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The structure and homological properties of generalized standard Auslander–Reiten components ☆



Piotr Malicki*, Andrzej Skowroński

*Faculty of Mathematics and Computer Science, Nicolaus Copernicus University,
Chopina 12/18, 87-100 Toruń, Poland*

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ABSTRACT

We describe the structure and homological properties of arbitrary generalized standard Auslander–Reiten components of artin algebras. In particular, we prove that for all but finitely many indecomposable modules in such components the Euler characteristic is defined and nonnegative. Further, we provide a handy criterion for an infinite Auslander–Reiten component of an artin algebra to be generalized standard. We solve also the long standing open problem concerning the structure of artin algebras admitting a separating family of Auslander–Reiten components.

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* Corresponding author.

E-mail addresses: pmalicki@mat.umk.pl (P. Malicki), skowron@mat.umk.pl (A. Skowroński).

1. Introduction and the main results

Throughout the paper, by an algebra we mean a basic indecomposable artin algebra over a commutative artin ring K . For an algebra A , we denote by $\text{mod } A$ the category of finitely generated right A -modules, by $\text{ind } A$ the full subcategory of $\text{mod } A$ formed by the indecomposable modules, and by D the standard duality on $\text{mod } A$. The radical rad_A of $\text{mod } A$ is the ideal generated by all nonisomorphisms between modules in $\text{ind } A$. Then the infinite radical rad_A^∞ of $\text{mod } A$ is the intersection of all powers rad_A^i , $i \geq 1$, of rad_A . By a result of Auslander [3], $\text{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, there are in $\text{mod } A$ only finitely many indecomposable modules up to isomorphism. On the other hand, if A is of infinite representation type then $(\text{rad}_A^\infty)^2 \neq 0$, by a result proved in [9].

An important combinatorial and homological invariant of the module category $\text{mod } A$ of an algebra A is its Auslander–Reiten quiver Γ_A . Recall that Γ_A is a valued translation quiver whose vertices are the isomorphism classes $\{X\}$ of modules X in $\text{ind } A$, the arrows correspond to irreducible homomorphisms between modules in $\text{ind } A$, and the translation is the Auslander–Reiten translation $\tau_A = D\text{Tr}$. We shall not distinguish between a module X in $\text{ind } A$ and the vertex $\{X\}$ of Γ_A . Moreover, by a component of Γ_A we mean a connected component of the quiver Γ_A . Frequently, algebras can be recovered from the shapes of the components of their Auslander–Reiten quiver. Further, very often the behavior of components of the Auslander–Reiten quiver Γ_A of an algebra A in the module category $\text{mod } A$ leads to essential information on A , allowing to determine A and $\text{mod } A$ completely. For a component \mathcal{C} of Γ_A , we denote by $\text{ann}_A(\mathcal{C})$ the annihilator of \mathcal{C} in A , that is, the intersection of the annihilators $\{a \in A \mid Ma = 0\}$ of all modules M in \mathcal{C} , and by $B(\mathcal{C})$ the quotient algebra $A/\text{ann}_A(\mathcal{C})$, called the faithful algebra of \mathcal{C} . We note that \mathcal{C} is a faithful component of $\Gamma_{B(\mathcal{C})}$.

In the representation theory of finite-dimensional algebras over an algebraically closed field k a prominent role is played by the standard Auslander–Reiten components (see [6,7,22,51,54,62,63] for some results characterizing tame algebras via standardness of their Auslander–Reiten components). Recall that, following [7,51], a component \mathcal{C} in the Auslander–Reiten quiver Γ_Λ of a finite-dimensional algebra Λ over k is called standard if the full subcategory of $\text{mod } \Lambda$ formed by all modules from \mathcal{C} is equivalent to the mesh-category $k(\mathcal{C})$ of \mathcal{C} . In particular, one knows [7,8] that, for Λ of finite representation type, Γ_Λ is standard if and only if Λ admits a simply connected Galois covering. Moreover, long time ago Ringel asked [52, Problem 3] if any standard regular component of the Auslander–Reiten quiver of an algebra over an algebraically closed field is either a stable tube or of the form $\mathbb{Z}\Delta$ for a finite acyclic quiver Δ . This was shown to be true, also in the wider context of artin algebras [58, Corollaries 2.4 and 2.5].

The second named author proposed in [58] a natural generalization of the concept of standard component, called generalized standard component, which is simpler and makes sense for any artin algebra. Namely, a component \mathcal{C} of the Auslander–Reiten quiver Γ_A of an algebra A is called generalized standard if $\text{rad}_A^\infty(X, Y) = 0$ for all modules X

and Y from \mathcal{C} . It follows from general theory [4] that every nonzero nonisomorphism $f : X \rightarrow Y$ with X and Y in a generalized standard component \mathcal{C} of Γ_A is a finite sum of compositions of irreducible homomorphisms between indecomposable modules from \mathcal{C} . Moreover, the additive category $\text{add}(\mathcal{C})$ of a generalized standard component \mathcal{C} of Γ_A is closed under extensions in $\text{mod } A$, and this allows to describe the degeneration-like orders for modules in $\text{add}(\mathcal{C})$ with the same composition factors (see [69]). We also mention that \mathcal{C} is a generalized standard component of Γ_A if and only if \mathcal{C} is a generalized standard component of $\Gamma_{B(\mathcal{C})}$. The Auslander–Reiten quiver of every algebra of finite representation type is generalized standard. Further, Liu proved in [35] that every standard component of an Auslander–Reiten quiver of a finite-dimensional algebra over an algebraically closed field is generalized standard. The converse implication is not true, because there are nonstandard Auslander–Reiten quivers for some algebras of finite representation type over algebraically closed fields of characteristic 2 [46]. The following results show that existence of a generalized standard component in the Auslander–Reiten quiver Γ_A of an algebra A may determine the algebra A . For example, it is known that an algebra A is a tilted algebra (respectively, double tilted algebra, generalized double tilted algebra) if and only if Γ_A admits a faithful generalized standard component \mathcal{C} with a section [34,57] (respectively, double section [44], multisection [45]). Moreover, by results established in [65,66], every self-injective algebra A for which Γ_A admits an acyclic generalized standard component is a socle deformation of the orbit algebra \widehat{B}/G of the repetitive category \widehat{B} of a tilted algebra B of infinite representation type and an infinite cyclic automorphism group G of \widehat{B} . We also mention that distinguished classes of generalized standard components are formed by the separating families of tubes of quasitilted algebras of canonical type [30], or more generally, the separating families of generalized multicoils of generalized multicoil algebras [38].

We are concerned with the structure of an arbitrary generalized standard component of an Auslander–Reiten quiver. It has been proved in [58] that every generalized standard component \mathcal{C} of the Auslander–Reiten quiver Γ_A of an algebra A is almost periodic, that is, all but finitely many τ_A -orbits in \mathcal{C} are periodic. This implies that \mathcal{C} may contain at most finitely many indecomposable modules of any given length. The acyclic generalized standard components were described completely in [57]. In particular, the acyclic generalized standard semiregular components of an Auslander–Reiten quiver Γ_A are the connecting components of quotient tilted algebras of A (see [34,58]). On the other hand, the description of generalized standard components with oriented cycles is an exciting but difficult problem. Namely, it was shown in [64] that every finite-dimensional algebra Λ over a field K is a quotient algebra of an algebra A with Γ_A having a faithful generalized standard stable tube (see also [31,39]). In general, one needs some extra information concerning interaction of a given component \mathcal{C} of Γ_A with other components of Γ_A . For example, it was shown in [21] that if a semiregular component \mathcal{C} of Γ_A is without external short paths (in the sense of [43]) in $\text{mod } A$ then \mathcal{C} is generalized standard and a component of a quasitilted quotient algebra of A .

The first main result of the paper provides a handy criterion for an infinite component of an Auslander–Reiten quiver Γ_A to be generalized standard. Let \mathcal{C} be an almost periodic component of the Auslander–Reiten quiver Γ_A of an algebra A . Following [37] an indecomposable projective module P in \mathcal{C} is said to be right coherent if there is an infinite sectional path $P = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_i \rightarrow X_{i+1} \rightarrow X_{i+2} \rightarrow \cdots$ (that is, $X_i \neq \tau_A X_{i+2}$ for all $i \geq 1$) in \mathcal{C} . Dually, an indecomposable injective module Q in \mathcal{C} is said to be left coherent if there is an infinite sectional path $\cdots \rightarrow Y_{j+2} \rightarrow Y_{j+1} \rightarrow Y_j \rightarrow \cdots \rightarrow Y_2 \rightarrow Y_1 = Q$ (that is, $Y_{j+2} \neq \tau_A Y_j$ for all $j \geq 1$) in \mathcal{C} . We denote by $P_{\mathcal{C}}$ (respectively, $P_{\mathcal{C}}^{\text{coh}}$) the direct sum of all indecomposable projective (respectively, right coherent projective) modules in \mathcal{C} , and by $Q_{\mathcal{C}}$ (respectively, $Q_{\mathcal{C}}^{\text{coh}}$) the direct sum of all indecomposable injective (respectively, left coherent injective) modules in \mathcal{C} . Applying [33], we may also distinguish in \mathcal{C} a left stable acyclic module $M_{\mathcal{C}}^{(l)}$ of \mathcal{C} , being the direct sum of indecomposable modules forming left sections of the left stable acyclic components of \mathcal{C} , and a right stable acyclic module $M_{\mathcal{C}}^{(r)}$ of \mathcal{C} , being the direct sum of indecomposable modules forming right sections of the right stable acyclic components of \mathcal{C} . We may also assume that there is no path in \mathcal{C} from a direct summand of $M_{\mathcal{C}}^{(r)}$ to a direct summand of $M_{\mathcal{C}}^{(l)}$. Further, applying [37], we may define a tubular module $M_{\mathcal{C}}^{(t)}$ of \mathcal{C} , being the direct sum of indecomposable modules forming the mouth of all stable tubes involved in constructing the maximal cyclic coherent full translation subquivers of \mathcal{C} . We refer to Section 2 for details.

We are now in position to formulate the first main result of the paper.

Theorem 1.1. *Let A be an algebra and \mathcal{C} be an infinite component of Γ_A . The following statements are equivalent:*

- (i) \mathcal{C} is generalized standard.
- (ii) \mathcal{C} is almost periodic and the following vanishing conditions hold

$$\begin{aligned} \text{Hom}_A(P_{\mathcal{C}} \oplus M_{\mathcal{C}}^{(t)} \oplus M_{\mathcal{C}}^{(r)}, M_{\mathcal{C}}^{(l)}) &= 0, \text{Hom}_A(M_{\mathcal{C}}^{(r)}, Q_{\mathcal{C}} \oplus M_{\mathcal{C}}^{(t)} \oplus M_{\mathcal{C}}^{(l)}) = 0, \\ \text{Hom}_A(M_{\mathcal{C}}^{(l)}, \tau_A M_{\mathcal{C}}^{(l)}) &= 0, \text{rad}_A(M_{\mathcal{C}}^{(t)}, M_{\mathcal{C}}^{(t)}) = 0, \text{Hom}_A(\tau_A^{-1} M_{\mathcal{C}}^{(r)}, M_{\mathcal{C}}^{(r)}) = 0, \\ \text{Hom}_A(P_{\mathcal{C}}, \text{soc}(Q_{\mathcal{C}}^{\text{coh}}) \oplus M_{\mathcal{C}}^{(t)}) &= 0, \text{Hom}_A(\text{top}(P_{\mathcal{C}}^{\text{coh}}) \oplus M_{\mathcal{C}}^{(t)}, Q_{\mathcal{C}}) = 0. \end{aligned}$$

The second main result of the paper describes the structure of an arbitrary infinite generalized standard component of an Auslander–Reiten quiver.

Theorem 1.2. *Let A be an algebra and \mathcal{C} an infinite generalized standard component of Γ_A . Then there are quotient algebras $A_{\mathcal{C}}^{(lt)}$, $A_{\mathcal{C}}^{(c)}$, $A_{\mathcal{C}}^{(rt)}$ of A such that the following statements hold:*

- (i) $A_{\mathcal{C}}^{(lt)} = A_1^{(lt)} \times \cdots \times A_m^{(lt)}$, where

- (a) For each $i \in \{1, \dots, m\}$, $A_i^{(lt)}$ is a tilted algebra of the form $\text{End}_{H_i^{(l)}}(T_i^{(l)})$, for a hereditary algebra $H_i^{(l)}$ and a tilting module $T_i^{(l)}$ in $\text{mod } H_i^{(l)}$ without indecomposable preinjective direct summands.
- (b) For each $i \in \{1, \dots, m\}$, the image $\mathcal{C}_i^{(l)}$ of the preinjective component $\mathcal{Q}(H_i^{(l)})$ of $\Gamma_{H_i^{(l)}}$ via the functor $\text{Hom}_{H_i^{(l)}}(T_i^{(l)}, -) : \text{mod } H_i^{(l)} \rightarrow \text{mod } A_i^{(lt)}$ is an acyclic full translation subquiver of \mathcal{C} which is closed under predecessors.
- (ii) $A_{\mathcal{C}}^{(rt)} = A_1^{(rt)} \times \dots \times A_n^{(rt)}$, where
 - (a) For each $j \in \{1, \dots, n\}$, $A_j^{(rt)}$ is a tilted algebra of the form $\text{End}_{H_j^{(r)}}(T_j^{(r)})$, for a hereditary algebra $H_j^{(r)}$ and a tilting module $T_j^{(r)}$ in $\text{mod } H_j^{(r)}$ without indecomposable postprojective direct summands.
 - (b) For each $j \in \{1, \dots, n\}$, the image $\mathcal{C}_j^{(r)}$ of the postprojective component $\mathcal{P}(H_j^{(r)})$ of $\Gamma_{H_j^{(r)}}$ via the functor $\text{Ext}_{H_j^{(r)}}^1(T_j^{(r)}, -) : \text{mod } H_j^{(r)} \rightarrow \text{mod } A_j^{(rt)}$ is an acyclic full translation subquiver of \mathcal{C} which is closed under successors.
- (iii) $A_{\mathcal{C}}^{(c)} = A_1^{(c)} \times \dots \times A_p^{(c)}$, where
 - (a) For each $k \in \{1, \dots, p\}$, $A_k^{(c)}$ is a generalized multicoil enlargement of an algebra $B_k^{(c)}$ with a faithful family $\mathcal{T}^{B_k^{(c)}}$ of pairwise orthogonal generalized standard stable tubes in $\Gamma_{B_k^{(c)}}$.
 - (b) For each $k \in \{1, \dots, p\}$, $\Gamma_{A_k^{(c)}}$ admits a generalized multicoil $\mathcal{C}_k^{(c)}$, obtained from a finite number of stable tubes in $\mathcal{T}^{B_k^{(c)}}$ by translation quiver admissible operations corresponding to the algebra admissible operations leading from $B_k^{(c)}$ to $A_k^{(c)}$, and the cyclic part ${}^c\mathcal{C}_k^{(c)}$ of $\mathcal{C}_k^{(c)}$ is a full translation subquiver of \mathcal{C} .
- (iv) The translation quivers $\mathcal{C}^{(l)} = \mathcal{C}_1^{(l)} \cup \dots \cup \mathcal{C}_m^{(l)}$ and $\mathcal{C}^{(r)} = \mathcal{C}_1^{(r)} \cup \dots \cup \mathcal{C}_n^{(r)}$ are disjoint and their union $\mathcal{C}^{(l)} \cup \mathcal{C}^{(r)}$ contains all but finitely many acyclic indecomposable modules of \mathcal{C} .
- (v) The translation quivers $\mathcal{C}_1^{(c)}, \dots, \mathcal{C}_p^{(c)}$ are pairwise disjoint and their union $\mathcal{C}^{(c)} = \mathcal{C}_1^{(c)} \cup \dots \cup \mathcal{C}_p^{(c)}$ contains all but finitely many indecomposable modules of the cyclic part ${}^c\mathcal{C}$ of \mathcal{C} .

The algebra $A_{\mathcal{C}}^{(lt)}$ is said to be the *left tilted algebra* of \mathcal{C} and the algebra $A_{\mathcal{C}}^{(rt)}$ is said to be the *right tilted algebra* of \mathcal{C} . Further, the algebra $A_{\mathcal{C}}^{(c)}$ is said to be the *coherent algebra* of \mathcal{C} . We mention that $A_{\mathcal{C}}^{(lt)}$ (respectively, $A_{\mathcal{C}}^{(rt)}$) is nontrivial provided \mathcal{C} admits left stable (respectively, right stable) acyclic part. Similarly, $A_{\mathcal{C}}^{(c)}$ is nontrivial provided the cyclic part ${}^c\mathcal{C}$ of \mathcal{C} is infinite.

For an algebra A and a module M in $\text{mod } A$, we denote by $|M|$ the length of M over the commutative artin ring K .

The following corollary is a direct consequence of Theorem 1.2 and [40, Theorem 1.3].

Corollary 1.3. *Let A be an algebra and \mathcal{C} be a generalized standard component of Γ_A . Then for all but finitely many indecomposable modules M in \mathcal{C} we have*

- (i) $|\mathrm{Ext}_A^1(M, M)| \leq |\mathrm{End}_A(M)|$.
- (ii) $\mathrm{Ext}_A^r(M, M) = 0$ for any $r \geq 2$.

