

Operations on Ring Structures Preserved by Normalized Automorphisms of Group Rings

Frauke M. Bleher and Ted Chinburg*

*Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania
19104-6395*

E-mail: frauke@math.upenn.edu, ted@math.upenn.edu

Communicated by Walter Feit

Received May 27, 1998

Let \mathcal{O} be a commutative ring, and suppose σ is a normalized \mathcal{O} -algebra automorphism of the group ring $\mathcal{O}G$ of a finite group G over \mathcal{O} . In this paper we consider the action of σ on various algebraic structures associated to G . Suppose \mathcal{O} is an integral domain of characteristic 0, and that no prime divisor of the order of G is invertible in \mathcal{O} . We show that σ preserves the λ -ring structure of $G_0(kG)$ when k is a field with a ring homomorphism $\mathcal{O} \rightarrow k$. If \mathcal{O} is the ring of integers of a number field, we show that σ preserves the $G_0^{\mathcal{O}}(\mathcal{O}G)$ -module structure of the class group $\text{Cl}(\mathcal{O}G)$ of $\mathcal{O}G$, where $G_0^{\mathcal{O}}(\mathcal{O}G)$ is the Grothendieck group of $\mathcal{O}G$ -lattices. © 1999 Academic Press

Key Words: normalized automorphisms; ring structures; cohomology.

1. INTRODUCTION

Many classical problems in algebraic number theory and ring theory concern the action of automorphisms of rings on objects associated to the ring. For example, the action of the automorphism group of a number field on its class group is the subject of many fundamental results and conjectures in number theory. A group theoretic example is provided by the Zassenhaus Conjecture [8, 10]. For simple groups G , this conjecture is equivalent to the statement that a normalized (i.e., augmentation preserving) automorphism of the integral group ring of G acts on the character group of G in the same way as an automorphism of G . In this paper we prove results concerning actions of automorphisms of group rings which pertain to both of these examples.

*The second author was supported in part by NSF Grant DMS-9701411.

Let \mathcal{O} be a commutative ring. Let $\text{Aut}_n(\mathcal{O}G)$ be the group of normalized \mathcal{O} -algebra automorphisms σ of the group ring $\mathcal{O}G$ of a finite group G . In Section 2 we suppose \mathcal{O} is an integral domain of characteristic 0, and that no prime divisor of the order of G is invertible in \mathcal{O} . Let $\mathcal{O} \rightarrow k$ be a ring homomorphism to a field k . We recall in Section 2 the definition of the λ -ring structure of $G_0(kG)$. Our first result is that $\sigma \in \text{Aut}_n(\mathcal{O}G)$ preserves this structure. The significance of this result to the Zassenhaus Conjecture for simple groups G is as follows. To prove the conjecture, it suffices to show that every character table automorphism which preserves the λ -ring structure of $G_0(kG)$ arises from an automorphism of G .

In Section 3 we suppose \mathcal{O} is the ring of integers of a number field F . Let $G_0^{\mathcal{O}}(\mathcal{O}G)$ be the Grothendieck group of $\mathcal{O}G$ -lattices. We recall in Section 3 the definitions of the class group $\text{Cl}(\mathcal{O}G)$ and kernel subgroup $\text{D}(\mathcal{O}G)$ of $\text{Cl}(\mathcal{O}G)$. The study of the action of automorphisms in $\text{Aut}_n(\mathcal{O}G)$ on $\text{Cl}(\mathcal{O}G)$ is an integral group ring counterpart of the study of the action of Galois automorphisms of number fields on their class groups. We show that the action of each element σ of $\text{Aut}_n(\mathcal{O}G)$ preserves the $G_0^{\mathcal{O}}(\mathcal{O}G)$ -module structures of $\text{Cl}(\mathcal{O}G)$, $\text{D}(\mathcal{O}G)$, $K_0(\mathcal{O}G)$ and of the torsion subgroup $G_0(\mathcal{O}G)_{\text{tor}}$ of $G_0(\mathcal{O}G)$. It remains an interesting open problem whether the action of σ preserves the ring structure of $G_0^{\mathcal{O}}(\mathcal{O}G)$.

2. THE λ -RING STRUCTURE OF $G_0(kG)$

Throughout this section, let G be a finite group and let k be an arbitrary field. Furthermore, let \mathcal{O} be an integral domain of characteristic zero such that no rational prime $p \leq \max\{\text{order}(g) | g \in G\}$ is invertible in \mathcal{O} , and such that there exists a ring homomorphism $f: \mathcal{O} \rightarrow k$. We want to show that normalized \mathcal{O} -algebra automorphisms of the group ring $\mathcal{O}G$ respect the λ -ring structure of the Grothendieck group $G_0(kG)$.

