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On the EKL-degree of a Weyl cover

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ABSTRACT

More than four decades ago, Eisenbud, Khimšiašvili, and Levine introduced an analogue in the algebro-geometric setting of the notion of local degree from differential topology. Their notion of degree, which we call the EKL-degree, can be thought of as a refinement of the usual notion of local degree in algebraic geometry that works over non-algebraically closed base fields, taking values in the Grothendieck-Witt ring. In this note, we compute the EKL-degree at the origin of certain finite covers $f: \mathbb{A}^n \rightarrow \mathbb{A}^n$ induced by quotients under actions of Weyl groups. We use knowledge of the cohomology ring of partial flag varieties as a key input in our proofs, and our computations give interesting explicit examples in the field of \mathbb{A}^1 -enumerative geometry.

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1. Introduction

We work over a field K , which is arbitrary with characteristic not equal to 2 unless stated otherwise. Associated to a finite morphism $f: \mathbb{A}^n \rightarrow \mathbb{A}^n$ of K -varieties, we have the usual notion of its degree at the origin of the source, denoted by $\deg_0 f$. Refining this notion, Eisenbud, Khimšiašvili, and Levine (see [8,9]) introduced a new notion of degree, namely the EKL-degree, which we denote by $\deg_0^{\text{EKL}} f$ and which is an element of the Grothendieck-Witt ring $\text{GW}(K)$.¹

If K is algebraically closed, then the rank homomorphism defines an isomorphism of rings $\text{GW}(K) \xrightarrow{\sim} \mathbb{Z}$, and $\deg_0^{\text{EKL}} f$ coincides with $\deg_0 f$. However, if $K = \mathbb{R}$, then the rank homomorphism $\text{GW}(\mathbb{R}) \rightarrow \mathbb{Z}$ has kernel isomorphic to \mathbb{Z} , reflecting the fact that $\deg_0^{\text{EKL}} f$ also contains the data of the Brouwer degree of the underlying map of \mathbb{R} -manifolds. In general, $\deg_0^{\text{EKL}} f$ can be viewed as an enrichment of $\deg_0 f$ that contains interesting arithmetic data.

In this paper, we compute EKL-degrees at the origin of maps $\mathbb{A}^n \rightarrow \mathbb{A}^n$ induced by actions of Weyl groups on root spaces. There are at least two motivations for performing this computation:

- (1) The work of Eisenbud, Khimšiašvili, and Levine from over four decades ago has experienced something of a revival in recent years through the field of \mathbb{A}^1 -enumerative geometry. By a result of Kass and Wickelgren [10], the algebraic notion of EKL-degree coincides with the local \mathbb{A}^1 -Brouwer-degree, which is a topological notion of degree that is of central importance to the field of \mathbb{A}^1 -enumerative geometry. Since the EKL-degree lends itself more readily to computation, their result provides a way of explicitly computing the local \mathbb{A}^1 -Brouwer-degree. In this regard, our computation constitutes an explicit example of a degree computation in \mathbb{A}^1 -enumerative geometry.
- (2) In all of the maps $\mathbb{A}^n \rightarrow \mathbb{A}^n$ that we consider, the preimage of the origin is a finite scheme supported at the origin. Thus, it is natural to expect that the local degree that we compute in this paper should agree with a suitable notion of (global) \mathbb{A}^1 -Brouwer-degree, if such a notion were to be discovered.

As a first example, consider the quotient map $\pi: \mathbb{A}^n \rightarrow \mathbb{A}^n/S_n \simeq \mathbb{A}^n$ of affine space by the action of the symmetric group on the coordinates. The usual degree of π is $\deg_0 \pi = n!$, and it turns out that $\deg_0^{\text{EKL}} \pi = \frac{n!}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle)$ for $n \geq 2$. This follows easily from the fact that S_n contains a simple reflection, leading us to the following preliminary observation.

Proposition 1. *Let G be a finite group acting linearly on a finite-dimensional K -vector space V . If the ring $K[V]^G$ of G -invariants of $K[V]$ is a polynomial ring and G contains a simple reflection, then the EKL-degree of $\pi: \text{Spec } K[V] \rightarrow \text{Spec } K[V]^G$ is given by*

¹ We give precise definitions of $\text{GW}(K)$ and $\deg_0^{\text{EKL}} f$ in Section 2.

$$\deg_0^{\text{EKL}} \pi = \frac{\deg_0 \pi}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle).$$

For instance, Proposition 1 applies to quotients of root spaces by Weyl groups when K is of characteristic zero by the Chevalley–Shephard–Todd theorem (see [5, (A)]) or in arbitrary characteristic when the Weyl group is of type A or C (see [6, Théorème]).

We can also compute EKL-degrees in situations where Proposition 1 does not apply. For example, we will show that the EKL-degree of the quotient map $\mathbb{A}^4/(S_2 \times S_2) \rightarrow \mathbb{A}^4/S_4$ is given by $4 \cdot \langle 1 \rangle + 2 \cdot \langle -1 \rangle$, so in particular, the EKL-degree is no longer a multiple of $\langle 1 \rangle + \langle -1 \rangle$. Generalizing this example, we prove the following:

Theorem 2. *Let n_1, \dots, n_r be positive integers satisfying $n = \sum_{i=1}^r n_i$. The EKL-degree of the map $\pi: \mathbb{A}_K^n / \prod_{i=1}^r S_{n_i} \rightarrow \mathbb{A}_K^n / S_n$ is given by*

$$\begin{aligned} \deg_0^{\text{EKL}} \pi &= \frac{\deg_0 \pi - a}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle) + a \cdot \langle 1 \rangle \\ &= \frac{1}{2} \left(\frac{n!}{\prod_{i=1}^r n_i!} + a \right) \cdot \langle 1 \rangle + \frac{1}{2} \left(\frac{n!}{\prod_{i=1}^r n_i!} - a \right) \cdot \langle -1 \rangle, \end{aligned}$$

where $a = \lfloor \frac{n}{2} \rfloor! / \prod_{i=1}^r \lfloor \frac{n_i}{2} \rfloor!$ if at most one n_i is odd and $a = 0$ otherwise.