Therefore, for all but finitely many modules M in a generalized standard component \mathcal{C} of an Auslander–Reiten quiver Γ_A , the Euler characteristic

$$\chi_A(M) = \sum_{i=0}^{\infty} (-1)^i |\mathrm{Ext}_A^i(M, M)|$$

is defined and nonnegative.

Recall that a component \mathcal{C} of an Auslander–Reiten quiver Γ_A is called regular if \mathcal{C} is without projective modules and injective modules. It is known that every regular generalized standard component \mathcal{C} of Γ_A is either a stable tube or of the form $\mathbb{Z}\Delta$ for a finite acyclic valued quiver Δ (see [58, Corollary 2.4]). In the later case, the faithful algebra $B(\mathcal{C})$ of \mathcal{C} is a tilted algebra $\mathrm{End}_H(T)$ for a wild hereditary algebra H and a regular tilting module T in $\mathrm{mod} H$ (see [58, Corollary 3.3]). We refer to [53] (see also [68, Section VIII.9]) for the existence of acyclic regular generalized standard components. On the other hand, the structure of the faithful algebra of a generalized standard stable tube is still an open problem. But the homological properties of indecomposable modules in generalized standard stable tubes are described in [60, Corollary 3.6].

We obtain then the following consequence of these results and Corollary 1.3.

Corollary 1.4. *Let A be an algebra such that every component in Γ_A is generalized standard. Then for all but finitely many modules M in $\mathrm{ind} A$ the Euler characteristic $\chi_A(M)$ is defined and*

$$\chi_A(M) = |\mathrm{End}_A(M)| - |\mathrm{Ext}_A^1(M, M)| \geq 0.$$

A prominent role in the representation theory of algebras is played by the algebras with separating families of Auslander–Reiten components. A concept of a separating family of tubes has been introduced by Ringel in [50, 51] who proved that they occur in the Auslander–Reiten quivers of hereditary algebras of Euclidean type, tubular algebras, and canonical algebras. In order to deal with wider classes of algebras, the following more general concept of a separating family of Auslander–Reiten components was proposed by Assem, the second named author and Tomé in [2] (see also [38]). A family $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ of components of the Auslander–Reiten quiver Γ_A of an algebra A is called separating in $\mathrm{mod} A$ if the components of Γ_A split into three disjoint families \mathcal{P}^A , $\mathcal{C}^A = \mathcal{C}$ and \mathcal{Q}^A such that:

- (S1) \mathcal{C}^A is a sincere family of pairwise orthogonal generalized standard components;
- (S2) $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\mathrm{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;
- (S3) any homomorphism from \mathcal{P}^A to \mathcal{Q}^A in $\mathrm{mod} A$ factors through the additive category $\mathrm{add}(\mathcal{C}^A)$ of \mathcal{C}^A .

Then we say that \mathcal{C}^A separates \mathcal{P}^A from \mathcal{Q}^A and write

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A.$$

We note that then \mathcal{P}^A and \mathcal{Q}^A are uniquely determined by \mathcal{C}^A (see [2, (2.1)] or [51, (3.1)]). Moreover, we have $\text{ann}_A(\mathcal{C}^A) = 0$, so \mathcal{C}^A is a faithful family of components of Γ_A . We note that if A is an algebra of finite representation type that $\mathcal{C}^A = \Gamma_A$ is trivially a unique separating component of Γ_A , with \mathcal{P}^A and \mathcal{Q}^A being empty. It is known that an algebra A is a generalized double tilted algebra if and only if Γ_A admits a separating almost acyclic component [45]. In [29] Lenzing and de la Peña proved that an Auslander–Reiten quiver Γ_A admits a separating family of stable tubes if and only if A is a concealed canonical algebra. Moreover, by a result proved in [30], Γ_A admits a separating family of semiregular tubes if and only if A is a quasitilted algebra of canonical type. This was extended in [38] to the following result: the Auslander–Reiten quiver Γ_A of an algebra A admits a separating family of almost cyclic coherent components if and only if A is a *generalized multicoil algebra*, that is, is a generalized multicoil enlargement of a product of quasitilted algebras of canonical type. We refer to the survey article [41] for more details on algebras with separating families of Auslander–Reiten components and their representation theory.

The next aim is to describe the structure of the Auslander–Reiten quiver Γ_A of an algebra A with a separating family of components. We need some notation.

Let H be a hereditary algebra, T a tilting module in $\text{mod } H$, and $B = \text{End}_H(T)$ the associated tilted algebra. Then Γ_B admits an acyclic component \mathcal{C}_T with the section Δ_T given by the images of the indecomposable injective modules in $\text{mod } H$ via the functor $\text{Hom}_H(T, -) : \text{mod } H \rightarrow \text{mod } B$. Moreover, Γ_B has a decomposition

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{C}^B \cup \mathcal{Q}^B,$$

where \mathcal{P}^B is the disjoint union of all components of Γ_B contained entirely in the torsion-free part $\mathcal{Y}(T) = \{Y \in \text{mod } B \mid \text{Tor}_1^B(Y, T) = 0\}$, \mathcal{Q}^B is the disjoint union of all components of Γ_B contained entirely in the torsion part $\mathcal{X}(T) = \{X \in \text{mod } B \mid X \otimes_B T = 0\}$, and $\mathcal{C}^B = \mathcal{C}_T$ separates \mathcal{P}^B from \mathcal{Q}^B (see [20, 68]).

Let Λ be a quasitilted algebra of canonical type. Then Γ_Λ has a decomposition

$$\Gamma_\Lambda = \mathcal{P}^\Lambda \cup \mathcal{C}^\Lambda \cup \mathcal{Q}^\Lambda,$$

where \mathcal{C}^Λ is a family of semiregular tubes separating \mathcal{P}^Λ from \mathcal{Q}^Λ (see [30]).

The next theorem provides solution of the problem concerning the structure of artin algebras admitting a separating family of Auslander–Reiten components, initiated by Ringel [50–52].

Theorem 1.5. *Let A be an algebra with a separating family \mathcal{C}^A of components in Γ_A , and $\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A$ the associated decomposition of Γ_A . Then there exist quotient algebras $A^{(l)}$ and $A^{(r)}$ of A such that the following statements hold.*

- (i) $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)} \times A_{m+1}^{(l)} \times \cdots \times A_{m+p}^{(l)}$, where
 - (a) For each $i \in \{1, \dots, m\}$, $A_i^{(l)}$ is a tilted algebra of the form $\text{End}_{H_i^{(l)}}(T_i^{(l)})$ for a hereditary algebra $H_i^{(l)}$ and a tilting module $T_i^{(l)}$ in $\text{mod } H_i^{(l)}$ without indecomposable preinjective direct summands.
 - (b) For each $i \in \{m+1, \dots, m+p\}$, $A_i^{(l)}$ is a quasitilted algebra of canonical type with a separating family of coray tubes in $\Gamma_{A_i^{(l)}}$.
- (ii) $A^{(r)} = A_1^{(r)} \times \cdots \times A_n^{(r)} \times A_{n+1}^{(r)} \times \cdots \times A_{n+q}^{(r)}$, where
 - (a) For each $j \in \{1, \dots, n\}$, $A_j^{(r)}$ is a tilted algebra of the form $\text{End}_{H_j^{(r)}}(T_j^{(r)})$ for a hereditary algebra $H_j^{(r)}$ and a tilting module $T_j^{(r)}$ in $\text{mod } H_j^{(r)}$ without indecomposable postprojective direct summands.
 - (b) For each $j \in \{n+1, \dots, n+q\}$, $A_j^{(r)}$ is a quasitilted algebra of canonical type with a separating family of ray tubes in $\Gamma_{A_j^{(r)}}$.
- (iii) $\mathcal{P}^A = \bigcup_{i=1}^{m+p} \mathcal{P}^{A_i^{(l)}}$ and every component in \mathcal{P}^A is either a postprojective component, a ray tube, or obtained from a component of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly zero) of ray insertions.
- (iv) $\mathcal{Q}^A = \bigcup_{j=1}^{n+q} \mathcal{Q}^{A_j^{(r)}}$ and every component in \mathcal{Q}^A is either a preinjective component, a coray tube, or obtained from a component of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly zero) of coray insertions.

The proof of the next theorem applies the representation theory of generalized double tilted algebras developed by Reiten and Skowroński in [44,45]. We also note that any algebra of finite representation type is a generalized double tilted algebra in the sense of [45].

Theorem 1.6. *Let A be an algebra. The following statements are equivalent:*

- (i) Γ_A admits a finite separating family of components.
- (ii) Γ_A admits a separating almost acyclic component.
- (iii) A is a generalized double tilted algebra.

We note that the equivalence of (ii) and (iii) follows from [45, Theorem 3.1].

The following corollary is a consequence of Theorem 1.6 and Proposition 5.5.

Corollary 1.7. *Let A be an algebra. The following statements are equivalent:*

- (i) Γ_A admits a separating family of components containing of at least two components.
- (ii) Γ_A admits a separating family of components containing infinitely many stable tubes.

The next theorem is a consequence of Theorems 1.5 and 1.6, and describes the supports of indecomposable modules in the module categories of algebras with separating families of Auslander–Reiten components.

Theorem 1.8. *Let A be an algebra with a separating family of components in Γ_A . Then there exist quotient algebras B_1, \dots, B_n of A such that the following statements hold.*

- (i) *For each $i \in \{1, \dots, n\}$, B_i is either a generalized double tilted algebra or a generalized multicoil algebra.*
- (ii) *The indecomposable modules in $\text{ind } B_i$, $i \in \{1, \dots, n\}$, exhaust all modules in $\text{ind } A$.*

In particular, we have the following direct consequence of Theorem 1.8 and results proved in [38,45].

Corollary 1.9. *Let A be an algebra with a separating family of components in Γ_A . Then for all but finitely many isomorphism classes of modules M in $\text{ind } A$ there exists a quotient algebra B of A such that $\text{gl.dim } B \leq 3$ and M is a module in $\text{ind } B$ with $\text{pd}_B M \leq 2$ and $\text{id}_B M \leq 2$.*

In [12,13] Crawley-Boevey introduced the concept of a generically tame algebra. An indecomposable right A -module M over an algebra A is called a generic module if M is of infinite length over A but of finite length over $\text{End}_A(M)$, called the endlength of M . Then an algebra A is called generically tame if, for any positive integer d , there are only finitely many isomorphism classes of generic right A -modules of endlength d . An algebra A is called generically finite if there are at most finitely many pairwise non-isomorphic generic right A -modules. Further, A is called generically of polynomial growth if there is a positive integer m such that for any positive integer d the number of isomorphism classes of generic right A -modules of endlength d is at most d^m . We note that every algebra A of finite representation type is generically trivial, that is, there is no generic right A -module. We also stress that by a theorem of Crawley-Boevey [12, Theorem 4.4], if A is an algebra over an algebraically closed field K , then A is generically tame if and only if A is tame in the sense of Drozd [16] (see also [10,56]).

Recall also that following [59] the component quiver Σ_A of an algebra A has the components of Γ_A as vertices and there is an arrow $\mathcal{C} \rightarrow \mathcal{D}$ in Σ_A if $\text{rad}_A^\infty(X, Y) \neq 0$, for some modules X in \mathcal{C} and Y in \mathcal{D} . In particular, a component \mathcal{C} of Γ_A is generalized standard if and only if there is no loop at \mathcal{C} in Σ_A .

The next result characterizes the generically tame algebras with separating families of Auslander–Reiten components.

Theorem 1.10. *Let A be an algebra with a separating family of components in Γ_A . The following statements are equivalent:*

- (i) *A is generically tame.*

- (ii) A is generically of polynomial growth.
- (iii) $A^{(l)}$ and $A^{(r)}$ are products of tilted algebras of Euclidean type or tubular algebras.
- (iv) Γ_A is almost periodic.
- (v) Σ_A is acyclic.

Corollary 1.11. *Let A be an algebra with a separating family of components in Γ_A . The following statements are equivalent:*

- (i) A is generically finite.
- (ii) $A^{(l)}$ and $A^{(r)}$ are products of tilted algebras of Euclidean type.
- (iii) All but finitely many components of Γ_A are stable tubes of rank one.

The final result provides homological characterizations of generically tame algebras with separating families of Auslander–Reiten components.

Theorem 1.12. *Let A be an algebra with a separating family of components in Γ_A . Then the following statements are equivalent:*

- (i) A is generically tame.
- (ii) For all but finitely many isomorphism classes of modules M in $\text{ind } A$ we have $|\text{Ext}_A^1(M, M)| \leq |\text{End}_A(M)|$.
- (iii) For all but finitely many isomorphism classes of modules M in $\text{ind } A$ we have $|\text{Ext}_A^1(M, M)| \leq |\text{End}_A(M)|$ and $\text{Ext}_A^r(M, M) = 0$ for any $r \geq 2$.
- (iv) For all but finitely many isomorphism classes of modules M in $\text{ind } A$ the Euler characteristic $\chi_A(M)$ is defined and nonnegative.

This paper is organized as follows. In Section 2 we present and prove several results applied in the proofs of main result of the paper. Sections 3 and 4 are devoted to the proofs of Theorems 1.2 and 1.1, and illustrating examples. In Sections 5 and 6 we prove Theorems 1.5 and 1.6. The final Section 7 is devoted to the proofs of Theorem 1.10, Corollary 1.11, and Theorem 1.12.

For general results on the relevant representation theory we refer to the books [1, 4, 51, 55, 56, 67, 68] and the survey articles [13, 41, 50, 52].

2. Preliminary results

The aim of this section is to present several key results for the proofs of Theorems 1.1 and 1.2, and relevant background.

Let A be an algebra and \mathcal{C} be an infinite almost periodic component of Γ_A . An indecomposable module X in \mathcal{C} lying on an oriented cycle of \mathcal{C} is said to be *cyclic*, and otherwise *acyclic*. Following [37], we denote by ${}_c\mathcal{C}$ the full translation subquiver of \mathcal{C} obtained by removing all acyclic modules and the arrows attached to them, and call it

the *cyclic part* of \mathcal{C} . The connected translation subquivers of ${}_{\mathcal{C}}\mathcal{C}$ are said to be *cyclic components* of \mathcal{C} . It was shown in [37, Proposition 5.1] that two modules X and Y in ${}_{\mathcal{C}}\mathcal{C}$ belong to the same cyclic component of \mathcal{C} if there is an oriented cycle in \mathcal{C} passing through X and Y . An indecomposable module X in \mathcal{C} is said to be *right coherent* if there is in \mathcal{C} an infinite sectional path

$$X = X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow X_{i+2} \longrightarrow \cdots$$

Dually, an indecomposable module Y in \mathcal{C} is said to be *left coherent* if there is in \mathcal{C} an infinite sectional path

$$\cdots \longrightarrow Y_{j+2} \longrightarrow Y_{j+1} \longrightarrow Y_j \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow Y_1 = Y.$$

A module Z in \mathcal{C} is said to be *coherent* if Z is left and right coherent. We denote by ${}_{\mathcal{C}}\mathcal{C}^{\text{coh}}$ the full translation subquiver of ${}_{\mathcal{C}}\mathcal{C}$ given by all coherent modules in ${}_{\mathcal{C}}\mathcal{C}$, and call it the *coherent cyclic part* of \mathcal{C} . We note that ${}_{\mathcal{C}}\mathcal{C}$ may have finite cyclic components, which are obviously not coherent. We will see below that ${}_{\mathcal{C}}\mathcal{C}^{\text{coh}}$ is the disjoint union of the coherent parts of all infinite cyclic components of \mathcal{C} .

In our paper [37] we introduced the concept of a generalized multicoil in order to describe the shape and combinatorial properties of almost cyclic components with all indecomposable projective modules (right) coherent and all indecomposable injective modules (left) coherent. Namely, a connected translation quiver Γ is called in [37] a generalized multicoil if Γ can be obtained from a finite family $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s$ of stable tubes by an iterated application of admissible operations (ad 1), (ad 2), (ad 3), (ad 4), (ad 5) and their duals (ad 1*), (ad 2*), (ad 3*), (ad 4*), (ad 5*). We refer to [37, Section 2] for a detailed description of these admissible operations and generalized multicoils. In particular, one knows that all arrows of a generalized multicoil have trivial valuation.

We have the following consequence of [37, Theorem A].

Proposition 2.1. *Let A be an algebra and \mathcal{C} an infinite almost periodic component of Γ_A , and Γ a component of ${}_{\mathcal{C}}\mathcal{C}^{\text{coh}}$. The following statements hold.*

- (i) Γ is a cyclic generalized multicoil.
- (ii) There is a finite subquiver $\Sigma_{\Gamma}^{(l)}$ of Γ which is a disjoint union of sectional paths such that every path in \mathcal{C} from a module in $\mathcal{C} \setminus \Gamma$ to a module in Γ intersects $\Sigma_{\Gamma}^{(l)}$.
- (iii) There is a finite subquiver $\Sigma_{\Gamma}^{(r)}$ of Γ which is a disjoint union of sectional paths such that every path in \mathcal{C} from a module in Γ to a module in $\mathcal{C} \setminus \Gamma$ intersects $\Sigma_{\Gamma}^{(r)}$.
- (iv) Every module in Γ is a successor of a module in $\Sigma_{\Gamma}^{(l)}$ and a predecessor of a module in $\Sigma_{\Gamma}^{(r)}$.