A commutative ring R is called a λ -ring [6, Sect. I.1] if there exists a family of maps

$$\lambda^i: R \rightarrow R$$

such that $\lambda^0(x) = 1$, $\lambda^1(x) = x$ for all $x \in R$, and such that

$$\lambda_r(x + y) = \sum_{i=0}^r \lambda^i(x) \lambda^{r-i}(y)$$

for all $r \geq 0$ and $x, y \in R$. The Grothendieck group $G_0(kG)$ has a ring structure induced by taking tensor products over k . It has a unique λ -ring structure such that for each finitely generated kG -module M , $\lambda^i([M])$ is the class $[\Lambda^i M]$ of the i th exterior power $\Lambda^i M$ over k .

Let σ be a normalized \mathcal{O} -algebra automorphism of $\mathcal{O}G$. Using the ring homomorphism $f: \mathcal{O} \rightarrow k$, σ induces a normalized k -algebra automorphism σ of kG . Since σ induces an automorphism of the category of finitely generated kG -modules, σ induces an action on $G_0(kG)$. Furthermore, this action is compatible with the ring structure of $G_0(kG)$. If k is sufficiently large, this is proved in [2, Proposition 2.1], while the general case follows from the fact that $G_0(kG) \rightarrow G_0(k'G)$ is an injective ring homomorphism if k is a subfield of a field k' .

The main result of this section is:

THEOREM 2.1. *Let σ be a normalized \mathcal{O} -algebra automorphism of $\mathcal{O}G$. Then σ respects the λ -ring structure of $G_0(kG)$, in the sense that*

$$\sigma(\lambda^i(c)) = \lambda^i(\sigma(c)) \quad (2.1)$$

for all $c \in G_0(kG)$ and all $i \geq 0$.

Proof. Define $R_k(G)$ to be the ring of virtual characters $\text{ch}(kG)$ if $\text{char}(k) = 0$, and to be the ring of virtual Brauer characters $\text{Bch}(kG)$ if $\text{char}(k) = p > 0$. We identify the Grothendieck group $G_0(kG)$ with $R_k(G)$. Thus (2.1) is equivalent to

$$\sigma(\lambda^i(\mu)) = \lambda^i(\sigma(\mu)) \quad (2.2)$$

for $\mu \in R_k(G)$.

Let $G_{p'}$ be the set of elements of G of order prime to p if $\text{char}(k) = p > 0$, and let $G_{p'} = G$ otherwise. We will use a well-known recursive formula for the λ -operators in terms of the Adams operators $\psi^m: G_0(kG) \rightarrow G_0(kG)$. Recall that ψ^m can be defined by the formula [4, Proposition 12.8]

$$\{\psi^m(\mu)\}(x) = \mu(x^m), \quad m = 1, 2, \dots \quad (2.3)$$

for all $\mu \in R_k(G)$ and $x \in G_{p'}$. Note that since we take Brauer characters in characteristic p , both sides of this formula lie in a field of characteristic 0.

The recursive formula which relates λ^i and ψ^m [4, (12.7)] can be rewritten in the following way:

$$\begin{cases} \lambda^1(\mu) = \psi^1(\mu) = \mu, \\ \lambda^2(\mu) = -\frac{1}{2}[\psi^2(\mu) - \lambda^1(\mu)\psi^1(\mu)], \\ \lambda^3(\mu) = \frac{1}{3}[\psi^3(\mu) - \lambda^1(\mu)\psi^2(\mu) + \lambda^2(\mu)\psi^1(\mu)], \\ \dots \\ \lambda^m(\mu) = (-1)^{m-1} \frac{1}{m} \left[\sum_{i=0}^{m-1} (-1)^{i-1} \lambda^i(\mu) \psi^{m-i}(\mu) \right], \\ \dots \end{cases} \quad (2.4)$$

The class sum $K(g)$ of an element $g \in G$ is the sum in $\mathcal{O}G$ of the conjugates of $g \in G$. Denote the conjugacy class of g in G by $C(g)$. Because of the class sum correspondence [9, Theorem 4.1], σ sends $K(g)$ to $K(h(g))$ where $h(g)$ is an element of G . Thus σ defines an action on the conjugacy classes of G by mapping $C(g)$ to $C(h(g))$. Note that since σ preserves the augmentation map $\epsilon: \mathcal{O}G \rightarrow \mathcal{O}$, $C(g)$ and $C(h(g))$ have the same length. It follows from [9, Proposition V.1.1] that if n is an integer then

$$\sigma(C(g^n)) = C(h(g)^n). \quad (2.5)$$

Suppose $\mu \in R_k(G)$ and $g \in G_p$. By [2, Proposition 2.1], the character μ^σ of the twist of μ by σ satisfies

$$\mu^\sigma(C(g^n)) = \mu(\sigma(C(g^n))). \quad (2.6)$$

Therefore (2.3), (2.5), and (2.6) imply σ preserves the power map on characters, i.e., $\sigma(\psi^n(\mu)) = \psi^n(\sigma(\mu))$ for all characters μ . Thus by using induction and the recursive formula (2.4), σ commutes with the λ -operators as stated in (2.2), which proves the theorem. ■

The operators γ^i (see [6, Sect. 3.1]) on a λ -ring R are defined by the power series

$$\sum_i \lambda^i(x) \left(\frac{t}{t-1} \right)^i = \sum_i \gamma^i(x) t^i.$$

The augmentation map $\eta: G_0(kG) \rightarrow \mathbb{Z}$ is defined by $\eta(\mu) = \mu(1)$. The Grothendieck γ -filtration of R is defined in the following way. For $n \leq 0$, we set $F^n R = R$. Let $F^1 R = \text{Ker}(\eta)$. For $n \geq 2$, define $F^n R$ to be the \mathbb{Z} -module generated by the elements $\gamma^{r_1}(x_1) \cdots \gamma^{r_k}(x_k)$ with $x_1, \dots, x_k \in F^1 R$ and $\sum_i r_i \geq n$.

