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Kazhdan–Lusztig basis, Wedderburn decomposition, and Lusztig's homomorphism for Iwahori–Hecke algebras

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Abstract

Let (W, S) be a finite Coxeter system and $A := \mathbb{Z}[\Gamma]$ be the group algebra of a finitely generated free abelian group Γ. Let $\mathcal{H}$ be an Iwahori–Hecke algebra of (W, S) over A with parameters v_s. Further let K be an extension field of the field of fractions of A and $K\mathcal{H}$ be the extension of scalars. In this situation Kazhdan and Lusztig have defined their famous basis and the so-called left cell modules.

In this paper, using the Kazhdan–Lusztig basis and its dual basis, formulae for a K-basis are derived that gives a direct sum decomposition of the right regular $K\mathcal{H}$-module into right ideals each being isomorphic to the dual module of a left cell module. For those left cells, for which the corresponding left cell module is a simple $K\mathcal{H}$-module, this gives explicit formulae for basis elements belonging to a Wedderburn basis of $K\mathcal{H}$. For the other left cells, similar relations are derived.

These results in turn are used to find preimages of the standard basis elements t_z of Lusztig's asymptotic algebra $\mathcal{J}$ under the Lusztig homomorphism from $\mathcal{H}$ into the asymptotic algebra $\mathcal{J}$. Again for those left cells, for which the corresponding left cell module is simple, explicit formulae for the preimages are given.

These results shed a new light onto Lusztig's homomorphism interpreting it as an inclusion of $\mathcal{H}$ into an A-subalgebra $\mathcal{L}$ of $K\mathcal{H}$. In the case that all left cell modules are simple (like for example in type A), $\mathcal{L}$ is isomorphic to a direct sum of full matrix rings over A.

Keywords: Coxeter group; Iwahori–Hecke algebra; Kazhdan–Lusztig basis; Asymptotic algebra; Lusztig homomorphism; Wedderburn decomposition

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1. Introduction

For the situation described in the abstract Kazhdan and Lusztig have constructed their famous basis $\{C_x \mid x \in W\}$, where W is the Weyl group. Using that $\mathcal{H}$ is a symmetric algebra they also defined the dual basis $\{D_{y^{-1}} \mid y \in W\}$. Later in [1] Lusztig defined the asymptotic algebra $\mathcal{J}$ together with the Lusztig homomorphism $\phi : \mathcal{H} \to \mathcal{J}$. All these objects have been used successfully to study the representation theory of $\mathcal{H}$.

In the present paper, we build on these results and obtain two theorems and a few corollaries, that help to explain the connections to the representation theory of $\mathcal{H}$.

After some preparations in Sections 2 to 4 we show in Section 5 that for a left cell Λ, for which the left cell module is simple, the elements $\{c^{-1} C_x D_{y^{-1}} \mid x, y \in \Lambda\}$ where c is some constant depending only on Λ fulfill the relations of the standard basis of a full matrix ring over A and span the isotypic component of $K\mathcal{H}$ corresponding to the simple left cell module.

For the case that all left cell modules are simple, this explicitly yields a Wedderburn basis of $K\mathcal{H}$, giving an explicit isomorphism of K-algebras between $K\mathcal{H}$ and a direct sum of full matrix rings over K. For the case of non-simple left cell modules the situation is not so good. However, the rest of Section 5 exhibits, how much of the above results can still be shown. For example, the right regular $K\mathcal{H}$-module is still the direct sum of (explicitly given) right ideals each of which is isomorphic to the dual of a left cell module.

Moreover, these results are used in Section 6 to construct preimages of the standard basis $(t_z)_{z \in W}$ of the asymptotic algebra $\mathcal{J}$ under the Lusztig homomorphism. This allows to interpret the Lusztig homomorphism $\phi : \mathcal{H} \to \mathcal{J}$ as the inclusion of $\mathcal{H}$ into an A-subalgebra of $K\mathcal{H}$.

2. Notation

The basic reference for the setting is [2]. For the base ring we use a slightly more general notation inspired by [3].

Let W be a finite Coxeter group with generating set S, and Γ a totally-ordered abelian group, written additively. Let $A := \mathbb{Z}[\Gamma]$ be the group algebra, written exponentially, that is, as a $\mathbb{Z}$-module, it is free with basis $\{v^\gamma \mid \gamma \in \Gamma\}$ and basis elements are multiplied by the rule $v^\gamma \cdot v^{\gamma'} := v^{\gamma+\gamma'}$. We denote by $\Gamma_{>0}$ the set of elements of Γ that are greater than 0 and set $A_{<0} := \langle v^\gamma \mid \gamma \in \Gamma_{<0}\rangle_{\mathbb{Z}}$. We define $\Gamma_{<0}$, $\Gamma_{\geqslant 0}$, $\Gamma_{\leqslant 0}$, $A_{>0}$, $A_{\geqslant 0}$, and $A_{\leqslant 0}$ analogously. For elements $a = \sum_{\gamma \in \Gamma} a_\gamma \cdot v^\gamma \in A$ we define the degree by $\deg a := \max\{\gamma \in \Gamma \mid a_\gamma \neq 0\}$.

