.10 there exists an integral domain $\overline{B_e}$ such that:

(1') $\overline{B_e}$ is a semi-local Dedekind domain that is a simple free separable integral extension domain of $\overline{R} = R/z$.

(2') $[\overline{B_e} : \overline{R}] = e$.

(3') The Rees integers of $\overline{IB_e}$ are all equal to m .

Let $f_e(X)$ be the pre-image in $R[X]$ of the irreducible monic polynomial $\overline{f_e}(X)$ (of degree e) in $\overline{R}[X]$ such that $\overline{B_e} = \overline{R}[X]/(\overline{f_e}(X)\overline{R}[X])$. Then $\overline{B_e} = R[X]/((f_e(X), z)R[X]) = R[x]/(zR[x])$, where $x = X + (f_e(X)R[X])$.

Let $B_e = R[x]$. Then, since $\overline{B_e} = R[x]/(zR[x])$ is an integral domain, it follows that zB_e is a prime ideal. Also, by hypothesis, there exists $r \in \mathbb{N}_{>0}$ such that $z^r = (0)$ in R , so it follows that zB_e is the only minimal prime ideal in B_e , so (4) holds. Therefore the Rees valuation rings and Rees integers of IB_e are the Rees valuation rings and Rees integers of $\overline{IB_e}$, by Remark 2.2.3, so (1)–(3) follow immediately from (1')–(3').

The proof of (4.11.3) is the same as the proof of (4.11.2), but use Remark 4.11.1 in place of Corollary 4.10. \square

The next corollary of Theorem 4.7 extends Corollary 4.10 to Noetherian rings.

Corollary 4.12. *Let I be a regular proper ideal in a Noetherian ring R , let e_1, \dots, e_n be the Rees integers of I , let m be the least common multiple of e_1, \dots, e_n , and let $e = km$ be a positive multiple of m . Then there exist rings R^* and B_e such that:*

- (1) $R \subseteq R^* \subseteq B_e$ and B_e is a finite integral extension ring of R .
- (2) IR^* and IB_e are regular proper ideals.
- (3) *There is a one-to-one correspondence between the minimal prime ideals w^* in B_e such that $IB_e + w^* \neq B_e$, the minimal prime ideals z^* in R^* such that $IR^* + z^* \neq R^*$, and the minimal prime ideals z in R such that $I + z \neq R$; namely, $w^* = z^*B_e$ and $z = z^* \cap R$.*
- (4) *The Rees integers of IB_e are all equal to e .*

Proof. Let $(0) = \cap \{q_h \mid h = 1, \dots, k\}$ be an irredundant primary decomposition of the zero ideal in R and let $\text{Rad}(q_h) = z_h$ ($h = 1, \dots, k$). Assume that the z_h are re-ordered so that z_1, \dots, z_{d_1} are the minimal prime ideals z in R such that $I + z \neq R$ and $z_{d_1+1}, \dots, z_{d_2}$ are the remaining minimal prime ideals in R (so $d_1 \leq d_2 \leq k$).

Rewrite $\cap \{q_h \mid h = 1, \dots, k\}$ as $\cap \{Z_h \mid h = 1, \dots, d_2\}$, where Z_h is the intersection of all q_i such that $z_i \supseteq z_h$, for $h = 1, \dots, d_2$. For $i = 1, \dots, d_2$, let $R_i = R/Z_i$, let $\overline{z_i} = z_i/Z_i$ (so the unique minimal prime ideal in R_i is $\overline{z_i}$, therefore, $\overline{z_i} = \text{Rad}(0_{R_i})$), and let $I_i = (I + Z_i)/Z_i$ (so I_i is a regular proper ideal in R_i for $i = 1, \dots, d_1$ and $I_i = R_i$ for $i = d_1 + 1, \dots, d_2$).

By Remark 2.2.3 the set of Rees valuation rings of I is the disjoint union of the sets of Rees valuation rings of the ideals I_i ($i = 1, \dots, d_1$), so the Rees integers of the I_i are among the Rees integers of I . Hence e is a multiple of the least common multiple of the Rees integers of the ideals I_i ($i = 1, \dots, d_1$).

Therefore, by Remark 4.11.3, for $i = 1, \dots, d_1$, there exists a simple free integral extension ring $B_{i,e}$ of R_i such that $[B_{i,e} : R_i] = e$, the Rees integers of $I_i B_{i,e}$ are all equal to e , and $\overline{z_i} B_{i,e}$ is the only minimal prime ideal in $B_{i,e}$.

