Simplicity of the automorphism groups of order and tournament expansions of homogeneous structures ☆

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A R T I C L E   I N F O

A B S T R A C T

We define the notions of a free fusion of structures and a weakly stationary independence relation and use them to prove simplicity for the automorphism groups of order and tournament expansions of homogeneous structures like the rational bounded Urysohn space, the random graph, and the random poset.

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1. Introduction

This article contributes to the study of the automorphism groups of countable structures. Such groups are natural examples of separable and completely metrisable topological groups. The richness of their topological properties have recently brought to light a crucial interplay between Fraïssé amalgamation theory and other areas of mathematics like topological dynamics, Ramsey theory, and ergodic theory. (See [1] and [5].)

The program of understanding the normal subgroup structure of these groups dates back at least to the '50s, when Higman [4] proved that $\mathrm{Aut}(\mathbb{Q}, <)$, the group of order-preserving permutations of the rational numbers, has very few normal subgroups.[1]

In recent years, Macpherson and Tent [8] proved simplicity for a large collection of groups that arise in a similar fashion as automorphism groups of homogeneous structures. Their methods encompass a number of examples that had been considered before by various authors: the random graph [14], the random K_n-free graphs and the random tournament [10], and many others. However, as the authors of [8] pointed out, their framework does not apply to ordered or even partially ordered structures, in particular it does not apply to the random poset whose automorphism group was proved to be simple in [3].

A few years later, Tent and Ziegler [12] introduced the notion of a *stationary independence relation* and investigated automorphism groups of structures allowing for such a relation. Their approach is very general: apart from recovering the cases from [8] it applies to the random poset and many homogeneous metric structures like the Urysohn space and its variations. However, ordered homogeneous structures like the ordered random graph and the random tournament do not carry such a stationary independence relation.

In this article we weaken the notion of a stationary independence relation from [12] to study the automorphism groups of many order and tournament expansions of structures arising naturally in Fraïssé amalgamation theory. We believe that such *weakly stationary independence relations* will also be useful in other expansions of homogeneous structures.

Before stating our main theorem, we introduce some terminology.

Definition 1.1. Let $L_i, i = 1, 2$, be disjoint relational languages and let $\mathbf{M}_i, i = 1, 2$, countable homogeneous L_i structures on the same universe M. We call an $\mathcal{L}^* = L_1 \cup L_2$ structure $\mathbf{M}^*$ on M the free fusion of $\mathbf{M}_1$ and $\mathbf{M}_2$ if $\mathbf{M}^* \restriction L_i = M_i$, $i = 1, 2$, and

(∗) for every non-algebraic L_i-type p_i over a finite set $A_i \subset M$ for $i = 1, 2$, their union $p_1 \cup p_2$ is realized in $\mathbf{M}^*$.

For any $\mathcal{L}^*$-type p and $L \subset \mathcal{L}^*$, we write p_L for its restriction to L.

All types considered in this article are types over finite sets.

Similar constructions were considered by Bodirsky [2] and Sokić in [11].

We are particularly interested in the following special cases:

I. Order expansion Let L_1 be a relational language and $\mathbf{M} = \mathbf{M}_1$ be a countable homogeneous L_1-structure on a set M. Let $L_2 = \{<\}$ and $\mathbf{M}_2 \cong \mathbb{Q}$ be a dense linear ordering on M. In this case we denote the free fusion of $\mathbf{M}_1$ and $\mathbf{M}_2$ by $\mathbf{M}_<$ and call it an *order expansion* of $\mathbf{M}$. Thus, an $\mathcal{L}^*$-structure $\mathbf{M}_<$ is an order expansion of $\mathbf{M}$ if $<$ is a total order on $\mathbf{M}$ without endpoints, $\mathbf{M}_< \upharpoonright L_1 = \mathbf{M}$, and $\mathbf{M}_<$ satisfies the following property:

$(*)$ For every non-algebraic 1-type p_{L_1} over a finite set A, and every interval $(a, b) \subseteq M$, there is a realization of p in (a, b).

II. Tournament expansion Let L_1 be a relational language and $\mathbf{M} = \mathbf{M}_1$ be a homogeneous L_1-structure on a set M. Let $L_2 = \{\rightarrow\}$ and $\mathbf{M}_2$ be a random tournament on M. In this case we denote the free fusion of $\mathbf{M}_1$ and $\mathbf{M}_2$ by $\mathbf{M}_\rightarrow$ and call it a *tournament expansion* of $\mathbf{M}$. Thus, an $\mathcal{L}^*$-structure $\mathbf{M}_\rightarrow$ is a tournament expansion of $\mathbf{M}$ if $\rightarrow$ is a tournament on M, $\mathbf{M}_\rightarrow \upharpoonright L_1 = \mathbf{M}$, and $\mathbf{M}_\rightarrow$ satisfies the following property:

$(*)$ For every non-algebraic 1-type p_{L_1} over a finite set X, and two disjoint finite subsets $A, B \subseteq M$, there is a realization x of p such that $x \rightarrow a$ for all $a \in A$ and $b \rightarrow x$ for all $b \in B$.

Remark 1.2. Note that if $\mathbf{M}_i, i = 1, 2$, is the Fraïssé limit of some L_i-class $\mathcal{C}_i, i = 1, 2$ having disjoint amalgamation, then a structure $\mathbf{M}^*$ is the free fusion of $\mathbf{M}_1$ and $\mathbf{M}_2$ if and only if $\mathbf{M}^*$ is the Fraïssé limit of the $\mathcal{L}^*$-class $\mathcal{C}^*$ where an $\mathcal{L}^*$-structure A^* is in $\mathcal{C}^*$ if and only if $A^* \upharpoonright L_i \in \mathcal{C}_i, i = 1, 2$.

Thus, a structure $\mathbf{M}^*$ is an order expansion of a Fraïssé limit $\mathbf{M}$ with disjoint amalgamation if and only if $\mathbf{M}^*$ is the Fraïssé limit of the class $\mathcal{C}_<$ where an $\mathcal{L}^*$-structure $A_<$ is in $\mathcal{C}_<$ if and only if $A_< \upharpoonright L_1 \in \mathcal{C}$. Equivalently, $\mathcal{C}_<$ consists of all $A \in \mathcal{C}$ expanded by all possible orderings. Similarly for the tournament expansion of a Fraïssé limit.

Our main theorem can now be stated as follows:

Theorem 1.3. *Assume that $\mathbf{M}$ is one of the following:*

(1) the Fraïssé limit of a free, transitive and nontrivial amalgamation class[2];

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[2] We call an amalgamation class nontrivial, if its limit is not an indiscernible set and transitive if the automorphism group of its Fraïssé limit is transitive.

(2) the bounded rational Urysohn space; or

(3) the random poset.

If $\mathbf{M}^*$ is an order expansion of $\mathbf{M}$, then $G := \mathrm{Aut}(\mathbf{M}^*)$ is simple. The same holds if $\mathbf{M}^*$ is a tournament expansion of (1) *or* (2).

Apart from the ordered bounded rational Urysohn space see (e.g. [13]), Theorem 1.3 implies simplicity of the automorphism groups of various countable structures including the ordered random poset, the ordered random graph, the ordered random n-hypergraphs, the ordered random K_n-free graphs and their hypergraph analogues.

Remark 1.4. In the same way as in [12] we can also conclude that for the ordered rational Urysohn space $\mathbb{U}_<$, the quotient of $\mathrm{Aut}(\mathbb{U}_<)$ modulo the normal subgroup of automorphisms of bounded displacement is a simple group.

Note also that with minor modifications the same proof applies to expansions by several independent tournaments or indeed for any binary homogeneous structure with the property that the union of 1-types over disjoint finite sets is always consistent. Furthermore, it is worth pointing out that the proof also adapts to several independent orderings. These generalisations were shown by Silke Meißner in her master's thesis [9].