Associated with the filtration F^n on R , we have the associated graded ring

$$\text{Gr}(R) = \bigoplus_{i=0}^{\infty} F^i R / F^{i+1} R.$$

From Theorem 2.1 we have the following result with respect to the Grothendieck γ -filtration of $G_0(kG)$.

THEOREM 2.2. *Let σ be a normalized \mathcal{O} -algebra automorphism of $\mathcal{O}G$. Then σ preserves the γ -filtration of $G_0(kG)$, in the sense that*

$$\sigma(F^n G_0(kG)) = F^n G_0(kG) \quad \text{for all } n.$$

In particular, σ induces an operation on $\text{Gr}(G_0(kG))$.

Proof. It follows directly from the definition of the γ -operators in terms of the λ -operators and from Theorem 2.1 that $\sigma(\gamma^i(\mu)) = \gamma^i(\sigma(\mu))$ for all $\mu \in G_0(kG)$. Since σ preserves the degree of characters, one has $\eta \circ \sigma = \eta$ so $\sigma(F^1 G_0(kG)) = F^1 G_0(kG)$. Because σ commutes with the γ -operators, it follows also that $\sigma(F^n G_0(kG)) = F^n G_0(kG)$ for all n . ■

3. ACTIONS ON GROTHENDIECK GROUPS OF $\mathcal{O}_F G$ -MODULES

In this section we will suppose that G is a finite group and that F is a number field. Let $\mathcal{O} = \mathcal{O}_F$ be the ring of integers of F . We will study the action of normalized \mathcal{O} -algebra automorphisms of $\mathcal{O}G$ on certain Grothendieck groups of $\mathcal{O}G$ -modules.

Let $G_0(\mathcal{O}G)$ (resp. $K_0(\mathcal{O}G)$) be the Grothendieck group of all finitely generated (resp. finitely generated projective) $\mathcal{O}G$ -modules. Define $G_0(\mathcal{O}G)_{\text{tor}}$ to be the subgroup of elements of $G_0(\mathcal{O}G)$ which have finite order. We identify the locally free class group $\text{Cl}(\mathcal{O}G)$ of $\mathcal{O}G$ with the torsion subgroup of $K_0(\mathcal{O}G)$. The kernel subgroup $D(\mathcal{O}G)$ of $\text{Cl}(\mathcal{O}G)$ is the kernel of the homomorphism $\text{Cl}(\mathcal{O}G) \rightarrow \text{Cl}(\mathcal{M})$ induced by tensoring projective $\mathcal{O}G$ -modules with a maximal \mathcal{O} -order \mathcal{M} in FG which contains \mathcal{O} . By [5, p. 24], $D(\mathcal{O}G)$ is independent of the choice of \mathcal{M} .

Let \mathcal{G} be one of the abelian groups $K_0(\mathcal{O}G)$, $\text{Cl}(\mathcal{O}G)$, $D(\mathcal{O}G)$, or $G_0(\mathcal{O}G)_{\text{tor}}$. Let $G_0^{\mathcal{O}}(\mathcal{O}G)$ be the Grothendieck group of all finitely generated $\mathcal{O}G$ -lattices. Via the tensor product of $\mathcal{O}G$ -modules over \mathcal{O} , $G_0^{\mathcal{O}}(\mathcal{O}G)$ is a ring, and \mathcal{G} becomes a module for this ring. We also have an action of the group $\text{Aut}_n(\mathcal{O}G)$ of normalized \mathcal{O} -algebra automorphisms of $\mathcal{O}G$ on \mathcal{G} via the twisting of $\mathcal{O}G$ -modules. The main result of this section is that these two actions on \mathcal{G} are compatible, in the following sense.

DEFINITION 3.1. Suppose R is a ring and N is an R -module. If σ is an automorphism of the additive groups underlying R and N , then σ is compatible with the R -module structure of N if $\sigma(rn) = \sigma(r)\sigma(n)$ for all $r \in R$ and $n \in N$.

THEOREM 3.2. Suppose \mathcal{G} denotes $K_0(\mathcal{O}G)$, $\text{Cl}(\mathcal{O}G)$, $D(\mathcal{O}G)$, or $G_0(\mathcal{O}G)_{\text{tor}}$. The action of each element of $\text{Aut}_n(\mathcal{O}G)$ is compatible with the structure of \mathcal{G} as a module for $G_0^{\mathcal{O}}(\mathcal{O}G)$.