We call $\Sigma_{\Gamma}^{(l)}$ and $\Sigma_{\Gamma}^{(r)}$ the *left border* and the *right border* of Γ , respectively. Further, we denote by \mathcal{T}_{Γ} the family of all indecomposable modules in Γ forming the stable tubes

used to create the cyclic generalized multicoil Γ by iterated application of admissible operations (ad 1)–(ad 5) and (ad 1*)–(ad 5*), and call it the *tubular part* of Γ . Moreover, we denote by $M_\Gamma^{(t)}$ the direct sum of all indecomposable modules lying on the mouth of the stable tubes in \mathcal{T}_Γ , and call it the *tubular module* of Γ . We note that $\text{rad}_A(M_\Gamma^{(t)}, M_\Gamma^{(t)}) = 0$ if and only if $M_\Gamma^{(t)}$ is a direct sum of pairwise orthogonal bricks.

The following consequence of [37, Sections 2 and 5] is also essential for our considerations.

Proposition 2.2. *Let A be an algebra, \mathcal{C} an infinite almost periodic component of Γ_A , Γ a component of ${}_c\mathcal{C}^{\text{coh}}$, and Ω a generalized multicoil enlargement of the family \mathcal{T}_Γ of stable tubes such that $\Gamma = {}_c\Omega$. Then the following statements hold.*

- (i) Ω is obtained from \mathcal{T}_Γ by iterated application of operations of type (ad 1) followed by operations of types (ad 1*)–(ad 5*).
- (ii) Ω is obtained from \mathcal{T}_Γ by iterated application of operations of type (ad 1*) followed by operations of types (ad 1)–(ad 5).

We note that an iterated application of operations of type (ad 1) (respectively, (ad 1*)) to a stable tube leads to a ray tube (respectively, a coray tube) in the sense of D’Este and Ringel [14].

In [38] we introduced the concept of a generalized multicoil enlargement of a product C of algebras with respect to a finite family $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s$ of pairwise orthogonal generalized standard stable tubes of Γ_C . Then such a generalized multicoil enlargement A of C is obtained from $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s$ by iterated application of admissible algebra operations (ad 1), (ad 2), (ad 3), (ad 4), (ad 5) and their duals (ad 1*), (ad 2*), (ad 3*), (ad 4*), (ad 5*). Then Γ_A admits a generalized standard generalized multicoil Γ obtained from the stable tubes $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s$ by iterated application of admissible translation quiver operations corresponding to the admissible algebra operations leading from C to A .

The following proposition follows from [38, Section 3].

Proposition 2.3. *Let A be an algebra, \mathcal{C} an infinite almost cyclic component of Γ_A , Γ a component of ${}_c\mathcal{C}^{\text{coh}}$, and $B(\Gamma)$ and $B(\mathcal{T}_\Gamma)$ the associated faithful algebras of Γ and \mathcal{T}_Γ . Then the following statements are equivalent:*

- (i) Γ is generalized standard.
- (ii) \mathcal{T}_Γ is generalized standard.
- (iii) \mathcal{T}_Γ is a finite faithful family of pairwise orthogonal generalized standard stable tubes in $\Gamma_{B(\mathcal{T}_\Gamma)}$, and $B(\Gamma)$ is a generalized multicoil enlargement of $B(\mathcal{T}_\Gamma)$ with respect to \mathcal{T}_Γ .

We have also the following consequence of [38, Sections 3 and 4].

Proposition 2.4. *Let B be a generalized multicoil enlargement of an algebra C (not necessarily indecomposable) with respect to a faithful family \mathcal{T} of pairwise orthogonal generalized standard stable tubes in Γ_C . Then the following statements hold.*

- (i) *B can be obtained from C by iterated application of algebra operations of type (ad 1) followed by algebra operations of types (ad 1*)–(ad 5*).*
- (ii) *B can be obtained from C by iterated application of algebra operations of type (ad 1*) followed by algebra operations of types (ad 1)–(ad 5).*

We note that an iterated application of algebra operations of type (ad 1) (respectively, (ad 1*)) to the family \mathcal{T} leads to a tubular extension (respectively, tubular coextension) of C in the sense of Ringel [51,56].

The following consequence of [58, Corollary 5.3] and [55, Theorem X.4.5] is relevant and provides a criterion for a stable tube to be generalized standard.

Proposition 2.5. *Let A be an algebra and \mathcal{T} be a stable tube of Γ_A . The following statements are equivalent:*

- (i) *\mathcal{T} is generalized standard.*
- (ii) *The mouth modules of \mathcal{T} are pairwise orthogonal bricks.*

We note that if A is an algebra and \mathcal{T} a faithful generalized standard stable tube of Γ_A , then $\text{pd}_A X \leq 1$ and $\text{id}_A X \leq 1$ for any module X in \mathcal{C} (see [58, Lemma 5.9]). But such an algebra A may have an arbitrary (finite or infinite) global dimension (see [64]). We refer also to [60] for results on the composition factors of modules lying in generalized standard stable tubes.

Let A be an algebra and \mathcal{C} be an almost periodic component of Γ_A . Recall that an indecomposable module X in \mathcal{C} is called *left stable* (respectively, *right stable*) if $\tau_A^n X$ is nonzero for all $n \geq 0$ (respectively, $n \leq 0$), and *stable* if it is left stable and right stable. Following [33], we denote by ${}_l\mathcal{C}$ the *left stable part* of \mathcal{C} , obtained by removing the τ_A -orbits containing projective modules, and by ${}_r\mathcal{C}$ the *right stable part* of \mathcal{C} , obtained by removing the τ_A -orbits containing injective modules. Moreover, we denote by ${}_l\mathcal{C}^{\text{ac}}$ (respectively, ${}_r\mathcal{C}^{\text{ac}}$) the union of components in ${}_l\mathcal{C}$ (respectively, ${}_r\mathcal{C}$) consisting entirely of acyclic modules. We call ${}_l\mathcal{C}^{\text{ac}}$ the *left stable acyclic part* of \mathcal{C} and ${}_r\mathcal{C}^{\text{ac}}$ the *right stable acyclic part* of \mathcal{C} . We note that all components in ${}_l\mathcal{C}^{\text{ac}}$ and ${}_r\mathcal{C}^{\text{ac}}$ have only finitely many τ_A -orbits, because \mathcal{C} is assumed to be almost periodic. Let Γ be a component of ${}_l\mathcal{C}^{\text{ac}}$. Then it follows from [33, Section 3] that there exists a finite connected acyclic full valued subquiver $\Delta_\Gamma^{(l)}$ of Γ which intersects every τ_A -orbit in Γ exactly once, and $\mathcal{Q}_\Gamma^{(l)} = \mathbb{N}\Delta_\Gamma^{(l)}$ is a full translation subquiver of Γ closed under predecessors. We denote by $M_\Gamma^{(l)}$ the direct sum of all indecomposable modules lying on $\Delta_\Gamma^{(l)}$. Dually, let Γ be a component of ${}_r\mathcal{C}^{\text{ac}}$. Then, applying [33, Section 3], we conclude that there exists a finite connected acyclic full valued subquiver $\Delta_\Gamma^{(r)}$ of Γ which intersects every τ_A -orbit in Γ exactly once,

and $\mathcal{D}_\Gamma^{(r)} = (-\mathbb{N})\Delta_\Gamma^{(r)}$ is a full translation subquiver of Γ closed under successors. We denote by $M_\Gamma^{(r)}$ the direct sum of all indecomposable modules lying on $\Delta_\Gamma^{(r)}$. In case Γ is a stable component, that is a component of ${}_l\mathcal{C}^{\text{ac}}$ and a component of ${}_r\mathcal{C}^{\text{ac}}$, we choose $\Delta_\Gamma^{(l)}$ and $\Delta_\Gamma^{(r)}$ such that $\mathcal{D}_\Gamma^{(l)}$ and $\mathcal{D}_\Gamma^{(r)}$ have no common modules.

Proposition 2.6. *Let A be an algebra, \mathcal{C} an almost periodic component of Γ_A with ${}_l\mathcal{C}^{\text{ac}}$ nonempty, and Γ a component of ${}_l\mathcal{C}^{\text{ac}}$. Then the following conditions are equivalent:*

- (i) $\mathcal{D}_\Gamma^{(l)}$ is generalized standard.
- (ii) $\text{Hom}_A(M_\Gamma^{(l)}, \tau_A M_\Gamma^{(l)}) = 0$.
- (iii) $A_\Gamma^{(l)} = B(M_\Gamma^{(l)})$ is a tilted algebra of the form $\text{End}_{H_\Gamma^{(l)}}(T_\Gamma^{(l)})$, for a hereditary algebra $H_\Gamma^{(l)}$ of type $\Delta_\Gamma^{(l)}$ and a tilting module $T_\Gamma^{(l)}$ in $\text{mod } H_\Gamma^{(l)}$ without indecomposable preinjective direct summands, and $\mathcal{D}_\Gamma^{(l)}$ is the image of the preinjective component $\mathcal{Q}_\Gamma^{(l)}$ of $\Gamma_{H_\Gamma^{(l)}}$ via the functor $\text{Hom}_{H_\Gamma^{(l)}}(T_\Gamma^{(l)}, -) : \text{mod } H_\Gamma^{(l)} \rightarrow \text{mod } A_\Gamma^{(l)}$.

Proof. (i) \implies (iii). Assume that $\mathcal{D}_\Gamma^{(l)}$ is generalized standard. Observe that the faithful algebra $A_\Gamma^{(l)}$ of $M_\Gamma^{(l)}$ is the faithful algebra $B(\mathcal{D}_\Gamma^{(l)})$ of $\mathcal{D}_\Gamma^{(l)}$. There exists a module N in the additive category $\text{add}(\mathcal{D}_\Gamma^{(l)})$ such that $\text{ann}_A(\mathcal{D}_\Gamma^{(l)}) = \text{ann}_A(N)$. Clearly, we have $\text{ann}_A(\mathcal{D}_\Gamma^{(l)}) \subseteq \text{ann}_A(M_\Gamma^{(l)})$. Consider a monomorphism $u : N \rightarrow E_A(N)$ from N to its injective envelope $E_A(N)$ in $\text{mod } A$. Since $\Delta_\Gamma^{(l)}$ is connected and intersects every τ_A -orbit in $\mathcal{D}_\Gamma^{(l)}$, we conclude that $\mathcal{D}_\Gamma^{(l)} \setminus \Delta_\Gamma^{(l)}$ does not contain an injective module. Then u factors through a module M from $\text{add}(M_\Gamma^{(l)})$, and hence there is a monomorphism $v : N \rightarrow M$. But then $\text{ann}_A(M_\Gamma^{(l)}) \subseteq \text{ann}_A(M) \subseteq \text{ann}_A(N) = \text{ann}_A(\mathcal{D}_\Gamma^{(l)})$. Therefore, indeed $A_\Gamma^{(l)} = B(\mathcal{D}_\Gamma^{(l)})$. Moreover, because $A_\Gamma^{(l)}$ is a quotient algebra of A , $\mathcal{D}_\Gamma^{(l)}$ is a full translation subquiver of the Auslander–Reiten quiver $\Gamma_{A_\Gamma^{(l)}}$ of $A_\Gamma^{(l)}$. Let $\mathcal{E}_\Gamma^{(l)}$ be the component of $\Gamma_{A_\Gamma^{(l)}}$ containing $\mathcal{D}_\Gamma^{(l)}$. We claim that $\Delta_\Gamma^{(l)}$ is a section of $\mathcal{E}_\Gamma^{(l)}$. It is enough to show that $\mathcal{E}_\Gamma^{(l)}$ does not contain an indecomposable projective $A_\Gamma^{(l)}$ -module. Suppose that $\mathcal{E}_\Gamma^{(l)}$ contains a projective module. We may choose minimal $m \in \mathbb{N}$ such that $\tau_{A_\Gamma^{(l)}}^{-m} \Delta_\Gamma^{(l)}$ contains an indecomposable direct summand, say R , of the radical of an indecomposable projective module P in $\mathcal{E}_\Gamma^{(l)}$. Then considering the projective cover of R in $\text{mod } A_\Gamma^{(l)}$ we conclude that there is an epimorphism from a module in $\text{add}(M_\Gamma^{(l)})$ to R , and hence a nonzero homomorphism $f : X \rightarrow R$ with X a module lying on $\Delta_\Gamma^{(l)}$. Further, since $M_\Gamma^{(l)}$ is a faithful $A_\Gamma^{(l)}$ -module, there is a monomorphism from P to a module in $\text{add}(M_\Gamma^{(l)})$, and hence $gf \neq 0$ for a homomorphism $g : R \rightarrow Y$ with Y a module lying on $\Delta_\Gamma^{(l)}$. This implies that gf is a nonzero homomorphism in $\text{rad}_{A_\Gamma^{(l)}}^\infty(X, Y)$, because there is no path in $\mathcal{E}_\Gamma^{(l)}$ from X to Y passing through R . Clearly, $\text{rad}_{A_\Gamma^{(l)}}^\infty(X, Y) \neq 0$ implies $\text{rad}_A^\infty(X, Y) \neq 0$, a contradiction because $\mathcal{D}_\Gamma^{(l)}$ is generalized standard. Therefore, $\Delta_\Gamma^{(l)}$ is a faithful section of the component of $\mathcal{E}_\Gamma^{(l)}$, and $\text{Hom}_{A_\Gamma^{(l)}}(M_\Gamma^{(l)}, \tau_{A_\Gamma^{(l)}} M_\Gamma^{(l)}) = \text{Hom}_A(M_\Gamma^{(l)}, \tau_A M_\Gamma^{(l)}) = 0$.

Applying now the criterion of Liu and Skowroński (see [1, Theorem VIII.5.6] or [68, Theorem VIII.7.7]) we conclude that $H_\Gamma^{(l)} = \text{End}_{A_\Gamma^{(l)}}(M_\Gamma^{(l)}) = \text{End}_A(M_\Gamma^{(l)})$ is a hereditary

algebra, $T_\Gamma^{(l)} = D(M_\Gamma^{(l)})$ is a tilting module in $\text{mod } H_\Gamma^{(l)}$, there is a canonical isomorphism of algebras $A_\Gamma^{(l)} \rightarrow \text{End}_{H_\Gamma^{(l)}}(T_\Gamma^{(l)})$, and $\mathcal{E}_\Gamma^{(l)}$ is the connecting component of $\Gamma_{A_\Gamma^{(l)}}$ with $\Delta_\Gamma^{(l)}$ the section determined by $T_\Gamma^{(l)}$. Moreover, $T_\Gamma^{(l)}$ is without an indecomposable preinjective $H_\Gamma^{(l)}$ -module, because $\mathcal{D}_\Gamma^{(l)}$ does not contain a projective module (see [68, Proposition VIII.6.9]). Finally, the translation quiver $\mathcal{D}_\Gamma^{(l)}$ is the image of the preinjective component $\mathcal{Q}_\Gamma^{(l)}$ of $\Gamma_{H_\Gamma^{(l)}}$ via the functor $\text{Hom}_{H_\Gamma^{(l)}}(T_\Gamma^{(l)}, -) : \text{mod } H_\Gamma^{(l)} \rightarrow \text{mod } A_\Gamma^{(l)}$ (see [68, Theorem VIII.6.7]).

(iii) \implies (ii). Assume that (iii) holds. Then the functor $\text{Hom}_{H_\Gamma^{(l)}}(T_\Gamma^{(l)}, -) : \text{mod } H_\Gamma^{(l)} \rightarrow \text{mod } A_\Gamma^{(l)}$ induces an equivalence of categories $\text{add}(\mathcal{Q}_\Gamma^{(l)}) \rightarrow \text{add}(\mathcal{D}_\Gamma^{(l)})$, with $\mathcal{Q}_\Gamma^{(l)}$ the preinjective component of $\Gamma_{H_\Gamma^{(l)}}$. Hence $\text{Hom}_{A_\Gamma^{(l)}}(M_\Gamma^{(l)}, \tau_{A_\Gamma^{(l)}} M_\Gamma^{(l)}) = 0$ follows.