Theorem 1.3 is proved in two main steps: we first define a notion of moving maximally adapted to free fusion structures and a notion of compatibility and prove:

Theorem 1.5. *Let $\mathbf{M}^*$ be the free fusion of a homogeneous L_1-structure $\mathbf{M}_1$ carrying a stationary independence relation with an L_2-structure $\mathbf{M}_2$. If $g \in G$ moves maximally and is compatible, then any element of G is the product of at most eight conjugates of g and g^{-1}.*

In Section 4 we prove simplicity of G for the ordered random posets and complete the proof of Theorem 1.3 by proving:

Proposition 1.6. *If $\mathbf{M}^*$ is an order or tournament expansion of a structure $\mathbf{M}$ as in Theorem 1.3* (1) *or* (2) *and $\mathrm{id} \neq h \in G = \mathrm{Aut}(\mathbf{M}^*)$, then there is some $g \in \langle h \rangle^G$ that moves maximally and is compatible.*

Clearly, Proposition 1.6 and Theorem 1.5 imply Theorem 1.3 for the cases (1) and (2).

2. Background and definitions

First we recall the definition of a stationary independence relation due to Tent and Ziegler. We also follow their convention by saying that a tuple a' from a countable

structure M realizes a type tp(a/A) over a finite set $A \subset M$ if there is an automorphism of $\mathbf{M}$ that maps a to a' and fixes A pointwise.

Definition 2.1. [12, Definition 2.1] Let $\mathbf{M}$ be a countable structure with universe M and let $\underset{}{\smile}$ be a ternary relation between finite subset of M. We say that $\underset{}{\smile}$ is a *stationary independence relation* on $\mathbf{M}$ if for all finite sets $A, B, C, D \subseteq M$ the following hold:

(i) (Invariance) The independence of A and B over C depends only on the type of ABC. In other words, if tp$(ABC) = $ tp$(A'B'C')$, then

$$A \underset{C}{\smile} B \quad \text{if and only if} \quad A' \underset{C'}{\smile} B'.$$

(ii) (Monotonicity) $A \underset{B}{\smile} CD$ implies that $A \underset{B}{\smile} C$ and $A \underset{BC}{\smile} D$.

(iii) (Transitivity)[3]

$$A \underset{B}{\smile} C \text{ and } A \underset{BC}{\smile} D \text{ implies } A \underset{B}{\smile} CD.$$

(iv) (Symmetry) $A \underset{B}{\smile} C$ if and only if $C \underset{B}{\smile} A$.

(v) (Existence) If p is a non-algebraic type over B and C is a finite set, there is some a realizing p such that $a \underset{B}{\smile} C$.

(vi) (Stationarity) If the tuples x and y have the same type over B and are both independent from C over B, then x and y have the same type over BC.

Moreover, we will write $A \underset{}{\smile} C$ for $A \underset{\emptyset}{\smile} C$.

Remark 2.2. As in [12, 2.4] we let $A \underset{C;D}{\smile} B$ denote the conjunction of $AC \underset{D}{\smile} B$ and $A \underset{C}{\smile} BD$. Then Transitivity and Monotonicity imply

$$x_1 x_2 \underset{A;B}{\smile} y_1 y_2 \text{ if and only if } \left[x_1 \underset{A;B}{\smile} y_1 \text{ and } x_2 \underset{Ax_1;By_1}{\smile} y_2 \right].$$

Remark 2.3. Recall from [12] that if $\mathbf{M}$ is the limit of a Fraïssé class of structures with free amalgamation, then $\mathbf{M}$ admits a stationary independence relation: define $A \underset{B}{\smile} C$ if and only if ABC is isomorphic to the free amalgam of A and C over B, i.e. if and only if $A \cap C = B$ and for every n-ary relation R in L, if $d_1, \ldots, d_n$ is an n-tuple in $A \cup B \cup C$ with some $d_i \in A \setminus B$ and $d_j \in C \setminus B$, then $R(d_1, \ldots, d_n)$ does not hold. (See [12, Example 2.2].)

For the stationary independence relation on the (bounded) Urysohn space and other metric spaces, we put $A \underset{C}{\smile} B$ if and only if for all $a \in A, b \in B$ there is some $c \in C$

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[3] As noted by several people Transitivity follows from the other axioms. We include it here for convenience.

such that $d(a,b) = d(a,c) + d(c,b)$, and $A \underset{}{\downarrow} B$ if and only if for all $a \in A, b \in B$ the distance $d(a,b)$ is maximal, see [12].

In the same vein, the random poset carries a natural stationary independence relation, namely $A \underset{C}{\downarrow} B$ if and only if $A \cap B \subset C$ and for all $a \in A \setminus C, b \in B \setminus C$ such that $a <_{\mathrm{po}} b$ or $b <_{\mathrm{po}} a$ there is some $c \in C$ such that $a <_{\mathrm{po}} c <_{\mathrm{po}} b$ or $b <_{\mathrm{po}} c <_{\mathrm{po}} a$, respectively, where we write $<_{\mathrm{po}}$ for the partial order of the random poset (see [6, 4.2.1]). Note that we have $A \underset{}{\downarrow} B$ if and only if no element of a is comparable in the partial order to any element of B, i.e. if for all $a \in A, b \in B$ we have $a \not<_{\mathrm{po}} b$ and $b \not<_{\mathrm{po}} a$.

Definition 2.4. Let $\mathbf{M}^*$ be the free fusion of countable L_i-structures $\mathbf{M}_i, i = 1, 2$, with universe M and let $\underset{}{\downarrow}$ be a ternary relation between finite subset of M. We say that $\underset{}{\downarrow}$ is a *weakly stationary independence relation* on $\mathbf{M}^*$ if it satisfies (Invariance), (Monotonicity), (Symmetry), (Existence) and

(v′) (Weak Stationarity) If x and y have the same $\mathcal{L}^*$-type over B and are both independent from C over B, then x and y have the same L_1-type over BC. Thus, if furthermore $\mathrm{tp}_{L_2}(x/BC) = \mathrm{tp}_{L_2}(y/BC)$, then x and y have the same $\mathcal{L}^*$-type over BC.

We first note the following:

Proposition 2.5. *Let $\mathbf{M}^*$ be the free fusion of a homogeneous L_1-structure $\mathbf{M}_1$ carrying a stationary independence relation $\underset{}{\downarrow}$ with an L_2-structure $\mathbf{M}_2$. Then on $\mathbf{M}^*$ the relation $\underset{}{\downarrow}$ is a weakly stationary independence relation.*

Proof. All properties except (Existence) follow immediately. To see that (Existence) holds for $\mathcal{L}^*$-types, let p be an $\mathcal{L}^*$-type over a finite set B and let C be a finite set. By (Existence) for L_1-types there is a realization a of p_{L_1} with $a \underset{B}{\downarrow} C$. By $(*)$, there is a realization b of $\mathrm{tp}_{L_1}(a/BC)$ realizing p_{L_2}. Then b realizes p and $b \underset{B}{\downarrow} C$. □

Remark 2.6. The proof shows in fact that for any finite sets A, B and C, any L_1-type p_{L_1} over a finite set A and any L_2-type q_{L_2} over a finite set B there is a realization of $p_{L_1} \cup q_{L_2}$ which is independent from C over A.

We first note the following adaptation from [12]:

Lemma 2.7. *If $\mathbf{M}^*$ is the free fusion of a homogeneous L_1-structure $\mathbf{M}_1$ carrying a stationary independence relation $\underset{}{\downarrow}$ with some L_2-structure $\mathbf{M}_2$. If $\mathrm{Aut}(\mathbf{M}_2)$ has a dense conjugacy class, then $G = \mathrm{Aut}(\mathbf{M}^*)$ has a dense conjugacy class as well.*

Proof. Clearly, G contains a dense conjugacy class if and only if for any finite tuples $\bar{x}, \bar{y}, \bar{a}, \bar{b}$ with $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$ and $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ there are tuples $\bar{x}', \bar{y}'$ such that

$\mathrm{tp}(\bar{x}'\bar{y}') = \mathrm{tp}(\bar{x}\bar{y})$ and $\mathrm{tp}(\bar{x}'\bar{a}) = \mathrm{tp}(\bar{y}'\bar{b})$. Since $\mathbf{M}^*$ carries a weakly stationary in-dependence relation, we can choose $\bar{x}'\bar{y}'$ realising $\mathrm{tp}(\bar{x}\bar{y})$ with $\bar{x}'\bar{y}' \underset{}{\downarrow} \bar{a}\bar{b}$. By condi-tion $(*)$ and since $\mathrm{Aut}(\mathbf{M}_2)$ has a dense conjugacy class, we can choose $\bar{x}'\bar{y}'$ so that $\mathrm{tp}_{L_2}(\bar{x}'\bar{a}) = \mathrm{tp}_{L_2}(\bar{y}'\bar{b})$. By Weak Stationarity we then have $\mathrm{tp}(\bar{x}'\bar{a}) = \mathrm{tp}(\bar{y}'\bar{b})$.   □

Notation. For $a, g \in G$, we write $g^a = a^{-1}ga$ and $[a, g] = a^{-1}g^{-1}ag$.

