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Fixed rings of generalized Weyl algebras

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A B S T R A C T

We study actions by filtered automorphisms on classical generalized Weyl algebras (GWAs). In the case of a defining polynomial of degree two, we prove that the fixed ring under the action of a finite cyclic group of filtered automorphisms is again a classical GWA, extending a result of Jordan and Wells. Partial results are provided for the case of higher degree polynomials. In addition, we establish a version of Auslander's theorem for finite cyclic groups of filtered automorphisms acting on classical GWAs.

1. Introduction

Throughout $\Bbbk$ is an algebraically closed field of characteristic zero. All algebras may be regarded as $\Bbbk$-algebras unless otherwise specified.

The main aim of this paper is to study invariant theory questions related to generalized Weyl algebras. Generalized Weyl algebras were named by Bavula [6] but include many

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classes of algebras that have been studied in other contexts. This includes the classical Weyl algebras, primitive quotients of $U(\mathfrak{sl}_2)$, and ambiskew polynomial rings.

Definition 1.1. Let D be a ring, $\sigma \in \mathrm{Aut}(D)$, and $a \in \mathcal{Z}(D)$, $a \neq 0$. The generalized Weyl algebra (of degree one) $D[x, y; \sigma, a]$ is the ring obtained by adjoining to D the variables x and y subject to the relations

$$xy = \sigma(a), \quad yx = a, \quad xd = \sigma(d)x, \quad yd = \sigma^{-1}(d)y,$$

for all $d \in D$. In the case that $D = \mathbb{k}[z]$ and $\sigma(z) = z - \alpha$ for some $\alpha \in \mathbb{k}^\times$ we call $D[x, y; \sigma, a]$ a classical GWA.

Let $R = \mathbb{k}[z][x, y; \sigma, a]$ be a classical GWA. If $\deg_z(a) = 0$, then $R \cong \mathbb{k}[x, x^{-1}]$ and if $\deg_z(a) = 1$, then $R \cong A_1(\mathbb{k})$, the first Weyl algebra over $\mathbb{k}$. The study of classical GWAs goes back at least as far as Joseph [14] and were also studied by Hodges under the name noncommutative deformations of Type-A Kleinian singularities [10]. Every classical GWA is isomorphic to one where $\sigma(z) = z - 1$ and, by [7, Theorem 4.2], where a is monic and 0 is a root of a. We assume these facts throughout without further comment.

The impetus for this study is a result of Smith stating that $A_1(\mathbb{k})^G \ncong A_1(\mathbb{k})$ for any nontrivial finite subgroup $G \subset \mathrm{Aut}(A_1(\mathbb{k}))$ [18]. This relies on an earlier result of Stafford: if P is a projective right ideal of $A_1(\mathbb{k})$, then $\mathrm{End}(P) \cong A_1(\mathbb{k})$ if and only if P is cyclic [19, Theorem 3.1]. Alev, Hodges, and Velez proved that, for two finite subgroups $G, H \subset \mathrm{Aut}(A_1(\mathbb{k}))$, $A_1(\mathbb{k})^G \cong A_1(\mathbb{k})^H$ if and only if $G \cong H$ [1]. Additionally, Alev and Polo extended Smith's theorem to the nth Weyl algebra and proved a similar result for the universal enveloping algebra of a semisimple Lie algebra [2].

A common technique to these papers is reduction modulo primes p, making use of the fact that, over a field of finite characteristic, the center of the nth Weyl algebra is a polynomial ring. Unfortunately, this is not the case for classical GWAs with $\deg_z(a) > 1$. Thus, while we cannot generalize these results entirely, we can give further insight into the study of the fixed rings of a classical GWA.

Let $R = D[x, y; \sigma, a]$ and let β be a primitive ℓth root of unity. Define the automorphism Θ_β of R by $\Theta_\beta(x) = \beta x$, $\Theta_\beta(y) = \beta^{-1} y$, and $\Theta_\beta(d) = d$ for all $d \in D$. In [12], Jordan and Wells prove $R^{\langle \Theta_\beta \rangle} = D[x^\ell, y^\ell; \sigma^\ell, h_\ell]$ where

$$h_\ell = \prod_{i=0}^{\ell-1} \sigma^{-i}(a).$$

When $D = \mathbb{k}[z]$, it is worth observing that the fixed ring is a classical GWA generated by $X = x^\ell$, $Y = y^\ell$, and $Z = \frac{1}{\ell} z$. In [15], Kirkman and Kuzmanovich considered fixed rings of GWAs $D[x, y; \sigma, \alpha]$ under automorphisms satisfying $\phi(x), \phi(y) \in \mathrm{span}_{\mathbb{k}}\{x, y\}$ and $\phi|_D \in \mathrm{Aut}(D)$, as well as their corresponding fixed rings. In several examples, they show that the fixed ring is again a GWA.

Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $a = a_0 + \cdots + a_n$, with $a_i \in \Bbbk[z]_i$ and $a_n \neq 0$. Setting $\deg x = \deg y = n$ and $\deg z = 2$ defines a filtration, called the standard filtration, on R. Unless otherwise noted, we assume throughout that this is the filtration on R. Under the standard filtration, $\operatorname{gr} R \cong \Bbbk[x, y, z]/(yx - a_n)$, a complete intersection domain. Our interest is in filtered automorphisms, i.e., maps $\phi \in \operatorname{Aut}(R)$ such that $\deg(\phi(r)) = \phi(\deg(r))$ for all $r \in R$. As R is assumed to be a classical GWA, ϕ is a filtered automorphism if and only if this holds for $r = x$, y, and z. We denote the group of filtered automorphisms of R by $\operatorname{Aut_{fl}}(R)$. In Section 2, we determine $\operatorname{Aut_{fl}}(R)$.

In Theorem 2.10, we show that when $\deg_z(a) = 2$ and $g \in \operatorname{Aut_{fl}}(R)$ is of finite order, then $R^{\langle g \rangle}$ is a classical GWA. We view this as a step toward a Shephard-Todd-Chevalley theorem for classical GWAs, but do not yet know a good analog for a reflection group in this setting. Also, one obtains a version of Smith's theorem (Corollary 2.11). It is then reasonable to conjecture that an analog of Smith's theorem is true for finite groups of filtered automorphisms acting on a classical GWA with $\deg_z(a) = 2$. When $\deg_z(a) > 2$, the group $\operatorname{Aut_{fl}}(R)$ is much more restricted (Theorems 2.4 and 3.1). In this case we are able to compute the fixed ring of R under the action of any filtered automorphism g of finite order (Theorem 3.2). However, there are certain cases in which we cannot detect whether or not $R^{\langle g \rangle}$ is a GWA.

We are also interested in the homological determinant and its connection to the above results. The homological determinant of a linear automorphism of a noncommutative algebra generalizes the notion of the determinant of a linear map— in fact, when applied to a commutative polynomial ring, the homological determinant restricts to the usual determinant.

We refer the reader to [13] for a full definition of the homological determinant, but we note an essential result that will be important for our analysis. Let A be a filtered, noetherian, AS-Gorenstein ring such that $\operatorname{gr} A$ is commutative and let $g \in \operatorname{Aut_{fl}}(A)$. By [11, Lemma 2.1 and Proposition 2.4], one may define the homological determinant of g to be

$$\operatorname{hdet}_A(g) = \operatorname{hdet}_{\operatorname{gr} A}(g) = \det_{\operatorname{gr} A}(g).$$

Using this result, we prove that all filtered automorphisms of classical GWAs act with homological determinant 1 (Theorems 2.4 and 2.6). We denote by $\operatorname{SL}(A)$ the subgroup of $\operatorname{Aut}(A)$ consisting of automorphisms of homological determinant 1.

A noetherian ring A of finite injective dimension is called Auslander-Gorenstein if for any (left or right) module M and submodule N of $\operatorname{Ext}^s_A(M, A)$, $s \in \mathbb{Z}_+$, we have $\operatorname{Ext}^i_A(N, A) = 0$ for $i < s$. By [10, Theorem 2.2], every classical GWA $R = \Bbbk[z][x, y; \sigma, a]$ with $\deg_z(a) \geq 2$ is Auslander-Gorenstein. If R is a classical GWA and G is a finite subgroup of $\operatorname{Aut_{fl}}(R)$, then $G \subset \operatorname{SL}(R)$ and so R^G is filtered Auslander-Gorenstein by [11, Theorem 3.5], that is, $\operatorname{gr}(R^G)$ is Auslander-Gorenstein. This will be useful in the case of $\deg_z(a) \geq 3$ when we are not able to determine whether or not $R^{\langle g \rangle}$ is a GWA for all $g \in \operatorname{Aut_{fl}}(R)$. We also recover this result as a consequence of Theorem 2.10 in the case of a classical GWA with $\deg_z(a) = 2$ and $G \subset \mathrm{Aut}_{\mathrm{fl}}(R)$ a finite cyclic group. Though this implies that R^G has finite left and right injective dimension, we will see that this is not enough to guarantee finite global dimension (Corollary 2.12).