(ii) \implies (i). Assume that (ii) holds. Suppose that there exist indecomposable modules X and Y in $\mathcal{D}_\Gamma^{(l)}$ such that $\text{rad}_A^\infty(X, Y) \neq 0$. Then it follows from [61, Lemma 2.1(i)] that there exist an infinite path

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{r-1}} X_{r-1} \xrightarrow{f_r} X_r \xrightarrow{f_{r+1}} \cdots$$

of irreducible homomorphisms between indecomposable modules in $\text{mod } A$ and $g_r \in \text{rad}_A^\infty(X_r, Y)$, $r \geq 1$, such that $g_r f_r \cdots f_1 \neq 0$ for any $r \geq 1$. Since $\mathcal{D}_\Gamma^{(l)}$ is an acyclic quiver of the form $\mathbb{N}\Delta_\Gamma^{(l)}$ and X belongs to $\mathcal{D}_\Gamma^{(l)}$, there is $s \geq 1$ such that X_s belongs to $\Delta_\Gamma^{(l)}$. Because $\text{rad}_A^\infty(X_s, Y) \neq 0$, applying [61, Lemma 2.1(ii)], we conclude that there exist an infinite path

$$\cdots \xrightarrow{h_{t+1}} Y_t \xrightarrow{h_t} Y_{t-1} \xrightarrow{h_{t-1}} \cdots \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = Y$$

of irreducible homomorphisms between indecomposable modules in $\text{mod } A$ and $u_t \in \text{rad}_A^\infty(X_s, Y_t)$, $t \geq 1$, such that $h_1 \cdots h_t u_t \neq 0$ for any $t \geq 1$. Then there exists $m \geq 1$ such that Y_t belongs to the subquiver $\mathbb{N}\tau_A \Delta_\Gamma^{(l)}$ of $\mathcal{D}_\Gamma^{(l)} = \mathbb{N}\Delta_\Gamma^{(l)}$, for any $t \geq m$. Let $U = X_s$, $V = Y_m$, and $f : U \rightarrow V$ a nonzero homomorphism in $\text{rad}_A^\infty(U, V)$. Consider now a monomorphism $v : V \rightarrow E_A(V)$, where $E_A(V)$ is an injective envelope of V in $\text{mod } A$. Observe that the subquiver $\mathbb{N}\tau_A \Delta_\Gamma^{(l)}$ has no injective module. Hence v factors through a module N in $\text{add}(\tau_A \Delta_\Gamma^{(l)})$, and then there is a monomorphism $w : V \rightarrow N$. Then $wu_m \neq 0$, and hence there is an indecomposable direct summand W of N such that $\text{Hom}_A(U, W) \neq 0$. But then $\text{Hom}_A(M_\Gamma^{(l)}, \tau_A M_\Gamma^{(l)}) \neq 0$, because U is a direct summand of $M_\Gamma^{(l)}$ and W is a direct summand of $\tau_A M_\Gamma^{(l)}$. This shows that (ii) implies (i).

We have the dual proposition.

Proposition 2.7. *Let A be an algebra, \mathcal{C} an almost periodic component of Γ_A with ${}_r\mathcal{C}^{\text{ac}}$ nonempty, and Γ a component of ${}_r\mathcal{C}^{\text{ac}}$. Then the following conditions are equivalent:*

- (i) $\mathcal{D}_\Gamma^{(r)}$ is generalized standard.

- (ii) $\text{Hom}_A(\tau_A^{-1}M_\Gamma^{(r)}, M_\Gamma^{(r)}) = 0$.
- (iii) $A_\Gamma^{(r)} = B(M_\Gamma^{(r)})$ is a tilted algebra of the form $\text{End}_{H_\Gamma^{(r)}}(T_\Gamma^{(r)})$, for a hereditary algebra $H_\Gamma^{(r)}$ of type $\Delta_\Gamma^{(r)}$ and a tilting module $T_\Gamma^{(r)}$ in $\text{mod } H_\Gamma^{(r)}$ without indecomposable postprojective direct summands, and $\mathcal{D}_\Gamma^{(r)}$ is the image of the postprojective component $\mathcal{P}_\Gamma^{(r)}$ of $\Gamma_{H_\Gamma^{(r)}}$ via the functor $\text{Ext}_{H_\Gamma^{(r)}}^1(T_\Gamma^{(r)}, -) : \text{mod } H_\Gamma^{(r)} \rightarrow \text{mod } A_\Gamma^{(r)}$.

3. Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2, using the results presented in the previous section.

Let A be an algebra and \mathcal{C} be an infinite generalized standard component of Γ_A . Then, by [58, Theorem 2.3], \mathcal{C} is almost periodic. We use the notation introduced above. We consider the following ideals of A :

- $I_\mathcal{C}^{(l)}$ the annihilator of the disjoint union of the translation quivers $\mathcal{D}_\Gamma^{(l)}$, for all components Γ of ${}_l\mathcal{C}^{\text{ac}}$,
- $I_\mathcal{C}^{(r)}$ the annihilator of the disjoint union of the translation quivers $\mathcal{D}_\Gamma^{(r)}$, for all components Γ of ${}_r\mathcal{C}^{\text{ac}}$,
- $I_\mathcal{C}^{(c)}$ the annihilator of the disjoint union of all components Γ of ${}_c\mathcal{C}^{\text{coh}}$.

Further, we consider the quotient algebras $A_\mathcal{C}^{(lt)} = A/I_\mathcal{C}^{(l)}$, $A_\mathcal{C}^{(rt)} = A/I_\mathcal{C}^{(r)}$, $A_\mathcal{C}^{(c)} = A/I_\mathcal{C}^{(c)}$. Then it follows from Propositions 2.1, 2.3, 2.6 and 2.7 that the required statements (i), (ii), (iii) are satisfied.

We prove that the statement (iv) holds. Indeed, let X be an acyclic module in \mathcal{C} which does not belong to a subquiver of the form $\mathcal{D}_\Gamma^{(l)}$ or $\mathcal{D}_\Gamma^{(r)}$. Then there exists a nonnegative integer m_X such that $\tau_A^{m_X}X$ is projective, or belongs to $\Delta_\Gamma^{(l)}$, for a component Γ of ${}_l\mathcal{C}^{\text{ac}}$, or belongs to the right border $\Sigma_\Gamma^{(r)}$ of a component Γ of ${}_c\mathcal{C}^{\text{coh}}$. Similarly, there is a nonnegative integer n_X such that $\tau_A^{-n_X}X$ is injective, or belongs to $\Delta_\Gamma^{(r)}$, for a component Γ of ${}_r\mathcal{C}^{\text{ac}}$, or belongs to the left border $\Sigma_\Gamma^{(l)}$ of a component Γ of ${}_c\mathcal{C}^{\text{coh}}$. Hence there are at most finitely many such acyclic modules in \mathcal{C} , and (iv) follows.

The statement (v) will follow from the facts below.

- (1) It follows from [45, Proposition 2.4 and Theorem 2.5] that every finite component of ${}_c\mathcal{C}$ contains an indecomposable projective module and indecomposable injective module. Hence there are at most finitely many indecomposable modules lying in finite cyclic components of ${}_c\mathcal{C}$.
- (2) An indecomposable module X in \mathcal{C} is said to be stable if it is left stable and right stable. We denote by ${}_s\mathcal{C}$ the stable part of \mathcal{C} obtained by removing all nonstable modules and the arrows attached to them. It is known (see [33], [71] or [68, Theorem IX.4.9]) that an infinite component Γ of ${}_s\mathcal{C}$ contains an oriented cycle if and only if Γ is a stable tube. Moreover, every stable tube of ${}_s\mathcal{C}$ consists of periodic coherent

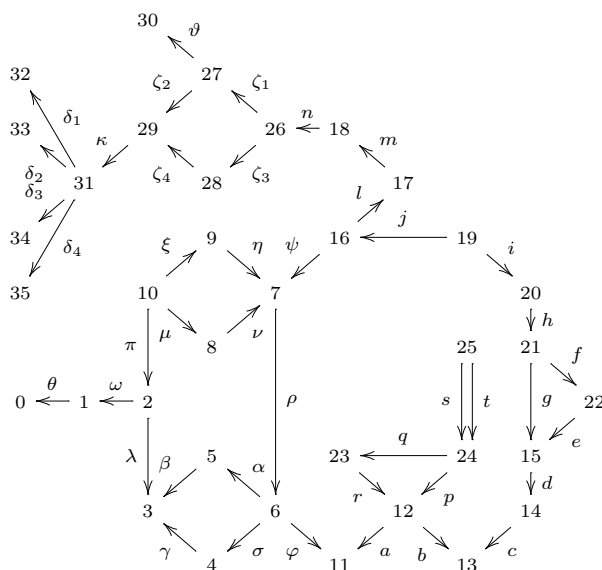
modules, and hence is contained in ${}_c\mathcal{C}^{\text{coh}}$. On the other hand, every finite component Γ of ${}_s\mathcal{C}$ containing an oriented cycle consists of periodic modules, contains an immediate predecessor of a projective module in \mathcal{C} (respectively, an immediate successor of an injective module in \mathcal{C}). Hence the number of indecomposable modules in ${}_c\mathcal{C} \setminus {}_c\mathcal{C}^{\text{coh}}$ lying in ${}_s\mathcal{C}$ is finite.

- (3) Let X be a module in ${}_c\mathcal{C} \setminus {}_c\mathcal{C}^{\text{coh}}$ such that $\tau_A^n X$ belongs to ${}_c\mathcal{C}^{\text{coh}}$ for some positive integer n . Then there exist $m \in \{1, \dots, n\}$ and a component Γ of ${}_c\mathcal{C}^{\text{coh}}$ such that $\tau_A^m X$ belongs to the right border $\Sigma_{\Gamma}^{(r)}$ of Γ .
- (4) Let X be a module in ${}_c\mathcal{C} \setminus {}_c\mathcal{C}^{\text{coh}}$ such that $\tau_A^{-n} X$ belongs to ${}_c\mathcal{C}^{\text{coh}}$ for some positive integer n . Then there exist $m \in \{1, \dots, n\}$ and a component Γ of ${}_c\mathcal{C}^{\text{coh}}$ such that $\tau_A^{-m} X$ belongs to the left border $\Sigma_{\Gamma}^{(l)}$ of Γ .
- (5) Let X be a nonperiodic, nonprojective module in ${}_c\mathcal{C} \setminus {}_c\mathcal{C}^{\text{coh}}$ such that, for any positive integer r , $\tau_A^r X$ does not belong to ${}_c\mathcal{C}^{\text{coh}}$. Then there exists a positive integer m such that $\tau_A^m X$ is projective or belongs to $\Delta_{\Gamma}^{(l)}$ for a component Γ of ${}_l\mathcal{C}^{\text{ac}}$.
- (6) Let X be a nonperiodic, noninjective module in ${}_c\mathcal{C} \setminus {}_c\mathcal{C}^{\text{coh}}$ such that, for any positive integer s , $\tau_A^{-s} X$ does not belong to ${}_c\mathcal{C}^{\text{coh}}$. Then there exists a positive integer n such that $\tau_A^{-n} X$ is injective or belongs to $\Delta_{\Gamma}^{(r)}$ for a component Γ of ${}_r\mathcal{C}^{\text{ac}}$.

Summing up, we conclude that ${}_c\mathcal{C} \setminus {}_c\mathcal{C}^{\text{coh}}$ is finite, which is the statement (v).

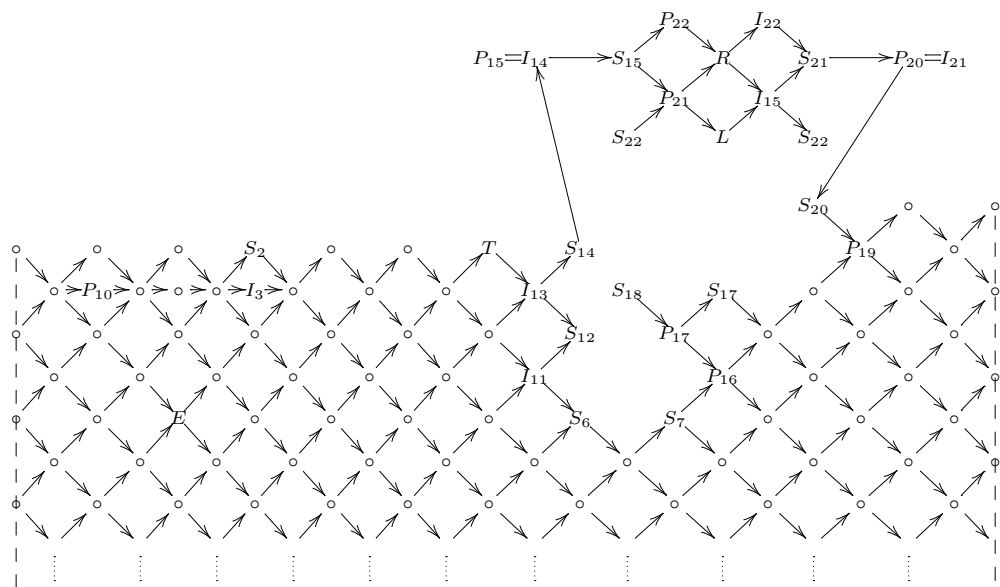
We end this section with an example illustrating the above considerations.

Example 3.1. Let K be an algebraically closed field, Q the quiver of the form

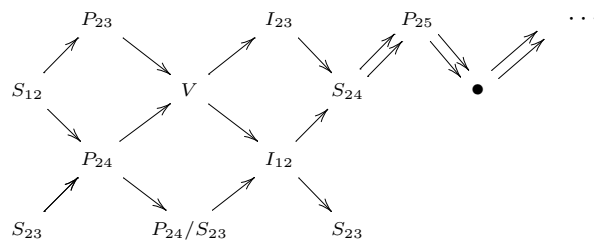


and I the ideal in the path algebra KQ of Q over K generated by the paths $\alpha\beta - \sigma\gamma$, $\xi\eta - \mu\nu$, $\pi\lambda - \xi\eta\rho\alpha\beta$, $\rho\varphi$, $\psi\rho$, jl , dc , ed , gd , hg , hf , fe , ih , pa , pb , ra , rb , qr , sq , sp , tg , tp , mn ,

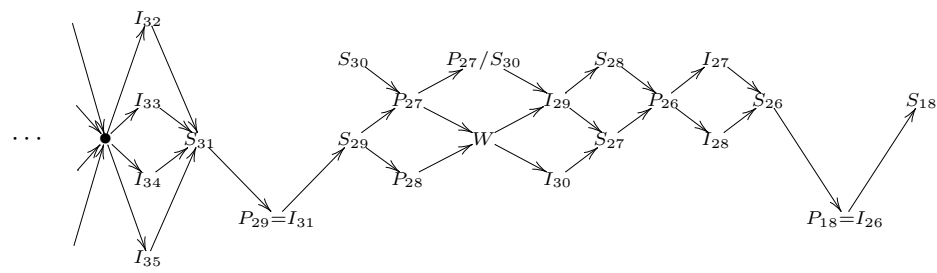
$n\zeta_1, n\zeta_3, \zeta_1\vartheta, \zeta_1\zeta_2, \zeta_3\zeta_4, \zeta_2\kappa, \zeta_4\kappa, \kappa\delta_1, \kappa\delta_2, \kappa\delta_3, \kappa\delta_4$, and $A = KQ/I$. We mention that this is an enlarged version of [36, Example 6.4]. The Auslander–Reiten quiver Γ_A admits a generalized standard component \mathcal{C} obtained by identification of the common simple modules S_{12} and S_{18} occurring in the following three translation quivers: Γ_1 of the form



Γ_2 of the form



and Γ_3 of the form



The cyclic part ${}_c\mathcal{C}$ of \mathcal{C} consists of one infinite component Γ and one finite component Γ' described as follows. The infinite cyclic component Γ is obtained by removing from \mathcal{C} the modules $S_{12}, S_{17}, S_{18}, P_{17}$ together with the arrows attached to them, and the translation quivers Γ_2, Γ_3 . The finite cyclic component Γ' is the full translation subquiver of \mathcal{C} given by the vertices $S_{23}, P_{24}, P_{24}/S_{23}, V, I_{12}$. The maximal cyclic coherent part ${}_c\Gamma^{\text{coh}}$ of Γ is the full translation subquiver of \mathcal{C} obtained by removing from \mathcal{C} the modules $S_{12}, I_{13}, T, S_{14}, P_{15} = I_{14}, S_{15}, P_{21}, S_{22}, L, P_{22}, R, I_{15}, I_{22}, S_{21}, P_{20} = I_{21}, S_{20}, S_{17}, P_{17}, S_{18}$ together with the arrows attached to them (see [36, Example 6.4]), and the translation quivers Γ_2, Γ_3 . We observe that S_{22} is a periodic module of Γ which does not belong to ${}_c\Gamma^{\text{coh}}$.

Moreover, we have

$$\begin{aligned} P_{\mathcal{C}} &= P_{10} \oplus P_{15} \oplus P_{16} \oplus P_{17} \oplus P_{18} \oplus P_{19} \oplus P_{20} \oplus P_{21} \oplus P_{22} \oplus P_{23} \oplus P_{24} \oplus P_{25} \\ &\quad \oplus P_{26} \oplus P_{27} \oplus P_{28} \oplus P_{29} \oplus P_{30}, \end{aligned}$$

$$P_{\mathcal{C}}^{\text{coh}} = P_{10} \oplus P_{16} \oplus P_{17} \oplus P_{19} \oplus P_{20},$$

$$\begin{aligned} Q_{\mathcal{C}} &= I_3 \oplus I_{11} \oplus I_{12} \oplus I_{13} \oplus I_{14} \oplus I_{15} \oplus I_{21} \oplus I_{22} \oplus I_{23} \oplus I_{26} \oplus I_{27} \\ &\quad \oplus I_{28} \oplus I_{29} \oplus I_{30} \oplus I_{31} \oplus I_{32} \oplus I_{33} \oplus I_{34} \oplus I_{35}, \end{aligned}$$

$$Q_{\mathcal{C}}^{\text{coh}} = I_3 \oplus I_{11} \oplus I_{13} \oplus I_{14},$$

$$M_{\mathcal{C}}^{(l)} = I_{32} \oplus I_{33} \oplus I_{34} \oplus I_{35} \oplus S_{31}, \quad M_{\mathcal{C}}^{(r)} = P_{25} \oplus S_{24}, \quad M_{\mathcal{C}}^{(t)} = S_6 \oplus S_7 \oplus E.$$

We note that P_{20} is a right coherent projective module and I_{14} is a left coherent injective module in Γ which do not belong to ${}_c\Gamma^{\text{coh}}$.