Definition 2.8. Suppose that G acts transitively on M. We say that $g \in G$ is L_2-*homogeneous* if for any element $x \in M$ and $a \in G$ we have $\mathrm{tp}_{L_2}(g(x)/x) = \mathrm{tp}_{L_2}(g^a(x)/x)$, or, equivalently, if for all $x, y \in M$ we have $\mathrm{tp}_{L_2}(xg(x)) = \mathrm{tp}_{L_2}(yg(y))$.

Example 2.9. Let $\mathbf{M}^*$ be the free fusion of a homogeneous L_1-structure $\mathbf{M}_1$ carrying a stationary independence relation with an L_2-structure $\mathbf{M}_2$ and let $G = \mathrm{Aut}(\mathbf{M}^*)$.

1. If $\mathbf{M}_2$ is the trivial structure, any fixed point free $g \in G$ is L_2-homogeneous.

2. If $\mathbf{M}_<$ is an order expansion of $\mathbf{M_1}$, then $g \in G$ is $<$-homogeneous if and only if g is the identity isomorphism, or g is strictly increasing or g is strictly decreasing.

3. If $\mathbf{M}_\rightarrow$ is a tournament expansion of $\mathbf{M_1}$, then $g \in G$ is $\rightarrow$-homogeneous if and only if g is the identity isomorphism, or $a \rightarrow g(a)$ for all $a \in M$, or $g(a) \rightarrow a$ for all $a \in M$.

4. If $\mathbf{M}_2$ is the random graph, then $g \in G$ is E-homogeneous if and only if g is the identity isomorphism, $E(a, g(a))$ for all $a \in M$ or $\neg E(a, g(a))$ for all $a \in M$.

Remark 2.10. Using property $(*)$ and back-and-forth it is easy to construct examples of L_2-homogeneous automorphisms in each of the cases above.

The free fusion of two structures both having a stationary independence relation has again a stationary independence relation. Therefore, while our methods will transfer, we do not consider e.g. expansions by graphs in this article.

Definition 2.11. Let $\mathbf{M}^*$ be the free fusion of a homogeneous L_1-structure $\mathbf{M}_1$ carrying a stationary independence relation with an L_2-structure $\mathbf{M}_2$ and let $G = \mathrm{Aut}(\mathbf{M}^*)$. We say that $g \in G$ *moves maximally* if

  (i) g is L_2-homogeneous; and
 (ii) every non-algebraic type over a finite set X has a realization x such that

$$x \underset{X;g(X)}{\downarrow} g(x).$$

When x is a realization as in (ii), we say that x is *moved maximally* by g over X.

Clearly if g moves maximally, then so do g^{-1} and all conjugates of g.
We will frequently use the following refinement of the maximal moving condition:

Proposition 2.12. *If $g \in G$ is moving maximally, then for every non-algebraic n-type p over a finite set X and non-algebraic L_2-type q_{L_2} over A such that $q_{L_2} \cup p_{L_2}$ is consistent there is some y realizing $p \cup q_{L_2}$ which is moved maximally by g over X.*

Proof. Suppose that g moves maximally and consider a non-algebraic type p over a finite set X and an L_2-type q_{L_2} over $A \subset M$ such that $q_{L_2} \cup p_{L_2}$ is consistent. Let y be a realization of p_{L_1} such that $y \underset{X}{\bigcup} A$. By $(*)$ we can choose a realisation z of $\mathrm{tp}_{L_1}(y/XA) \cup q_{L_2}$. Thus $z \underset{X}{\bigcup} A$. Since g moves maximally, there is a realization c of $\mathrm{tp}(z/XA)$ such that $c \underset{XA;g(XA)}{\bigcup} g(c)$. Furthermore, since $c \underset{XA}{\bigcup} g(XA)g(c)$ and $c \underset{X}{\bigcup} A$, we get $c \underset{X}{\bigcup} g(X)g(c)$ by Transitivity. Similarly, $g(c) \underset{g(X)}{\bigcup} g(A)$ and $cXA \underset{g(XA)}{\bigcup} g(c)$ imply $cX \underset{g(X)}{\bigcup} g(c)$ and hence we see that $c \underset{X;g(X)}{\bigcup} g(c)$.

Thus c realizes $p \cup q_{L_2}$ and is moved maximally by g over X. $\quad\square$

Notation. For a tuple x we denote the components by x^i.

Definition 2.13. Let $\mathbf{M}^*$ be the free fusion of a homogeneous L_1-structure $\mathbf{M}_1$ carrying a stationary independence relation with an L_2-structure $\mathbf{M}_2$ and let $g \in G = \mathrm{Aut}(\mathbf{M}^*)$ move maximally. We say that g is *compatible* if the following holds:

Any finite set X_0 has a finite extension X depending only on X_0 and g such that the following holds:

(1) for all tuples x, y such that $g(\mathrm{tp}(x/X)) = \mathrm{tp}(y/Y)$ with $Y = g(X)$, $\mathrm{tp}_{L_2}(y^i/x^i) = \mathrm{tp}_{L_2}(g(x^i)/x^i)$ and $x \underset{X;Y}{\bigcup} y$ there is some $a \in \mathrm{Fix}(XY)$ such that $g^a(x) = y$. In this case we call X *full* for g.

(2) Suppose that $b \in G$, Z is a finite set such that $g^b(Z) = Y$, x, z are tuples such that $g^b(\mathrm{tp}(z/Z)) = g(\mathrm{tp}(x/X))$, $z \underset{Z}{\bigcup} YX$, $x \underset{X}{\bigcup} YZ$. Then there is some $b' \in \mathrm{Fix}(YZ)$, and some finite extension Z^+ of Z which is full for g^b and such that with $Y^+ = g^b(Z^+)$ we have $\mathrm{tp}(g^{bb'}(z)/Y) = \mathrm{tp}(g^b(z)/Y)$,

$$g^{bb'}(z) \underset{Y}{\bigcup} (b')^{-1}(Y^+) \quad \text{and} \quad b'(z) \underset{Z^+}{\bigcup} Y^+$$

and the L_2-type

$$p = \mathrm{tp}_{L_2}(g^{bb'}(z)/b'^{-1}(Y^+)) \cup \bigcup_i \mathrm{tp}_{L_2}(g(x^i)/x^i) \cup \bigcup_i \mathrm{tp}_{L_2}(g^{bb'}(z^i)/z^i)$$

is consistent.

Note that our notion L_2-homogeneity only refers to 2-types consisting of a single element and its image. This is suitable only for binary languages. While a similar notion can be defined for higher arities, we leave the details for future investigation.

Proposition 2.14. *Let* $\mathbf{M}^*$ *be the free fusion of a homogeneous* L_1*-structure* $\mathbf{M}_1$ *carrying a stationary independence relation with an* L_2*-structure* M_2 *and let* $g \in G = \mathrm{Aut}(\mathbf{M}^*)$ *move maximally. If* g *is* compatible *and* X *is full for* g, $Y = g(X)$, *then the following holds:*

For any finite set Z *such that* $g(X) = Y = g^b(Z)$ *for some* $b \in G$ *and all tuples* x, z *such* $\mathrm{tp}(g(x)/Y) = \mathrm{tp}(g^b(z)/Y)$ *and* $x \underset{X}{\smile} YZ$ *and* $XY \underset{Z}{\smile} z$ *there are* $a_1 \in \mathrm{Fix}(XY), a_2 \in \mathrm{Fix}(YZ)$ *such that* $g^{-ba_2}g^{a_1}(x) = z$.