The proof of Theorem 2 involves applying the Definition of the EKL-degree together with knowledge of the cohomology ring of partial flag varieties of type A . Motivated by this, we extend Theorem 2 to apply to Weyl groups of other types as follows:

Theorem 3. *Let K be a field of characteristic 0. Let G be a simple complex Lie group with root space V/K , and let $P \subset G$ be a parabolic subgroup. Let W be the Weyl group of G , and let $W_P \subset W$ be the associated parabolic subgroup. Then the EKL-degree of the map $\pi: \text{Spec } K[V]^{W_P} \rightarrow \text{Spec } K[V]^W$ is given by*

$$\deg_0^{\text{EKL}} \pi = \frac{\deg_0 \pi - a_P}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle) + a_P \cdot \langle \alpha \rangle,$$

where $\alpha \in K^\times$, and a_P is equal to the number of cosets $\omega W_P \in W/W_P$ for which $\omega^{-1} \omega_0 \omega \in W_P$, where $\omega_0 \in W$ is the longest word.

The element α in the statement of Theorem 3 depends on the choice of identifications of $\text{Spec}(K[V]^W)$ and $\text{Spec}(K[V]^{W_P})$ with $\mathbb{A}^{\dim(V)}$. Such identifications are equivalent to choosing generators of $\text{Spec}(K[V]^W)$ and $\text{Spec}(K[V]^{W_P})$ as polynomial rings over K . In particular, scaling a generator of $\text{Spec}(K[V]^W)$ by α' scales $\deg_0^{\text{EKL}} \pi$ by $(\alpha')^{-1}$, so there is always a choice of generators making α in Theorem 3 equal to 1.

In the type- A case (i.e., Theorem 2), we show that taking the obvious choice of generators using elementary symmetric functions yields $\alpha = 1$. On the other hand, the number a_P in the statement of Theorem 3 can be computed explicitly in all cases, as we demonstrate in the following result:

Proposition 4. *We have $a_P = 0$ in Theorem 3 except in the following cases, tabulated according to the Dynkin diagrams of W and the parabolic subgroup W_P :*

W	W_P	a_P
A_n	$\Pi_{i=1}^r A_{n_i}$ with $n = \sum_{i=1}^r n_i$ and $\#\{\text{odd } n_i\} \leq 1$	$\lfloor \frac{n}{2} \rfloor! / \prod_{i=1}^r \lfloor \frac{n_i}{2} \rfloor!$
D_{2n+1}	D_{2n}	2
E_6	D_5	3
E_6	D_4	6

In particular, for all pairs (G, P) not tabulated in Proposition 4, the EKL-degree of the map $\pi: \text{Spec } K[V]^{W_P} \rightarrow \text{Spec } K[V]^W$ is given by

$$\deg_0^{\text{EKL}} \pi = \frac{\deg_0 \pi}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle).$$

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2. Background material

Before we prove our results, we provide a brief exposition on Grothendieck-Witt rings and on the EKL-degree in the case of finite maps between affine spaces.

2.1. The Grothendieck-Witt ring

We recall the definition of the Grothendieck-Witt ring of K and introduce notation for specifying its elements. For more information about Grothendieck-Witt rings, see [7, 12, 17].

Definition 5. Denoted by $\text{GW}(K)$, the Grothendieck-Witt ring of K is defined to be the group completion of the semi-ring (under the operations of direct sum and tensor product) of isomorphism classes of symmetric nondegenerate bilinear forms on finite-dimensional vector spaces valued in K .

Definition 6. For $u \in K^\times$, define $\langle u \rangle \in \text{GW}(K)$ to be the class of the nondegenerate symmetric bilinear form that sends $(x, y) \in K^2$ to $u \cdot xy \in K$.

The Grothendieck-Witt ring is generated by the classes $\langle u \rangle$ for $u \in K^\times$ and its relations can be written down explicitly [7, Theorem 4.7], giving a concrete presentation.

The class of the bilinear form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $\text{GW}(K)$ is known as the hyperbolic form and can be expressed as $\langle 1 \rangle + \langle -1 \rangle$. It is easy to deduce from [7, Theorem 4.7, p. 23] that the product of the class of the hyperbolic form with any element in $\text{GW}(K)$ is an integral multiple of the class of the hyperbolic form.

2.2. The EKL-degree

In this subsection, we recall the definition of the EKL-degree, where we will largely follow [10, Section 1]. Let $f: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be a morphism sending 0 to 0 such that $0 \in f^{-1}(0)$ is isolated in its fiber, and let (f_1, \dots, f_n) be its component functions.

Definition 7. The local algebra of f at $0 \in \mathbb{A}^n$ is $Q(f) := K[x_1, \dots, x_n]_{m_0} / (f_1, \dots, f_n)$, where m_0 is the maximal ideal of the point 0. The distinguished socle element is $E(f) := \det(a_{ij}) \in Q(f)$ where $a_{ij} \in K[x_1, \dots, x_n]$ are polynomials such that $f_i = \sum_j a_{ij} \cdot (x_j - a_j)$.

Definition 8. To a linear functional $\phi: Q(f) \rightarrow K$, we can associate a symmetric bilinear form β_ϕ on $Q(f)$ defined by $\beta_\phi(a, b) = \phi(ab)$.

Definition 9. The EKL-degree of f at $0 \in \mathbb{A}^n$, denoted by $\deg_0^{\text{EKL}} f$, is given by the class of the bilinear form β_ϕ in $\text{GW}(K)$, where ϕ is any linear functional sending the distinguished socle element $E(f)$ to 1.

Definition 9 does not depend on the choice of ϕ [10, Lemma 6]. In the proof of Theorem 2 only, we will make use of the Jacobian element, which is defined as follows:

Definition 10. The Jacobian element is $J(f) := \det \left(\frac{\partial f_i}{\partial x_j} \Big|_0 \right) \in Q(f)$.

The Jacobian element and distinguished socle element are related to each other by the equation $J(f) = (\dim_K Q(f)) \cdot E(f)$ [16, (4.7) Korollar], so $J(f)$ contains the same information as $E(f)$ if the characteristic of K does not divide the dimension of $Q(f)$ as a K -vector space.