We end in Section 4 with a note regarding Auslander's theorem for GWAs. Given an algebra A and a group G acting as automorphisms on A, the skew group algebra $A\#G$ is defined to be the $\Bbbk$-vector space $A \otimes \Bbbk G$ with multiplication,

$$(a\#g)(b\#h) = ag(b)\#gh \quad \text{for all } a, b \in A,\ g, h \in G.$$

For a filtered algebra A and a finite group G acting as filtered automorphisms on A, the Auslander map is given by

$$\begin{aligned} \gamma_{A,G} : A\#G &\longrightarrow \mathrm{End}_{A^G}(A) \\ a\#g &\mapsto \begin{pmatrix} A & \longrightarrow & A \\ b & \mapsto & ag(b) \end{pmatrix}. \end{aligned}$$

If G is a finite group that contains no reflections acting linearly on $A = \Bbbk[x_1, \dots, x_t]$, then a theorem of Auslander asserts that $\gamma_{A,G}$ is an isomorphism [3]. In this setting, Auslander's theorem is a sort of dual result to Shephard-Todd-Chevalley. However, this is not the case for filtered actions on classical GWAs. That is, there are classical GWAs R and finite groups $G \subset \mathrm{Aut}_{\mathrm{fl}}(R)$ such that R^G is again a classical GWA and for which $\gamma_{R,G}$ is an isomorphism. For example, if G is a finite group acting linearly on the first Weyl algebra $A_1(\Bbbk)$, then the action of G is outer and so $\gamma_{A_1(\Bbbk),G}$ is an isomorphism [17, Theorems 2.4] and if G is cyclic, then $A_1(\Bbbk)^G$ is a classical GWA (Proposition 2.1). Similarly, since the units of a classical GWA R all live in the degree zero component (under the $\mathbb{Z}$-grading), $\Bbbk[z]$, the units of R are just $\Bbbk^\times$. Thus, every finite group action on R is outer and so if R is simple, then we may apply the same theorem. We present another method that will include classical GWAs that are not simple.

Let A be an affine algebra generated in degree 1 and G a finite subgroup of $\mathrm{GL}_n(\Bbbk)$ acting on A_1. The pertinency of the G-action on A is defined to be

$$\mathsf{p}(A, G) = \mathrm{GKdim}\, A - \mathrm{GKdim}(A\#G)/(f_G)$$

where (f_G) is the two sided ideal of $A\#G$ generated by $f_G = \sum_{g\in G} 1\#g$ and GKdim is the Gelfand-Kirillov (GK) dimension. The notion of pertinency was developed by Bao, He, and Zhang as a way to study the Auslander map for noncommutative algebras [4,5]. It is possible to define pertinency in terms of any dimension function on right A-modules, but GK dimension is sufficient for our purpose. Under suitable conditions, the Auslander map is an isomorphism for (A, G) if and only if $\mathsf{p}(A, G) \geq 2$. We show that this holds for (R, G) where R is a classical GWA R and $G \subset \mathrm{Aut}_{\mathrm{fl}}(R)$ is a finite cyclic group (Theorem 4.4).

2. Group actions preserving the standard filtration

Throughout this section, assume $R = \Bbbk[z][x, y; \sigma, a]$ is a classical GWA. Our primary goal will be to compute $\mathrm{Aut}_{\mathrm{fl}}(R)$ and prove that $R^{\langle g\rangle}$ is a classical GWA for every $g \in \mathrm{Aut}_{\mathrm{fl}}(R)$ with $|g| < \infty$. Although our interest is primarily in fixed rings of higher-degree classical GWAs, as a warm-up we compute the fixed rings for cyclic subgroups of filtered automorphisms acting on the first Weyl algebra, $A_1(\Bbbk)$.

Proposition 2.1. *Let $g \in \mathrm{Aut}_{\mathrm{fl}}(A_1(\Bbbk))$ have finite order, then the fixed ring $A_1(\Bbbk)^{\langle g\rangle}$ is a classical GWA.*

Proof. A filtered map $g : A_1(\Bbbk) \longrightarrow A_1(\Bbbk)$ given by

$$g(x) = a_1x + a_2y + a_3, \quad g(y) = b_1x + b_2y + b_3,$$

for some $a_i, b_j \in \Bbbk$, is an automorphism if and only if $a_1b_2 - a_2b_1 = 1$. Assume first that $b_2 - a_1 \pm \sqrt{w^2 - 4} \neq 0$. We diagonalize the action by setting $w = a_1 + b_2$ and

$$X = r\left(\left(\frac{2b_1a_2(w-2+\sqrt{w^2-4})}{b_2-a_1+\sqrt{w^2-4}}\right)x + (w-2+\sqrt{w^2-4})y \right.$$
$$\left. + ((a_1-b_2)a_3 + 2b_3a_2 + a_3\sqrt{w^2-4})\right),$$

$$Y = s\left(\left(\frac{2b_1a_2(w-2-\sqrt{w^2-4})}{b_2-a_1-\sqrt{w^2-4}}\right)x + (w-2-\sqrt{w^2-4})y \right.$$
$$\left. + ((a_1-b_2)a_3 + 2b_3a_2 - a_3\sqrt{w^2-4})\right),$$

for any $r, s \in \Bbbk^\times$. Then X and Y generate $A_1(\Bbbk)$ and we may choose r, s such that $XY - YX + 1 = 0$. Moreover, if we let $\beta = \frac{1}{2}(w + \sqrt{w^2-4})$, then one can check that $g(X) = \beta X$ and $g(Y) = \beta Y$, so $|\beta| = \ell < \infty$. Set $Z = YX$. By [12, Theorem 2.6], $A_1(\Bbbk)^{\langle g\rangle} = \Bbbk[Z][X^\ell, Y^\ell; \varsigma^\ell, A(Z)]$ where $\varsigma : \Bbbk[Z] \to \Bbbk[Z]$ is given by $\varsigma(Z) = Z - 1$ and $A(Z) = \prod_{i=0}^{\ell-1} \varsigma^{-i}(Z)$.

In the case that $b_2 - a_1 \pm \sqrt{w^2-4} = 0$ we have $b_2 = a_1^{-1}$, hence $b_1a_2 = 0$. In this case the analysis simplifies significantly. Assuming $b_1 = 0$ (the case $a_2 = 0$ is similar), we may take

$$X = r\left((a_1-1)^3(a_1+1)x + (a_1-1)a_1a_2y + (a_1^2a_3 + a_1a_2b_3 - a_3)\right),$$
$$Y = s\left((a_1-1)y - a_1b_3\right),$$

for any $r, s \in \Bbbk^\times$. Again, X and Y generate $A_1(\Bbbk)$ and we may choose r, s such that $XY - YX + 1 = 0$. Here we set $\beta = a_1$. $\square$

Let $n = \deg_z(a)$, $\lambda \in \Bbbk$, $\beta \in \Bbbk^\times$, $m \in \mathbb{N}$, and let Δ_m be the linear map $\Bbbk[z] \to \Bbbk[z]$ given by $\sigma^m - 1$. Bavula and Jordan [7] define the following maps:

$$\Theta_\beta : x \mapsto \beta x, y \mapsto \beta^{-1} y, z \mapsto z,$$

$$\Psi_{m,\lambda} : x \mapsto x, y \mapsto y + \sum_{i=1}^{n} \frac{\lambda^i}{i!} \Delta_m^i(a) x^{im-1}, z \mapsto z - m\lambda x^m,$$

$$\Phi_{m,\lambda} : x \mapsto x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{i!} y^{im-1} \Delta_m^i(a), y \mapsto y, z \mapsto z + m\lambda y^m.$$

Note that $\Psi_{0,\lambda}$ and $\Phi_{0,\lambda}$ are both the identity map. If there exists some $\rho \in \Bbbk$ such that $a(\rho - z) = (-1)^n a(z)$, then a is said to be reflective. By [7, Theorem 3.29], $\operatorname{Aut}(R)$ is generated by Θ_β, $\Psi_{m,\lambda}$, and $\Phi_{m,\lambda}$, $\beta \in \Bbbk^\times$, $\lambda \in \Bbbk$, and $m \in \mathbb{N}$, when a is not reflective. On the other hand, when a is reflective, then $\operatorname{Aut}(R)$ has an additional generator Ω given by

$$\Omega(x) = y, \quad \Omega(y) = (-1)^n x, \quad \Omega(z) = 1 + \rho - z.$$

Below, we consider some relations between the generators of $\operatorname{Aut}(R)$.