Proof. Since g is compatible, there is a finite extension Z^+ of Z which is full for g^b and there is $b' \in \mathrm{Fix}(YZ)$ such that with $Y^+ = g^b(Z^+)$, $Y^* = b'^{-1}(Y^+)$ and $Z^* = b'^{-1}(Z^+)$ we have $\mathrm{tp}(g^{bb'}(z)/Y) = \mathrm{tp}(g^b(z)/Y)$,

$$g^{bb'}(z) \underset{Y}{\smile} Y^* \quad \text{and} \quad Y^* \underset{Z^*}{\smile} z$$

and the L_2-type

$$p_{L_2} = \mathrm{tp}_{L_2}(g^{bb'}(z)/Y^*) \cup \bigcup_i \mathrm{tp}_{L_2}(g(x^i)/x^i) \cup \bigcup_i \mathrm{tp}_{L_2}(g^{bb'}(z^i)/z^i)$$

is consistent. Note that Y^* is full for $g^{bb'}$.

Let y be a realization of $p_{L_2} \cup \mathrm{tp}_{L_1}(g^{bb'}(z)/Y)$ independent from $xXzY^*Z^*$ over Y. Since $y \underset{Y}{\smile} Y^*$ and $\mathrm{tp}_{L_1}(y/Y) = \mathrm{tp}_{L_1}(g^{bb'}(z)/Y)$ and $g^{bb'}(z) \underset{Y}{\smile} Y^*$. Stationarity implies $\mathrm{tp}_{L_1}(y/Y^*) = \mathrm{tp}_{L_1}(g^{bb'}(z)/Y^*)$. Since $\mathrm{tp}_{L_1}(g^b(z)/Y) = \mathrm{tp}_{L_1}(g^{bb'}(z)/Y)$, we have $\mathrm{tp}_{L_1}(y/Y)) = \mathrm{tp}_{L_1}(g(x)/Y))$, and by the assumption $\mathrm{tp}_{L_2}(y^i/x^i) = \mathrm{tp}_{L_2}(g(x^i)/x^i)$ and $\mathrm{tp}_{L_2}(y^i/z^i) = \mathrm{tp}_{L_2}(g^{bb'}(z^i)/z^i)$. Moreover,

$$x \underset{X;Y}{\smile} y \quad \text{and} \quad y \underset{Y^*;Z^*}{\smile} z.$$

Since X is full for g and Y^* is full for $g^{bb'}$, we can find the required a_1, a_2. $\quad\square$

3. Proof of the main result

In this section we prove Theorem 1.5 using the general strategy of [7] and [12]. Let $\mathbf{M}^*$ be as in the hypothesis of Theorem 1.5 and $G = \mathrm{Aut}(\mathbf{M}^*)$. For $A \subseteq M$, let $\mathrm{Fix}(A)$ denote the *pointwise stabiliser* of A. For $A, B \subseteq M$, we write AB for their union $A \cup B$.

Lemma 3.1. *Let* $g \in G$ *move maximally, let* X, Y, C *be finite sets such that* $g(X) = Y$ *and* $X \underset{Y}{\smile} C$ *and let* x *be a tuple. Then there is some* $a \in \mathrm{Fix}(XY)$ *such that*

$$g^a(x) \underset{Y}{\smile} C.$$

Proof. This follows as in [12, 3.5]. $\quad\square$

Proposition 3.2. *Consider $g_1, \ldots, g_4 \in G$ that move maximally and finite sets $X_0, \ldots, X_4$ such that $g_i(X_{i-1}) = X_i$. Assume that g_2 is compatible. Then for $i = 1, \ldots, 4$ there are extensions $Y_i \supset X_i$ and $a_i \in \mathrm{Fix}(X_{i-1}X_i)$ (with $a_2 = a_3 = \mathrm{id}$) such that*

(1) Y_1 is full for g_2;
(2) $g_i^{a_i}(Y_{i-1}) = Y_i$,
(3) $Y_0 \mathop{\downarrow}_{Y_1} Y_2 Y_3$ and $Y_1 Y_2 \mathop{\downarrow}_{Y_3} Y_4$.

Proof. Put $Y_2' = g_2(X_0 X_1) \cup g_3^{-1}(X_3 X_4)$ and $Y_1' = g_2^{-1}(Y_2')$. Note that Y_1' contains $X_0 X_1$. Now let Y_1 be an extension of Y_1' which is full for g_2 and put $Y_2 = g_2(Y_1), Y_3 = g_3(Y_2)$. Note that Y_3 contains $X_3 X_4$. Put $Y_0' = g_1^{-1}(Y_1)$ and let Y_0 realize $\mathrm{tp}(Y_0'/Y_1)$ independent from $Y_2 Y_3$. Let $a_1 \in \mathrm{Fix}(Y_1)$ such that $a_1(Y_0) = Y_0'$. Then $g_1^{a_1}(Y_0) = Y_1$. Now put $Y_4' = g_4(Y_3)$. Let Y_4 realize $\mathrm{tp}(Y_4'/Y_3)$ independent from $Y_1 Y_2$ and let $a_4 \in \mathrm{Fix}(Y_3)$ such that $a_4(Y_4) = Y_4'$. Then $g_4^{a_4}(Y_3) = Y_4$. $\quad\square$

Proposition 3.3. *Let $g_1, \ldots, g_4 \in G$ be moving maximally and assume that g_2 is compatible and conjugate to g_3^{-1}. Let $Y_0, \ldots, Y_4$ be finite sets such that $g_i(Y_{i-1}) = Y_i$ for $i = 1, \ldots, 4$. Assume also that*

$$Y_0 \mathop{\downarrow}_{Y_1} Y_2 Y_3 \quad and \quad Y_1 Y_2 \mathop{\downarrow}_{Y_3} Y_4$$

and that Y_1 is full for g_2. Let x_0 and x_4 be two tuples such that $g_4 g_3 g_2 g_1$ maps $\mathrm{tp}(x_0/Y_0)$ to $\mathrm{tp}(x_4/Y_4)$. Then for $i = 1, \ldots, 4$, there are $a_i \in \mathrm{Fix}(Y_{i-1}Y_i)$ such that $g_4^{a_4} \ldots g_1^{a_1}(x_0) = x_4$.

Proof of Proposition 3.3. Since g_1 and g_4 move maximally, using Lemma 3.1 we find $a_1 \in \mathrm{Fix}(Y_0 Y_1)$ and $a_4 \in \mathrm{Fix}(Y_3 Y_4)$ such that for $x_1 = g_1^{a_1}(x_0)$ and $x_3 = (g_4^{-1})^{a_4}(x_4)$ we have

$$x_1 \mathop{\downarrow}_{Y_1} Y_2 Y_3 \quad and \quad Y_1 Y_2 \mathop{\downarrow}_{Y_3} x_3.$$

Since g_2 is compatible and Y_1 is full for g_2, by Proposition 2.14 we find the required $a_2 \in \mathrm{Fix}(Y_1 Y_2), a_3 \in \mathrm{Fix}(Y_2 Y_3)$ such that $g_3^{a_3} g_2^{a_2}(x_1) = x_3$. Thus,

$$g_4^{a_4} g_3^{a_3} g_2^{a_2} g_1^{a_1}(x_0) = x_4. \quad\square$$

Proposition 3.4. *Let $g_1, \ldots, g_4 \in G$ be compatible (hence moves maximally), and assume that g_2 is conjugate to g_3^{-1}. Then, for any non-empty open set $U \subseteq G^4$, there is some non-empty open set $W \subseteq G$ such that the image $\phi(U)$ under the map*

$$\phi \colon G^4 \to G \colon (h_1, \ldots, h_4) \mapsto g_4^{h_4} g_3^{h_3} g_2^{h_2} g_1^{h_1},$$

is dense in W.