This result is not immediate from the definition of the twisting of modules by $\sigma \in \text{Aut}_n(\mathcal{O}G)$. Suppose, for example, that T_1 and T_2 are

$\mathcal{O}G$ -modules. While there is a natural identification of the underlying abelian groups of $T_1^\sigma \otimes_{\mathcal{O}} T_2^\sigma$ and $(T_1 \otimes_{\mathcal{O}} T_2)^\sigma$, this identification need not respect the action of $\mathcal{O}G$ on these twists.

Conjecture 3.3. The elements of $\text{Aut}_n(\mathcal{O}G)$ induce ring automorphisms of $G_0^\mathcal{O}(\mathcal{O}G)$.

Remark 3.4. It is known [2, Proposition 2.1] that the elements of $\text{Aut}_n(\mathcal{O}G)$ induce ring automorphisms of $G_0(FG)$. Conjecture 3.3 is stronger than this statement, and consistent with Theorem 3.2, in the following way. The forgetful homomorphism $G_0^\mathcal{O}(\mathcal{O}G) \rightarrow G_0(\mathcal{O}G)$ is an isomorphism by [4, Theorem 38.42]. By [4, Theorem 39.14], tensoring $\mathcal{O}G$ -modules with F over \mathcal{O} gives an exact sequence

$$0 \rightarrow G_0(\mathcal{O}G)_{\text{tor}} \rightarrow G_0^\mathcal{O}(\mathcal{O}G) \xrightarrow{\delta} G_0(FG) \rightarrow 0 \quad (3.1)$$

in which δ is a ring homomorphism. Thus $G_0(\mathcal{O}G)_{\text{tor}}$ is an ideal of $G_0^\mathcal{O}(\mathcal{O}G)$. Theorem 3.2 shows the action of each element of $\text{Aut}_n(\mathcal{O}G)$ is compatible with the structure of $G_0(\mathcal{O}G)_{\text{tor}}$ as a module for $G_0^\mathcal{O}(\mathcal{O}G)$. Note that since the square of the ideal $G_0(\mathcal{O}G)_{\text{tor}}$ in $G_0^\mathcal{O}(\mathcal{O}G)$ is 0 by [4, Theorem 39.16], the action of $G_0^\mathcal{O}(\mathcal{O}G)$ on $G_0(\mathcal{O}G)_{\text{tor}}$ factors through an action of $G_0(FG)$.

The following lemma will lead to a first reduction step (Corollary 3.6) in the proof of Theorem 3.2.

LEMMA 3.5. *Suppose L is an $\mathcal{O}G$ -lattice and $\sigma \in \text{Aut}_n(\mathcal{O}G)$. Then $(L \otimes_{\mathcal{O}} \mathcal{O}G)^\sigma$ is isomorphic to $L^\sigma \otimes_{\mathcal{O}} (\mathcal{O}G)^\sigma$.*

Proof. Let L_0 be the underlying \mathcal{O} -module associated to L , with trivial G action. In the following we will identify the elements of L_0 with those of L . We have an isomorphism of $\mathcal{O}G$ -modules $L_0 \otimes_{\mathcal{O}} \mathcal{O}G \rightarrow L \otimes_{\mathcal{O}} \mathcal{O}G$ defined by $l_0 \otimes g \rightarrow gl_0 \otimes g$ for $g \in G$ and $l_0 \in L_0$. Since σ is a normalized automorphism, $(L^\sigma)_0$ is isomorphic to L_0 . Because $(\mathcal{O}G)^\sigma$ is a free rank one $\mathcal{O}G$ -module, it must be isomorphic to $\mathcal{O}G$. Thus to prove Lemma 3.5, it will suffice to show $(L_0 \otimes_{\mathcal{O}} \mathcal{O}G)^\sigma$ is isomorphic to $L_0 \otimes_{\mathcal{O}} \mathcal{O}G$. Let us first show that there is a unique \mathcal{O} -linear isomorphism $\psi: (L_0 \otimes_{\mathcal{O}} \mathcal{O}G)^\sigma \rightarrow L_0 \otimes_{\mathcal{O}} \mathcal{O}G$ such that

$$\psi(l_0 \otimes \beta) = l_0 \otimes \sigma^{-1}(\beta)$$

for all $l_0 \in L_0$ and $\beta \in \mathcal{O}G$, where on the left side $l_0 \otimes \beta$ is identified with an element of the underlying abelian group of $(L_0 \otimes_{\mathcal{O}} \mathcal{O}G)^\sigma$. The existence and uniqueness of such a ψ is clear if L_0 is a free \mathcal{O} -module. In general, L_0 is projective as an \mathcal{O} -module because L is an $\mathcal{O}G$ -lattice. Since σ respects direct sums, we can deduce the existence and uniqueness

of ψ for arbitrary L_0 from the case in which L_0 is free. Suppose $h \in G$ and $\sigma(h) = \sum_{\tau \in G} a_\tau \tau$. Then

$$\begin{aligned} \psi(h \cdot_\sigma (l_0 \otimes g)) &= \psi(\sigma(h) \cdot (l_0 \otimes g)) = \psi\left(\sum_{\tau} a_\tau (\tau l_0 \otimes \tau g)\right) \\ &= \psi\left(l_0 \otimes \sum_{\tau} a_\tau \tau g\right) = l_0 \otimes \sigma^{-1}(\sigma(h)g) \\ &= h l_0 \otimes h \sigma^{-1}(g) = h \cdot \psi(l_0 \otimes g) \end{aligned} \quad (3.2)$$

because σ and σ^{-1} are ring automorphisms of $\mathcal{O}G$ and L_0 has trivial G -action, so Lemma 3.5 follows. ■