4. Proof of Theorem 1.1

Let A be an algebra and \mathcal{C} be an infinite component of Γ_A . Since every generalized standard component is almost periodic, we may assume that \mathcal{C} is almost periodic. We use the notation introduced in Section 2. Hence

- $M_{\mathcal{C}}^{(l)}$ is the direct sum of all modules $M_{\Gamma}^{(l)}$ given by the quivers $\Delta_{\Gamma}^{(l)}$ associated to all components Γ of the left stable acyclic part ${}_l\mathcal{C}^{\text{ac}}$ of \mathcal{C} ,
- $M_{\mathcal{C}}^{(r)}$ is the direct sum of all modules $M_{\Gamma}^{(r)}$ given by the quivers $\Delta_{\Gamma}^{(r)}$ associated to all components Γ of the right stable acyclic part ${}_r\mathcal{C}^{\text{ac}}$ of \mathcal{C} ,
- $M_{\mathcal{C}}^{(t)}$ is the direct sum of all modules $M_{\Gamma}^{(t)}$ given by the mouth of the tubular parts \mathcal{T}_{Γ} of all components Γ of the cyclic coherent part ${}_c\mathcal{C}^{\text{coh}}$ of \mathcal{C} .

Further, let

- $P_{\mathcal{C}}$ be the direct sum of all projective modules in \mathcal{C} , and $P_{\mathcal{C}}^{\text{coh}}$ the direct sum of all right coherent projective modules in \mathcal{C} ,

- $Q_{\mathcal{C}}$ be the direct sum of all injective modules in \mathcal{C} , and $Q_{\mathcal{C}}^{\text{coh}}$ the direct sum of all left coherent injective modules in \mathcal{C} .

Moreover, let $B(\mathcal{C})$ be the faithful algebra $A/\text{ann}_A(\mathcal{C})$ of \mathcal{C} . Then \mathcal{C} is a faithful component of $\Gamma_{B(\mathcal{C})}$ and $\text{Hom}_A(X, Y) = \text{Hom}_{B(\mathcal{C})}(X, Y)$ for any modules X and Y in \mathcal{C} . By general theory, we know that \mathcal{C} is a generalized standard component in Γ_A if and only if \mathcal{C} is a generalized standard component in $\Gamma_{B(\mathcal{C})}$. Therefore, we may assume that $A = B(\mathcal{C})$.

We first prove that (i) implies (ii). Assume that \mathcal{C} is generalized standard. The any nonzero homomorphism $f : X \rightarrow Y$ between two indecomposable modules X and Y in \mathcal{C} is a finite sum of compositions of irreducible homomorphisms between indecomposable modules in \mathcal{C} (see [4, Proposition V.7.5] or [68, Proposition VII.3.9]). In particular, $\text{rad}_A(X, Y) \neq 0$, for X and Y in \mathcal{C} , implies that there is a path in \mathcal{C} from X to Y .

- (1) It follows from the definition of $M_{\mathcal{C}}^{(l)}$ that there is no path in \mathcal{C} from an indecomposable direct summand of $P_{\mathcal{C}} \oplus M_{\mathcal{C}}^{(t)} \oplus M_{\mathcal{C}}^{(r)}$ to an indecomposable direct summand of $M_{\mathcal{C}}^{(l)}$, and hence $\text{Hom}_A(P_{\mathcal{C}} \oplus M_{\mathcal{C}}^{(t)} \oplus M_{\mathcal{C}}^{(r)}, M_{\mathcal{C}}^{(l)}) = 0$. Similarly, it follows from the definition of $M_{\mathcal{C}}^{(r)}$ that there is no path in \mathcal{C} from an indecomposable direct summand of $M_{\mathcal{C}}^{(r)}$ to an indecomposable direct summand of $Q_{\mathcal{C}} \oplus M_{\mathcal{C}}^{(t)} \oplus M_{\mathcal{C}}^{(l)}$, and hence $\text{Hom}_A(M_{\mathcal{C}}^{(r)}, Q_{\mathcal{C}} \oplus M_{\mathcal{C}}^{(t)} \oplus M_{\mathcal{C}}^{(l)}) = 0$.
- (2) For a component Γ of ${}_l\mathcal{C}^{\text{ac}}$, the translation quiver $\mathcal{D}_{\Gamma}^{(l)}$ is generalized standard, and then $\text{Hom}_A(M_{\Gamma}^{(l)}, \tau_A M_{\Gamma}^{(l)}) = 0$, by Proposition 2.6. Similarly, for a component Γ of ${}_r\mathcal{C}^{\text{ac}}$, the translation quiver $\mathcal{D}_{\Gamma}^{(r)}$ is generalized standard, and then $\text{Hom}_A(\tau_A^{-1} M_{\Gamma}^{(r)}, M_{\Gamma}^{(r)}) = 0$, by Proposition 2.7. Therefore, we conclude that $\text{Hom}_A(M_{\mathcal{C}}^{(l)}, \tau_A M_{\mathcal{C}}^{(l)}) = 0$ and $\text{Hom}_A(\tau_A^{-1} M_{\mathcal{C}}^{(r)}, M_{\mathcal{C}}^{(r)}) = 0$.
- (3) Let Γ be a component of ${}_c\mathcal{C}^{\text{coh}}$ and \mathcal{T}_{Γ} the tubular part of Γ . Since \mathcal{C} is generalized standard, we conclude that Γ is also generalized standard. Then it follows from Proposition 2.3 that \mathcal{T}_{Γ} is a finite faithful family of pairwise orthogonal generalized standard stable tubes in $\Gamma_{B(\mathcal{T}_{\Gamma})}$ and $B(\Gamma)$ is a generalized multicoil enlargement of $B(\mathcal{T}_{\Gamma})$ with respect to the family \mathcal{T}_{Γ} . In particular, applying Proposition 2.5, we obtain that $\text{rad}_A(M_{\Gamma}^{(t)}, M_{\Gamma}^{(t)}) = 0$. Moreover, the simple composition factors of $M_{\Gamma}^{(t)}$ are not direct summands of $\text{top}(P_{\mathcal{C}}) \oplus \text{soc}(Q_{\mathcal{C}})$, and hence $\text{Hom}_A(P_{\mathcal{C}}, M_{\Gamma}^{(t)}) = 0$ and $\text{Hom}_A(M_{\Gamma}^{(t)}, Q_{\mathcal{C}}) = 0$. We also note that, for two different components Γ and Ω of ${}_c\mathcal{C}^{\text{coh}}$, the tubular parts \mathcal{T}_{Γ} and \mathcal{T}_{Ω} are disjoint and orthogonal, by Propositions 2.1 and 2.3, and hence $\text{rad}_A(M_{\Gamma}^{(t)}, M_{\Omega}^{(t)}) = 0$ and $\text{rad}_A(M_{\Omega}^{(t)}, M_{\Gamma}^{(t)}) = 0$. Summing up we have

$$\text{Hom}_A(P_{\mathcal{C}}, M_{\mathcal{C}}^{(t)}) = 0, \quad \text{Hom}_A(M_{\mathcal{C}}^{(t)}, Q_{\mathcal{C}}) = 0, \quad \text{rad}_A(M_{\mathcal{C}}^{(t)}, M_{\mathcal{C}}^{(t)}) = 0.$$

- (4) We shall prove that $\text{Hom}_A(\text{top}(P_{\mathcal{C}}^{\text{coh}}), Q_{\mathcal{C}}) = 0$. Let $S = \text{top}(P)$ for an indecomposable direct summand P of $P_{\mathcal{C}}^{\text{coh}}$. Assume first that S does not belong to \mathcal{C} . Since \mathcal{C} is generalized standard and there is a canonical epimorphism $P \rightarrow S$, we conclude

that $\text{Hom}_A(S, \text{soc}(Q_{\mathcal{C}})) = \text{Hom}_A(S, Q_{\mathcal{C}}) = 0$. Therefore, assume that S belongs to \mathcal{C} . Since P is right coherent, there exists an infinite sectional path

$$P = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow X_{i+2} \longrightarrow \cdots$$

with all but finitely many modules from a cyclic generalized multicoil Γ of ${}_c\mathcal{C}^{\text{coh}}$ (see Proposition 2.1). Moreover, by Propositions 2.1 and 2.3, $B(\Gamma)$ is a generalized multicoil enlargement of $B(\mathcal{T}_{\Gamma})$ with respect to the tubular family \mathcal{T}_{Γ} of Γ , and Γ is the cyclic part of the generalized multicoil Ω , created from \mathcal{T}_{Γ} by iterated application of translation quiver admissible operations corresponding to the algebra admissible operations leading from $B(\mathcal{T}_{\Gamma})$ to $B(\Gamma)$. We have also $B(\Gamma) = B(\Omega)$. By Proposition 2.4 we may obtain $B(\Omega)$ from $B(\mathcal{T}_{\Gamma})$ applying first a sequence of admissible operations of type (ad 1), creating a tubular extension B of $B(\mathcal{T}_{\Gamma})$, and then applying a sequence of admissible operations of types (ad 1*)–(ad 5*). Then the projective module P and its top S are acyclic modules in one of the created ray tubes of Γ_B . In particular, we conclude that the injective envelope $E_{B(\Omega)}(S)$ of S in $\text{mod } B(\Omega)$ does not belong to Ω , and consequently $\text{rad}_{B(\Omega)}^{\infty}(S, E_{B(\Omega)}(S)) \neq 0$. Since $B(\Omega)$ is a quotient algebra of A , we conclude that $E_{B(\Omega)}(S)$ is a submodule of an injective envelope $E_A(S)$ of S in $\text{mod } A$, and hence $\text{rad}_A^{\infty}(S, E_A(S)) \neq 0$. This shows that $E_A(S)$ does not belong to \mathcal{C} , because \mathcal{C} is generalized standard. Hence $\text{Hom}_A(S, Q_{\mathcal{C}}) = 0$. Therefore, we obtain $\text{Hom}_A(\text{top}(P_{\mathcal{C}}^{\text{coh}}), Q_{\mathcal{C}}) = 0$.

- (5) Applying dual arguments to those in (4), we conclude that $\text{Hom}_A(P_{\mathcal{C}}, \text{soc}(Q_{\mathcal{C}}^{\text{coh}})) = 0$.

This finishes the proof of the implication (i) \implies (ii).

Assume now that the conditions of (ii) are satisfied. We shall show that \mathcal{C} is generalized standard in few steps. We use the notation introduced in Section 2. Moreover, we denote by $\mathcal{D}_{\mathcal{C}}^{(l)}$ the disjoint union of the translation quivers $\mathcal{D}_{\Gamma}^{(l)}$ for all components Γ of ${}_l\mathcal{C}^{\text{ac}}$, and by $\mathcal{D}_{\mathcal{C}}^{(r)}$ the disjoint union of the translation quivers $\mathcal{D}_{\Gamma}^{(r)}$ for all components Γ of ${}_r\mathcal{C}^{\text{ac}}$.

- (a) Let Γ be a component of ${}_l\mathcal{C}^{\text{ac}}$. Since $\text{Hom}_A(M_{\mathcal{C}}^{(l)}, \tau_A M_{\mathcal{C}}^{(l)}) = 0$, we have $\text{Hom}_A(M_{\Gamma}^{(l)}, \tau_A M_{\Gamma}^{(l)}) = 0$ and hence the translation quiver $\mathcal{D}_{\Gamma}^{(l)}$ is generalized standard, by Proposition 2.6.
- (b) Let Γ be a component of ${}_r\mathcal{C}^{\text{ac}}$. Since $\text{Hom}_A(\tau_A^{-1} M_{\mathcal{C}}^{(r)}, M_{\mathcal{C}}^{(r)}) = 0$, we have $\text{Hom}_A(\tau_A^{-1} M_{\Gamma}^{(r)}, M_{\Gamma}^{(r)}) = 0$ and hence the translation quiver $\mathcal{D}_{\Gamma}^{(r)}$ is generalized standard, by Proposition 2.7.
- (c) Let Γ be a component of ${}_c\mathcal{C}^{\text{coh}}$. Then the condition $\text{rad}_A(M_{\mathcal{C}}^{(t)}, M_{\mathcal{C}}^{(t)}) = 0$ implies $\text{rad}_A(M_{\Gamma}^{(t)}, M_{\Gamma}^{(t)}) = 0$. Let \mathcal{T}_{Γ} be the tubular family of Γ , $C = B(\mathcal{T}_{\Gamma})$ and $B = B(\Gamma)$. Since $\text{Hom}_A(P_{\mathcal{C}}, M_{\Gamma}^{(t)}) = 0$ (respectively, $\text{Hom}_A(M_{\Gamma}^{(t)}, Q_{\mathcal{C}}) = 0$), we have $\text{Hom}_A(P_{\mathcal{C}}, Z) = 0$ (respectively, $\text{Hom}_A(Z, Q_{\mathcal{C}}) = 0$) for any indecomposable module Z in \mathcal{T}_{Γ} . Hence \mathcal{T}_{Γ} is a faithful family of stable tubes in Γ_C with $\text{rad}_C(M_{\Gamma}^{(t)}, M_{\Gamma}^{(t)}) = 0$.

Then it follows from Proposition 2.5 that \mathcal{T}_Γ is a family of generalized standard stable tubes in Γ_C . Moreover, the tubes in \mathcal{T}_Γ are pairwise orthogonal. Indeed, if $\text{Hom}_C(U, V) \neq 0$ for two modules in different tubes, say \mathcal{T}_1 containing U and \mathcal{T}_2 containing V , then $\text{rad}_C(M_1, M_2) = \text{Hom}_C(M_1, M_2) \neq 0$ for some modules M_1 from the mouth of \mathcal{T}_1 and M_2 from the mouth of \mathcal{T}_2 , and this contradicts the assumption $\text{rad}_A(M_{\mathcal{C}}^{(t)}, M_{\mathcal{C}}^{(t)}) = 0$. Then it follows from Proposition 2.3 that Γ is a generalized standard cyclic multicoil and B is a generalized multicoil enlargement of C with respect to the family \mathcal{T}_Γ . Therefore, we conclude that ${}_C\mathcal{C}^{\text{coh}}$ consists of generalized standard components, which are cyclic generalized multicoils (Proposition 2.1).

- (d) It follows from the arguments applied in the proofs of statements (iv) and (v) of Theorem 1.2 that the translation quiver $\mathcal{D}_{\mathcal{C}}^{(l)} \cup {}_C\mathcal{C}^{\text{coh}} \cup \mathcal{D}_{\mathcal{C}}^{(r)}$ is a cofinite subquiver of \mathcal{C} . Hence, if N is the direct sum of all indecomposable modules of \mathcal{C} which do not belong to this translation subquiver, then $\text{rad}_A^l(N, N) = 0$ for some positive integer l .
- (e) Assume now that $\text{rad}_A^\infty(X, Y) \neq 0$ for some indecomposable modules X and Y in \mathcal{C} . It follows from [61, Lemma 2.1(i)] that there exist an infinite path

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{s-1}} X_{s-1} \xrightarrow{f_s} X_s \xrightarrow{f_{s+1}} \cdots$$

of irreducible homomorphisms between indecomposable modules in \mathcal{C} and homomorphisms $g_s \in \text{rad}_A^\infty(X_s, Y)$, $s \geq 1$, such that $g_s f_s \cdots f_1 \neq 0$ for any $s \geq 1$. Then it follows from (d) that there exists a positive integer m such that X_s belongs to ${}_C\mathcal{C}^{\text{coh}} \cup \mathcal{D}_{\mathcal{C}}^{(r)}$ for any $s \geq m$. Hence, we may assume that X belongs to ${}_C\mathcal{C}^{\text{coh}} \cup \mathcal{D}_{\mathcal{C}}^{(r)}$. Applying [61, Lemma 2.1(ii)], we conclude that there exist an infinite path

$$\cdots \xrightarrow{h_{r+1}} Y_r \xrightarrow{h_r} Y_{r-1} \xrightarrow{h_{r-1}} \cdots \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = Y$$

of irreducible homomorphisms between indecomposable modules in \mathcal{C} and homomorphisms $u_r \in \text{rad}_A^\infty(X, Y_r)$, $r \geq 1$, such that $h_1 \cdots h_r u_r \neq 0$ for any $r \geq 1$. Then it follows from (d) that there exists a positive integer n such that Y_r belongs to $\mathcal{D}_{\mathcal{C}}^{(l)} \cup {}_C\mathcal{C}^{\text{coh}}$ for any $r \geq n$. Hence, we may also assume that Y belongs to $\mathcal{D}_{\mathcal{C}}^{(l)} \cup {}_C\mathcal{C}^{\text{coh}}$.