Proof. Using Proposition 3.2 and compatibility of the g_i, the proposition follows exactly as in [12, 2.13]. □

Theorem 3.5. *If $g \in G$ is compatible, then any element of G is the product of at most eight conjugates of g and g^{-1}.*

Proof. We can use Proposition 3.4 and follow the proof in [12, 2.7]. □

4. Obtaining compatible automorphisms

In this section we prove Proposition 1.6 assuming that $\mathbf{M}$ is either the random poset, the Fraïssé limit of a nontrivial free amalgamation class or the rational bounded Urysohn space and that $\mathbf{M}_<$ is an order expansion of $\mathbf{M}$. We obtain the same results for the tournament expansion $\mathbf{M}_\to$ of the Fraïssé limit of a nontrivial free amalgamation class and the rational bounded Urysohn space.

We first prove that for tournament expansions any automorphism that moves maximally is automatically compatible.

Proposition 4.1. *If $\mathbf{M}_\to$ is a tournament expansion of a homogeneous L_1-structure $\mathbf{M}_1$ carrying a stationary independence relation, any $g \in G = \mathrm{Aut}(\mathbf{M}_\to)$ that moves maximally is compatible.*

We first prove the following lemma which uses the fact that in the random tournament the union of any two 1-types over disjoint finite sets is consistent.

Lemma 4.2. *Let $\mathbf{M}_\to$ be a tournament expansion of a homogeneous L_1-structure $\mathbf{M}_1$ carrying a stationary independence relation and let g be an automorphism moving maximally. Then any finite set X_0 has a finite extension X which is full for g.*

Proof. Let g move maximally and let X_0 be a finite set. We will show that any extension X of X_0 such that $g^{-1}(X) \cap g(X) \subset X$ is full for g. Put

$$X = X_0 \cup \{x \in M : x = g(a) \text{ and } g(x) = b \text{ for some } a, b \in X_0\}.$$

Then X is finite and $g^{-1}(X) \cap g(X) \subset X$. Now assume that x, y are as in (1) and put $Y = g(X)$. If $x_0 \subset x$ is contained in X, then for the corresponding coordinates y_0 of y we have $g^a(x_0) = y_0$ for all $a \in \mathrm{Fix}(XY)$. Since $x \underset{X;Y}{\downarrow} y$, we may assume that $x \cap XY = \emptyset$. Let x' be a realization of $\mathrm{tp}(x/XY)$ such that $g(x'), g^{-1}(x'), g^2(x')$ and $g^{-2}(x')$ do not intersect X and let $a \in \mathrm{Fix}(XY)$ such that $a(x) = x'$. Thus replacing g by g^a if necessary, we may assume that $g^2(x)$ and $g^{-2}(x)$ do not intersect X. Note that this implies that for any coordinate x_0 of x the set $X \cup \{x_0\}$ again satisfies $g^{-1}(Xx_0) \cap g(Xx_0) \subset X$. This will be used below in the induction.

Now we do induction on the length of x, y. First suppose that x and y are single elements.

Since $g^{-1}(X) \cap g(X) \subset X$, the L_2-type $\mathrm{tp}_{L_2}(x/XY) \cup \mathrm{tp}_{L_2}(g^{-1}(y)/g^{-1}(XY))$ is consistent. This uses the fact that in the random tournament the union of any two types over disjoint finite sets is consistent. By Proposition 2.12 we can choose a realisation x' of $\mathrm{tp}(x/X) \cup \mathrm{tp}_{L_2}(x/XY) \cup \mathrm{tp}_{L_2}(g^{-1}(y)/g^{-1}(XY))$ which is moved maximally by g over X. Since $x' \underset{X}{\downarrow} Y$, we have $\mathrm{tp}(x'/XY) = \mathrm{tp}(x/XY)$ by weak stationary independence. Choose $a_1 \in \mathrm{Fix}(XY)$ with $a_1(x) = x'$. Then g^{a_1} moves x maximally over X. We have $\mathrm{tp}_{L_1}(g^{a_1}(x)/xXY) = \mathrm{tp}_{L_1}(y/xXY)$ by Stationarity. From the L_2-homogeneity of g, we have $\mathrm{tp}_{L_2}(g^{a_1}(x)/x) = \mathrm{tp}_{L_2}(y/x)$ and therefore, since tournaments have a binary language, we obtain $\mathrm{tp}_{L_2}(g^{a_1}(x)/xXY) = \mathrm{tp}_{L_2}(y/xXY)$. Using weak stationary independence we conclude that $\mathrm{tp}(g^{a_1}(x)/xXY) = \mathrm{tp}(y/xXY)$. Choose $a_2 \in \mathrm{Fix}(xXY)$ with $a_2(y) = g^{a_1}(x)$. Then $g^{a_1 a_2}(x) = y$.

For the induction step assume that the claim is proved for tuples of length $n-1$ and let x, y be tuples of length n. Write x, y as $x_0 x'$ and $y_0 y'$, respectively, where x', y' are tuples of length $n-1$. By the first step of the induction we may assume that $g(x_0) = y_0$ so that $\mathrm{tp}(g(x')/y_0 Y) = \mathrm{tp}(y'/y_0 Y)$ and $x' \underset{x_0 X; y_0 Y}{\downarrow} y'$. Since $X x_0$ is again full, we can apply the induction hypothesis and find $b \in \mathrm{Fix}(x_0 X y_0 Y)$ such that $g^b(x') = y'$. □

We can now prove Proposition 4.1:

Proof of Proposition 4.1. Let x, z, Z, b be such that $\mathrm{tp}(g(x)/Y) = \mathrm{tp}(g^b(z)/Y)$, $x \underset{X}{\downarrow} YZ$ and $XY \underset{Z}{\downarrow} z$, $g(X) = Y$, $g^b(Z) = Y$, and X is full for g. Clearly we may assume that x does not intersect X (and so z does not intersect Z) and since $x \underset{X}{\downarrow} YZ$ we see that x is in fact disjoint from XYZ. Let Z^+ be the finite extension of Z full for g^b as above, i.e.

$$Z^+ = Z \cup \{x \in M : x = g^b(c) \quad \text{and} \quad g^b(x) = d \quad \text{for some} \quad c, d \in Z\}$$

and put $Y^+ = g^b(Z^+)$. Then $Z^+ \subset YZ$ and $Y^+ \subset YZ$. Hence $Y^+ \underset{Z^+}{\downarrow} z$ and $g^b(z) \underset{Y}{\downarrow} Y^+$ by Monotonicity and Invariance and since x, z are disjoint from Y^+. Since g is $\rightarrow$-homogeneous, the type

$$\mathrm{tp}_{\rightarrow}(g^b(z)/Y^+) \cup \bigcup_i \mathrm{tp}_{\rightarrow}(g(x^i)/x^i) \cup \bigcup_i \mathrm{tp}_{\rightarrow}(g^b(z^i)/z^i)$$

is consistent as required. (Note that in this case we can choose $b' = \mathrm{id}$.) □

We now turn to the case of order expansions. We call an element $g \in \mathrm{Aut}(\mathbf{M}_<)$ *unboundedly increasing (resp. decreasing)* if it is increasing (resp. decreasing) and for every $a < b$ in $\mathbf{M}_<$ there is $m \in \mathbb{N}$ such that $g^m(a) > b$ (resp. $g^m(b) < a$).

The following observation will be helpful later on:

Lemma 4.3. *If* $\mathbf{M}_<$ *is an order expansion of a homogeneous structure and* $g \in G$ *is unboundedly increasing, then we can identify* $\mathbf{M}_<$ *with the rationals* $\mathbb{Q}$ *in such a way that* $g(x) = x + 1$ *for all* $x \in \mathbf{M}_<$.

Proof. Pick some $x_0 \in \mathbf{M}_<$, identify the interval $[x_0, g(x_0)] \subset \mathbf{M}_<$ in an order preserving way with $[0, 1] \subset \mathbb{Q}$ and extend. $\square$

Clearly, in the same way we can identify an unboundedly decreasing function on $\mathbf{M}_<$ with $g(x) = x - 1$ for all $x \in \mathbf{M}_<$.