2.3. Lie theory notation

We fix notation regarding complex Lie groups and their associated Weyl groups. In this paper, G denotes a simple complex Lie group and $P \subset G$ denotes a parabolic subgroup of G . The Weyl group W of G is generated by a set $\Sigma \subset W$, consisting of simple reflections, which is in bijection with the nodes of the Dynkin diagram of G . If B is a minimal parabolic subgroup of G , then there is a bijection between parabolic

subgroups P containing B and subsets Σ_P of Σ . The parabolic subgroup $W_P \subset W$ determined by such a P is the subgroup of W generated by the corresponding subset of Σ . A parabolic subgroup $P \subset G$ is *proper* if $P \neq G$, or equivalently if $\Sigma_P \neq \Sigma$. A proper parabolic subgroup W_P is said to be *maximal* if it is not properly contained in any other proper parabolic subgroup.

The *length* $\ell(\omega)$ of an element $\omega \in W$ is defined to be the length of the shortest expression of ω as a product of the elements of the generating set Σ . The *longest word*, denoted by ω_0 , is the unique element of maximal length in W .

Remark. In this paper, we restrict our consideration to the case of simple complex Lie groups. However, the classification of simple complex Lie groups is the same as the classification of split simple algebraic groups over fields of arbitrary characteristic (see [14, Theorem 23.25 and Theorem 23.55]). Furthermore, the actions of the Weyl groups on the root spaces are the same for both simple complex Lie groups and split simple algebraic groups. Thus, our results can be extended to hold for the case of split simple algebraic groups, with the caveat that Chow groups, as opposed to singular cohomology groups, must be used in the proof.

3. Proofs of the results

For all of the maps $\pi: \mathbb{A}^n \rightarrow \mathbb{A}^n$ that we consider, $\pi^{-1}(0)$ is supported at the origin. This is because the orbit of $0 \in \mathbb{A}^n$ under a linear group action is just the origin.

3.1. Proof of Proposition 1

Here, K is an arbitrary field of characteristic not equal to 2. Since G contains a simple reflection r , the map π factors through $\text{Spec}(K[V]^r)$:

$$\pi: \text{Spec}(K[V]) \rightarrow \text{Spec}(K[V]^r) \rightarrow \text{Spec}(K[V]^G).$$

It is easy to check (for example using [10, Section 1]) that

$$\deg_0^{\text{EKL}}(\text{Spec}(K[V]) \rightarrow \text{Spec}(K[V]^r)) = \langle 1 \rangle + \langle -1 \rangle.$$

By the fact that EKL-degrees are multiplicative in compositions,² we have that

$$\deg_0^{\text{EKL}} \pi = (\langle 1 \rangle + \langle -1 \rangle) \cdot \deg_0^{\text{EKL}}(\text{Spec}(K[V]^r \rightarrow \text{Spec}(K[V]^G)).$$

² Given the connection between the EKL-degree and the local \mathbb{A}^1 -Brouwer-degree established in [10], the multiplicativity of the EKL-degree in compositions can be deduced from the corresponding statement for the local \mathbb{A}^1 -Brouwer-degree in the 1-dimensional case (see [11, Section 2]). We suspect that this fact is known in arbitrary dimension to experts, but we have included a purely algebraic proof in Appendix A for the sake of clarity.

As mentioned in Section 2.1, it follows from the presentation of the Grothendieck-Witt ring in [7, Theorem 4.7] that any product with the class of the hyperbolic form $\langle 1 \rangle + \langle -1 \rangle$ is actually an integral multiple of the class of the hyperbolic form. Thus, there is some integer N such that $\deg_0^{\text{EKL}} \pi = N \cdot (\langle 1 \rangle + \langle -1 \rangle)$. Taking the rank of $\deg_0^{\text{EKL}} \pi$, we find that $2N = \deg_0 \pi$, which is the desired result. \square

It turns out to be more efficient from an expository standpoint to prove Theorem 3 and Proposition 4 before Theorem 2, so we order the remaining proofs accordingly.

3.2. Proof of Theorem 3

Here, K is an arbitrary field of characteristic 0. We setup the proof in Section 3.2.1 and prove the result in Section 3.2.2.

3.2.1. The idea of the proof

Consider the algebra

$$Q := Q(\pi) \simeq K[V]^{W_P} / (K[V]^W)^+,$$

where for a graded ring R , we denote by R^+ its irrelevant ideal. Suppose that we can produce a K -linear functional $\phi: Q \rightarrow K$ sending the distinguished socle element $E \in Q$ to 1. Then because $\pi^{-1}(0) = \{0\}$, it follows from Definition 9 that $\deg_0^{\text{EKL}} \pi$ is the class in $\text{GW}(k)$ of the symmetric bilinear form $\beta_\phi: Q \times Q \rightarrow K$ defined by $\beta_\phi(a_1, a_2) := \phi(a_1 a_2)$.

We now briefly sketch our idea for producing the desired functional ϕ . The key observation is that Q can be identified with the singular cohomology ring of a certain partial flag variety, so we can choose ϕ to be a certain scalar multiple of the integration map. Because the cohomology of this partial flag variety has a basis given by classes of Schubert varieties, and because each Schubert variety has a dual Schubert variety, this forces β_ϕ to be a direct sum of copies of the bilinear forms (α) and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where the multiplicity of each form depends on how many Schubert varieties are cohomologically equivalent to their dual Schubert variety.

By [2, Proposition 29.2(a)] and the assumption in Theorem 3 that $\text{char}(K) = 0$, the partial flag variety $F := G/P$ has cohomology³

$$H^\bullet(F, K) = K[V]^{W_P} / (K[V]^W)^+ = Q.$$

Integration over G/P gives a functional $H^\bullet(F, K) \rightarrow K$. To understand the integration map explicitly, $H^\bullet(F, K)$ is graded by degree and is generated by the elements of degree 2. Integration maps the 1-dimensional K -vector space $H^{\dim_{\mathbb{R}}(F)}(F, K)$ isomorphically to

³ Note that our partial flag variety is defined over \mathbb{C} , but we take its cohomology with coefficients in K .