Proposition 2.2. *Let R be a classical GWA. The following relations hold in* $\operatorname{Aut}(R)$*. For all $m \in \mathbb{N}$ and all $\lambda, \mu \in \Bbbk$, $\beta, \gamma \in \Bbbk^\times$,*

1. $\Phi_{m,\mu} \circ \Phi_{m,\lambda} = \Phi_{m,\mu+\lambda}$ *and* $\Psi_{m,\mu} \circ \Psi_{m,\lambda} = \Psi_{m,\mu+\lambda}$*,*
2. $\Theta_\beta \circ \Phi_{m,\lambda} = \Phi_{m,\lambda\beta^{-m}} \circ \Theta_\beta$ *and* $\Theta_\beta \circ \Psi_{m,\lambda} = \Psi_{m,\lambda\beta^m} \circ \Theta_\beta$*, and*
3. $\Theta_\beta \circ \Theta_\gamma = \Theta_{\beta\gamma}$*.*

When a is reflective we have the following additional relations. For all $m \in \mathbb{N}$ and all $\lambda \in \Bbbk$, $\beta \in \Bbbk^\times$,

1. $\Omega \circ \Theta_\beta = \Theta_{\beta^{-1}} \circ \Omega$ *and*
2. $\Phi_{m,\lambda} \circ \Omega = \Omega \circ \Psi_{m,\lambda}$*.*

Proof. By [7], the maps $\operatorname{ad} x^m$ and $\operatorname{ad} y^m$ are locally nilpotent derivations of R and hence $\Phi_{m,\lambda} = e^{\lambda \operatorname{ad} x^m}$ is an automorphism of R. It then follows easily that

$$\Phi_{m,\mu} \circ \Phi_{m,\lambda} = e^{\mu \operatorname{ad} y^m} \circ e^{\lambda \operatorname{ad} y^m} = e^{(\mu+\lambda) \operatorname{ad} y^m} = \Phi_{m,\mu+\lambda}.$$

The claim for the maps $\Psi_{m,\lambda}$ is similar. Thus, (1) holds.

We will check the first claim of (2) by verifying that the relation holds on the generators. Observe that

$$\begin{aligned}
\Theta_\beta(\Phi_{m,\lambda}(x)) &= \Theta_\beta\left(x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{i!} y^{im-1}\Delta_m^i(a)\right) \\
&= \beta x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{i!} \beta^{1-im} y^{im-1}\Delta_m^i(a) \\
&= \beta x + \sum_{i=1}^{n} \frac{(-\lambda\beta^{-m})^i}{i!} \beta y^{im-1}\Delta_m^i(a) \\
&= \Phi_{m,\lambda\beta^{-m}}(\beta x) = \Phi_{m,\lambda\beta^{-m}}(\Theta_\beta(x)) \\
\Theta_\beta(\Phi_{m,\lambda}(y)) &= \Theta_\beta(y) = \beta^{-1}y = \Phi_{m,\lambda\beta^{-m}}(\Theta_\beta(y)) \\
\Theta_\beta(\Phi_{m,\lambda}(z)) &= \Theta_\beta(z + m\lambda y^m) = z + m\lambda(\beta^{-1}y)^m = z + m(\lambda\beta^{-m})y^m \\
&= \Phi_{m,\lambda\beta^{-m}}(z) = \Phi_{m,\lambda\beta^{-m}}(\Theta_\beta(z)).
\end{aligned}$$

Thus, $\Theta_\beta \circ \Phi_{m,\lambda} = \Phi_{m,\lambda\beta^{-m}} \circ \Theta_\beta$ as claimed. The second relation in (2) holds similarly.

We leave the claims in (3) and (4) to the reader and finish by checking (5). Assume that a is reflective. By [7, Equations (7) and (9)] we have

$$(\operatorname{ad} x^m)^i(y) = \Delta_m^i(a)x^{im-1} \quad \text{and} \quad (\operatorname{ad} y^m)^i(x) = (-1)^i y^{im-1}\Delta_m^i(a).$$

Thus,

$$\Omega((\operatorname{ad} x^m)^i(y)) = (-1)^n(\operatorname{ad} y^m)^i(x).$$

Using this, we check that the relation holds on the generators of R:

$$\begin{aligned}
\Phi_{m,\lambda}(\Omega(x)) &= \Phi_{m,\lambda}(y) = y = \Omega(x) = \Omega(\Psi_{m,\lambda}(x)) \\
\Phi_{m,\lambda}(\Omega(y)) &= \Phi_{m,\lambda}((-1)^n x) = (-1)^n\left(x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{i!} y^{im-1}\Delta_m^i(a)\right) \\
&= (-1)^n\left(x + \sum_{i=1}^{n} \frac{\lambda^i}{i!}(\operatorname{ad} y^m)^i(x)\right) = \Omega\left(y + \sum_{i=1}^{n} \frac{\lambda^i}{i!}(\operatorname{ad} x^m)^i(y)\right) \\
&= \Omega\left(y + \sum_{i=1}^{n} \frac{\lambda^i}{i!}\Delta_m^i(a)x^{im-1}\right) = \Omega(\Psi_{m,\lambda}(y)) \\
\Phi_{m,\lambda}(\Omega(z)) &= \Phi_{m,\lambda}(1 + \rho - z) = 1 + \rho - (z + m\lambda y^m) \\
&= (1 + \rho - z) - m\lambda y^m = \Omega(z - m\lambda x^m) = \Omega(\Psi_{m,\lambda}(z)).
\end{aligned}$$

Hence, $\Phi_{m,\lambda} \circ \Omega = \Omega \circ \Psi_{m,\lambda}$ as claimed. $\square$

We next give criteria for identifying filtered automorphisms based on the action on z. This will allow us to completely determine $\operatorname{Aut}_{\mathrm{fl}}(R)$ when R is a classical GWA with $\deg_z(a) > 2$. It will also be a useful step in the case of $\deg_z(a) = 2$.

Lemma 2.3. *Let ϕ be a filtered automorphism of a classical GWA $R = \Bbbk[z][x, y; \sigma, a]$.*

1. *If $\phi(z) = kz + c$ for some $c \in \Bbbk$ and $k \in \Bbbk^\times$, then either*
   - *$\phi = \Theta_\beta$ for some $\beta \in \Bbbk^\times$; or*
   - *a is reflective and $\phi = \Omega$ or $\phi = \Omega \circ \Theta_{-1}$.*
2. *If $\phi(z) \neq kz + c$ for some $c \in \Bbbk$ and $k \in \Bbbk^\times$, then* $\deg_z(a) \leq 2$.

Proof. (1) Suppose $\phi(z) = kz + c$ for some $c \in \Bbbk$ and $k \in \Bbbk^\times$. As ϕ is a filtered map, we may write

$$\begin{aligned}\phi(x) &= k_{11}x + k_{12}y + p_1(z),\\ \phi(y) &= k_{21}x + k_{22}y + p_2(z),\end{aligned}$$

where $k_{ij} \in \Bbbk$ and $p_i(z)$ are polynomials in z of degree at most $n/2$. Then

$$\begin{aligned}0 &= \phi([x, z] + x)\\ &= [k_{11}x + k_{12}y + p_1(z), kz + c] + (k_{11}x + k_{12}y + p_1(z))\\ &= k_{11}(k[x, z] + x) + k_{12}(k[y, z] + y) - p_1(z)\\ &= k_{11}(1 - k)x + k_{12}(1 + k)y - p_1(z).\end{aligned}$$

A similar computation shows that

$$0 = \phi([y, z] - y) = -k_{21}(1 + k)x - k_{22}(1 - k)y - p_2(z).$$

If $k_{11} = k_{12} = 0$, then $\phi(x) \in \Bbbk[z]$, violating the surjectivity of ϕ. Similarly, we may not have $k_{21} = k_{22} = 0$. If $k_{12} = 0$, then $k_{11} \neq 0$ and $k = 1$, so $k_{21} = 0$. Otherwise, $k_{11} = 0$ so $k = -1$ and $k_{22} = 0$. In either case, $p_1(z) = p_2(z) = 0$.

In the first case,

$$0 = \phi(yx - a(z)) = k_{11}k_{22}a(z) - a(z + c).$$

We may assume without loss of generality that $a(z) = z(z - t_1)\cdots(z - t_{n-1})$ for some $t_i \in \Bbbk$. Thus, $k_{22} = k_{11}^{-1}$ and $c = 0$, so $\phi = \Theta_{k_{11}}$.