Proposition 4.4. *If* $\mathbf{M}_<$ *is an order expansion of a homogeneous* L_1*-structure* $\mathbf{M}_1$ *carrying a stationary independence relation, any unboundedly increasing* $g \in G = \mathrm{Aut}(\mathbf{M}_<)$ *that moves maximally is compatible.*

As a first step towards obtaining full extensions for finite sets we prove the following weaker version (which nevertheless turns out to be sufficient) because the automorphism $b \in \mathrm{Fix}(Xg(X))$ obtained in the proof below is only an $<$-automorphism of M, so does not necessarily respect the L_1-structure on M:

Lemma 4.5. *Let* $\mathbf{M}_<$ *be an order expansion of a homogeneous* L*-structure and let* g *be unboundedly increasing. Then for any finite set* $X_0 \subset M$ *there is a finite set* X *such that for all* n*-tuples* $x, y \in M$ *with* $\mathrm{tp}(g(x)/g(X)) = \mathrm{tp}(y/g(X))$ *and* $\mathrm{tp}_<(g(x^i)/x^i) = \mathrm{tp}_<(y^i/x^i), 1 \le i \le n$, *there is some* $<$*-automorphism* $b \in \mathrm{Fix}(Xg(X))$ *of* M *such that* $g^b(x) = y$.

In fact, the proof of Proposition 4.4 will show that the extension X obtained here is in fact full for g.

Proof. If $X_0 = \emptyset$, the conclusion holds. So let X_0 be a nonempty finite set, $X_0' = X_0 \cup \{g(x)\}$ for some $x \in X_0$, let $x_{\min} = \min X_0'$, $x_{\max} = \max X_0'$. Let

$$X = \{g^m(x) \colon x \in X_0, m \in \mathbb{Z} \text{ and } x_{\min} \le g^m(x) \le x_{\max}\}$$

and put $Y = g(X)$.

We claim that the conclusion holds for X and g. We may identify $\mathbf{M}$ with the rationals in such a way that $X = \{0, \dots, m\}$, $Y = g(X) = \{k, \dots, k+m\}$ and $g(x) = x + k$ is a shift by k (see Lemma 4.3). Let x, y be increasing n-tuples such that $\mathrm{tp}(g(x)/Y) = \mathrm{tp}(y/Y)$ and $\mathrm{tp}_<(x^i + k/x^i) = \mathrm{tp}_<(y^i/x^i)$. We now define $b \in \mathrm{Fix}(XY)$ such that $g^b(x) = y$.

Let b be an arbitrary isomorphism defined over $[0, k]$ fixing XY. Then, in turn, we extend b to $[k, k+1], \dots, [k+m-1, k+m]$ in such a way that for all $y^i \in [k+j, k+j+1]$, $j = 0, \dots, m-1$, we have $g^b(x^i) = y^i$. We now extend the definition of b from $[0, k+m]$ to $[0, \infty)$ by extending it stepwise to adjacent intervals using the shift function $g(x) = x + k$.

Let i be maximal such that $x^i \in [0, m+k]$. We extend b to $[m+k, y^i]$. Assume that $x^{i+1}, \ldots x^q \in [m+k, y^i]$. Then for $j = i+1, \ldots q$ we have an ordered tuple $y^j > y^i$ and corresponding $g(b(x^j)) > g(b(x^i))$. So setting $b(y^j) = g(b(x^j)) = b(x^j) + k > b(x^i) + k$ for $j = i = 1, \ldots q$ preserves the order. Now extend b to all of $[m+k, y^q]$. In the next step, b is already defined on any $x^k \in [y^i, y^q]$ and we continue by defining $b(y^k)$ exactly as before until b is defined on $[0, y^n]$. Then we extend b to $[0, \infty)$.

It is left to argue that we can extend b to $(-\infty, 0]$. The argument is essentially symmetric to the one in the previous paragraph (replacing g by g^{-1} and switching the roles of x^j and y^j). $\quad \square$

Proof of Proposition 4.4. Identify $\mathbf{M}$ with the rationals in such a way that the orders agree. Put $Y = g(X)$. By Lemma 4.5 there is some $<$-automorphism $d \in \mathrm{Fix}(XY)$ such that $y = g^d(x)$. Clearly $\mathrm{tp}_<(g^d(x)/xXY) = \mathrm{tp}_<(y/xXY)$. It follows that $\mathrm{tp}_<(g^a(x)/xXY) = \mathrm{tp}_<(y/xXY)$ for any $a \in \mathrm{Fix}(XY)$ such that $|a(xy) - d(xy)| = \max\{|a(z) - d(z)| \colon z \in xy\}$ is sufficiently small. Choose a realisation x' of $\mathrm{tp}(x/X) \cup \mathrm{tp}_<(x/XY)$ with $|x' - d(x)|$ sufficiently small which is moved maximally by g over X. Then as in the corresponding proof for tournaments we have $\mathrm{tp}(x'/XY) = \mathrm{tp}(x/XY)$. Let $a_1 \in \mathrm{Fix}(XY)$ with $a_1(x) = x'$ and $|a_1(xy) - d(xy)|$ sufficiently small (which is possible by property (∗)). Thus we have $\mathrm{tp}_<(g^{a_1}(x)/xXY) = \mathrm{tp}_<(y/xXY)$ and $tp_{L_1}(g^{a_1}(x)/xXY) = tp_{L_1}(y/xXY)$ by Stationarity. Choose $a_2 \in \mathrm{Fix}(xXY)$ with $a_2(y) = g^{a_1}(x)$. Then $g^{a_1 a_2}(x) = y$.

Now let X be as given by Lemma 4.5, $Y = g(X)$, x, z, Z, and b be such that $g^b(Z) = Y$, $x \underset{X}{\perp} YZ$ and $XY \underset{Z}{\perp} z$, $\mathrm{tp}(g(x)/Y) = \mathrm{tp}(g^b(z)/Y)$. In particular, X is full for g. Let Z^+ be a finite extension for Z which is full for g^b defined as above, i.e.

$$Z^+ = \{(g^b)^m(x) \colon x \in Z, m \in \mathbb{Z} \text{ and } z_{\min} \le (g^b)^m(x) \le z_{\max}\}$$

where $z_{\min}, z_{\max}$ are the minimal and the maximal element of Z, respectively, and put $Y^+ = g^b(Z^+)$. First note that

$$p_< = \mathrm{tp}_<(g^b(z)/Y^+) \cup \bigcup_i \mathrm{tp}_<(g(x^i)/x^i) \cup \bigcup_i \mathrm{tp}_<(g^b(z^i)/z^i)$$

is consistent. Indeed, observe first that $y_{\min} \le Y^+ \setminus Y \le y_{\max}$, where $y_{\min}$ and $y_{\max}$ are the minimal and maximal element of Y. Let also $x_{\min}$ and $x_{\max}$ be the minimal and maximal element of X. If $x^i < x_{\min}$, then $z^i < z_{\min}$ and $g(x^i), g^b(z^i) < y_{\min}$. So we can find a corresponding coordinate y^i such that $x^i, z^i < y^i < y_{\min}$. Similarly, if $x^i > x_{\max}$, then $z^i > z_{\max}$ and $g(x^i), g^b(z^i) > y_{\max}$ and we can find a corresponding coordinate y^i such that $y_{\max} < x^i, z^i < y^i$. Finally, since X is full for g, if $x_{\min} < x^i < x_{\max}$, we have $x^i < w$ for every w which realizes $\mathrm{tp}_<(g^b(z^i)/Y^+)$. Therefore $y^i = g^b(z^i)$, which clearly satisfies $\mathrm{tp}_<(g^b(z^i)/Y^+)$ and so $z^i < y^i$, works.

Note that for any L^*-automorphism b' sufficiently close to the identity (in the sense of $<$) the type

$$p_{b'} = \mathrm{tp}_<(g^{bb'}(z)/b'^{-1}(Y^+)) \cup \bigcup_i \mathrm{tp}_<(g(x^i)/x^i) \cup \bigcup_i \mathrm{tp}_<(g^{bb'}(z^i)/z^i)$$

is also consistent.