K and the other graded pieces (which are of smaller degree) of $H^\bullet(F, K)$ to zero. The class of a point $[\text{pt}] \in H^{\dim_{\mathbb{R}}(F)}(F, K)$ integrates to 1.

We might try to take the functional ϕ to be the integration map on $H^\bullet(F, K)$, but to make this work, we would need to verify that the distinguished socle element E , viewed as an element of $H^\bullet(F, K)$, integrates to 1. This also depends on the choice and ordering of polynomial generators of $K[V]^{W_P}$ and $K[V]^W$ providing isomorphisms $\text{Spec}(K[V]^{W_P}) \simeq \text{Spec}(K[V]^W) \simeq \mathbb{A}^{\dim_K V}$. We verify this in the case where $G = \text{SL}_n$ in Section 3.4, where we used elementary symmetric functions as the generators in the invariant rings.

3.2.2. The proof

Consider the integration map

$$H^\bullet(F, K) \simeq Q \rightarrow K.$$

Let $\alpha \in K^\times$ be such that $\frac{1}{\alpha}$ is the integral of E , and let ϕ be such that $\frac{1}{\alpha} \cdot \phi$ is the integration map. We now compute the intersection pairing β_ϕ on Q . To do this, we use the following three facts (see [4, Section 2.1]):

- (1) the cohomology $H^\bullet(F, K)$ of F has a basis given by the classes of the Schubert varieties;
- (2) Schubert varieties are indexed by cosets of W/W_P ; and
- (3) the basis of Schubert varieties has a dual basis under the integration pairing, also given by Schubert varieties. The Schubert variety dual to the Schubert variety associated to the coset ωW_P is given by the coset $\omega_0 \omega W_P$, where $\omega_0 \in W$ is the longest word.

It follows that the matrix of β_ϕ with respect to the basis of Schubert classes is block-diagonal, where the blocks are of two types: (α) arising from self-dual Schubert classes, and $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ arising from all other dual pairs. Note that the class of (α) in $\text{GW}(K)$ is given by $\langle \alpha \rangle$ and that [7, part (2) of Remark 1.15] implies that the class of $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ is given by $\langle 1 \rangle + \langle -1 \rangle$.

Let a_P be the number of self-dual Schubert classes. Then the number of other dual pairs of Schubert classes is simply given by $\frac{1}{2}(\dim_K Q - a_P) = \frac{1}{2}(\deg_0 \pi - a_P)$. The theorem now follows upon observing that a_P is equal to the number of cosets ωW_P such that $\omega_0 \omega$ belongs to the same coset, which is equivalent to saying that $\omega^{-1} \omega_0 \omega \in W_P$. \square

3.3. Proof of Proposition 4

Here, K is an arbitrary field of characteristic 0. Suppose that the Dynkin diagram of G is not any one of A_n , D_n for n odd, or E_6 . Then the longest word ω_0 is in the center

of W (see [3, part (XI) of each of Planches I–IX, p. 250–275]), and the support of ω_0 is full (in the sense that one requires every generator of W to express ω_0). It follows that $\omega^{-1}\omega_0\omega = \omega_0$ is not contained in any parabolic subgroup of W , so we must have that $a_P = 0$. We treat the remaining cases separately as follows.

3.3.1. The A_n case

In this case, the Weyl group of G is $W = S_n$, and any parabolic subgroup $W_P \subset W$ is of the form $W_P = \prod_{i=1}^r S_{n_i}$, where $n = \sum_{i=1}^r n_i$. The longest word $\omega_0 \in S_n$ is the permutation that sends i to $n - i$ for every i . Recall that the number a_P of self-dual Schubert classes is equal to the number of cosets $\omega \cdot \prod_{i=1}^r S_{n_i}$ such that $\omega_0\omega$ belongs to the same coset, which is further equal to the number of elements in the set P of partitions of $\{1, \dots, n\}$ into blocks B_1, \dots, B_r (not necessarily contiguous) of sizes n_1, \dots, n_r such that swapping $n - i$ for each i preserves those blocks. If all of the blocks are of even size, then $\#P$ is equal to the number of partitions of $\{1, \dots, \frac{n}{2}\}$ into blocks of size $\frac{n_i}{2}$. If some block has odd size, then that block must contain $\frac{n+1}{2}$ (in particular, n must be odd) and must therefore be the only block of odd size. Thus, if there is a single block of odd size that contains $\frac{n+1}{2}$, then $\#P$ is equal to the number of partitions of $\{1, \dots, \frac{n-1}{2}\}$ into blocks of size $\lfloor \frac{n_i}{2} \rfloor$, and $\#P = 0$ otherwise. So $a_P = \#P = \lfloor \frac{n}{2} \rfloor! / \prod_{i=1}^r \lfloor \frac{n_i}{2} \rfloor!$ if at most one n_i is odd and $a_P = \#P = 0$ otherwise, as desired.

3.3.2. The D_n case

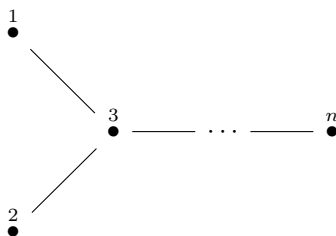
We take $n \geq 5$ to be odd (note that when $n = 3$, we have $D_3 = A_3$). In this case, the Weyl group of G has presentation

$$W = \langle r_1, \dots, r_n : (r_i r_j)^{m_{ij}} = 1 \rangle,$$

where the m_{ij} are defined by

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, & \text{if } (i, j) = (1, 2) \text{ or } |i - j| > 1 \text{ for } (i, j) \neq (1, 3), (3, 1) \\ 3, & \text{if } |i - j| = 1 \text{ for } i, j \geq 2, \text{ or if } (i, j) = (1, 3), (3, 1) \end{cases}$$

The generator r_k of W corresponds to the node k of the Dynkin diagram of D_n labeled below.