Let $R^* = R_1 \oplus \dots \oplus R_{d_2}$, let $I^* = I_1 \oplus \dots \oplus I_{d_2}$ so $I^* = IR^*$, and for $i = 1, \dots, d_2$, let $z_i^* = R_1 \oplus \dots \oplus R_{i-1} \oplus \overline{z_i} \oplus R_{i+1} \oplus \dots \oplus R_{d_2}$. Then R^* is a finite integral extension ring of R , I^* is a regular proper ideal in R^* , and for $i = 1, \dots, d_1$, the z_i^* are the minimal prime ideals z^* in R^* such that $I^* + z^* \neq R^*$. Also, $z_i^* \cap R = z_i$ and $R^*/z_i^* = R_i/\overline{z_i} = R/z_i$ ($i = 1, \dots, d_2$), so the Rees valuation rings of I^* are the Rees valuation rings of I , so the ideals I and I^* have the same Rees integers.

Let $B_e = B_{1,e} \oplus \dots \oplus B_{d_2,e}$, where, for notational convenience, we let $B_{h,e} = R_h$ for $h = d_1 + 1, \dots, d_2$. Also, for $i = 1, \dots, d_2$, let $w_i^* = B_{1,e} \oplus \dots \oplus B_{i-1,e} \oplus \overline{z_i} B_{i,e} \oplus B_{i+1,e} \oplus \dots \oplus B_{d_2,e}$, so $w_i^* \cap R^* = z_i^*$ and $w_i^* = z_i^* B_e$. Then B_e is a finite R^* -module (and is also a finite integral extension ring of R), $IB_e = I^* B_e = IB_{1,e} \oplus \dots \oplus IB_{d_1,e} \oplus$

$B_{d_1+1,e} \oplus \cdots \oplus B_{d_2,e}$ is a regular proper ideal in B_e , the w_i^* are d_2 minimal prime ideals in B_e , and $B_e/w_i^* = B_{i,e}/(\overline{z_i}B_{i,e})$ for $i = 1, \dots, d_2$.

Since the ideals z_i^* are the minimal prime ideals z^* in R^* such that $I^* + z^* \neq R^*$ (for $i = 1, \dots, d_1$), and since w_i^* is a minimal prime ideal in B_e such that $w_i^* = z_i^*B_e$, it follows that the w_i^* ($i = 1, \dots, d_1$) are the minimal prime ideals w^* in B_e such that $IB_e + w^* = I^*B_e + w^* \neq B_e$. So the set of Rees valuation rings of I^*B_e is the disjoint union of the sets of Rees valuation rings of the ideals $(I^*B_e + w_i^*)/w_i^* = (I^*B_e + z_i^*B_e)/w_i^* = (B_{1,e} \oplus \cdots \oplus B_{i-1,e} \oplus (I_i + \overline{z_i})B_{i,e} \oplus B_{i+1,e} \oplus \cdots \oplus B_{d_2,e})/w_i^* = (I_iB_{i,e} + \overline{z_i}B_{i,e})/(\overline{z_i}B_{i,e})$. Therefore, since $\overline{z_i}B_{i,e}$ is the unique minimal prime ideal in $B_{i,e}$, it follows that, for $i = 1, \dots, d_1$, $I_iB_{i,e}$ and $(I_iB_{i,e} + \overline{z_i}B_{i,e})/(\overline{z_i}B_{i,e})$ have the same Rees valuation rings and the same Rees integers.

Finally, for $i = 1, \dots, d_1$, the Rees integers of $I_iB_{i,e}$ are all equal to e (by the last sentence in the second preceding paragraph), so it follows that the Rees integers of IB_e are all equal to e . \square

To state an additional corollary, we need the following definition.

Definition 4.13. Let I and J be ideals in a ring R . Then:

(4.13.1) I and J are **projectively equivalent** in case there exist $i, j \in \mathbb{N}_{>0}$ such that $(I^i)_a = (J^j)_a$ (see (2.1.2)).

(4.13.2) I is **projectively full** in case, for each ideal J in R that is projectively equivalent to I (see (4.13.1)), $J_a = (I^k)_a$ for some $k \in \mathbb{N}_{>0}$.

Remark 4.14. With Definition 4.13 in mind, it should be noted that Theorem 4.7.6 shows that the Jacobson radical J of \mathbf{E} is projectively equivalent to $I\mathbf{E}$, and since \mathbf{E} is a semi-local Dedekind domain, it follows that J is a projectively full radical ideal whose Rees integers are all equal to one.

The next corollary of Theorem 4.7 was proved in [4, Theorem 2.8(2)] in the case when R is a Noetherian domain of altitude one.

Corollary 4.15. *If R is a Noetherian ring of altitude one, then for each regular proper ideal I in R there exists a finite integral extension ring A of R with an ideal J such that J and IA are projectively equivalent, J is a projectively full radical ideal, and the Rees integers of I are all equal to one.*

Proof. It is shown in [4, Theorem 2.8(2)] that this result holds for nonzero proper ideals in Noetherian domains of altitude one. So a proof similar to the proof of Corollary 4.12 shows that it continues to hold for regular proper ideals in Noetherian rings of altitude one. \square

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