Now by Existence and $(*)$, let c be a realization of $\mathrm{tp}_{L_1}(z/YZ)$, sufficiently close to z and such that $c \underset{YZ}{\downharpoonleft} Y^+Z^+$ and let $b' \in \mathrm{Fix}(YZ)$ with $b'(z) = c$ be sufficiently close to the identity such that $p_{b'}$ is consistent. Since $g^b(Z) = Y$ and $\mathrm{tp}_{L_1}(z/Z) = \mathrm{tp}_{L_1}(b'(z)/Z)$, we have $\mathrm{tp}_{L_1}(g^b(z)/Y) = \mathrm{tp}_{L_1}(g^b(b'(z))/Y)$, and therefore $\mathrm{tp}(g^{bb'}(z)/Y) = \mathrm{tp}(g^b(z)/Y)$. We claim that $g^{bb'}(z) \underset{Y}{\downharpoonleft} b'^{-1}(Y^+)$, i.e. $c \underset{Z}{\downharpoonleft} Z^+$, and $c \underset{Z^+}{\downharpoonleft} Y^+$. Indeed, we have $c \underset{YZ}{\downharpoonleft} Z^+$ and $z \underset{Z}{\downharpoonleft} Y$, and therefore $c \underset{Z}{\downharpoonleft} Y$. By Transitivity we obtain $c \underset{Z}{\downharpoonleft} Z^+$. Since $c \underset{YZ}{\downharpoonleft} Y^+Z^+$ and $c \underset{Z}{\downharpoonleft} Y$, by Transitivity we obtain $c \underset{Z}{\downharpoonleft} Y^+Z^+$. Hence by Monotonicity we get $c \underset{Z^+}{\downharpoonleft} Y^+$, as we wanted. $\square$

To prove Proposition 1.6 for $\mathbf{M}_<$, we construct an unboundedly increasing automorphism $g \in \langle h \rangle^G$ moving maximally starting from an arbitrary $h \in G$. This is done in four steps.

(1) construct a fixed point free $h_1 = [h, f_1] \in \langle h \rangle^G$;
(2) construct a strictly increasing $h_2 = [h_1, f_2] \in \langle h_1 \rangle^G$;
(3) construct an unboundedly increasing $h_3 \in \langle h_2 \rangle^G$;
(4) construct an unboundedly increasing $h_3 = [h_2, f_3] \in \langle h_1 \rangle^G$ moving maximally.

We take care of each of these steps in the Lemmas 4.8, 4.9, 4.10 and 4.11.

Lemma 4.6. (cf. [8, 3.4(ii)]) *No element of $G \setminus \{\mathrm{id}\}$ fixes any interval pointwise.*

Proof. Suppose otherwise. Choose $h \in G \setminus \{1\}$ and $a, c, d \in M$ such that h fixes $[c, d]$ pointwise and $a \neq h(a)$. We first claim that there exists a finite set $Y \subset M$ such that $\mathrm{tp}_L(a/Y) \neq \mathrm{tp}_L(h(a)/Y)$ or, equivalently, $\mathrm{tp}_L(Ya) \neq \mathrm{tp}(Yh(a))$. Suppose not. Then by homogeneity of M we can construct an L-automorphism of M that fixes $M \setminus \{a, h(a)\}$ pointwise and swaps a and $h(a)$, contradicting [8, 2.10].

By property $(*)$, $\mathrm{tp}(Y/ah(a))$ is realized by some finite set $B \subseteq (c, d)$. Since h fixes B pointwise, we see that $\mathrm{tp}(h(a)/B) = \mathrm{tp}(a/B)$, a contradiction.

If $\mathbf{M}_<$ is the ordered rational bounded Urysohn space, then the L_1-type $p_{L_1}(x)$ expressing $d(x, a) = 1 \wedge d(x, h(a)) = 1/2$ is consistent. Using $(*)$ we can pick a realization b of $p_{L_1}(x)$ in the interval (c, d), contradicting the assumption that h is an isometry fixing b and taking a to $h(a)$.

If $\mathbf{M}_<$ is the ordered random poset, assume $a \not\preceq_{\mathrm{po}} h(a)$. Using $(*)$ we can pick some $x \in (c, d)$ with $x <_{\mathrm{po}} a$ and $x \not\prec_{\mathrm{po}} h(a)$. Then $\mathrm{tp}(h(a)/x) \neq \mathrm{tp}(a/x)$, contradicting the assumption that h fixes x. $\square$

The previous lemma can easily be adapted to many other fusion structures. However, it seems difficult to give a uniform proof extending the result from [8] to arbitrary fusion structures.

Corollary 4.7. *A nontrivial element of G does not fix the set of realizations $D = p(\mathbf{M}_<)$ of any non-algebraic $\mathcal{L}^*$-type p over a finite set A.*

Proof. We can assume that $p = q \cup \{a < x < b\}$ for some $a, b \in \mathbf{M} \cup \{-\infty, \infty\}$ where q is a complete L-type over A. Let $D' = q(\mathbf{M})$, so $D = D' \cap (a, b)$. By condition $(*)$ $D' \cap (a, b)$ is dense in (a, b), so if h fixes D pointwise, then h is the identity on (a, b) and hence $h = \mathrm{id}$ by Lemma 4.6. $\square$

Lemma 4.8. *For any $h \in G$ there is some $g = [h, f] \in \langle h \rangle^G$ which is fixed point free.*

Proof. This follows from Corollary 4.7 as in [8, 2.11]. Note that we can use condition $(*)$ to ensure that f preserves the ordering. $\square$

Lemma 4.9. *If $h \in G$ has no fixed point, then there is some $g = [h, f] \in \langle h \rangle^G$ which is strictly increasing.*

Proof. Write $\mathbf{M}_<$ as an ordered union of intervals $J_i, i \in I$, on which h is either strictly increasing or strictly decreasing. By further subdividing we may assume that on each interval J_i the automorphism h is either unboundedly increasing or unboundedly decreasing.

If h is unboundedly increasing on J_i, then as in Lemma 4.3 we identify J_i with (a copy of) $\mathbb{Q}$ in such a way that we have $h(x) = x + 1$ for $x \in J_i$. Similarly, if h is unboundedly decreasing on J_j, we identify J_j with (a copy of) $\mathbb{Q}$ in such a way that we have $h(x) = x - 1$ for $x \in J_j$.

Fix a positive $\epsilon < \frac{1}{2}$. We construct an element $f \in G$ by back and forth leaving each J_i invariant and such that if h is increasing on J_i, then we choose $f(x) \in [\frac{x}{2}, \frac{x}{2} + \epsilon)$ for $x \in J_i$, or, equivalently, $f^{-1}(x) \in (2x - 2\epsilon, 2x]$. Similarly, if h is decreasing on J_i we choose $f(x) \in [2x, 2x + \epsilon)$ for $x \in J_i$, or, equivalently, $f^{-1}(x) \in (\frac{x}{2} - \frac{\epsilon}{2}, \frac{x}{2}]$. Clearly, such an $f \in G$ can easily be constructed by back-and-forth thanks to condition $(*)$. We then have $h(f(x)) > f(h(x))$, and so $[h, f](x) > x$ for any $x \in \mathbf{M}_<$. $\square$

Lemma 4.10. *If $h \in G$ is strictly increasing, there is an unboundedly increasing $g \in \langle h \rangle^G$.*

Proof. Since h is strictly increasing, we may identify $\mathbf{M}_<$ with a union of copies of Q such that on each copy we have $h(x) = x + 1$.

First assume that the union of copies of $\mathbb{Q}$ is infinite in both directions. Divide each copy of $\mathbb{Q}$ into an ordered union of two copies $Q_1 \cup Q_2$ (each Q_i being again isomorphic to $\mathbb{Q}$). Using assumption $(*)$, we define $f \in G$ by back-and forth such that each half-copy of Q is moved to the next one above: so $f(Q_1) = Q_2$ and $f(Q_2) = Q_1$ in the next copy

of $\mathbb{Q}$. If there is a first copy $Q_1 \cup Q_2$ we define f in such a way that $f(Q_1) = Q_1 \cup Q_2$. And if there is a last copy $Q = Q_1 \cup Q_2$ we define f in such a way that $f(Q_1 \cup Q_2) = Q_2$. Then $g = h \cdot h^f$ is unboundedly increasing. $\square$

Lemma 4.11. *If* $\mathbf{M}_<$ *is as in Theorem 1.3 (1) or (2) and* $h \in G$ *is unboundedly increasing, there is some unboundedly increasing* $g \in \langle h \rangle^G$ *that moves maximally.*

Proof. Since h is unboundedly increasing, we may identify $\mathbf{M}_<$ with $\mathbb{Q}$ in such a way that we have $h(x) = x + 1$ for all $x \in \mathbf{M}_<$. Fix a positive $\epsilon < \frac{1}{2}$.