We now restate the following useful facts from Section 2.3 in terms of the generators r_k :

- The length $\ell(\omega)$ of $\omega \in W$ is the length of the shortest expression of ω as a product of the generators r_k .
- The longest word ω_0 is an involution satisfying $\ell(\omega_0) = n^2 - n$, and when n is odd, ω_0 acts by conjugation on the generators as follows: $\omega_0 r_1 \omega_0^{-1} = r_2$, and $\omega_0 r_i \omega_0^{-1} = r_i$ for $i \geq 3$ (see [3, part (XI) of Planche IV, p. 257]).
- The proper parabolic subgroups of W are precisely those subgroups of the form $W_{P_I} = \langle r_i : i \in I \rangle$, where $I \subsetneq \{1, \dots, n\}$ is any subset.
- The maximal parabolic subgroup W_{P_I} is given by taking $\#I = n - 1$.

The following lemma tells us that $a_{P_I} = 0$ unless $I = \{1, \dots, n - 1\}$:

Lemma 11. *If $\omega^{-1}\omega_0\omega \in W_{P_I}$ for some $\omega \in W$ and proper parabolic subgroup W_{P_I} , then $I = \{1, \dots, n - 1\}$.*

Proof. Note that for such an ω as in the statement of Lemma 11, any element $\omega' \in Z\omega$ also satisfies $(\omega')^{-1}\omega_0\omega' \in W_{P_I}$, where Z is the centralizer of ω_0 in W .

The following table shows how to reduce the length of a coset representative of $Z\omega$ by left-multiplication with elements of Z . In each row, the leftmost entry is a possible starting segment b for ω expressed as a word $\omega = b \cdot c$, the middle entry is a re-expression of the starting segment b that is more convenient for the purpose of length reduction, and the rightmost entry is a shortened segment $b' \in Zb$ with $\ell(b') < \ell(b)$. Before we demonstrate how to perform the length reduction, we narrow down the list of starting segments that we need to work with:

- First, because $\{r_i : i \geq 3\} \subset Z$, it is sufficient to consider starting segments b that begin with r_1 or r_2 .
- Moreover, because $r_2 r_1 \in Z$, and $r_2 r_1 r_2 = r_1$, it is sufficient to consider starting segments b that begin with r_1 .
- Finally, because $r_1 r_2 \in Z$ and because $\{r_i : i \geq 4\}$ is contained in the centralizer of r_1 , it is sufficient to consider starting segments b that begin with $r_1 r_3$.

Let $\sigma_{i,k} \in W$ be defined by

$$\sigma_{i,k} = \begin{cases} \prod_{j=i}^k r_j, & \text{if } i \leq k \\ 1, & \text{if } i > k \end{cases}$$

In terms of the $\sigma_{i,k}$, one verifies using the three itemized observations above that the possible starting segments that we need to work with are given by $r_1 r_3 r_1$, $r_1 r_3 r_2$, $r_1 \sigma_{3,k} r_j$, $r_1 \sigma_{3,k} r_j$, and $r_1 r_3 r_j$. We now present the table indicating how to reduce the lengths of words with these initial segments:

Starting Segment	Re-expression of Starting Segment	Shortened Segment	Conditions
$r_1 r_3 r_1$	$r_3 \cdot r_1 r_3$	$r_1 r_3$	n/a
$r_1 r_3 r_2$	$r_1 r_3 r_2 \cdot (r_2 r_3)^3 = (r_1 r_2 r_3) \cdot r_2 r_3$	$r_1 r_3$	n/a
$r_1 \sigma_{3,k} r_j$	$(r_1 r_3 r_j) \cdot \sigma_{4,k}$	$r_1 \sigma_{3,k}$	$1 \leq j \leq 2$
$r_1 \sigma_{3,k} r_j$	$r_1 r_3 \sigma_{4,j+1} r_j \sigma_{j+2,k} = r_{j+1} \cdot r_1 r_3 \sigma_{4,k}$	$r_1 \sigma_{3,k}$	$3 \leq j \leq k-1$
$r_1 r_3 r_j$	$r_j \cdot r_1 r_3$	$r_1 r_3$	$j > 4$

For example, the reduction in row 4 of the table is justified as follows: the defining relations of W imply that $\sigma_{j+2,k} r_j = r_j \sigma_{j+2,k}$, and that $r_j r_{j+1} r_j = r_{j+1} r_j r_{j+1}$.

Since each row of the table constitutes a reduction in length, we have shown that if the conjugacy class of ω_0 meets W_{P_I} , then there is an element

$$\omega \in S := \{r_1 \sigma_{3,k} : 3 \leq k \leq n\}$$

such that $\omega^{-1} \omega_0 \omega \in W_{P_I}$. The length of the longest element of S is $n-1$, so we deduce that

$$\ell(\omega^{-1} \omega_0 \omega) \geq \ell(\omega_0) - 2 \cdot \ell(\omega) \geq (n^2 - n) - 2 \cdot (n-1) = n^2 - 3n + 2.$$

To finish the proof of the claim, it is enough to see that $\ell_k < n^2 - 3n + 2$, where $k < n$ and ℓ_k is the length of the longest element of the (unique) maximal parabolic subgroup not containing r_k . The maximal lengths in a Weyl group of type A_r or D_r are $\binom{r+1}{2}$ and $r^2 - r$, respectively (see [3, part (I) of each of Planche I, p. 250 and Planche IV, p. 256]). Using this fact together with the additivity of maximal lengths in products of Coxeter groups, we find that

$$\ell_k = \begin{cases} \binom{n}{2}, & \text{if } 1 \leq k \leq 2 \\ (k-1)^2 - (k-1) + \binom{n-k+1}{2}, & \text{if } 2 < k < n \end{cases}$$

It is easy to check in each case that $f(n, k) := n^2 - 3n + 2 - \ell_k > 0$ when $n \geq 5$. For example, in the case $2 < k < n$, one readily checks that $f(n, k)$ is minimal when $k = n-1$, in which case $f(n, n-1) = 4n - 10 > 0$. Thus we have proven the claim. \square