- (f) Assume that X belongs to $\mathcal{D}_{\mathcal{C}}^{(r)}$ and Y belongs to $\mathcal{D}_{\mathcal{C}}^{(l)}$. Let X belongs to $\mathcal{D}_\Gamma^{(r)}$ and Y belongs to $\mathcal{D}_\Omega^{(l)}$ for some components Γ of ${}_r\mathcal{C}^{\text{ac}}$ and Ω of ${}_l\mathcal{C}^{\text{ac}}$. Then there are an epimorphism $(M_\Gamma^{(r)})^a \rightarrow X$ and a monomorphism $Y \rightarrow (M_\Omega^{(l)})^b$ for some positive integers a, b , and $\text{rad}_A^\infty(X, Y) \neq 0$ implies that $\text{Hom}_A(M_\Gamma^{(r)}, M_\Omega^{(l)}) \neq 0$. But this contradicts the assumption $\text{Hom}_A(M_{\mathcal{C}}^{(r)}, M_{\mathcal{C}}^{(l)}) = 0$.
- (g) Assume that X belongs to $\mathcal{D}_{\mathcal{C}}^{(r)}$ and Y belongs to ${}_C\mathcal{C}^{\text{coh}}$. Let X belongs to $\mathcal{D}_\Gamma^{(r)}$ for a component Γ of ${}_r\mathcal{C}^{\text{ac}}$ and Y belongs to a component Ω of ${}_C\mathcal{C}^{\text{coh}}$. It follows from (f) that we may also assume that X belongs to $\Delta_\Gamma^{(r)}$. We know from [37, Corollary B] that all but finitely many modules in the generalized multicoil Ω have exactly

two direct predecessors and two direct successors. Hence we may assume that there is an infinite sectional path

$$Y = Z_0 \xrightarrow{v_1} Z_1 \xrightarrow{v_2} \cdots \xrightarrow{v_{i-1}} Z_{i-1} \xrightarrow{v_i} Z_i \xrightarrow{v_{i+1}} \cdots$$

of irreducible homomorphisms between indecomposable modules in Ω such that, for every $i \geq 1$, there is a mesh complete translation subquiver of the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & U_{j+1}^{(i)} & \longrightarrow & U_j^{(i)} & \longrightarrow & \cdots \longrightarrow U_1^{(i)} \longrightarrow U_0^{(i)} = Z_{i-1} \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & V_{j+1}^{(i)} & \longrightarrow & V_j^{(i)} & \longrightarrow & \cdots \longrightarrow V_1^{(i)} \longrightarrow V_0^{(i)} = Z_i \end{array}$$

Then it follows from [32, Corollary 1.6] (see also [68, Corollary IX.3.15]) that the homomorphisms v_i , $i \geq 1$, are of infinite right degree, and hence $v_i \cdots v_1 f \neq 0$ for any nonzero homomorphism $f : X \rightarrow Y$ and $i \geq 1$. It follows from the structure of the generalized multicoil (see [37, Section 2] and [38, Section 3]) that infinitely many modules Z_i belong to the tubular part \mathcal{T}_Ω of Ω . Hence $\text{Hom}_A(X, Z) \neq 0$ for a module Z in \mathcal{T}_Ω . But then $\text{Hom}_A(X, W) \neq 0$ for a module W lying on the mouth of a stable tube of \mathcal{T}_Ω . Therefore, we have $\text{Hom}_A(M_\Gamma^{(r)}, M_\Omega^{(t)}) \neq 0$, which contradicts the condition $\text{Hom}_A(M_\mathcal{C}^{(r)}, M_\mathcal{C}^{(t)}) = 0$.

- (h) Assume that X belongs to ${}_{\mathcal{C}}\mathcal{C}^{\text{coh}}$ and Y belongs to $\mathcal{D}_\mathcal{C}^{(l)}$. Then, applying dual arguments to those in (g), we conclude that $\text{Hom}_A(M_\mathcal{C}^{(t)}, M_\mathcal{C}^{(l)}) \neq 0$, again a contradiction.
- (i) Assume now that the both X and Y belong to ${}_{\mathcal{C}}\mathcal{C}^{\text{coh}}$. Then, taking into account (c), we conclude that there are different components Γ and Ω of ${}_{\mathcal{C}}\mathcal{C}^{\text{coh}}$ such that X belongs to Γ and Y belongs to Ω . Then we may choose X and Y such that there exist a nonzero homomorphism $f : X \rightarrow Y$, an infinite sectional path

$$\cdots \xrightarrow{p_{j+1}} W_j \xrightarrow{p_j} W_{j-1} \xrightarrow{p_{j-1}} \cdots \xrightarrow{p_2} W_1 \xrightarrow{p_1} W_0 = X$$

of irreducible homomorphisms of infinite left degree between indecomposable modules in Γ , and infinite sectional path

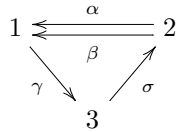
$$Y = Z_0 \xrightarrow{v_1} Z_1 \xrightarrow{v_2} \cdots \xrightarrow{v_{i-1}} Z_{i-1} \xrightarrow{v_i} Z_i \xrightarrow{v_{i+1}} \cdots$$

of irreducible homomorphisms of infinite right degree between indecomposable modules in Ω . Then we have $v_i \cdots v_1 f p_1 \cdots p_j \neq 0$ for any $i, j \geq 1$. Since Γ and Ω are generalized multicoils, there are $j \geq 1$ with W_j in \mathcal{T}_Γ and $i \geq 1$ with Z_i in \mathcal{T}_Ω . Thus $\text{Hom}_A(W, Z) \neq 0$ for some $W \in \mathcal{T}_\Gamma$ and $Z \in \mathcal{T}_\Omega$, and consequently $\text{rad}_A(M_\Gamma^{(t)}, M_\Omega^{(t)}) \neq 0$, contradicting the condition $\text{rad}_A(M_\mathcal{C}^{(t)}, M_\mathcal{C}^{(t)}) = 0$.

Summing up, we proved that \mathcal{C} is generalized standard.

We present now two example showing that the vanishing conditions (ii) of Theorem 1.1 are necessary for the generalized standardness of an Auslander–Reiten component.

Example 4.1. Let K be an algebraically closed field, Q the quiver of the form

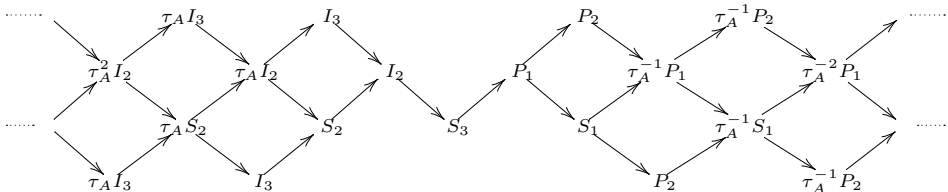


and I the ideal in the path algebra KQ of Q over K generated by the paths $\alpha\gamma$, $\gamma\sigma$, $\sigma\alpha$, and $A = KQ/I$. Moreover, let H be the path algebra $K\Delta$ of the subquiver Δ of Q given by the vertices 1, 2 and the arrows α, β . Then A is a 10-dimensional K -algebra and H is a quotient algebra of A . Applying [55, Section X.4] and [56, Section XVI.1] we conclude that the Auslander–Reiten quiver Γ_A has a disjoint union decomposition

$$\Gamma_A = \mathcal{C} \cup \mathcal{T} \cup \left(\bigcup_{\lambda \in K} \mathcal{T}_\lambda \right),$$

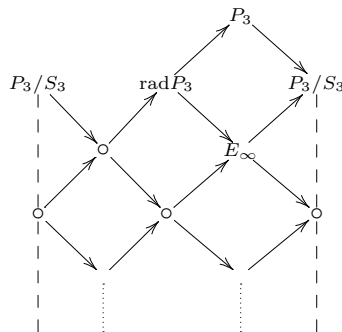
where

(1) \mathcal{C} is the component of the form



containing all indecomposable postprojective modules and all indecomposable preinjective modules over the Kronecker algebra H ;

(2) \mathcal{T} is the quasi-tube of the form



obtained from the stable tube \mathcal{T}_∞^H of Γ_H of rank 1, with the mouth module

$$E_\infty : K \begin{smallmatrix} \xleftarrow{0} \\ \xleftarrow{1} \end{smallmatrix} K,$$

by one ray insertion and one coray insertion, and the modules along the dashed lines have to be identified;

- (3) For each $\lambda \in K$, $\mathcal{T}_\lambda = \mathcal{T}_\lambda^H$ is the stable tube of rank 1 in Γ_H with the mouth module

$$E_\lambda : K \begin{smallmatrix} \xleftarrow{1} \\ \xleftarrow{\lambda} \end{smallmatrix} K.$$

We observe that the stable tubes \mathcal{T}_λ , $\lambda \in K$, are generalized standard in $\text{mod } A$. On the other hand, for the quasi-tube \mathcal{T} we have

$$M_{\mathcal{T}}^{(t)} = E_\infty, \quad P_{\mathcal{T}} = P_3 = P_{\mathcal{T}}^{\text{coh}}, \quad Q_{\mathcal{T}} = P_3 = Q_{\mathcal{T}}^{\text{coh}}.$$

Further, P_3 is an indecomposable projective-injective module with $\text{top}(P_3) = S_3 = \text{soc}(P_3)$, and S_3 lies in the component \mathcal{C} . Hence,

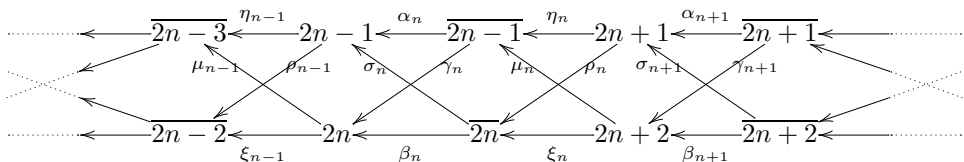
$$\text{Hom}_A(\text{top}(P_{\mathcal{T}}^{\text{coh}}), Q_{\mathcal{T}}^{\text{coh}}) = \text{Hom}_A(\text{top}(P_{\mathcal{T}}^{\text{coh}}), \text{soc}(Q_{\mathcal{T}}^{\text{coh}})) = \text{Hom}_A(S_3, S_3) \neq 0.$$

Clearly, we have $\text{rad}_A^\infty(P_3, P_3) = \text{rad}_A(P_3, P_3) \neq 0$, and so \mathcal{T} is not generalized standard. We also note that $\text{Hom}_A(P_{\mathcal{T}}, M_{\mathcal{T}}^{(t)}) = 0$ and $\text{Hom}_A(M_{\mathcal{T}}^{(t)}, Q_{\mathcal{T}}) = 0$. Consider now the acyclic component \mathcal{C} . We may take $M_{\mathcal{C}}^{(l)} = S_2 \oplus I_2 \oplus I_3$ and $M_{\mathcal{C}}^{(r)} = S_1 \oplus P_1 \oplus P_2$. We have

$$\text{Hom}_A(P_2, I_2) = \text{rad}_A^\infty(P_2, I_2) \neq 0,$$

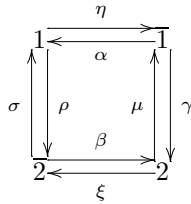
and hence $\text{Hom}_A(M_{\mathcal{C}}^{(r)}, M_{\mathcal{C}}^{(l)}) \neq 0$ and \mathcal{C} is not generalized standard.

Example 4.2. Let K be an algebraically closed field, $a \in K \setminus \{0, 1\}$, and R be the locally bounded K -category given by the infinite quiver

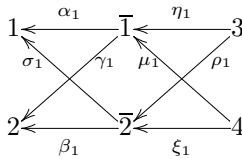


and the relations $\eta_n \alpha_n = \rho_n \sigma_n$, $\xi_n \beta_n = a \mu_n \gamma_n$, $\alpha_{n+1} \eta_n = \gamma_{n+1} \mu_n$, $\beta_{n+1} \xi_n = a \sigma_{n+1} \rho_n$, $\eta_n \alpha_n \eta_{n-1} = 0$, $\alpha_{n+1} \eta_n \alpha_n = 0$, $\xi_n \beta_n \xi_{n-1} = 0$, $\beta_{n+1} \xi_n \beta_n = 0$, $\rho_n \sigma_n \rho_{n-1} = 0$, $\sigma_{n+1} \rho_n \sigma_n = 0$, $\mu_n \gamma_n \mu_{n-1} = 0$, $\gamma_{n+1} \mu_n \gamma_n = 0$, $\eta_n \gamma_n \xi_{n-1} = 0$, $\gamma_{n+1} \xi_n \sigma_n = 0$,

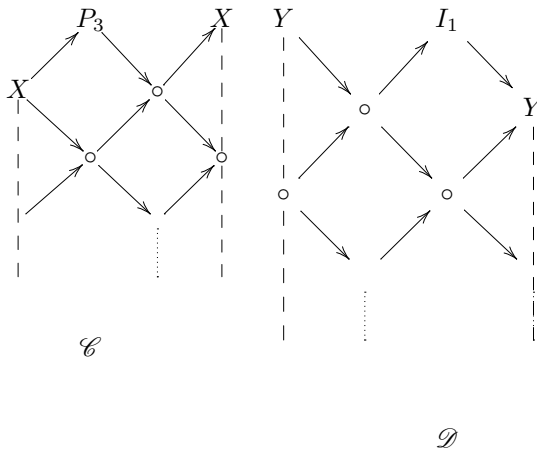
$\xi_n \sigma_n \eta_{n-1} = 0$, $\sigma_{n+1} \eta_n \gamma_n = 0$, for all $n \in \mathbb{Z}$. Moreover, let G be the infinite cyclic group of automorphisms of R generated by the shift g given by $g(2n-1) = 2n+1$, $g(2n) = 2n+2$, $g(\overline{2n-1}) = \overline{2n+1}$, $g(\overline{2n}) = \overline{2n+2}$, $g(\alpha_n) = \alpha_{n+1}$, $g(\beta_n) = \beta_{n+1}$, $g(\gamma_n) = \gamma_{n+1}$, $g(\sigma_n) = \sigma_{n+1}$, $g(\eta_n) = \eta_{n+1}$, $g(\xi_n) = \xi_{n+1}$, $g(\rho_n) = \rho_{n+1}$, for any $n \in \mathbb{Z}$. Consider now the orbit algebras $A = R/G$ and the push-down functor $F_\lambda : \text{mod } R \rightarrow \text{mod } A$ associated to the Galois covering $F : R \rightarrow R/G = A$. Since the group G is torsion-free, it follows from [17] that F_λ preserves indecomposable modules and almost split sequences. The orbit algebra A is given by the quiver



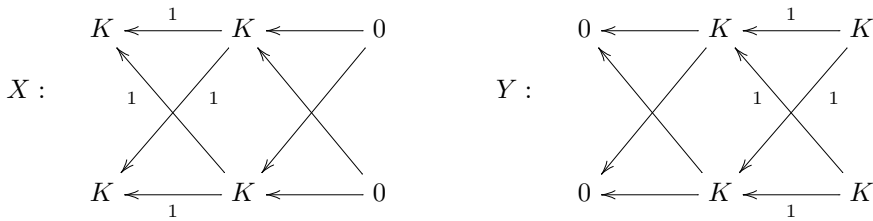
and the relations $\eta\alpha = \alpha\rho\sigma$, $\xi\beta = \mu\gamma$, $\alpha\eta = \alpha\gamma\mu$, $\beta\xi = \sigma\rho$, $\eta\alpha\eta = 0$, $\alpha\eta\alpha = 0$, $\xi\beta\xi = 0$, $\beta\xi\beta = 0$, $\rho\sigma\rho = 0$, $\sigma\rho\sigma = 0$, $\mu\gamma\mu = 0$, $\gamma\mu\gamma = 0$, $\eta\gamma\xi = 0$, $\gamma\xi\sigma = 0$, $\xi\sigma\eta = 0$, $\sigma\eta\gamma = 0$. We denote by B the full subcategory of R given by the convex subquiver



Then B is a tubular algebra of tubular type $(2, 2, 2, 2)$ in the sense of [51]. Then it follows from [51, 5.2] that Γ_B admits a generalized standard ray tube \mathcal{C} and a generalized standard coray tube \mathcal{D} of the forms



where X and Y are the indecomposable B -modules



In particular, we have $M_{\mathcal{C}}^{(t)} = X$ and $M_{\mathcal{D}}^{(t)} = Y$. Observe that $\text{Hom}_B(P_{\mathcal{C}}, M_{\mathcal{C}}^{(t)}) = \text{Hom}_B(P_3, X) = 0$ and $\text{Hom}_B(M_{\mathcal{D}}^{(t)}, Q_{\mathcal{C}}) = \text{Hom}_B(Y, I_1) = 0$. Since \mathcal{C} and \mathcal{D} are components in Γ_R , we have the induced ray tube $\mathcal{C}^* = F_{\lambda}(\mathcal{C})$, containing the indecomposable projective module $F_{\lambda}(P_3) = F_{\lambda}(P_1)$, and coray tube $\mathcal{D}^* = F_{\lambda}(\mathcal{D})$, containing the indecomposable injective module $F_{\lambda}(I_1) = F_{\lambda}(I_3)$. Hence, we obtain $P_{\mathcal{C}^*} = P_{\mathcal{C}^*}^{\text{coh}} = F_{\lambda}(P_1)$ and $Q_{\mathcal{D}^*} = Q_{\mathcal{D}^*}^{\text{coh}} = F_{\lambda}(I_3)$. Moreover, $M_{\mathcal{C}^*}^{(t)} = F_{\lambda}(X)$ and $M_{\mathcal{D}^*}^{(t)} = F_{\lambda}(Y)$, with $\text{rad}_A(F_{\lambda}(X), F_{\lambda}(X)) = \text{rad}_A(X, X) = 0$ and $\text{rad}_A(F_{\lambda}(Y), F_{\lambda}(Y)) = \text{rad}_A(Y, Y) = 0$. We also note that there are nonzero homomorphisms $F_{\lambda}(P_1) \rightarrow F_{\lambda}(S_1) \rightarrow F_{\lambda}(X)$ and $F_{\lambda}(Y) \rightarrow F_{\lambda}(S_1) \rightarrow F_{\lambda}(I_1)$ and the simple module $F_{\lambda}(S_1)$ is neither in \mathcal{C}^* nor in \mathcal{D}^* . Therefore, \mathcal{C}^* , \mathcal{D}^* are not generalized standard components of Γ_A , and $\text{Hom}_A(P_{\mathcal{C}^*}, M_{\mathcal{C}^*}^{(t)}) \neq 0$, $\text{Hom}_A(M_{\mathcal{D}^*}^{(t)}, Q_{\mathcal{D}^*}) \neq 0$.