Case I: If $\mathbf{M}_<$ is the ordered Fraïssé limit of a free amalgamation class, we define f by a back and forth construction like in [12, 5.1], with the additional requirement that we have

$$f(x) \in \left(\frac{x}{2} - \epsilon, \frac{x}{2} \right] \text{ for each } x \in \mathbf{M}_<.$$

Since this implies that $f^{-1}(x) \in [2x, 2x + 2\epsilon)$, it follows that

$$[h, f](x) > x + \delta$$

with $\delta = 2(\frac{1}{2} - \epsilon) > 0$. Hence the commutator $[h, f]$ will again be unboundedly increasing.

So suppose that f' is already defined on a finite set A and let p be a type over a finite set X. It suffices to show that f' has an extension f such that $[h, f]$ moves p maximally.

By possibly extending f' we can assume that $[h, f']$ is defined on X and that $f'^{-1}hf'(X) \subseteq A$. Now pick a realisation a of p independent from $X' = A \cup h(X) \cup [h, f'](X)$ over X and such that $h(a) \neq a$ which is possible by Corollary 4.7. Let $B = f'(A)$ and pick a realisation b of $f'(\text{tp}(a/A))$ in such a way that $b \underset{B}{\downarrow} h^{-1}(B)$ and $b \in (\frac{a}{2} - \epsilon, \frac{a}{2}]$. Extend f' to Aa by setting $f'(a) = b$. Next pick a realisation c of $f'^{-1}(\text{tp}(h(b)/Bb))$ such that c is independent from $h(a)h(X)$ over Aa, and $c \in [2h(b), 2h(b) + 2\epsilon)$. Extend f' by setting $f'(c) = h(b)$. Since weak stationary independence agrees with stationary independence on subsets of $\mathbf{M}_<$ the proof of Lemma 5.1 in [12] shows that $a \underset{X;h(X)}{\downarrow} [h, f'](a)$.

Case II: Now suppose that $\mathbf{M}_<$ is the ordered bounded Urysohn space. If there is no $a \in \mathbf{M}_<$ with $d(a, h(a)) = 1$, then as in [13, 1.3] and using condition $(*)$ we construct some unboundedly increasing $h_1 \in \langle h \rangle^G$ as a product of conjugates of h such that there is some $b \in \mathbf{M}_<$ with $d(b, h_1(b)) = 1$: let $0 < \epsilon < 1$ and $a \in \mathbf{M}_<$ such that $d(a, h(a)) = \epsilon$. Assume $h(a) > a$ (the other case being similar). Pick some $b \in (a, \infty)$ with $d(a, b) = 1$. Let $k > 1$ be such that $k\epsilon \geq 1$. Put $a_0 = a, a_k = b$ and, using $(*)$, pick $a_i \in (a_{i-1}, b)$ such that $d(a_{i-1}, a_i) = \epsilon, i = 1, \ldots k$. Let $f_i \in G$ with $f_i(a_{i-1}, a_i) = (a_i, a_{i+1})$ and put $h_1 = h^{f_1} \cdot \ldots \cdot h^{f_k}$.

In the same way we can adapt [13, 2.4] to construct iterated commutators $[h, f]$ using
(∗) to make sure that f preserves the order and additionally satisfies

$$f(x) \in \left(\frac{x}{2} - \epsilon, \frac{x}{2}\right]$$

for each $x \in M$. Thus, after each step, the commutator will again be unboundedly
increasing and we end up as in [13, 2.5] with an automorphism $g' \in \langle h \rangle^G$ which is un-
boundedly increasing and moves *almost maximally*, i. e. every nonalgebraic type p over
a finite set X has a realization a such that $a \underset{X}{\bigcup} g'(a)$. An application of the previous
argument as in [12, 5.3] then yields the required $g \in \langle h \rangle^G$ which is unboundedly increas-
ing and moves maximally. This concludes the proof in the case of the ordered bounded
Urysohn space and thus of Proposition 1.6. □

To complete the proof of Theorem 1.3 in the case of order expansions, it is left to
prove the following two propositions:

Proposition 4.12. *If* $\mathbf{M}^*$ *is the ordered random poset, then* $G = \mathrm{Aut}(\mathbf{M}^*)$ *is simple.*

Proof. Let $h \in G$. By Lemmas 4.8, 4.9 and 4.10 we may assume that h is unboundedly
increasing in the sense of $<$. Now we can follow the steps of [3, Sec. 3] to construct
some $g \in \langle h \rangle^G$ which is unboundedly increasing in the sense of the partial order $<_{\mathrm{po}}$.
Using property (∗) we can make sure that at each step the result is again unboundedly
increasing in the sense of the order $<$. It is easy to see using (∗) that any two elements
of G that are unboundedly increasing both in the sense of $<_{\mathrm{po}}$ and in the sense of $<$ are
conjugate. Adapting the proof of [3, 3.4] using (∗), any element f of G can be written
as a product $f = g_1^{-1} g_2$ with g_1, g_2 unboundedly increasing in the sense of $<_{\mathrm{po}}$ and $<$.
Thus G is simple also in this case. □

Proposition 4.13. *Assume that* $\mathbf{M}$ *is the Fraïssé limit of a nontrivial free amalgamation
class such that* G *acts transitively on* M *or the bounded rational Urysohn space and*
$\mathbf{M}_{\rightarrow}$ *is a tournament expansion of* $\mathbf{M}$. *For any* $h \in G$, *there is some* $g \in \langle h \rangle^G$ *moving
maximally.*

Proof. As in Lemma 4.6, Corollary 4.7 and [8, 3.4(ii)] we see that a nontrivial element
of G does not fix pointwise the set of realizations of any nonalgebraic type over a finite
set. Thus as in Lemma 4.8 we can replace h by some fixed point free element in $\langle h \rangle^G$.
Then as in Lemma 4.11 we can follow the construction of [12, 5.1] and [13] respectively
to construct an element $f \in G$ such that $g = [h, f]$ moves maximally in the sense of
the stationary independence relation of $\mathbf{M}_1$. By the axioms of the random tournament
and property (∗), we can construct f in such a way to ensure that $a \rightarrow [h, f](a)$ for all
$a \in M$. Thus, $g = [h, f]$ moves maximally. □

Remark 4.14. An indiscernible set $\mathbf{M}$ carries a stationary independence relation by setting A to be independent from B over C if $A \cap B \subset C$. Any fixed point free permutation of $\mathbf{M}$ moves maximally with respect to this stationary independence relation (see [12]). Then $\mathbf{M}_<$ is isomorphic to $(\mathbb{Q}, <)$, and we conclude that given any unboundedly increasing automorphism g of $(\mathbb{Q}, <)$, any $h \in \mathrm{Aut}(\mathbb{Q})$ can be written as a product of at most eight conjugates of g and g^{-1}.

Similarly, in this case the tournament expansion $\mathbf{M}_\to$ is just the countable random tournament and we conclude that for any automorphism g such that $a \to g(a)$ any $h \in G$ can be written as a product of at most eight conjugates of g and g^{-1}.

Hence Theorem 1.5 applies to any unboundedly increasing automorphism of $(\mathbb{Q}, <)$ and to any automorphism g of the random tournament such that $a \to g(a)$ for all a. We thus obtain as a corollary the following result which yields more specific information about Higman's theorem on $\mathrm{Aut}(\mathbb{Q}, <)$ and the result from [8] on the random tournament $(\mathbb{T}, \to)$:

Corollary 4.15. *If g is an unboundedly increasing automorphism of $(\mathbb{Q}, <)$ or an automorphism of the random tournament $(\mathbb{T}, \to)$ such that $a \to g(a)$ for all $a \in M$, then any element of $\mathrm{Aut}(\mathbb{Q}, <)$ or of $\mathrm{Aut}(\mathbb{T}, \to)$ is the product of at most eight conjugates of g and g^{-1}.*

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