COROLLARY 3.6. *To prove Theorem 3.2 for $\mathcal{G} = K_0(\mathcal{O}G)$, it will suffice to prove it for $\mathcal{G} = \text{Cl}(\mathcal{O}G)$.*

Proof. This follows directly from Lemma 3.5 and the fact (cf. [12]) that each element of $K_0(\mathcal{O}G)$ can be written in the form $a \cdot [\mathcal{O}G] + c$ for some $a \in \mathbb{Z}$ and some class $c \in \text{Cl}(\mathcal{O}G)$. ■

To continue the proof of Theorem 3.2 we now recall the Fröhlich Hom descriptions of $\text{Cl}(\mathcal{O}G)$, $\text{D}(\mathcal{O}G)$, and $G_0(\mathcal{O}G)_{\text{tor}}$.

Let \bar{F} be an algebraic closure of F , and define $\Omega_F = \text{Gal}(\bar{F}/F)$. Let $J(\bar{F})$ be the ideles of \bar{F} . Let R_G be the ring $K_0(\bar{F}G)$, i.e., the absolute character ring of G . Let v be a (finite or infinite) place of F . Define

$$(\bar{F})_v^* = \varinjlim_L (L \otimes_F F_v)^*,$$

where the limit is over the finite extensions of F in \bar{F} . Then in a natural way we can view $(\bar{F})_v^*$ as a subgroup of $J(\bar{F})$. If v is finite, let \mathcal{O}_v be the integers of F_v , and let $(\bar{\mathcal{O}})_v^*$ be the multiplicative subgroup of elements of $(\bar{F})_v^*$ which are integral over \mathcal{O}_v . Thus $(\bar{\mathcal{O}})_v^*$ is the group of integral units of $(\bar{F})_v^*$. If v is infinite, let $\mathcal{O}_v = F_v$ and $(\bar{\mathcal{O}})_v^* = (\bar{F})_v^*$. Define $U(\bar{\mathcal{O}}) = \prod_v (\bar{\mathcal{O}})_v^*$ where the product is over all places v of F .

Suppose v is a place of F , and that $u_v \in F_v G$. If $\chi \in R_G$ is the character of an irreducible representation of G , let $\det_\chi(u_v) \in (\bar{F})_v^*$ be the determinant of the image of u_v under a matrix representation of χ over $\bar{\mathbb{Q}}$. One then defines $\det_\chi(u_v)$ for all $\chi \in R_G$ by requiring this function to be multiplicative in χ . By [5, p. 19], the function $\text{Det}(u_v): \chi \rightarrow \det_\chi(u_v)$ lies in $\text{Hom}_{\Omega_F}(R_G, (\bar{F})_v^*)$. Let $\text{Det}((\mathcal{O}_v G)^*)$ be the subgroup $\{\text{Det}(u_v): u_v \in (\mathcal{O}_v G)^*\}$ of $\text{Hom}_{\Omega_F}(R_G, (\bar{F})_v^*)$. We define the unit ideles of $\mathcal{O}G$ to be $U(\mathcal{O}G) = \prod_v (\mathcal{O}_v G)^*$, where the product is over all places v of F . Let $\text{Det}(U(\mathcal{O}G)) = \prod_v \text{Det}((\mathcal{O}_v G)^*) \subset \text{Hom}_{\Omega_F}(R_G, J(\bar{F}))$.

For finite v , an element of G is called v -singular if its order is divisible by the residue characteristic of v . If v is finite, let H_v be the multiplicative group of $f \in \text{Hom}_{\Omega_F}(R_G, (\bar{F})_v^*)$ such that $f(\chi) \in (\mathcal{O})_v^*$ for all $\chi \in R_G$ which vanish on the v -singular elements of G . For v infinite, let H_v be the group of $f \in \text{Hom}_{\Omega_F}(R_G, (\bar{F})_v^*)$ which are totally positive on the symplectic characters of G .