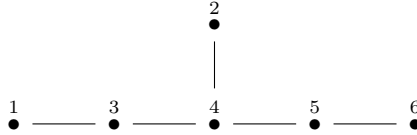
Let $I = \{1, \dots, n-1\}$, and let $P_I \subset G = \mathrm{SO}(2n)$ be an associated parabolic subgroup. We can realize the flag variety G/P_I as a smooth quadric hypersurface X of dimension $2n-2$. Indeed, under the obvious transitive action of G on such a quadric, the subgroup $M = \mathrm{SO}(2n-2) \subset P_I \subset G$ (embedded in the standard way by acting as the identity on the last two coordinates) stabilizes a point $p \in X$. Since the stabilizer $M' \subset G$ of p is parabolic and contains M , it follows by inspecting its Dynkin diagram that $M' = P_I$. By [15, Proof of Theorem 1.13], we have that

$$H^{n-1}(X, \mathbb{Z}) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2,$$

where the L_i are classes of linear subspaces on X satisfying $L_1^2 = L_2^2 = 1$ and $L_1 \cdot L_2 = 0$. Thus, there are exactly two self-dual classes, meaning that $a_{P_I} = 2$, as desired.

3.3.3. The E_6 case

We will use **SageMath** to compute this case. We label the nodes of the Dynkin diagram of E_6 as below:



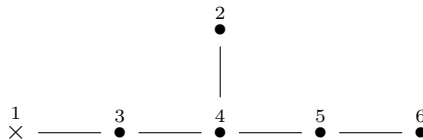
Each parabolic subgroup W_P of the Weyl group W associated to E_6 corresponds to deleting a subset of the nodes in the Dynkin diagram. For each such W_P , we are interested in computing the number a_P of cosets ωW_P such that $\omega^{-1}\omega_0\omega \in W_P$, which is given as follows:

$$a_P = \frac{\#\{\omega \in W : \omega^{-1}\omega_0\omega \in W_P\}}{\#W_P}. \quad (1)$$

To compute the quantity (1), we start with the following code, which initializes our Weyl group W and longest word ω_0 in **SageMath**:

```
W=WeylGroup(["E", 6]);
w0=E6.w0;
```

The first case to consider is where W_P is a maximal parabolic subgroup; in terms of the Dynkin diagram, such a subgroup corresponds to deleting a single node from the Dynkin diagram of E_6 . The number of distinct diagrams that arise from deleting a single node from the Dynkin diagram of E_6 is four, since deleting node 1 is equivalent to deleting node 6 and deleting node 3 is equivalent to deleting node 5 by symmetry. Deleting node 1 yields the Dynkin diagram



which is the Dynkin diagram of D_5 . For this choice of the maximal parabolic subgroup W_P , the following code computes $\#\{\omega \in W : \omega^{-1}\omega_0\omega \in W_P\}$ to be 5760:

```
INPUT:
i=0;
for w in W:
    if len((w.inverse()*w0*w).coset_representative([2,3,4,5,6]).reduced_word())==0:
        i=i+1;
print i;

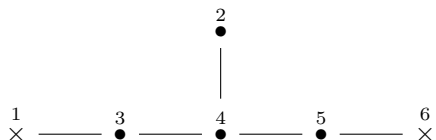
OUTPUT:
5760
```

Moreover, by [3, part (X) of Planche IV, p. 257], we have that $\#W_P = \#W_{D_5} = 2^4 \cdot 5! = 1920$ in this case, so using (1), we find that

$$a_P = \frac{\#\{\omega \in W : \omega^{-1}\omega_0\omega \in W_P\}}{\#W_P} = \frac{5760}{1920} = 3.$$

In the remaining three cases where W_P is a maximal parabolic arising from deleting nodes 2, 3, and 4, respectively, a similar block of code tells us that $\#\{\omega \in W : \omega^{-1}\omega_0\omega \in W_P\} = 0$, implying that $a_P = 0$.

For the non-maximal parabolic subgroups W_P , we have that $a_P = 0$ whenever W_P is contained in a maximal parabolic subgroup $W_{P'}$ for which $a_{P'} = 0$. Thus, it remains to consider those W_P that are *not* contained in any maximal parabolic subgroup $W_{P'}$ with $a_{P'} = 0$. Since there is only one maximal parabolic subgroup P' with $a_{P'} \neq 0$, one verifies by inspection that the only such P having the required property is obtained by deleting the nodes labeled 1 and 6 from the Dynkin diagram of E_6 , as illustrated below:



For this choice of W_P , the following code computes $\#\{\omega \in W : \omega^{-1}\omega_0\omega \in W_P\}$ to be 1152:

```

INPUT:
i=0;
for w in W:
    if len((w.inverse()*w0*w).coset_representative([2,3,4,5]).reduced_word())==0:
        i=i+1;
print i;

OUTPUT:
1152

```

Moreover, by [3, part (X) of Planche IV, p. 257], we have that $\#W_P = \#W_{D_4} = 2^3 \cdot 4! = 192$, so using (1), we find that

$$a_P = \frac{\#\{\omega \in W : \omega^{-1}\omega_0\omega \in W_P\}}{\#W_P} = \frac{1152}{192} = 6.$$

This completes the proof of Proposition 4. \square

3.4. Proof of Theorem 2

Here, K is an arbitrary field of characteristic not equal to 2. The idea is to use the same strategy as in the proof of Theorem 3. For convenience, let $m_i = \sum_{j=1}^i n_j$. Consider the partial flag variety $F := F(m_1, \dots, m_r)$ parametrizing flags of \mathbb{C} -vector spaces $0 \subset V_1 \subset \dots \subset V_r = \mathbb{C}^n$ where V_i has dimension m_i . By [2, Proposition 31.1], the integral cohomology ring of F is given by

$$H^\bullet(F, \mathbb{Z}) = \frac{\mathbb{Z}[x_1, \dots, x_n]^{\prod_{i=1}^r S_{n_i}}}{(\mathbb{Z}[x_1, \dots, x_n]^{S_n})^+}. \quad (2)$$

For any field K (regardless of characteristic), we have that

$$Q := Q(\pi) = \frac{K[x_1, \dots, x_n]^{\prod_{i=1}^r S_{n_i}}}{(K[x_1, \dots, x_n]^{S_n})^+} = H^\bullet(F, K).$$

We want to take the functional ϕ to be the integration map on $H^\bullet(F, K)$, so we need to verify that the distinguished socle element $E := E(\pi)$, viewed as an element of $H^\bullet(F, K)$, integrates to 1. To do this, consider the element $\tilde{E} \in \mathbb{Z}[x_1, \dots, x_n]$ defined by the formula for the distinguished socle element in Definition 7. Viewing \tilde{E} as an element of $H^\bullet(F, \mathbb{Z})$ via the identification (2), it is easy to see that the image of \tilde{E} under the map $H^\bullet(F, \mathbb{Z}) \rightarrow H^\bullet(F, K)$ is equal to E . It now suffices to show that \tilde{E} is equal to the class of a point in $H^\bullet(F, \mathbb{Z})$.