5. Proof of Theorem 1.5

The aim of this section is to prove Theorem 1.5 and recall the relevant facts.

Following Happel, Reiten and Smalø [19] an algebra B is called quasitilted if B is the endomorphism algebra $\text{End}_{\mathcal{H}}(T)$ of a tilting object T in an abelian hereditary category \mathcal{H} . In the case when \mathcal{H} is equivalent (as a triangulated category) to the derived category $D^b(\text{mod } \Lambda)$ of the module category $\text{mod } \Lambda$ of a canonical algebra Λ (in the sense of Ringel [51, 54]), $B = \text{End}_{\mathcal{H}}(T)$ is called a quasitilted algebra of canonical type. It was shown in [19] that an algebra B is quasitilted if and only if $\text{gl.dim } B \leq 2$ and every module X in $\text{ind } B$ satisfies $\text{pd}_B X \leq 1$ or $\text{id}_B X \leq 1$. In [18] Happel and Reiten proved that an algebra A is quasitilted if and only if A is tilted or quasitilted of canonical type.

The prominent class of quasitilted algebras of canonical type is formed by concealed canonical algebras [27], which are the postprojective tilts of canonical algebras. In [29] Lenzing and de la Peña proved that the concealed canonical algebras form the class of all algebras whose Auslander–Reiten quiver admits a separating family of stable tubes. Then we have the following characterization of arbitrary quasitilted algebras of canonical type established by Lenzing and Skowroński [30].

Proposition 5.1. *Let A be an algebra. The following statements are equivalent:*

- (i) *A is a quasitilted algebra of canonical type.*

- (ii) Γ_A admits a separating family of semiregular tubes (ray tubes and coray tubes).
- (iii) A is a semiregular branch enlargement of a concealed canonical algebra.

It follows that the quasitilted algebras of canonical type are very special generalized multicoil enlargements of concealed canonical algebras, invoking only admissible operations of types (ad 1) and (ad 1*), and applied to disjoint families of stable tubes. We refer to [19,30,41,51,54] for basic representation theory of quasitilted algebras of canonical type. We note that the separating family of semiregular tubes in the Auslander–Reiten quiver of a quasitilted algebra of canonical type contains infinitely many stable tubes.

In the paper, by a generalized multicoil algebra, we mean a generalized multicoil enlargement B of a product C of quasitilted algebras of canonical type with respect to a family of stable tubes of Γ_C . We have the following characterization of generalized multicoil algebras established in [38, Theorem A].

Proposition 5.2. *For an algebra B the following statements are equivalent:*

- (i) B is a generalized multicoil algebra.
- (ii) Γ_B admits a separating family of generalized multicoils.

The following consequences of [38, Theorem C and Corollary D] describe the structure of the Auslander–Reiten quiver of a generalized multicoil algebra.

Proposition 5.3. *Let B be a generalized multicoil algebra, \mathcal{C}^B a separating family of generalized multicoils in Γ_B , and $\Gamma_B = \mathcal{P}^B \cup \mathcal{C}^B \cup \mathcal{Q}^B$ the associated decomposition of Γ_B . Then the following statements hold.*

- (i) *There is a unique quotient algebra $B^{(l)}$ of B which is a quasitilted algebra of canonical type having a separating family $\mathcal{T}^{B^{(l)}}$ of coray tubes in $\Gamma_{B^{(l)}}$ such that $\Gamma_{B^{(l)}} = \mathcal{P}^{B^{(l)}} \cup \mathcal{T}^{B^{(l)}} \cup \mathcal{Q}^{B^{(l)}}$ with $\mathcal{P}^{B^{(l)}} = \mathcal{P}^B$.*
- (ii) *There is a unique quotient algebra $B^{(r)}$ of B which is a quasitilted algebra of canonical type having a separating family $\mathcal{T}^{B^{(r)}}$ of ray tubes in $\Gamma_{B^{(r)}}$ such that $\Gamma_{B^{(r)}} = \mathcal{P}^{B^{(r)}} \cup \mathcal{T}^{B^{(r)}} \cup \mathcal{Q}^{B^{(r)}}$ with $\mathcal{Q}^{B^{(r)}} = \mathcal{Q}^B$.*
- (iii) *Every component in $\mathcal{P}^B = \mathcal{P}^{B^{(l)}}$ is either a postprojective component, a ray tube, or obtained from a component of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly zero) of ray insertions.*
- (iv) *Every component in $\mathcal{Q}^B = \mathcal{Q}^{B^{(r)}}$ is either a preinjective component, a coray tube, or obtained from a component of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly zero) of coray insertions.*

Then $B^{(l)}$ is called the *left quasitilted algebra* of B and $B^{(r)}$ the *right quasitilted algebra* of B .

The following results from [38, Theorem E] describes homological properties of generalized multicoil algebras.

Proposition 5.4. *Let B be a generalized multicoil algebra, \mathcal{C}^B a separating family of generalized multicoils in Γ_B , and $\Gamma_B = \mathcal{P}^B \cup \mathcal{C}^B \cup \mathcal{Q}^B$ the associated decomposition of Γ_B . Then the following statements hold.*

- (i) $\text{pd}_B X \leq 1$ for any module X in \mathcal{P}^B .
- (ii) $\text{id}_B X \leq 1$ for any module X in \mathcal{Q}^B .
- (iii) $\text{pd}_B X \leq 2$ and $\text{id}_B X \leq 2$ for any module X in \mathcal{C}^B .
- (iv) $\text{gl.dim } B \leq 3$.

The following proposition is a crucial ingredient for the proofs of Theorems 1.5 and 1.6.

Proposition 5.5. *Let A be an algebra, $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ a separating family of components in Γ_A , and assume that the cyclic part ${}_{\mathcal{C}}\mathcal{C}$ of \mathcal{C} is infinite. Moreover, let Γ be a component of the coherent part ${}_{\mathcal{C}}\mathcal{C}^{\text{coh}}$ of ${}_{\mathcal{C}}\mathcal{C}$. Then the following statements hold.*

- (i) *The faithful algebra $C = B(\mathcal{T}_{\Gamma})$ of the tubular part \mathcal{T}_{Γ} of Γ is a product of concealed canonical algebras, and \mathcal{T}_{Γ} is a finite family of stable tubes in Γ_C .*
- (ii) *The faithful algebra $B = B(\Gamma)$ of Γ is a generalized multicoil enlargement of C with respect to \mathcal{T}_{Γ} , and hence a generalized multicoil algebra.*
- (iii) *Every stable tube of Γ_C which is not in the tubular part \mathcal{T}_{Ω} of a component Ω of ${}_{\mathcal{C}}\mathcal{C}^{\text{coh}}$ belongs to the family \mathcal{C} .*

In particular, the separating family \mathcal{C} contains infinitely many stable tubes, and hence I is infinite.

Proof. Let $\Gamma_A = \mathcal{P} \cup \mathcal{C} \cup \mathcal{Q}$ be the decomposition of Γ_A induced by the separating family \mathcal{C} . Let \mathcal{C}_i be a component in \mathcal{C} containing Γ , that is, Γ is a component of ${}_{\mathcal{C}_i}\mathcal{C}_i^{\text{coh}}$. We note that \mathcal{C}_i is a generalized standard, hence almost periodic, component of Γ_A . It follows from Proposition 2.3 that \mathcal{T}_{Γ} is a faithful family of pairwise orthogonal generalized standard stable tubes in Γ_C , and B is a generalized multicoil enlargement of C with respect to \mathcal{T}_{Γ} . We claim that C is a product of concealed canonical algebras, and consequently B is a generalized multicoil algebra. By [43, Theorem 3.1] (see also [21, Proposition 3.2]) it is enough to show that the family \mathcal{T}_{Γ} has no external short paths in $\text{ind } C$. Suppose that \mathcal{T}_{Γ} admits an external short path in $\text{ind } C$, that is, a sequence $X \rightarrow Y \rightarrow Z$ of nonzero nonisomorphisms in $\text{ind } C$ such that X and Z belong to \mathcal{T}_{Γ} but Y is not in \mathcal{T}_{Γ} . Because X and Z belong to the separating family \mathcal{C} , the module Y must belong to \mathcal{C} . Moreover, \mathcal{C} is generalized standard, and then there is a finite path of irreducible homomorphisms

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m = Y$$

in $\text{mod } A$ with $f_m \dots f_1 \neq 0$, and hence Y belongs to the component \mathcal{C}_i . We observe also that Y does not belong to the cyclic generalized multicoil Γ , because $B = B(\Gamma)$ is a generalized multicoil enlargement of C and Y is a C -module not lying in \mathcal{T}_Γ . Then we conclude that there is $s \in \{1, \dots, m\}$ such that X_s belongs to $\Sigma_\Gamma^{(r)}$. But it follows from the structure of a generalized multicoil that all modules in $\Sigma_\Gamma^{(r)}$ have no simple composition factors from $\text{mod } C$, and then $f_s \dots f_1 = 0$. This proves the claim.

We shall prove now the statement (iii). Without loss of generality, we may assume that C is indecomposable. Let $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda \in \Lambda}$ be the separating family of stable tubes in Γ_C and

$$\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C$$

the associated decomposition of Γ_C . It is known (see [29]) that \mathcal{T}^C has the strongly separation property: every homomorphism from \mathcal{P}^C to \mathcal{Q}^C factors through the additive category $\text{add}(\mathcal{T}_\lambda^C)$ for any $\lambda \in \Lambda$. Let \mathcal{T}_λ be a stable tube of the tubular family \mathcal{T}_Γ of a component Γ of ${}^c\mathcal{C}^{\text{coh}}$. For any indecomposable module X in \mathcal{P}^C there exists a nonzero homomorphism $f : X \rightarrow Y$ with Y in $\text{add}(\mathcal{T}_\lambda)$, and hence $\text{rad}_A^\infty(X, Y) = \text{rad}_C^\infty(X, Y) \neq 0$. Similarly, for any indecomposable module Z in \mathcal{Q}^C there is a nonzero homomorphism $g : V \rightarrow Z$ with V in $\text{add}(\mathcal{T}_\lambda)$, and hence $\text{rad}_A^\infty(V, Z) = \text{rad}_C^\infty(V, Z) \neq 0$. This shows that all indecomposable modules from \mathcal{P}^C are contained in \mathcal{P} and all indecomposable modules from \mathcal{Q}^C are contained in \mathcal{Q} . Let Λ' be the set of all $\xi \in \Lambda$ such that \mathcal{T}_ξ^C occurs in the tubular family \mathcal{T}_Ω of a component Ω of ${}^c\mathcal{C}^{\text{coh}}$. We also note that if a stable tube \mathcal{T}_μ^C has an indecomposable module lying in \mathcal{C} , then all indecomposable modules of \mathcal{T}_μ^C lie in \mathcal{C} , because $\text{Hom}_\Lambda(\mathcal{C}, \mathcal{P}) = 0$ and $\text{Hom}_\Lambda(\mathcal{Q}, \mathcal{C}) = 0$, and hence $\mu \in \Lambda'$. Hence $\mathcal{T}_\xi^C \subset \mathcal{C}$ if $\mathcal{T}_\xi^C \cap \mathcal{C} \neq \emptyset$. Suppose now $\Lambda \neq \Lambda'$ and $\sigma \in \Lambda \setminus \Lambda'$. Then $\mathcal{T}_\sigma^C \cap \mathcal{C} = \emptyset$. Since $\Gamma_A = \mathcal{P} \cup \mathcal{C} \cup \mathcal{Q}$ we may assume that \mathcal{T}_σ^C contains a module U from \mathcal{P} . Take a nonzero homomorphism $h : U \rightarrow E$, where E is an indecomposable injective C -module. Since E lies in \mathcal{Q}^C , it lies also in \mathcal{Q} . Then h factors through a module W in $\text{add}(\mathcal{C})$, and so there is a nonzero homomorphism $p : U \rightarrow T$ with T an indecomposable module in \mathcal{C} . Then $p(U)$ is contained in the largest C -submodule of T . But it follows from the structure of indecomposable modules in a generalized multicoil of a generalized multicoil algebra (see [38, Section 3]), and the above discussion, that the largest C -submodules of indecomposable modules in \mathcal{C} are the indecomposable modules from \mathcal{T}_ξ , $\xi \in \Lambda'$. Then $p = 0$, because different tubes in \mathcal{T}^C are orthogonal, a contradiction. As a result, we conclude that $\Lambda = \Lambda'$ and the statement (iii) holds.

We need also the following facts on tilted algebras.

Proposition 5.6. *Let H be an indecomposable hereditary algebra, T a tilting module in $\text{mod } H$ without indecomposable preinjective direct summands, $B = \text{End}_H(T)$ the associ-*

ated tilted algebra, and $\mathcal{Y}\Gamma_B$ the full translation subquiver of Γ_B formed by all components contained entirely in the torsion-free part $\mathcal{Y}(T) = \{Y \in \text{mod } B \mid \text{Tor}_1^B(Y, T) = 0\}$. Then the following statements hold.

- (i) If H is of Euclidean type, then $\mathcal{Y}\Gamma_B$ consists of a postprojective component \mathcal{P}_B and an infinite family of ray tubes, and the faithful algebra $B(\mathcal{P}_B)$ of \mathcal{P}_B is a tame concealed algebra.
- (ii) If H is of wild type, then $\mathcal{Y}\Gamma_B$ consists of a postprojective component \mathcal{P}_B and an infinite family of components obtained from components of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly zero) of ray insertions, and the faithful algebra $B(\mathcal{P}_B)$ of \mathcal{P}_B is a wild concealed algebra.

Proof. The statement (i) follows from the theory of tilted algebras of Euclidean type developed in [11,15,47,51]. The statement (ii) follows from the representation theory of tilted algebras of wild type developed in [22–24,47,48,53,70].

We have also the dual proposition.

Proposition 5.7. Let H be an indecomposable hereditary algebra, T a tilting module in $\text{mod } H$ without indecomposable postprojective direct summands, $B = \text{End}_H(T)$ the associated tilted algebra, and $\mathcal{X}\Gamma_B$ the full translation subquiver of Γ_B formed by all components contained entirely in the torsion part $\mathcal{X}(T) = \{X \in \text{mod } B \mid X \otimes_B T = 0\}$. Then the following statements hold.

- (i) If H is of Euclidean type, then $\mathcal{X}\Gamma_B$ consists of a preinjective component \mathcal{Q}_B and an infinite family of coray tubes, and the faithful algebra $B(\mathcal{Q}_B)$ of \mathcal{Q}_B is a tame concealed algebra.
- (ii) If H is of wild type, then $\mathcal{X}\Gamma_B$ consists of a preinjective component \mathcal{Q}_B and an infinite family of components obtained from components of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly zero) of coray insertions, and the faithful algebra $B(\mathcal{Q}_B)$ of \mathcal{Q}_B is a wild concealed algebra.

We are now able to complete the proof of Theorem 1.5.