Further, let $L : W \to \Gamma$ be a weight function in the sense of Lusztig (see [2, 3.1]), that is, we have $L(ww') = L(w) + L(w')$ whenever $\ell(ww') = \ell(w) + \ell(w')$ for $w, w' \in W$, where ℓ denotes the usual length function on W. We shall assume $L(s) > 0$ for $s \in S$.

Let $v_w := v^{L(w)}$ for $w \in W$. We can now define the Iwahori–Hecke algebra $\mathcal{H}$ over A with parameters v_s to be the associative A-algebra with generators $\{T_w \mid w \in W\}$ subject to the relations

$$T_s^2 = T_{\mathrm{id}} + \left(v_s - v_s^{-1}\right) T_s \quad \text{for all } s \in S,$$

$$T_w = T_{s_1} \cdot \cdots \cdot T_{s_k} \quad \text{for every reduced expression } w = s_1 \cdot \cdots \cdot s_k \text{ in } W \text{ with } s_i \in S,$$

where $\mathrm{id} \in W$ denotes the identity element. Note that we use here a slightly more general base ring than in [2], however, all proofs go through without change. In addition, our setup here has already been used in [4], [5], and [3].

The algebra $\mathcal{H}$ is free as an A-module with basis $\{T_w \mid w \in W\}$ (see [2, 3.3]) and has a symmetrizing trace map $\tau : \mathcal{H} \to A$, $T_{\mathrm{id}} \mapsto 1$, $T_w \mapsto 0$ for $\mathrm{id} \neq w \in W$, which makes $\mathcal{H}$ into a symmetric algebra in the sense of [5, 7.1.1]. Note that this means in particular that $\tau(h \cdot h') = \tau(h' \cdot h)$ for all $h, h' \in \mathcal{H}$. The dual basis of $\{T_w \mid w \in W\}$ with respect to τ is $\{T_w^\vee \mid w \in W\}$ with $T_w^\vee = T_{w^{-1}}$ (for all of this, see [2, 10.3,10.4]).

Let $\tilde{K} \subseteq \mathbb{C}$ be a splitting field for W. Then $K := \tilde{K}(v_s \mid s \in S)$ contains A as a subring and is a splitting field for the extension of scalars $K\mathcal{H} := K \otimes_A \mathcal{H}$. This follows from [5, 9.3.5] and the fact that our parameters v_s are square roots of the parameters u_s there. In addition, all irreducible characters of $K\mathcal{H}$ can be realized over K.

Note that here and in the sequel we mean by a character of $\mathcal{H}$ an A-linear form on $\mathcal{H}$ with values in K that comes from evaluating the trace of representing matrices of elements of $\mathcal{H}$. We denote the set of irreducible characters of $K\mathcal{H}$ by $\mathrm{Irr}(K\mathcal{H})$.

3. Kazhdan–Lusztig basis and cells

In this section we briefly repeat the definition of the famous Kazhdan–Lusztig basis and cells and collect a few results for further reference.

We denote by $\bar{\ }: A \to A, a \mapsto \bar{a}$ the ring automorphism mapping v^γ to $v^{-\gamma}$ for $\gamma \in \Gamma$. This involution can be extended to an involution of $\mathcal{H}$ by setting

$$\overline{\sum_{w \in W} a_w \cdot T_w} := \sum_{w \in W} \overline{a_w} \cdot T_{w^{-1}}^{-1}.$$

The involution $\bar{\ }: \mathcal{H} \to \mathcal{H}$ is in fact a ring homomorphism (see [2, Chapter 4]), but of course not a homomorphism of A-algebras, as it is only A-semilinear with respect to $\bar{\ }$ and not A-linear.

The algebra $\mathcal{H}$ has an A-basis $\{C_w \mid w \in W\}$, defined by Kazhdan and Lusztig in [6, §1] by the two properties that

$$\overline{C_w} = C_w \quad \text{and} \quad C_w = \sum_{y \leqslant w} p_{y,w} \cdot T_y$$

for all $w \in W$, where $p_{y,w} \in A_{<0}$ for $y < w$ and $p_{w,w} = 1$. Here and in the following, "$\leqslant$" for elements of W denotes the Bruhat–Chevalley order. For a proof of existence and uniqueness see [2, 5.2]. Note that our elements C_w are denoted by c_w in [2] and by C'_w in [4–6].