Notice that $\tilde{E} \in H^{\text{top}}(F, \mathbb{Z})$ and that $H^{\text{top}}(F, \mathbb{Z}) \simeq \mathbb{Z}$. By [16, proof of Korollar 4.7] (see also [10, proof of Lemma 4]), E is nonzero independent of K , so we can vary $K = \mathbb{F}_p$ over all primes p to see that the image of \tilde{E} in $H^{\text{top}}(F, \mathbb{F}_p)$ must be nonzero for each prime p . It follows that \tilde{E} is a generator of $H^{\text{top}}(F, \mathbb{Z}) \simeq \mathbb{Z}$ and therefore agrees with the class of a point up to sign.

To determine the sign, it suffices to compute the sign of the Jacobian element $J := J(\pi)$, taking $K = \mathbb{Q}$. We first consider the case where $n_i = 1$ for every i . In this case, the Jacobian element is $J = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ by [13, Equation (1)]; notice that J is a Vandermonde determinant and can be expressed using the Leibniz formula as

$$J = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot \prod_{i=1}^n x_{\sigma(i)}^{n-i}. \quad (3)$$

On the other hand, the class of a point in F is by definition given by $\prod_{i=1}^n x_i^{n-i}$ (see [1, Section 1]). For any $\sigma \in S_n$, we have that

$$\sigma \cdot \prod_{i=1}^n x_i^{n-i} = \prod_{i=1}^n x_{\sigma(i)}^{n-i} = \text{sign}(\sigma) \cdot \prod_{i=1}^n x_i^{n-i}. \quad (4)$$

It follows from combining (3) and (4) that $J = n! \cdot \prod_{i=1}^n x_i^{n-i} \in H^{\text{top}}(F, \mathbb{Z})$, so the signs agree.

We next consider the general case where not every n_i is equal to 1. Consider the composition of maps

$$\text{Spec}(\mathbb{Q}[x_1, \dots, x_n]) \rightarrow \text{Spec}(\mathbb{Q}[x_1, \dots, x_n]^{\prod_{i=1}^r S_{n_i}}) \rightarrow \text{Spec}(\mathbb{Q}[x_1, \dots, x_n]^{S_n}). \quad (5)$$

The Jacobian element of the first map in (5) is the product of the Jacobian elements of the maps $\text{Spec}(\mathbb{Q}[x_{m_{k-1}+1}, \dots, x_{m_k}] \rightarrow \text{Spec}(\mathbb{Q}[x_{m_{k-1}+1}, \dots, x_{m_k}]^{S_k})$ over $1 \leq k \leq r$. It then follows from the Chain Rule that the Jacobian element of $\text{Spec}(\mathbb{Q}[x_1, \dots, x_n]^{\prod_{i=1}^r S_{n_i}}) \rightarrow \text{Spec}(\mathbb{Q}[x_1, \dots, x_n]^{S_n})$ is

$$\begin{aligned}
 J &= \prod_{1 \leq i < j \leq n} (x_i - x_j) \bigg/ \prod_{k=1}^r \prod_{m_{k-1}+1 \leq i < j \leq m_k} (x_i - x_j) \\
 &= \prod_{1 \leq k < \ell \leq r} \prod_{i=1}^{n_k} \prod_{j=1}^{n_\ell} (x_{m_{k-1}+i} - x_{m_{\ell-1}+j}).
 \end{aligned} \tag{6}$$

In words, this Jacobian element takes the same form as the product of differences $\prod_{1 \leq i < j \leq n} (x_i - x_j)$, but instead of taking all pairwise differences $x_i - x_j$ for $i < j$, we instead take the pairs $i < j$ such that i and j are from different blocks, where we partition $\{1, \dots, n\}$ into contiguous blocks of size n_1, \dots, n_r . Now, we want to compare the Jacobian element with the class of a point in F . As before, we visibly see that swapping two variables from different blocks switches the sign and swapping two variables from the same block preserves the Jacobian element. Therefore, the same is true for the formula for the class of a point in F .

By [4, Section 2.1], the class of a point in F is the Schubert polynomial associated to the permutation

$$m_{r-1} + 1, \dots, m_r, m_{r-2} + 1, \dots, m_{r-1}, \dots, 1, \dots, m_1 \tag{7}$$

of the list $1, \dots, n$. In words, the permutation (7) takes the numbers $1, \dots, n$, splits them up into contiguous blocks of size n_1, \dots, n_r and reverses the order of the blocks (keeping the order within each block fixed). By [1, Block decomposition formula], the Schubert polynomial associated to (7) is given by

$$\prod_{i=1}^r \left(\prod_{j=1}^{n_i} x_j \right)^{\sum_{k=i+1}^r n_k}$$

Expanding out (6) and keeping track of the signs, we find that

$$J = \frac{n!}{\prod_{i=1}^r n_i!} \cdot \prod_{i=1}^r \left(\prod_{j=1}^{n_i} x_j \right)^{\sum_{k=i+1}^r n_k}$$

so the signs agree.

We deduce that $\alpha = 1$, so the theorem now follows from Theorem 3 and Proposition 4. \square

Appendix A. EKL-degrees and compositions

In this section, we present a purely algebraic proof that the EKL-degree is multiplicative in compositions. As mentioned in Section 3.1, it is possible to prove this by utilizing the relationship between EKL-degrees and local A^1 -Brouwer-degrees proven

in [10]; nonetheless, we think the following proof is of independent interest. Here, K is an arbitrary field of characteristic not equal to 2.