Let $\beta = \prod \beta_v$ be an idele of FG . Thus $\beta_v \in (F_v G)^*$ for all v , and $\beta_v \in (\mathcal{O}_v G)^*$ for all but finitely many finite v . We can define a locally free rank 1 module $(\mathcal{O}G)_\beta \subset FG$ for $\mathcal{O}G$ by

$$(\mathcal{O}G)_\beta = \bigcap_{v \text{ finite}} (FG \cap \mathcal{O}G\beta_v). \quad (3.3)$$

Let c_β (resp. d_β) be the class $[(\mathcal{O}G)_\beta] - [\mathcal{O}G]$ in $K_0(\mathcal{O}G)$ (resp. $G_0(\mathcal{O}G)$). Define

$$f_\beta \in \text{Hom}_{\Omega_F}(R_G, J(\bar{F}))$$

to be the function which sends $\tau \in R_G$ to the idele $\prod_v \det_\tau(\beta_v)$. Let $J(FG)$ be the group of ideles of FG , and let $J_f(FG)$ be the subgroup of ideles β such that $\beta_v = 1$ for all infinite places v of G .

THEOREM 3.7 (Fröhlich [5, pp. 22–23]). *As β runs over $J_f(FG)$, c_β runs over all classes in $\text{Cl}(\mathcal{O}G)$. There is a unique isomorphism*

$$\psi_C: \text{Cl}(\mathcal{O}G) \rightarrow \frac{\text{Hom}_{\Omega_F}(R_G, J(\bar{F}))}{\text{Hom}_{\Omega_F}(R_G, \bar{F}^*) \cdot \text{Det}(U(\mathcal{O}G))}$$

for which $\psi_C(c_\beta)$ is the class of f_β for all $\beta \in J_f(FG)$. This induces an isomorphism

$$\psi_D: D(\mathcal{O}G) \rightarrow \frac{\text{Hom}_{\Omega_F}(R_G, U(\bar{\mathcal{O}}))}{\text{Hom}_{\Omega_F}(R_G, \bar{F}^*) \cdot \text{Det}(U(\mathcal{O}G))}.$$

THEOREM 3.8 (Queyrut [7]). *As β runs over $J_f(FG)$, d_β runs over all classes in $G_0(\mathcal{O}G)_{\text{tor}}$. There is a unique isomorphism*

$$\psi_G: G_0(\mathcal{O}G)_{\text{tor}} \rightarrow \frac{\text{Hom}_{\Omega_F}(R_G, J(\bar{F}))}{\text{Hom}_{\Omega_F}(R_G, \bar{F}^*) \cdot \prod_v H_v}$$

for which $\psi_G(d_\beta)$ is the class of f_β .

Proof. Define S to be a finite set of finite places of F . In [7, Definition 1.1], Queyrut lets $\mathcal{C}(\mathcal{O}G)$ be the category of all finitely generated $\mathcal{O}G$

modules which are torsion-free as \mathcal{O} -modules. He lets $\mathcal{Z}(\mathcal{E}(\mathcal{O}G))$ be the free abelian group on the isomorphism classes of objects in $\mathcal{E}(\mathcal{O}G)$. He defines $\mathcal{Z}_S(\mathcal{E}(\mathcal{O}G))$ to be the subgroup of $\mathcal{Z}(\mathcal{E}(\mathcal{O}G))$ generated by classes of the form $(M) - (M') - (M'')$, where M , M' , and M'' are finitely generated $\mathcal{O}G$ -modules which are torsion free as \mathcal{O} -modules and lie in an exact sequence

$$0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$$

such that the localized sequence

$$0 \rightarrow M_{\mathcal{P}} \rightarrow M'_{\mathcal{P}} \rightarrow M''_{\mathcal{P}} \rightarrow 0$$

is split for all prime ideals \mathcal{P} of \mathcal{O} which are not in S . (The latter condition is called “splitting outside S .”) On [7, p. 236] Queyruat defines $\mathcal{Z}_{\oplus}^S(\mathcal{O}G)$ to be $\mathcal{Z}(\mathcal{E}(\mathcal{O}G))/\mathcal{Z}_S(\mathcal{E}(\mathcal{O}G))$. Let $S(G)$ be the set of places of F over the prime divisors of $\#G$. If \mathcal{P} is a prime of \mathcal{O} not in $S(G)$, then \mathcal{P} has residue characteristic prime to $\#G$. Hence by [11, Proposition 43], each finitely generated $(\mathcal{O})_{\mathcal{P}}G$ -module which has no $\mathcal{O}_{\mathcal{P}}$ -torsion is projective (since it is free over the discrete valuation ring $\mathcal{O}_{\mathcal{P}}$). Thus exact sequences of finitely generated torsion free $\mathcal{O}G$ -modules automatically split outside $S(G)$. Therefore $\mathcal{Z}_{\oplus}^{S(G)}(\mathcal{O}G)$ is naturally isomorphic to the Grothendieck group $G_0^{\mathcal{O}}(\mathcal{O}G)$ of $\mathcal{O}G$ -lattices with respect to exact sequences. It is clear from the proof of [7, Corollary 1.6] that Queyruat’s group $\tilde{\mathcal{Z}}_{\oplus}^S(\mathcal{O}G)$ is the kernel of the natural map

$$\mathcal{Z}_{\oplus}^S(\mathcal{O}G) \rightarrow G_0(FG) = K_0(FG)$$

which results from tensoring modules with F over \mathcal{O} . Thus by Eq. (3.1) of Remark 3.4, Queyruat’s $\tilde{\mathcal{Z}}_{\oplus}^{S(G)}(\mathcal{O}G)$ is equal to $G_0(\mathcal{O}G)_{\text{tor}}$. Hence [7, Theorems 1.13, 4.1, 6.1] show Theorem 3.8. (In fact, Queyruat chooses to identify f_{β} with the class $-c_{\beta}$, so the isomorphism we want is the composition of Queyruat’s isomorphism with multiplication by -1 .) ■