In the second case,

$$0 = \phi(yx - a(z)) = k_{12}k_{21}a(z - 1) - a(-z + c).$$

As $a(z-1)$ is monic and the leading coefficient of $a(-z+c)$ is $(-1)^n$, then $k_{12}k_{21} = (-1)^n$. It follows that a is reflective and $\phi = \Omega$ or $\phi = \Omega \circ \Theta_{-1}$.

(2) If $\phi \in \mathrm{Aut}_{\mathrm{fl}}\, R$, then $\phi(z) = kz + p(x, y)$ for some polynomial p in x and y. In the filtration, $\deg(x) = \deg(y) = \deg(a) = n$, but $\deg(z) = 2$. By the hypothesis and part (1), we must have $\deg(p(x, y)) \geq 1$, whence $n = 2$. □

Theorem 2.4. *Suppose $R = \Bbbk[z][x, y; \sigma, a]$ is a classical GWA with $\deg_z(a) > 2$. If a is not reflective, then $\mathrm{Aut}_{\mathrm{fl}}(R)$ is generated by the maps Θ_λ. If a is reflective, then $\mathrm{Aut}_{\mathrm{fl}}(R)$ is generated by Ω and the maps Θ_λ. Moreover, we have $\mathrm{hdet}\, g = 1$ for all $g \in \mathrm{Aut}_{\mathrm{fl}}(R)$.*

Proof. This follows almost entirely from Lemma 2.3. Let $g \in \mathrm{Aut}_{\mathrm{fl}}(R)$ and recall that we have $\mathrm{gr}\, R = \Bbbk[x, y, z]/(xy - a_n)$. By the discussion in the introduction and a routine check, $\mathrm{hdet}_R(g) = \mathrm{hdet}_{\mathrm{gr}\, R}(g) = \det_{\Bbbk[x,y,z]}(g) = 1$. $\square$

In Theorem 3.1 we completely determine the finite subgroups of $\mathrm{Aut}_{\mathrm{fl}}(R)$ in the case when $n > 3$.

Assume $n = 2$. Without loss of generality, $a = z(z - t)$ for some $t \in \Bbbk$. Then $\Delta(a) = -2z + t + 1$ and $\Delta^2(a) = 2$. The generators of $\mathrm{Aut}(R)$ above that are also filtered maps can be stated explicitly.

$$\begin{aligned}
\Theta_\lambda : &\quad x \mapsto \lambda x, \quad y \mapsto \lambda^{-1} y, \quad z \mapsto z, \\
\Psi_{1,\lambda} : &\quad x \mapsto x, \quad y \mapsto y - 2\lambda z + \lambda^2 x + \lambda(t + 1), \quad z \mapsto z - \lambda x, \\
\Phi_{1,\lambda} : &\quad x \mapsto x + 2\lambda z + \lambda^2 y - \lambda(t + 1), \quad y \mapsto y, \quad z \mapsto z + \lambda y, \\
\Omega : &\quad x \mapsto y, \quad y \mapsto x, \quad z \mapsto 1 + t - z.
\end{aligned}$$

Let G be the group generated by these automorphisms. We will show below that $\mathrm{Aut}_{\mathrm{fl}}(R) = G$ in this case.

Before proving our main result regarding $\mathrm{Aut}_{\mathrm{fl}}(R)$, we need one more technical lemma.

Lemma 2.5. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $\deg_z(a) = 2$. Suppose $\Gamma \in \mathrm{Aut}_{\mathrm{fl}}(R)$ and $\Gamma(z) = k_1 x + k_2 y + k_4$ for some $k_i \in \Bbbk$. Then $k_1 k_2 \neq 0$.*

Proof. It is clear that we may not have $k_1 = k_2 = 0$. Suppose $k_2 = 0$. The case $k_1 = 0$ follows similarly. Write

$$\begin{aligned}
\Gamma(x) &= \ell_1 x + \ell_2 y + \ell_3 z + \ell_4, \\
\Gamma(y) &= m_1 x + m_2 y + m_3 z + m_4,
\end{aligned}$$

for $\ell_i, m_i \in \Bbbk$. Then

$$\begin{aligned}
\Gamma(x) &= [\Gamma(z), \Gamma(x)] = [k_1 x + k_4, \ell_1 x + \ell_2 y + \ell_3 z + \ell_4] = [k_1 x, \ell_2 y + \ell_3 z] \\
\Gamma(y) &= [\Gamma(y), \Gamma(z)] = [m_1 x + m_2 y + m_3 z + m_4, k_1 x + k_4] = [m_2 y + m_3 z, k_1 x].
\end{aligned}$$

In both cases, the image of the commutator is in the subalgebra generated by x and z, implying $\ell_2 = m_2 = 0$. This contradicts the surjectivity of Γ. $\square$

Theorem 2.6. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $\deg_z(a) = 2$. Then $\mathrm{Aut}_{\mathrm{fl}}(R) = G$. Moreover, if $g \in \mathrm{Aut}_{\mathrm{fl}}(R)$, then $\mathrm{hdet}(g) = 1$.*

Proof. The statement on homological determinant follows analogously to Theorem 2.4 once we have shown that $\mathrm{Aut}_{\mathrm{fl}}(R) = G$. Clearly, $G \subset \mathrm{Aut}_{\mathrm{fl}}(R)$. Let $\Gamma \in \mathrm{Aut}_{\mathrm{fl}}(R)$. We may write $\Gamma(z) = k_1 x + k_2 y + k_3 z + k_4$ for some $k_i \in \Bbbk$.

If $k_1 = k_2 = 0$, then $\Gamma \in G$ by Lemma 2.3. Suppose that $k_1 = 0$ but $k_2 \neq 0$. By Lemma 2.5, $k_3 \neq 0$. Then

$$\Phi_{1,-k_2/k_3}(\Gamma(z)) = k_2 y + k_3\left(z - (k_2/k_3)y\right) + k_4 = k_3 z + k_4.$$

Thus, $\Phi_{1,-k_2/k_3} \circ \Gamma \in G$ again by Lemma 2.3, so $\Gamma \in G$. Similarly, if $k_2 = 0$ but $k_1 \neq 0$, then

$$\Psi_{1,k_1/k_3}(\Gamma(z)) = k_1 x + k_3\left(z - (k_1/k_3)x\right) + k_4 = k_3 z + k_4.$$

Finally, suppose $k_1, k_2 \neq 0$. Note that we may have $k_3 = 0$ in this case. Set λ to be a root of $k_1\lambda^2 + k_3\lambda + k_2 = 0$. Then

$$\begin{aligned}\Phi_{1,\lambda}(\Gamma(z)) &= k_1(x + 2\lambda z + \lambda^2 y - \lambda(t+1)) + k_2 y + k_3(z + \lambda y) + k_4 \\ &= k_1 x + (k_1\lambda^2 + k_3\lambda + k_2)y + (2k_1\lambda + k_3)z + (k_4 - k_1\lambda(t+1)) \\ &= k_1 x + (2k_1\lambda + k_3)z + (k_4 - k_1\lambda(t+1)).\end{aligned}$$

Note that $2k_1\lambda + k_3 \neq 0$ by Lemma 2.5. We now defer to the above computation. $\square$

Using the techniques of Theorem 2.6, or straightforward computation, we achieve our last relation between the generators of G. Given $\lambda, \mu \in \Bbbk$, set $\eta = 1 - \lambda\mu$. Then

$$\Phi_{1,\mu} \circ \Psi_{1,\lambda} = \Psi_{1,\lambda\eta^{-1}} \circ \Phi_{1,\mu\eta} \circ \Theta_{\eta^{-2}}. \tag{2.7}$$

For $\lambda, \mu \in \Bbbk$ and $\beta \in \Bbbk^\times$, set $\tau_{\lambda,\mu,\beta} = \Psi_{1,\lambda} \circ \Phi_{1,\mu} \circ \Theta_\beta$. These maps satisfy

$$\begin{aligned}\tau_{\lambda,\mu,\beta} : \quad & x \mapsto \beta((\lambda\mu - 1)^2 x + \mu^2 y + 2\mu(1 - \lambda\mu)z + \mu(\lambda\mu - 1)(t+1)) \\ & y \mapsto \beta^{-1}(y + \lambda^2 x - 2\lambda z + \lambda(t+1)) \\ & z \mapsto (1 - 2\lambda\mu)z + \lambda(\lambda\mu - 1)x + \mu y + \lambda\mu(t+1).\end{aligned}$$

Note that $\tau_{\lambda,\mu,\beta} = \mathrm{id}$ if and only if $\lambda = \mu = 0$ and $\beta = 1$.