Let A be an algebra with a separating family \mathcal{C}^A of components in Γ_A , and $\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A$ the associated decomposition of Γ_A . Let $\mathcal{C}^A = (\mathcal{C}_i)_{i \in I}$. Applying Theorem 1.6 we may assume that I is infinite. Since every regular generalized standard component is a stable tube, the family \mathcal{C}^A has a disjoint decomposition

$$\mathcal{C}^A = \mathcal{C} \cup \mathcal{T},$$

where \mathcal{C} is the finite family of all nonregular components in \mathcal{C}^A and \mathcal{T} consists of all stable tubes in \mathcal{C}^A . Consider the faithful algebra $C = B(\mathcal{T})$ of \mathcal{T} . Then \mathcal{T} is a faithful

family of pairwise orthogonal generalized standard stable tubes in Γ_C . Moreover, because \mathcal{T} is a part of the separating family \mathcal{C}^A of Γ_A and C is a quotient algebra of A , we conclude that \mathcal{T} is without external short paths in $\text{ind } C$, and hence C is a product $C_1 \times \dots \times C_m$ of concealed canonical algebras. Let J be the finite subset of I such that $\mathcal{C} = (\mathcal{C}_i)_{i \in J}$. For each $i \in J$, we take the left tilted quotient algebra $A_{\mathcal{C}_i}^{(lt)}$ and the right tilted quotient algebra $A_{\mathcal{C}_i}^{(rt)}$ of A satisfying the statements (i) and (ii) of Theorem 1.2 for the component \mathcal{C}_i . Then we define $A^{(lt)}$ as the product of all algebras $A_{\mathcal{C}_i}^{(lt)}$, $i \in J$, and $A^{(rt)}$ as the product of all algebras $A_{\mathcal{C}_i}^{(rt)}$, $i \in J$. Further, for each $i \in J$, we choose the quotient algebra $A_{\mathcal{C}_i}^{(c)}$ of A satisfying the statement (iii) of Theorem 1.2, and denote by $A^{(c)}$ the product of these algebras $A_{\mathcal{C}_i}^{(c)}$, $i \in J$. Consider also the disjoint union $\mathcal{T}_{\mathcal{C}}$ of the tubular parts \mathcal{T}_{Γ} of all components Γ of ${}^c\mathcal{C}^{\text{coh}}$. Applying Proposition 5.5 to all these components Γ we conclude that the faithful algebra $B(\mathcal{T}_{\mathcal{C}})$ of $\mathcal{T}_{\mathcal{C}}$ coincides with C and $A^{(c)}$ is a multicoil enlargement of C using the family of stable tubes $\mathcal{T}_{\mathcal{C}}$ of Γ_C . We note that since $\mathcal{T}_{\mathcal{C}}$ is the disjoint union, $\mathcal{T}_{\mathcal{C}} \cup \mathcal{T}$ is a separating family of stable tubes in Γ_C . We denote by $A^{(lc)}$ the left quasitilted algebra $(A^{(c)})^{(l)}$ of $A^{(c)}$ and by $A^{(rc)}$ the right quasitilted algebra $(A^{(c)})^{(r)}$ of $A^{(c)}$. Finally, we set

$$A^{(l)} = A^{(lt)} \times A^{(lc)} \quad \text{and} \quad A^{(r)} = A^{(rt)} \times A^{(rc)}.$$

We note that $A^{(l)}$ and $A^{(r)}$ are quotient algebras of A . It follows now from Theorem 1.2 and Propositions 5.3, 5.6, 5.7 that $A^{(l)}$ and $A^{(r)}$ are the algebras satisfying the statements (i)–(iv) of Theorem 1.5.

6. Proof of Theorem 1.6

The aim of this section is to prove Theorem 1.6 and recall the relevant facts.

Let A be an algebra and \mathcal{C} a component of Γ_A . Then \mathcal{C} is said to be almost acyclic if the cyclic part ${}^c\mathcal{C}$ of \mathcal{C} is finite. It was shown in [45, Theorem 2.5] that \mathcal{C} is almost acyclic if and only if \mathcal{C} admits a multisection Δ , which is a full connected valued subquiver of \mathcal{C} satisfying the following conditions:

- (1) Δ is almost acyclic.
- (2) Δ is convex in \mathcal{C} .
- (3) For each τ_A -orbit \mathcal{O} in \mathcal{C} , we have $1 \leq |\Delta \cap \mathcal{O}| < \infty$.
- (4) $|\Delta \cap \mathcal{O}| = 1$ for all but finitely many τ_A -orbits \mathcal{O} in \mathcal{C} .
- (5) No proper full convex valued subquiver of Δ satisfies the conditions (1)–(4).

To a multisection Δ of \mathcal{C} one associates the left border Δ_l , the right border Δ_r , and the core Δ_c being a finite subquiver of Δ containing the cyclic part ${}^c\mathcal{C}$ of \mathcal{C} (see [45, Section 3] for definitions and properties of these quivers). We also note that if \mathcal{C} is additionally almost periodic, then every multisection of \mathcal{C} is finite.

Following [45], an algebra B is said to be a *generalized double tilted algebra* if the following conditions are satisfied:

- (1) Γ_B admits a component \mathcal{C} with a faithful multisection Δ .
- (2) There exists a tilted quotient algebra $B^{(l)}$ of B (not necessarily indecomposable) such that the left border Δ_l of Δ is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(l)}$, and the category of all predecessors of Δ_l in $\text{ind } B$ coincides with the category of all predecessors of Δ_l in $\text{ind } B^{(l)}$.
- (3) There exists a tilted quotient algebra $B^{(r)}$ of B (not necessarily indecomposable) such that the right border Δ_r of Δ is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(r)}$, and the category of all successors of Δ_r in $\text{ind } B$ coincides with the category of all successors of Δ_r in $\text{ind } B^{(r)}$.

Then $B^{(l)}$ is called the *left tilted algebra* and $B^{(r)}$ the *right tilted algebra* of B .

The following characterization of generalized double tilted algebras from [45, Theorem 3.1] is essential for the proof of Theorem 1.6.

Proposition 6.1. *Let B be an algebra. Then the following statements are equivalent:*

- (i) B is a generalized double tilted algebra.
- (ii) Γ_B admits a faithful generalized standard almost acyclic component.

We note that the class of generalized double tilted algebras contains all tilted algebras and all algebras of finite representation type. We refer to [45] for basic properties and representation theory of arbitrary generalized double tilted algebras. We only mention that, if B is a generalized double tilted algebra, then for all but finitely modules X in $\text{ind } B$ we have $\text{pd}_B X \leq 1$ or $\text{id}_B X \leq 1$ but B may have arbitrary (finite or infinite) global dimension.

We may now complete the proof of Theorem 1.6. The equivalence of the statements (ii) and (iii) follows from Proposition 6.1 and definition of a generalized double tilted algebra. The implication (ii) \implies (i) is obvious. The following proposition shows that also (i) implies (iii).

Proposition 6.2. *Let A be an algebra such that Γ_A admits a finite separating family \mathcal{C} of components. Then A is a generalized double tilted algebra.*

Proof. Let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be the components of \mathcal{C} . It follows from Proposition 5.5 that the cyclic part \mathcal{C}^c of \mathcal{C} is finite. Hence $\mathcal{C}_1, \dots, \mathcal{C}_m$ are almost acyclic components of Γ_A . Therefore, it is enough to prove that $m = 1$, by Proposition 6.1.

Assume $m \geq 2$. We shall prove that this contradicts the imposed indecomposability of A . For each $i \in \{1, \dots, m\}$, we may choose a finite multisection $\Delta^{(i)}$ of \mathcal{C}_i , because \mathcal{C}_i is generalized standard, and hence almost periodic. Further, let $\Delta_l^{(i)}$ be the left border

of $\Delta^{(i)}$, $\Delta_r^{(i)}$ the right border of $\Delta^{(i)}$, and $\Delta_c^{(i)}$ the core of $\Delta^{(i)}$. Then it follows from [45, Proposition 2.4] that every indecomposable module X in \mathcal{C}_i is in $\Delta_c^{(i)}$ or is a predecessor of $\Delta_l^{(i)}$ or is a successor of $\Delta_r^{(i)}$. We denote by $M_i^{(l)}$ the direct sum of all indecomposable modules lying on $\Delta_l^{(i)}$. Moreover, let $M^{(l)} = M_1^{(l)} \oplus \dots \oplus M_m^{(l)}$, and $B^{(l)} = B(M^{(l)})$, the faithful algebra of $M^{(l)}$. We note that $\text{Hom}_A(M_i^{(l)}, \tau_A M_i^{(l)}) = 0$ for any $i \in \{1, \dots, m\}$, because \mathcal{C}_i is generalized standard and there is no path in \mathcal{C}_i from an indecomposable direct summand of $M_i^{(l)}$ to an indecomposable direct summand of $\tau_A M_i^{(l)}$. Moreover, for any $i \neq j$ in $\{1, \dots, m\}$, we have $\text{Hom}_A(M_i^{(l)}, \tau_A M_j^{(l)}) = 0$, because the components \mathcal{C}_i and \mathcal{C}_j are orthogonal. Therefore, we conclude that $\text{Hom}_A(M^{(l)}, \tau_A M^{(l)}) = 0$. Let $H^{(l)} = \text{End}_A(M^{(l)})$ and $T^{(l)} = D(M^{(l)})$, where D is the standard duality on $\text{mod } A$. Then applying arguments from the proof of [1, Theorem VIII.5.6], or [68, Theorem VIII.7.7], we conclude that:

- (1) $H^{(l)}$ is a hereditary algebra (not necessarily indecomposable),
- (2) $T^{(l)}$ is a tilting module in $\text{mod } H^{(l)}$,
- (3) There is a canonical isomorphism of algebras $B^{(l)} \rightarrow \text{End}_{H^{(l)}}(T^{(l)})$.

We note that $B^{(l)}$ (respectively, $H^{(l)}$) is a product of indecomposable algebras corresponding to the connected components of the left border Δ_l of multisection Δ . In particular, we conclude that

$$B^{(l)} = B_1^{(l)} \times \dots \times B_m^{(l)},$$

where $B_i^{(l)} = B(M_i^{(l)})$ for $i \in \{1, \dots, m\}$. For each $i \in \{1, \dots, m\}$, let P_i be the direct sum of all indecomposable projective modules in $\text{mod } B_i^{(l)}$ and Q_i the direct sum of all projective modules in \mathcal{C}_i which are not predecessors of $\Delta_l^{(i)}$. Moreover, let $D_i = \text{End}_A(Q_i)$ and $N_i = \text{Hom}_A(P_i, Q_i)$, for any $i \in \{1, \dots, m\}$. We note that $B_i^{(l)} = \text{End}_A(P_i)$, and N_i is a $D_i - B_i^{(l)}$ -bimodule. Further, since $M_i^{(l)}$ is a faithful $B_i^{(l)}$ -module, there is a monomorphism $P_i \rightarrow (M_i^{(l)})^{n_i}$ for some $n_i \geq 1$. Since \mathcal{C}_i is generalized standard, we have $\text{Hom}_A(Q_i, M_i^{(l)}) = 0$, by the choice of Q_i , and hence $\text{Hom}_A(Q_i, P_i) = 0$. Similarly, for $i \neq j$ in $\{1, \dots, m\}$, we have $\text{Hom}_A(Q_j, P_i) = 0$, because \mathcal{C}_i and \mathcal{C}_j are orthogonal. Finally, let $P = P_1 \oplus \dots \oplus P_m$, $Q = Q_1 \oplus \dots \oplus Q_m$, $N = \text{Hom}_A(P, Q)$, and $D = \text{End}_A(Q)$. We note that $B^{(l)} = \text{End}_A(P) = B_1^{(l)} \times \dots \times B_m^{(l)}$, $D = D_1 \times \dots \times D_m$. Moreover, for each $i \in \{1, \dots, m\}$, we have an isomorphisms of algebras

$$B(\mathcal{C}_i) \cong \begin{bmatrix} D_i & N_i \\ 0 & B_i^{(l)} \end{bmatrix}.$$

Since $\text{Hom}_A(Q, P) = 0$, we obtain isomorphisms of algebras

$$A \cong \text{End}(A_A) \cong \begin{bmatrix} D & N \\ 0 & B^{(l)} \end{bmatrix} \cong B(\mathcal{C}_1) \times \dots \times B(\mathcal{C}_m),$$

and this contradicts the indecomposability of A . This finishes the proof of Theorem 1.6.

7. Generically tame algebras

The aim of this section is to prove Theorem 1.10, Corollary 1.11, Theorem 1.12 using the known results on generically tame quasitilted algebras.

Recall that, if B is a quasitilted algebra, then $\text{gl.dim } B \leq 2$ and every module X in $\text{ind } B$ satisfies $\text{pd}_B X \leq 1$ or $\text{id}_B X \leq 1$. In particular, we have a well defined Euler characteristic

$$\chi_B : K_0(B) \rightarrow \mathbb{Z}$$

such that $\chi_B(M) = |\text{End}_B(M)| - |\text{Ext}_B^1(M, M)|$ for any module M in $\text{mod } B$. We note that we may have χ_B indefinite but with the well defined Euler characteristic $\chi_B(M)$ of any module M in $\text{mod } B$ satisfying $\chi_B(M) = |\text{End}_B(M)| - |\text{Ext}_B^1(M, M)| \geq 0$.

Recall that following [12,13] an algebra A is called generically tame if for any positive integer d , there are only finitely many isomorphism classes of generic modules of endolength d . It is known that if A is generically tame and M is a generic A -module then the endomorphism ring $\text{End}_A(M)$ is a *PI* ring [13, Section 7]. Further, following [13], an algebra A is called generically wild if there is a generic A -module M whose endomorphism ring $\text{End}_A(M)$ is not a *PI* ring. We also note that, if H is an indecomposable hereditary algebra, then H is a finite dimensional algebra over a field (the center of H). It was already shown by Ringel [49, Section 6] that a hereditary algebra H of Euclidean type admits a unique generic module, and its endomorphism ring is a *PI* ring (by [5, Corollary 6.12]). Further, Crawley-Boevey proved in [13, Corollary 8.4] that an indecomposable hereditary algebra is generically tame if and only if H is of Dynkin or Euclidean type.

Then, applying Propositions 5.6 and 5.7, we obtain the following fact.

Proposition 7.1. *Let H be an indecomposable hereditary algebra, T a tilting module in $\text{mod } H$ without indecomposable preinjective direct summands (respectively, postprojective direct summands), and $B = \text{End}_H(T)$ the associated tilted algebra. Then the following statements are equivalent:*

- (i) B is generically tame.
- (ii) B is generically finite.
- (iii) B is of Euclidean type.

We also recall the known characterizations of representation-infinite tilted algebras of Euclidean type.

Proposition 7.2. *Let H be an indecomposable hereditary algebra, T a tilting module in $\text{mod } H$ without indecomposable preinjective direct summands (respectively, postprojec-*

tive direct summands), and $B = \text{End}_H(T)$ the associated tilted algebra. The following statements are equivalent:

- (i) B is of Euclidean type.
- (ii) χ_B is positive semidefinite.
- (iii) Γ_B is almost periodic.
- (iv) Σ_B is acyclic.

It follows also from general theory that, if B is a tilted algebra of Euclidean type as above, then Γ_B consists of a postprojective component, a preinjective component, and an infinite family ray tubes (respectively, coray tubes), and all but finitely many of them are stable tubes of rank one.

Let B be a quasitilted algebra of canonical type with a separating family of ray tubes (respectively, coray tubes). Then we have three possibilities for B with respect to behavior of its Euler form χ_B (see [30]):

- (1) χ_B is positive semidefinite of corank 1, and B is a tilted algebra of Euclidean type,
- (2) χ_B is positive semidefinite of corank 2, and B is a tubular algebra in the sense of Ringel [51,54] (see also [25]),
- (3) χ_B is indefinite.

The following proposition follows from results established by Meltzer in [42].

Proposition 7.3. *Let B be a quasitilted algebra of canonical type with a separating family of ray tubes (respectively, coray tubes), and assume that χ_B is indefinite. Then B admits a wild concealed quotient algebra C .*

The following fact was proved by Lenzing [26].

Proposition 7.4. *Let B be a tubular algebra. Then B is generically infinite of polynomial growth.*

Summing up, we obtain the following fact.

Proposition 7.5. *Let B be a quasitilted algebra of canonical type with a separating family of ray tubes (respectively, coray tubes). Then the following statements are equivalent:*

- (i) B is generically tame.
- (ii) B is generically of polynomial growth.
- (iii) χ_B is positive semidefinite.
- (iv) B is a tilted algebra of Euclidean type or a tubular algebra.

Furthermore, applying [28,42] and the known results on the module categories of tubular algebras, we obtain the following proposition.

Proposition 7.6. *Let B be a quasitilted algebra of canonical type with a separating family of ray tubes (respectively, coray tubes). Then the following statements are equivalent:*

- (i) B is a tilted algebra of Euclidean type or a tubular algebra.
- (ii) Γ_B is almost periodic.
- (iii) Σ_B is acyclic.

It follows also from general theory that, if B is a tubular algebra, then Γ_B consists of a postprojective component, a preinjective component and infinitely many ray tubes and coray tubes, and there are infinitely many stable tubes of rank at least two.

Finally, we recall also the following known fact.

Proposition 7.7. *Let B be a wild concealed algebra. Then there are infinitely many isomorphism classes of modules M in $\text{ind } B$ such that $|\text{End}_B(M)| < |\text{Ext}_B^1(M, M)|$.*

Now Theorem 1.10, Corollary 1.11, and Theorem 1.12 follow from Theorem 1.5 and the results presented above.

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