LEMMA 3.9. *Let M be an $\mathcal{O}G$ -lattice with character χ . The classes $[M] \cdot c_{\beta} \in \text{Cl}(\mathcal{O}G)$ and $[M] \cdot d_{\beta} \in G_0(\mathcal{O}G)_{\text{tor}}$ are defined to be*

$$[M \otimes_{\mathcal{O}} (\mathcal{O}G)_{\beta}] - [M \otimes_{\mathcal{O}} \mathcal{O}G]$$

in $\text{Cl}(\mathcal{O}G)$ and $G_0(\mathcal{O}G)_{\text{tor}}$, respectively. The action of $G_0^{\mathcal{O}}(\mathcal{O}G)$ and $\text{Cl}(\mathcal{O}G)$ and on $G_0(\mathcal{O}G)_{\text{tor}}$ factors through the surjective homomorphism $\delta: G_0^{\mathcal{O}}(\mathcal{O}G) \rightarrow G_0(FG)$ induced by tensoring with F over \mathcal{O} . Hence we can define $\chi \cdot c_{\beta} = [M] \cdot c_{\beta}$ and $\chi \cdot d_{\beta} = [M] \cdot d_{\beta}$. Define $\chi \cdot f_{\beta} \in \text{Hom}_{\Omega_F}(R_G, J(\bar{F}))$ by

$$(\chi \cdot f_{\beta})(\tau) = f_{\beta}(\chi \cdot \tau) \quad (3.4)$$

for all $\tau \in R_G$. Then $\chi \cdot f_\beta$ represents $\chi \cdot c_\beta$ and $\chi \cdot d_\beta$ relative to the Hom descriptions in Theorems 3.7 and 3.8.

Proof. We will prove the assertions in this lemma concerning the action of $G_0^\mathcal{C}(\mathcal{O}G)$ on $\text{Cl}(\mathcal{O}G)$; the assertions concerning $G_0(\mathcal{O}G)_{\text{tor}}$ are proved similarly. In [13; 5, p. 65], Ullom and Fröhlich work with right modules rather than left ones. We will identify a right G -module M_r with the left G -module M_l having the same underlying abelian group and G action defined by $g \cdot m = m \cdot g^{-1}$ for $g \in G$ and $m \in M_r = M_l$. This converts a right G module M_r with character χ into a left G -module M_l with character the dual $\bar{\chi}$ of χ . We now get (3.4) from formula 3.6 on [5, p. 65], which also shows the action of $G_0^\mathcal{C}(\mathcal{O}G)$ on $\text{Cl}(\mathcal{O}G)$ factors through δ . ■

Suppose $\sigma \in \text{Aut}_n(\mathcal{O}G)$. Extend $\tau \in R_G$ by linearity to an F -linear function $\tau: FG \rightarrow \bar{F}$. The twist τ^σ is defined by $\tau^\sigma(g) = \tau(\sigma(g))$ for $g \in G$, where $\sigma(g) \in \mathcal{O}G$. For $c \in \text{Cl}(\mathcal{O}G)$ (resp. $d \in G_0(\mathcal{O}G)_{\text{tor}}$), let c^σ (resp. d^σ) be the class resulting by twisting c (resp. d) by σ . We will also extend σ first to an F -algebra automorphism of FG , and then to an automorphism of the adèles of FG which fixes F and the places of F .

LEMMA 3.10. *The map $m \mapsto \sigma^{-1}(m)$ induces an $\mathcal{O}G$ -module isomorphism between the σ -twist of $(\mathcal{O}G)_\beta$ and $(\mathcal{O}G)_{\sigma^{-1}(\beta)}$.*

Proof. The underlying abelian group of the σ -twist of $(\mathcal{O}G)_\beta$ is $(\mathcal{O}G)_\beta$ and the G action is defined by $\xi \cdot_\sigma m = \sigma(\xi) \cdot m$ for $\xi \in \mathcal{O}G$ and $m \in (\mathcal{O}G)_\beta$. From (3.3) we see that σ^{-1} carries the additive group $(\mathcal{O}G)_\beta$ isomorphically to $(\mathcal{O}G)_{\sigma^{-1}(\beta)}$. So we just have to check that

$$\sigma^{-1}(\xi \cdot_\sigma m) = \sigma^{-1}(\sigma(\xi) \cdot m) = \xi \cdot \sigma^{-1}(m)$$

and this is true because σ induces an F -algebra automorphism of FG . ■

Proof of Theorem 3.2. We will show that the action of each element of $\text{Aut}_n(\mathcal{O}G)$ is compatible with the structure of $\mathcal{Z} = \text{Cl}(\mathcal{O}G)$ as a module for $G_0^\mathcal{C}(\mathcal{O}G)$. The proof when $\mathcal{Z} = G_0(\mathcal{O}G)_{\text{tor}}$ or $\mathcal{Z} = D(\mathcal{O}G)$ is similar, using Theorem 3.8 and the second equality in Theorem 3.7, respectively. By Corollary 3.6 the case $\mathcal{Z} = K_0(\mathcal{O}G)$ will follow from the case $\mathcal{Z} = \text{Cl}(\mathcal{O}G)$.