Corollary 2.8. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be classical GWA with $\deg_z(a) = 2$. If $g \in \mathrm{Aut}_{\mathrm{fl}}(R)$, then either $g = \tau_{\lambda,\mu,\beta}$ or $\tau_{\lambda,\mu,\beta} \circ \Omega$ for an appropriate choice of λ, μ, β.*

Proof. This follows from (2.7), Proposition 2.2, and Theorem 2.6. $\square$

Now that we understand $\mathrm{Aut}_{\mathrm{fl}}(R)$ when $\deg_z(a) = 2$, we are ready to consider fixed rings of R by its cyclic subgroups. We first give two examples that illustrate our methods before stating our main theorem.

Example 2.9. Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $a = z(z - t)$. Let $\alpha \in \Bbbk$ and $\beta \in \Bbbk^\times$.

(1) This example is similar to [15, Example 2.7], in which the authors compute $A_1(\Bbbk)^{\langle\Omega\rangle}$. Define

$$X = \frac{i}{2}(x - y) - \left(z - \frac{1+t}{2}\right), \quad Y = \frac{i}{2}(x - y) + \left(z - \frac{1+t}{2}\right), \quad \text{and} \quad Z = \frac{i}{2}(x + y).$$

Set $k = \frac{1}{4}(1 - t^2)$, $K_\pm = \frac{1}{2}(-1 \pm \sqrt{1 - 4k})$, and $a'(Z) = (Z - K_+)(Z - K_-)$. Let $\varsigma : \Bbbk[Z] \longrightarrow \Bbbk[Z]$ be the automorphism mapping Z to $Z - 1$. Then $R = \Bbbk[Z][X, Y; \sigma, a'(Z)]$ and we have $\Omega(X) = -X$, $\Omega(Y) = -Y$, $\Omega(Z) = Z$. Thus, by [12, Theorem 2.6], $R^{\langle\Omega\rangle} = \Bbbk[Z][X^2, Y^2; \varsigma^2, a'(Z)\varsigma^{-1}(a'(Z))]$.

(2) Let β be a primitive ℓth root of unity for some $\ell \geq 2$. Set $\pi_{\alpha,\beta} = \Phi_{1,\alpha} \circ \Theta_\beta$ and note that $|\pi_{\alpha,\beta}| = \ell$ by Proposition 2.2. Define

$$X = x + \frac{\alpha^2\beta^2}{(\beta-1)^2}y + \frac{2\alpha\beta}{\beta-1}z - \frac{\alpha\beta(r+1)}{\beta-1}, \quad Y = y, \quad \text{and} \quad Z = z + \frac{\alpha\beta}{\beta-1}y.$$

Let $\varsigma : \Bbbk[Z] \longrightarrow \Bbbk[Z]$ be the automorphism mapping Z to $Z - 1$. Then $R = \Bbbk[Z][X, Y; \varsigma, a(Z)]$ and we have $\pi_{\alpha,\beta}(X) = \beta X$, $\pi_{\alpha,\beta}(Y) = \beta^{-1}Y$ and $\pi_{\alpha,\beta}(Z) = Z$. Thus, $R^{\langle\pi_{\alpha,\beta}\rangle} = \Bbbk[Z][X^\ell, Y^\ell; \varsigma^\ell, A(Z)]$ where $A(Z) = \prod_{i=0}^{\ell-1} \varsigma^{-i}(a(Z))$.

We now show that, given an appropriate generating set, one can diagonalize the action of $\tau_{\lambda,\mu,\beta}$ and $\tau_{\lambda,\mu,\beta} \circ \Omega$ when the maps have finite order. Computations for the next theorem were done using Maple and the NCAlgebra package for Macaulay2.

Theorem 2.10. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $a = z(z - t)$. If $g \in \mathrm{Aut}_{\mathrm{fl}}(R)$ with $|g| = \ell$, $2 \leq \ell < \infty$, then the action of g is diagonalizable and hence $R^{\langle g\rangle}$ is again a GWA.*

Proof. First suppose that $g = \tau_{\lambda,\mu,\beta}$. Set $w = \beta\lambda\mu - \beta - 1$ and

$$\begin{aligned} &C_1 = (2\sqrt{\lambda\mu\beta(w^2 - 4\beta)})^{-1}, \qquad C_2 = (-1\sqrt{w^2 - 4\beta})^{-1}, \\ &K_\pm = \frac{(t+1)(w+2)}{2\sqrt{w^2 - 4\beta}} \pm \frac{t-1}{2}. \end{aligned}$$

Let

$$\begin{aligned} X &= C_1\Big(\lambda\left(w + 2 + \sqrt{w^2 - 4\beta}\right)x + \beta\mu\left(w + 2 - \sqrt{w^2 - 4\beta}\right)y \\ &\qquad - 4\lambda\mu\beta z + 2\lambda\mu\beta(t+1)\Big) \\ Y &= C_1\Big(\lambda\left(w + 2 - \sqrt{w^2 - 4\beta}\right)x + \beta\mu\left(w + 2 + \sqrt{w^2 - 4\beta}\right)y \end{aligned}$$

$$- 4\lambda\mu\beta z + 2\lambda\mu\beta(t+1)\Big)$$

$$Z = C_2\left(\lambda x + \beta\mu y - (w+2)z\right).$$

Next, suppose $g = \tau_{\lambda,\mu,\beta} \circ \Omega$. In the generic case, $\lambda\mu \neq 1$, set $w = \lambda + \mu\beta$ and

$$C_1 = \frac{\sqrt{\beta(1-\lambda\mu)}}{2w\sqrt{w^2-4\beta}}, \qquad C_2 = -\frac{1}{w^2-4\beta}, \qquad K_\pm = \frac{(t+1)(\lambda-\beta\mu)}{2\sqrt{w^2-4\beta}} \pm \frac{t-1}{2}.$$

Let

$$X = C_1\left(\left(\lambda^2 - (\mu\beta)^2 + w\sqrt{w^2-4\beta}\right)x + \left(\frac{(\mu\beta)^2 - \lambda^2 + w\sqrt{w^2-4\beta}}{\beta(\lambda\mu-1)}\right)y\right.$$
$$\left. - 4wz + 2w(t+1)\right)$$
$$Y = C_1\left(\left(\lambda^2 - (\mu\beta)^2 - w\sqrt{w^2-4\beta}\right)x + \left(\frac{(\mu\beta)^2 - \lambda^2 - w\sqrt{w^2-4\beta}}{\beta(\lambda\mu-1)}\right)y\right.$$
$$\left. - 4wz + 2w(t+1)\right)$$
$$Z = C_2\left(-\beta(\lambda\mu-1)x + y + (\beta\mu-\lambda)z\right).$$

We consider the special case when $\lambda\mu = 1$ at the end.

Let $X, Y, Z \in R$ be defined as above depending on the case. In either case, let $\varsigma : \Bbbk[Z] \longrightarrow \Bbbk[Z]$ be the automorphism mapping Z to $Z-1$ and set $a'(Z) := (Z-K_+)(Z-K_-)$. Direct computations show that $XZ = (Z-1)X$, $YZ = (Z+1)Y$, $YX = a'(Z)$, and $XY = \varsigma(a'(Z))$. Since X, Y, and Z generate R as an algebra, then R has a presentation as the classical GWA $\Bbbk[Z][X, Y; \varsigma, a'(Z)]$.

Let

$$\gamma = \frac{1}{2\beta}\left(w^2 - 2\beta + w\sqrt{w^2-4\beta}\right)$$

and note that if $w^2 - 4\beta = 0$, then $\gamma = 1$. A check shows that $\tau_{\lambda,\mu,\beta}(X) = \gamma X$, $\tau_{\lambda,\mu,\beta}(Y) = \gamma^{-1}Y$ and $\tau_{\lambda,\mu,\beta}(Z) = Z$. Thus, the action of g is diagonal with respect to this presentation and so by [12, Theorem 2.6], $R^{\langle g\rangle} = \Bbbk[Z][X^\ell, Y^\ell; \varsigma^\ell, A(Z)]$ where $A(Z) = \prod_{i=0}^{n-1} \varsigma^{-i}(a'(Z))$.

Finally, suppose we are in the case $g = \tau_{\lambda,\mu,\beta} \circ \Omega$ but $\lambda\mu = 1$. Here, we set $K_+ = t$, $K_- = 0$, and let

$$X = \left(\beta - \lambda^2\right)x + \left(\frac{\lambda^2}{\beta - \lambda^2}\right)y + 2\lambda z - (t+1)\lambda,$$

$$Y = \left(\frac{1}{\beta - \lambda^2}\right) y, \quad Z = \left(\frac{\lambda}{\beta - \lambda^2}\right) y + z.$$

The same argument as before works with $\gamma = \lambda^2/\beta$. □

The next theorem is analogous to the main result in [18], as well as [2, Theorem 2]. That is, these GWAs are *rigid* with respect to cyclic group actions.