For any $y, w \in W$ and $s \in S$ with $sy < y < w < sw$ we define $\mu^s_{y,w}$ as in [2, 6.3] by the requirements that

$$\bar{\mu}^s_{y,w} = \mu^s_{y,w} \quad \text{and} \quad \sum_{\substack{z \in W \\ y \leqslant z < w;\, sz < z}} p_{y,z} \cdot \mu^s_{z,w} - v_s \cdot p_{y,w} \in A_{<0}$$

for all such y, w, s. Then we have (see [2, 6.6 and 6.7]):

$$C_s \cdot C_w = \begin{cases} (v_s + v_s^{-1}) \cdot C_w & \text{for } sw < w, \\ C_{sw} + \sum\limits_{\substack{y \in W \\ sy < y < w}} \mu^s_{y,w} \cdot C_y & \text{for } sw > w, \end{cases} \tag{1}$$

and

$$C_w \cdot C_s = \begin{cases} (v_s + v_s^{-1}) \cdot C_w & \text{for } ws < w, \\ C_{ws} + \sum\limits_{\substack{y \in W \\ ys < y < w}} \mu^s_{y^{-1},w^{-1}} \cdot C_y & \text{for } ws > w. \end{cases} \tag{2}$$

We denote the structure constants of $\mathcal{H}$ with respect to the basis $\{C_w \mid w \in W\}$ by $h_{x,y,z}$, that is, we have

$$C_x \cdot C_y =: \sum_{z \in W} h_{x,y,z} \cdot C_z \quad \text{for all } x, y \in W.$$

Thus, Eq. (1) means, that we have

$$h_{s,w,y} = \begin{cases} \mu^s_{y,w} & \text{if } sy < y < w < sw, \\ 1 & \text{if } y = sw > w, \\ v_s + v_s^{-1} & \text{if } y = w > sw, \\ 0 & \text{otherwise.} \end{cases}$$

The following definitions are from [2, 8.1]. We define $y \preccurlyeq_L w$ to mean: either $y = w$ or there is an $s \in S$ such that the coefficient of C_y in the expansion of $C_s \cdot C_w$ in the C-basis in Eq. (1) is non-zero. Let $\leqslant_L$ be the transitive closure of the relation $\preccurlyeq_L$ and denote by $\sim_L$ the associated equivalence relation on W. The equivalence classes are called "left cells" and $\leqslant_L$ induces a partial order on the set of left cells.

We define $y \leqslant_R w$ analogously and denote the equivalence relation induced by $\leqslant_R$ with $\sim_R$ and call the equivalence classes "right cells." For $y, w \in W$ we write $y \leqslant_{LR} w$ if there is a sequence $y = y_0, y_1, \ldots, y_n = w$ of elements of W, such that, for $i = 0, 1, \ldots, n-1$ we have $y_i \leqslant_L y_{i+1}$ or $y_i \leqslant_R y_{i+1}$. We denote the equivalence relation on W corresponding to the transitive relation $\leqslant_{LR}$ on W by $\sim_{LR}$ and call the equivalence classes "two-sided cells."

Note that $y \leqslant_R w$ is equivalent to $y^{-1} \leqslant_L w^{-1}$ (see [2, 8.1]).

If $\Omega \subseteq W$ is a disjoint union of left cells, we call Ω "complete," if for every left cell $\Lambda \subseteq \Omega$ and for every left cell $\Lambda' \leqslant_L \Lambda$ we have $\Lambda' \subseteq \Omega$.

The set $\{C_s \mid s \in S\}$ generates $\mathcal{H}$ as an A-algebra. Thus the definition of $\leqslant_L$ shows that if Ω is a complete, disjoint union of left cells, the A-linear span $\langle C_w \mid w \in \Omega \rangle_A$ is a left ideal in $\mathcal{H}$ and thus a left $\mathcal{H}$-module. We call such an ideal a "left cell module" and denote it by $\mathrm{LC}^{(\Omega)}$.

If $\Sigma \subseteq \Omega$ are both complete, then $\mathrm{LC}^{(\Sigma)}$ is an A-pure $\mathcal{H}$-submodule of $\mathrm{LC}^{(\Omega)}$ and we denote the quotient module by $\mathrm{LC}^{(\Omega \setminus \Sigma)} := \mathrm{LC}^{(\Omega)} / \mathrm{LC}^{(\Sigma)}$. In particular, this definition covers the cell modules $\mathrm{LC}^{(\Lambda)}$ involving a single left cell Λ, which already appeared in [6, 1.4].

We denote by $\{D_{w^{-1}} \mid w \in W\}$ the dual basis of $\{C_w \mid w \in W\}$ with respect to the symmetrizing trace τ, i.e. we have

$$\tau(C_w D_{y^{-1}}) = \begin{cases} 0 & \text{for } w \neq y, \\ 1 & \text{for } w = y \end{cases}$$

(cf. [7, (5.1.10)]).