By Theorem 3.7, each class in $\text{Cl}(\mathcal{O}G)$ has the form c_β for some idele $\beta \in J_f(FG)$. By Lemma 3.10, the twist c_β^σ of c_β is the class

$$[\mathcal{O}G]_{\sigma^{-1}(\beta)} - [\mathcal{O}G].$$

From the formula $\tau^\sigma(g) = \tau(\sigma(g))$ we find $\det_\tau(\sigma^{-1}(\beta)) = \det_{\tau \circ \sigma^{-1}}(\beta)$. Therefore c_β^σ is represented by the character function f_β^σ defined by

$$f_\beta^\sigma(\tau) = f_\beta(\tau^{\sigma^{-1}}) \quad (3.5)$$

for all $\tau \in R_G$. This formula is consistent with the way σ acts on $\text{Hom}_{\Omega_F}(R_G, J(\bar{F}))$ given that σ acts trivially on $J(\bar{F})$.

To prove Theorem 3.2 when $\mathcal{G} = \text{Cl}(\mathcal{O}G)$, we must show

$$(\chi \cdot c_\beta)^\sigma = \chi^\sigma \cdot c_\beta^\sigma, \quad (3.6)$$

when χ is the character of an $\mathcal{O}G$ -lattice. By Lemma 3.9 and (3.5), the left hand side of (3.6) is represented by the character function which sends $\tau \in R_G$ to

$$(\chi \cdot f_\beta)^\sigma(\tau) = (\chi \cdot f_\beta)(\tau^{\sigma^{-1}}) = f_\beta(\chi \cdot \tau^{\sigma^{-1}}). \quad (3.7)$$

The right hand side of (3.6) is represented by the character function which sends τ to

$$(\chi^\sigma \cdot f_\beta^\sigma)(\tau) = f_\beta^\sigma(\chi^\sigma \cdot \tau) = f_\beta((\chi^\sigma \cdot \tau)^{\sigma^{-1}}) = f_\beta(\chi \cdot \tau^{\sigma^{-1}}). \quad (3.8)$$

Since (3.7) equals (3.8), this completes the proof of (3.6) and of Theorem 3.2. ■

REFERENCES

1. D. J. Benson, "Representations and Cohomology, I," Cambridge Studies in Advanced Mathematics, Vol. 30, Cambridge Univ. Press, Cambridge, UK, 1991.
2. F. M. Bleher, Tensor products and a conjecture of Zassenhaus, *Arch. Math.* **64** (1995), 289–298.
3. F. M. Bleher, G. Hiss, and W. Kimmerle, Autoequivalences of blocks and a conjecture of Zassenhaus, *J. Pure Appl. Algebra* **103** (1995), 23–43.
4. C. W. Curtis and I. Reiner, "Methods of Representation Theory," Vols. I and II, Wiley, New York, 1981, 1987.
5. A. Fröhlich, "Galois Module Structure of Algebraic Integers," *Ergeb. Math. Grenzgeb.* (3), Vol. 1, Springer-Verlag, New York/Berlin, 1983.
6. W. Fulton and S. Lang, "Riemann–Roch Algebra," *Grundlehren Math. Wiss.*, Vol. 277, Springer-Verlag, New York/Berlin, 1985.
7. J. Queyrut, S-groupes des classes d'un ordre arithmétique, *J. Algebra* **76** (1982), 234–260.
8. K. W. Roggenkamp, The isomorphism problem for integral group rings of finite groups, in "Proc. Int. Cong. Math., Kyoto, 1990," pp. 369–380, Springer-Verlag, New York/Berlin, 1991.
9. K. W. Roggenkamp and M. J. Taylor, "Group Rings and Class Groups," Birkhäuser, Basel, 1992.

10. S. K. Sehgal, "Units in Integral Group Rings, with an Appendix by Al Weiss," Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 69, Longman, Harlow, 1993.
11. J. P. Serre, "Linear Representations of Finite Groups," Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, New York/Berlin, 1977.
12. R. G. Swan, Induced representations and projective modules, *Ann. of Math. (3)* **71** (1960), 552–578.
13. S. V. Ullom, Character action on the class group of Fröhlich, in "Algebraic K-Theory, Part II, Oberwolfach, 1980," Lecture Notes in Math., Vol. 967, pp. 371–375, Springer-Verlag, New York/Berlin, 1982.