Corollary 2.11. *Let R be a classical GWA with $\deg_z(a) = 2$ and let $G, H \subset \mathrm{Aut}_{\mathrm{fl}}(R)$ be finite cyclic groups. If $R^G \cong R^H$, then $G \cong H$. In particular, if $R^H \cong R$, then H is trivial.*

Proof. An isomorphism of classical GWAs must preserve the degree of the defining polynomial [7, Theorem 3.28]. By Theorem 2.10, the degree of the defining polynomial of R^G (resp. R^H) is $2|G|$ (resp. $2|H|$). Thus, if $R^G \cong R^H$, then $|G| = |H|$. □

Let $R = \Bbbk[z][x, y; \sigma, a]$ be a GWA, not necessarily classical. Two roots α, β of a are said to be congruent if there exists an $i \in \mathbb{Z}$ such that, as ideals of $\Bbbk[z]$, $\big(\sigma^i(z - \alpha)\big) = (z - \beta)$. By [8, Theorem 1.6] and [10, Theorem 4.4], the global dimension of R satisfies

$$\mathrm{gldim}\, R = \begin{cases} \infty & \text{if } a \text{ has a multiple root} \\ 2 & \text{if } a \text{ has a congruent root and no multiple roots} \\ 1 & \text{if } a \text{ has no congruent roots and no multiple roots.} \end{cases}$$

Corollary 2.12. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $a = z(z - t)$ and let $H \subset \mathrm{Aut}_{\mathrm{fl}}(R)$ be a finite cyclic group with $|H| > 2$.*

1. *If $\mathrm{gldim}(R) = \infty$, then $\mathrm{gldim}(R^H) = \infty$.*
2. *If $\mathrm{gldim}(R) = 1$, then $\mathrm{gldim}(R^H) = 1$.*
3. *If $\mathrm{gldim}(R) = 2$, then $t \in \mathbb{Z}$ and*

$$\mathrm{gldim}(R^H) = \begin{cases} 2 & \textit{if } |t| \geq |H| \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. By Theorem 2.10, it suffices to consider the fixed ring by a diagonal action on R. Note that in Theorem 2.10, we have $|K_+ - K_-| = t$ and so the change of generating set does not affect the difference between the roots. We freely use the notation from that theorem in this proof.

Recall that $A(Z) = \prod_{i=0}^{n} \varsigma^{-i}(a'(Z))$. If a has a multiple root then so does A, proving (1). Suppose that $t > 0$. The case $t < 0$ is similar. Then the roots of $A(Z)$ are $0, 1, \ldots, n-1, t, t+1, \ldots, t+(n-1)$. In this case, $\mathrm{gldim}(R^H) = \infty$ if and only if $0 < t \leq n-1$. Because the automorphism associated to R^H is ς^n, where $\varsigma(Z) = Z - 1$, then it follows that R^H

has congruent roots if and only if R has congruent roots, proving (2). Furthermore, if $\text{gldim}(R) = 2$, so $t \in \mathbb{Z}$ and $t \neq 0$, then $\text{gldim}(R^H) \geq 2$ and $A(Z)$ has multiple roots if and only if $t \geq |H|$. □

It is clear that Corollary 2.12 also applies to higher degree classical GWAs under the action of Θ_β. As another application of Theorem 2.10, we consider the Calabi-Yau property for fixed rings of classical GWAs with $\deg_z(a) = 2$.

For an algebra A, we denote the enveloping algebra of A by $A^e = A \otimes A^{op}$, where A^{op} is the opposite algebra of A. The algebra A is homologically smooth if it has a finitely generated projective resolution of finite length in A^e. If, further, there exists $d \in \mathbb{N}$ such that $\text{Ext}^i_{A^e}(A, A^e) \cong \delta_{i,d} A$, where δ is the Kronecker-delta function, then A is said to be Calabi-Yau of dimension d.

By a result of Liu, a classical GWA $R = \Bbbk[z][x, y; \sigma, a]$ is Calabi-Yau if and only if R has finite global dimension [16, Theorem 1.1]. The Ext condition holds for all classical GWAs, but Liu proves that the finite global dimension hypothesis implies homological smoothness. It is noted in the discussion that for a homologically smooth algebra A, the Calabi-Yau dimension is bounded below by the global dimension. Consequently, if $\text{gldim}(R) = \infty$, then R is not homologically smooth. The next result now follows from the Corollary 2.12.

Corollary 2.13. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $a = z(z - t)$ and let $H \subset \text{Aut}_{\text{fl}}(R)$ be a finite cyclic group with $|H| > 2$. Then R^H is Calabi-Yau if and only if R is Calabi-Yau and either* $\text{gldim}(R) = 1$ *or* $\text{gldim}(R) = 2$ *and $|t| \geq |H|$.*

In most cases, we are unable to say whether the fixed ring of a classical GWA by a non-cyclic group of filtered automorphisms is a GWA. However, we can in one special case.

Corollary 2.14. *Let $R = A_1(\Bbbk)$ and $H = \langle \Theta_{-1}, \Omega \rangle \subset \text{Aut}_{\text{fl}}(R)$. Then R^H is a classical GWA.*

Proof. By Proposition 2.1, $R^{\langle \Theta_{-1} \rangle}$ is a classical GWA of degree 2. Since Ω is a filtered automorphism on $R^{\langle \Theta_{-1} \rangle}$, then $R^H = \left(R^{\langle \Theta_{-1} \rangle}\right)^{\langle \Omega \rangle}$. The result now follows from Theorem 2.10. □

3. The case $n \geq 3$

Let $R = \Bbbk[z][x, y; \sigma, a]$ with $n = \deg(a) \geq 3$. By Theorem 2.4, if a is not reflective then $\text{Aut}_{\text{fl}}(R)$ is generated by the maps Θ_β and if a is reflective than $\text{Aut}_{\text{fl}}(R)$ is generated by Ω and the maps Θ_β. In the former case, any finite subgroup H of $\text{Aut}_{\text{fl}}(R)$ will be cyclic, generated by some Θ_β where β is a root of unity. In this case, by [12, Theorem 2.6], R^H is again a generalized Weyl algebra.

We now study the finite subgroups of $\text{Aut}_{\text{fl}}(R)$ when a is reflective.

Theorem 3.1. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $a \in \Bbbk[z]$ a reflective polynomial of degree $n \geq 3$. Let H be a nontrivial finite subgroup of $\mathrm{Aut}_{\mathfrak{fl}}(R)$. If n is even, then one of the following holds:*

1. *$H = \langle \Theta_\beta \circ \Omega \rangle$ for some $\beta \in \Bbbk^\times$ and $H \cong C_2$,*
2. *$H = \langle \Theta_\lambda \rangle$ for some $\lambda \in \Bbbk^\times$ an mth root of unity and $H \cong C_m$, or*
3. *$H = \langle \Theta_\beta \circ \Omega, \Theta_\lambda \rangle$ for some $\beta, \lambda \in \Bbbk^\times$ with λ an mth root of unity and $H \cong D_{2m}$, the dihedral group of order $2m$.*

If n is odd, then one of the following holds:

1. *$H = \langle \Theta_\beta \circ \Omega \rangle$ for some $\beta \in \Bbbk^\times$ and $H \cong C_4$,*
2. *$H = \langle \Theta_\lambda \rangle$ for some $\lambda \in \Bbbk^\times$ an mth root of unity and $H \cong C_m$, or*
3. *$H = \langle \Theta_\beta \circ \Omega, \Theta_\lambda \rangle$ for some $\beta, \lambda \in \Bbbk^\times$ with λ an 2mth root of unity and $H \cong \mathrm{Dic}_m$, the binary dihedral group of order $4m$.*

Proof. Using Proposition 2.2, each element of $\mathrm{Aut}_{\mathfrak{fl}}(R)$ can be written as either Θ_β or $\Theta_\beta \circ \Omega$ for some $\beta \in \Bbbk^\times$. The automorphisms $\Theta_\beta \circ \Omega$ have order 2 if n is even and order 4 if n is odd. The automorphisms Θ_β have finite order if and only if β is a root of unity.

We consider the case that n is even. The case of n odd is similar. Let H be a finite subgroup of $\mathrm{Aut}_{\mathfrak{fl}}(R)$. Suppose first that H does not contain Θ_β for any $\beta \neq 1$. Because $\Theta_\beta \circ \Omega \circ \Theta_\gamma \circ \Omega = \Theta_{\beta\gamma^{-1}}$ for all $\gamma \in \Bbbk^\times$, this means that H is generated by a single $\Theta_\beta \circ \Omega$ and we are in Case 1.