The key building block in the proof is the following lemma.

Lemma 12. *Let $f, g: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be morphisms that send the origin to itself with $0 \in f^{-1}(0)$ and $0 \in g^{-1}(0)$ isolated in their fibers. Let $A \in K^{n \times n}$ be a unipotent matrix, and let L be the map defined by*

$$\begin{aligned} L: \mathbb{A}^n &\rightarrow \mathbb{A}^n \\ x &\mapsto Ax, \end{aligned}$$

where I_n is the $n \times n$ identity matrix. Then we have that

$$\deg_0^{\text{EKL}}(f \circ g) = \deg_0^{\text{EKL}}(f \circ L \circ g).$$

Proof. Let t be a parameter, and consider the family of linear maps L_t defined by the matrices $I_n + t(A - I_n)$. Since A is unipotent, $\det(I_n + t(A - I_n)) = 1$, so $I_n + t(A - I_n)$ is invertible as a matrix with entries in the polynomial ring $K[t]$. Composing on the right and on the left with g and f , respectively, yields the family of composite maps $f \circ L_t \circ g$. Upon applying the definition of EKL-degree (see Definitions 7 to 9) to each map in this family, we obtain a family of local algebras $Q(f \circ L_t \circ g)$, each equipped with a bilinear form β_{ϕ_t} . Note that when $t = 0$, the form β_{ϕ_0} represents the class $\deg_0^{\text{EKL}}(f \circ g) \in \text{GW}(K)$, and when $t = 1$, the form β_{ϕ_1} represents the class $\deg_0^{\text{EKL}}(f \circ L \circ g) \in \text{GW}(K)$. By a version of Harder's theorem (see [10, Lemma 30]), the class in $\text{GW}(K)$ represented by β_{ϕ_t} is independent of $t \in K$, so $\deg_0^{\text{EKL}}(f \circ g) = \deg_0^{\text{EKL}}(f \circ L \circ g)$ for each $t \in K$. \square

Remark. Although Lemma 12 was stated with A unipotent for maximal generality, we only need the case where A is upper or lower triangular and unipotent in what follows.

We are now ready to prove the desired result on the EKL-degree of a composition.

Theorem 13. *Let $f, g: \mathbb{A}^n \rightarrow \mathbb{A}^n$ be morphisms sending the origin to itself with $0 \in f^{-1}(0)$ and $0 \in g^{-1}(0)$ isolated in their fibers. Then, the EKL-degree of the composition $f \circ g$ is given by the product*

$$\deg_0^{\text{EKL}}(f \circ g) = (\deg_0^{\text{EKL}} f) \cdot (\deg_0^{\text{EKL}} g)$$

in $\text{GW}(K)$.

Proof. The idea is to use Lemma 12 to reduce to the case where f and g act on separate variables. To execute this idea, we first pad f and g with n extra coordinates to obtain morphisms

$$\tilde{f}, \tilde{g}: \mathbb{A}^n \times \mathbb{A}^n \rightarrow \mathbb{A}^n \times \mathbb{A}^n$$

where \tilde{f}, \tilde{g} send (x, y) to $(f(x), y)$ and $(g(x), y)$ respectively. One readily verifies using Definition 9 that the EKL-degree of a product of two morphisms $\mathbb{A}^n \rightarrow \mathbb{A}^n$ is the product of the EKL-degrees, so we have

$$\deg_0^{\text{EKL}}(\tilde{f} \circ \tilde{g}) = \deg_0^{\text{EKL}}(f \circ g).$$

Therefore, it suffices to show that

$$\deg_0^{\text{EKL}}(\tilde{f} \circ \tilde{g}) = (\deg_0^{\text{EKL}} f) \cdot (\deg_0^{\text{EKL}} g). \quad (8)$$

To this end, repeated applications of Lemma 12 imply that the following four compositions of maps all have the same EKL-degree (\deg_0^{EKL}) at the origin. Each map takes in a point $(x, y) \in \mathbb{A}^n \times \mathbb{A}^n$ and outputs a point in $\mathbb{A}^n \times \mathbb{A}^n$, and each line contains a composition of three maps read from left to right.

$$\begin{aligned} (x, y) &\mapsto (g(x), y) & (x, y) &\mapsto (x, y) & (x, y) &\mapsto (f(x), y) \\ (x, y) &\mapsto (g(x), y) & (x, y) &\mapsto (x, x + y) & (x, y) &\mapsto (f(x), y) \\ (x, y) &\mapsto (g(x), y) & (x, y) &\mapsto (x - y, x) & (x, y) &\mapsto (f(x), y) \\ (x, y) &\mapsto (g(x), y) & (x, y) &\mapsto (-y, x) & (x, y) &\mapsto (f(x), y). \end{aligned}$$

The first composition above is given by $\tilde{f} \circ \tilde{g}$, while the last composition sends (x, y) to $(-f(y), g(x))$. To finish, we apply Lemma 12 to the following four compositions of maps:

$$\begin{aligned} (x, y) &\mapsto (-f(y), g(x)) & (x, y) &\mapsto (x, y) \\ (x, y) &\mapsto (-f(y), g(x)) & (x, y) &\mapsto (x + y, y) \\ (x, y) &\mapsto (-f(y), g(x)) & (x, y) &\mapsto (y, -x + y) \\ (x, y) &\mapsto (-f(y), g(x)) & (x, y) &\mapsto (y, -x). \end{aligned}$$

The first composition sends (x, y) to $(-f(y), g(x))$, while the last composition sends (x, y) to $(g(x), f(y))$. Therefore, we have shown that $\tilde{f} \circ \tilde{g}$ has the same EKL-degree at the origin as the completely decoupled map that sends (x, y) to $(g(x), f(y))$, which can be thought of as a product of two morphisms $\mathbb{A}^n \rightarrow \mathbb{A}^n$. It follows that (8) holds. \square

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