Now if H contains Θ_β, then β must be a root of unity. Consider the subgroup of C of H consisting of elements of the form Θ_β. Since $\Theta_\beta \circ \Theta_\gamma = \Theta_{\beta\gamma}$ for any $\gamma \in \Bbbk^\times$, then C is generated by a single Θ_λ where λ is an mth root of unity. If $H = C$, then we are in Case 2.

So now suppose that $H \neq C$, so H contains some $\Theta_\beta \circ \Omega$. If $\Theta_\gamma \circ \Omega \in H$ for some $\gamma \in \Bbbk^\times$, then since $\Theta_\beta \circ \Omega \circ \Theta_\gamma \circ \Omega = \Theta_{\beta\gamma^{-1}}$, we must have that $\beta\gamma^{-1} = \lambda^j$ for some $0 \leq j < m$, so $\gamma = \beta\lambda^{-j}$ and hence $\Theta_\gamma \circ \Omega = \Theta_\beta \circ \Omega \circ \Theta_{\lambda^j}$, and we are in Case 3. Therefore, H is generated by $\Theta_\beta \circ \Omega$ and Θ_λ. Finally,

$$\Theta_\beta \circ \Omega \circ \Theta_\lambda = \Theta_\lambda^{-1} \circ \Theta_\beta \circ \Omega$$

so $H \cong D_{2m}$, as claimed. □

The classical GWA $R = \Bbbk[z][x, y; \sigma, a]$ is naturally $\mathbb{Z}$-graded by letting $\deg x = 1$, $\deg y = -1$ and $\deg f = 0$ for all $f \in \Bbbk[z]$. Under this grading, the maps Θ_β are graded automorphisms and the map Ω reverses the grading on R. In what follows, we exploit this $\mathbb{Z}$-grading on R.

Theorem 3.2. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $n = \deg_z(a)$. Suppose that a is a reflective polynomial so there exists $\rho \in \Bbbk$ with $a(\rho - z) = (-1)^n a(z)$. If n is even,* *then $R^{\langle\Theta_\beta\circ\Omega\rangle}$ is generated over $\Bbbk[z(1+\rho-z)]$ by $x+\beta y$ and $zx+\beta(1+\rho-z)y$. If n is odd, then $R^{\langle\Theta_\beta\circ\Omega\rangle}$ is generated over $\Bbbk[z(1+\rho-z)]$ by $x^2+\beta^2y^2$ and $zx^2+\beta^2(1+\rho-z)y^2$.*

Proof. By using the relations in R, each element r of R can be written

$$r=\sum_{i=0}^{m} f_i(z)x^i+\sum_{i=1}^{m} g_i(z)y^i$$

for some $f_i(z), g_i(z)\in\Bbbk[z]$. Since $\Theta_\beta\circ\Omega$ reverses the $\mathbb{Z}$-grading on R, if r is fixed by $\Theta_\beta\circ\Omega$, then we must have, for each $1\le i\le m$,

$$\Theta_\beta\circ\Omega(f_i(z)x^i)=g_i(z)y^i \quad\text{and}\quad \Theta_\beta\circ\Omega(g_i(z)y^i)=f_i(z)x^i$$

and hence

$$\beta^{-i}f_i(1+\rho-z)=g_i(z) \quad\text{and}\quad (-1)^{ni}\beta^{-i}f_i(1+\rho-z)=g_i(z).$$

Hence, the only nonzero summands of r occur when ni is even.

In particular, when $i=0$, we must have that $f_0(1+\rho-z)=f_0(z)$. By an induction argument, each $g(z)\in\Bbbk[z]$ can be written as h_1+zh_2 for some $h_1,h_2\in\Bbbk[z(1+\rho-z)]$. Hence, if $g(1+\rho-z)=g(z)$, then $h_2=0$ so $f_0\in\Bbbk[z(1+\rho-z)]$.

If n is even, then each invariant is a sum of terms of the form $f(z)x^m+\beta^m f(1+\rho-z)y^m$ where $m\ge 0$ and $f(z)\in\Bbbk[z]$. We claim that each of these elements is generated over $\Bbbk[z(1+\rho-z)]$ by $x+\beta y$ and $zx+\beta(1+\rho-z)y$. Since each $f(z)\in\Bbbk[z]$ can be written as h_1+zh_2 for some $h_1,h_2\in\Bbbk[z(1+\rho-z)]$, therefore we can write any

$$f(z)x^j+\beta^j f(1+\rho-z)y^j$$

as a $\Bbbk[z(1+\rho-z)]$-linear combination of $x^j+\beta^jy^j$ and $zx^j+\beta^j(1+\rho-z)y^j$.

It therefore suffices to show that for any $j\ge 0$, we can generate any $x^j+\beta^jy^j$ and $zx^j+\beta^j(1+\rho-z)y^j$. Now observe that

$$\left(x^{j-1}+\beta^{j-1}y^{j-1}\right)(x+\beta y)=x^j+\beta x^{j-1}y+\beta^{j-1}y^{j-1}x+\beta^jy^j$$

and since $a(z)=a(\rho-z)$,

$$\begin{aligned}\beta x^{j-1}y+\beta^{j-1}y^{j-1}x &= \beta x^{j-2}a(z-1)+\beta^{j-1}y^{j-2}a(z)\\ &= \beta a(z-j+1)x^{j-2}+\beta^{j-1}a(z+j-2)y^{j-2}\\ &= \beta a(z-j+1)x^{j-2}+\beta^{j-1}a(\rho-(z+j-2))y^{j-2}\\ &= \beta\left[a(z-j+1)x^{j-2}+\beta^{j-2}a((1+\rho-z)-j+1))y^{j-2}\right]\end{aligned}$$

so by induction we can generate any $x^j + \beta^j y^j$. By a similar argument, we can generate any $zx^j + \beta^j(1+\rho-z)y^j$. Therefore, the invariant ring has the claimed generators. The proof when n is odd is similar. □

Corollary 3.3. *Let $R = \Bbbk[z][x,y;\sigma,a]$ be a classical GWA with $n = \deg_z(a)$ and a reflective. Let*

$$A = \begin{cases} x+\beta y & n \text{ even} \\ x^2+\beta^2 y^2 & n \text{ odd}, \end{cases} \qquad B = \begin{cases} zx+\beta(1+\rho-z)y & n \text{ even} \\ zx^2+\beta^2(1+\rho-z)y^2 & n \text{ odd}, \end{cases}$$

and $C = z(1+\rho-z)$ so that A, B, and C generate $R^{\langle\Omega\rangle}$. If n is even, then the generators satisfy the following relations

$$\begin{aligned} [A,C] &= 2B-(2+\rho)A, & [B,A] &= A^2+\beta f(C), \\ [B,C] &= \rho B - 2CA, & B^2 &= \rho BA - CA^2 + \beta g(C). \end{aligned}$$

If n is odd, then the generators satisfy the following relations

$$\begin{aligned} [A,C] &= 4B-2(3+\rho)A, & [B,A] &= 2A^2+\beta f(C), \\ [B,C] &= 2(\rho-1)B - 4CA, & B^2 &= (\rho-1)BA - CA^2 + \beta g(C). \end{aligned}$$

In both cases, $f(C)$ and $g(C)$ represent polynomials in C with

$$\deg_C(f) = \begin{cases} n & n \text{ even} \\ 2n & n \text{ odd}, \end{cases} \quad \text{and} \quad \deg_C(g) = \begin{cases} 1+\frac{n}{2} & n \text{ even} \\ 2n+1 & n \text{ odd}. \end{cases}$$

Proof. Assume n is even. The case of n odd is similar. This is largely direct computation and we omit those for $[A,C]$ and $[B,C]$. Next we have,

$$\begin{aligned} [B,A] &= [zx+\beta(1+\rho-z)y, x+\beta y] \\ &= [z,x]x + \beta[1+\rho-z,y]y + \beta[zx,y] + \beta^2[(1+\rho-z)y,x] \\ &= (x^2+\beta^2y^2) + \beta(\rho-2z)yx + \beta(2z-2+\rho)xy \\ &= A^2 + \beta\left((\rho-1-2z)a + (2z-3-\rho)\sigma(a)\right). \end{aligned}$$

Observe that $(\rho-1-2z)a + (2z-3-\rho)\sigma(a) \in \Bbbk[z]$ and

$$\Omega((\rho-1-2z)a + (2z-3-\rho)\sigma(a)) = (2z-3-\rho)\sigma(a) + (\rho-1-2z)a.$$

Hence, it must be possible to express this as a polynomial in C. Finally we have

$$B^2 = \rho BA - CA^2 + \beta((3+\rho)z - 2z^2)\sigma(a) + \beta((1-\rho^2) + (3\rho+1)z - 2z^2)a.$$

As in the computation for $[B,A]$, the remaining polynomial in z is fixed by Ω. □

It is not clear to us whether $R^{\langle\Omega\rangle}$ is a GWA for $\deg_z(a) \geq 3$. One piece of evidence against is the following. A classical GWA $R = \Bbbk[z][x, y; \sigma, a]$ with $\deg_z(a) = 2$ can be presented with two generators by solving the relation $yx - xy = a - \sigma(a)$ for z and substituting into the other relations. When $\deg_z(a) > 2$, one cannot generate R using only x and y, but whether one can use a different pair of generators for R is unclear. We would expect that, were $R^{\langle\Omega\rangle}$ to be a classical GWA for $\deg_z(a) \geq 3$, then the degree of the corresponding defining polynomial would be higher and thus not able to be presented with two generators. However, one observes from Corollary 3.3 that it is possible to take

$$B = \begin{cases} \frac{1}{2}\left(AC - CA + (2+\rho)A\right) & n \text{ even} \\ \frac{1}{4}\left(AC - CA + 2(3+\rho)A\right) & n \text{ odd.} \end{cases}$$

4. Auslander's Theorem

In this final section we consider Auslander's theorem. As stated in the introduction, it is sufficient in many cases to show that $\mathsf{p}(A, G) \geq 2$ for an algebra A and a group G acting on A. In particular, by various results in [4,5], this applies when

1. A is noetherian, connected graded AS regular, and Cohen-Macaulay of GK dimension at least two, and G is a group acting linearly on A;
2. A is a noetherian PI and Kdim-CM algebra of Krull dimension at least 2;
3. A is *congenial* and G preserves the filtration on A.

Our focus will be on the last condition. We refer to [4] for a full definition.

Lemma 4.1. *Suppose F is a field of characteristic $p > 0$ and $R = F[z][x, y; \sigma, a]$ a classical GWA. Then $F[x^p, y^p] \subset \mathcal{Z}(R)$.*

Proof. It is clear that $\sigma^p = \mathrm{id}$ and, moreover, $\sigma^k = \mathrm{id}$ if and only if $p \mid k$. Thus, $[x^p, z] = [y^p, z] = 0$. Since $xy - yx = a - \sigma^{-1}(a)$, then it follows by induction that

$$x^k y - y x^k = (\sigma^{k-1}(a) - \sigma^{-1}(a))x^{k-1},$$
$$x y^k - y^k x = (a - \sigma^{-k}(a))y^{k-1}.$$

Setting $k = p$ gives $[x^p, y] = [x, y^p] = 0$ and the claim holds. $\square$

Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA and write

$$a = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0, \quad c_i \in \Bbbk.$$

Set $D = \mathbb{Z}[c_0, \ldots, c_{n-1}]$, then it is not difficult to see that $R_D = D[z][x, y; \sigma, a]$ is again a GWA. Moreover, R_D is free over D with a basis consisting of the standard monomials and

$$R_D = D[z][x, y; \sigma, a] \otimes_D \Bbbk = (D \otimes_D \Bbbk)[z][x, y; \sigma, a] = \Bbbk[z][x, y; \sigma, a] = R.$$

That is, R_D is an *order* of R. Next, we check the conditions of congeniality.

1. Under the standard filtration, R is a noetherian locally finite filtered algebra with the standard filtration.
2. The algebra R_D is also noetherian locally finite filtered (over D) and the standard filtration on R induces a filtration on R_D.
3. It is clear that $\operatorname{gr} R_D$ is an order of $\operatorname{gr} R$.
4. It is well-known that $\operatorname{gr} R_D = D[x, y, z]/(xy - z^n)$ is strongly noetherian and a locally finite graded algebra over D.
5. Let F be a factor ring of D that is a finite field of characteristic p. Then

$$R_D \otimes_D F \cong (D \otimes_D F)[z][x, y; \sigma, a]$$

and hence $R_D \otimes_D F$ is noetherian. Moreover, by Lemma 4.1, it is module finite over the commutative subalgebra $F[x^p, y^p]$.

We now adapt the methods of [9] to show that the pertinency condition is satisfied for a classical GWA and a cyclic subgroup of filtered automorphisms.

Lemma 4.2. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA. Set $G = \langle \Theta_\beta \rangle$ with β a primitive ℓth root of unity, $\ell \geq 2$. Then the Auslander map is an isomorphism for the pair (R, G).*

Proof. Set $S = \operatorname{gr}(R)$ under the standard filtration. Then G acts as graded automorphisms on S. We claim first that the theorem holds for the pair (S, G). Define

$$f = \sum_{i=0}^{\ell-1} 1\#(\Theta_\beta)^i \in S\#G.$$

Now observe that

$$xf - f(\beta x) = \sum_{i=0}^{\ell-2} (1 - \beta^{i+1}) x\#(\Theta_\beta)^i \in (f).$$

Repeating this process we find that $x^{\ell-1}\#e \in (f)$. Similarly, $y^{\ell-1}\#e \in (f)$ and so

$$y^{\ell-1}x^{\ell-1} = (a_2)^{\ell-1} \in (f).$$

Through the natural embedding $S \hookrightarrow S\#G$ given by $s \mapsto s\#e$, we have

$$\operatorname{GKdim} S\#G/(f) = \operatorname{GKdim} S/((f) \cap S).$$

It follows from the above computation that $S/((f) \cap S)$ is finite-dimensional and so $\mathsf{p}(S, G) = 2$. Thus, the Auslander map is an isomorphism for (S, G) [4, Theorem 0.2].

The action of Θ_β respects the standard filtration on R, both R and S are noetherian, and as S is a commutative complete intersection ring, it is CM and thus R is CM by [20, Lemma 4.4]. Hence, $\mathsf{p}(R, G) \geq \mathsf{p}(S, G) = 2$ by [4, Proposition 3.6] and so the theorem holds for (R, G) by [4, Theorem 3.3]. $\square$

Lemma 4.3. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA with $a \in \Bbbk[z]$ reflective, $\deg_z(a) \geq 3$, and $\beta \in \Bbbk^\times$. The Auslander map is an isomorphism for the pair $(R, \langle \Theta_\beta \circ \Omega \rangle)$.*

Proof. This follows similarly to Theorem 4.4. Set $\phi = \Theta_\beta \circ \Omega$ and $H = \langle \phi \rangle$. Throughout, let $S = \mathrm{gr}(R)$ and $f = 1\#e + 1\#\phi \in S\#H$.

First we consider the case of n odd. We have $xf + f\beta^{-1}y = (x + \beta^{-1}y)\#e \in (f)$. On the other hand, $yf - f(\beta x) = (y - \beta x)\#e \in (f)$. It follows that $x\#e, y\#e \in (f)$.

Next we suppose n is even. Then $xf - f(\beta^{-1}y) = (x - \beta^{-1}y)\#e \in (f)$. Similarly, $(x^2 - \beta^{-2}y^2)\#e \in (f)$ and $zf + fz = 2z\#e \in (f)$. It now follows that $x^2\#e, y^2\#e \in (f)$.

Hence, in either case, we have $S/((f) \cap S)$ is finite-dimensional and so $\mathsf{p}(R, H) \geq \mathsf{p}(S, H) = 2$. $\square$

Theorem 4.4. *Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA and let G be a finite nontrivial cyclic subgroup of $\mathrm{Aut}_{\mathrm{fl}}(R)$. Then the Auslander map is an isomorphism for the pair (R, G).*

Proof. The case $\deg_z(a) = 1$ is a consequence of [17, Theorem 2.4]. If $\deg_z(a) = 2$, then we apply Theorem 2.10 and the result follows from Theorem 4.2. Finally, if $\deg_z(a) > 2$, then by Theorem 2.4, $G = \langle \Theta_\beta \rangle$ or $G = \langle \Theta_\beta \circ \Omega \rangle$ and so the result follows from Lemma 4.2 and Lemma 4.3. $\square$

We end with a brief remark on the structure of the skew group ring appearing in the above results. Let $R = \Bbbk[z][x, y; \sigma, a]$ be a classical GWA and let $G = \langle \Theta_\beta \rangle$, $2 \leq |\beta| < \infty$. Then $R\#G \cong RG[x, y; \hat{\sigma}, \hat{a}]$ where RG is the group algebra of G with coefficients in R and $\hat{\sigma}, \hat{a}$ are naturally extended to RG from R. That is, $\hat{a} = a\#e$, and $\hat{\sigma}(p\#\Theta_\beta^k) = \beta^{-k}(\sigma(p)\#\Theta_\beta^k)$. When $\deg_z(a) = 2$, one can apply Theorem 2.10 and achieve the same result for any finite cyclic group acting linearly on R. Theorem 4.4 now implies, by way of the Auslander map, that the corresponding endomorphism ring has the structure of a GWA.

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