Using both bases and τ, we can now write every element of $\mathcal{H}$ as a linear combination in either basis:

$$h = \sum_{w \in W} \tau(D_{w^{-1}} \cdot h) C_w = \sum_{w \in W} \tau(C_w \cdot h) D_{w^{-1}} \quad \text{for all } h \in \mathcal{H}. \tag{3}$$

To proof this equation just set $h = \sum_{w \in W} \alpha_w C_w = \sum_{w \in W} \beta_w D_{w^{-1}}$ for $\alpha_w, \beta_w \in A$, multiply both sides by $D_{z^{-1}}$ or by C_z and apply τ.

We can now give a formula for the character afforded by a cell module of a single left cell Λ:

$$\chi_\Lambda(h) = \sum_{z \in \Lambda} \tau(D_{z^{-1}} \cdot h \cdot C_z) \quad \text{for all } h \in \mathcal{H}. \tag{4}$$

This holds, because the set $\{C_z + \mathrm{LC}^{(\Sigma)} \mid z \in \Lambda\}$ is a basis of the left cell module $\mathrm{LC}^{(\Lambda)} = \mathrm{LC}^{(\Omega)} / \mathrm{LC}^{(\Sigma)}$ where Ω is the union of all left cells which are $\leqslant_L$ than Λ and $\Sigma = \Omega \setminus \Lambda$. The entries of the representing matrix of h on $\mathrm{LC}^{(\Lambda)}$ with respect to this basis are $\tau(D_{x^{-1}} \cdot h \cdot C_z)$ for $x, z \in \Lambda$.

All these definitions can be done analogously for right cells Λ and right cell modules $\mathrm{RC}^{(\Lambda)}$.

For the convenience of the reader we give the formula for the character value of an algebra element with respect to the right cell module of a single right cell Λ:

$$\chi_\Lambda(h) = \sum_{z \in \Lambda} \tau(C_z \cdot h \cdot D_{z^{-1}}) \quad \text{for all } h \in \mathcal{H}.$$

This uses the basis $\{C_z + \mathrm{RC}^{(\Sigma)} \mid z \in \Lambda\}$ of the right cell module $\mathrm{RC}^{(\Lambda)}$, where Σ is the union of the right cells that are $<_R \Lambda$.

3.1. Definition *($\mathbf{a}$-values, $\mathcal{D}$, see [2, Chapter 13]).* For $z \in W$ let

$$\mathbf{a}(z) := \max\{\deg h_{x,y,z} \mid x, y \in W\} \quad \text{and} \quad \Delta(z) := -\deg p_{1,z},$$

such that $\Delta(z) \geqslant 0$.

For $x, y, z \in W$ let $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ be the coefficient of $v^{\mathbf{a}(z)}$ in $h_{x,y,z}$ and set

$$\mathcal{D} := \{d \in W \mid \mathbf{a}(d) = \Delta(d)\}.$$

Further, let $n_z \in \mathbb{Z} \setminus \{0\}$ denote the coefficient of $v^{-\deg p_{1,z}}$ in $p_{1,z}$.

4. Some results for later reference

In this section we recall some results and statements for later reference and convenience of the reader. The first few are for arbitrary type of the Coxeter group W. Then some stronger results for type A are presented and finally, we repeat a subset of Conjectures **P1** to **P15** from [2, Chapter 14], under the assumption of which we will later prove some results.

4.1. Lemma *(Characterization of $\leqslant_L$ and $\leqslant_R$, see [7, (5.1.14)]). We have $y \leqslant_L w$ if $C_w D_{y^{-1}} \neq 0$ and we have $y \leqslant_R w$ if $D_{y^{-1}} C_w \neq 0$.*

Comment. In both statements also "only if" is proved in [7, (5.1.14)] for the equal parameter case using that all $\mu^s_{y,w}$ are non-negative.

If W is of type A_{n-1}, we have the following strong result:

4.2. Theorem *(One-cell modules are simple, see [6, Theorem 1.4]). Let W be of type A_{n-1} and $K = \mathbb{Q}(\Gamma)$ be the field of fractions of A. Then for each left cell Λ, the extension of scalars $K\mathrm{LC}^{(\Lambda)}$ of the corresponding left cell module $\mathrm{LC}^{(\Lambda)}$ is a simple $K\mathcal{H}$-module. Further, the cell modules coming from two left cells are isomorphic if and only if they lie in the same two-sided cell, and all isomorphism classes of simple $K\mathcal{H}$-modules arise as left cell modules.*

Comment on the proof. All this is shown in [6, Theorem 1.4]. For a more detailed exposition see [8, Kapitel VI]. There also the connections to the Robinson–Schensted correspondence are described in detail. □

Note. This result implies especially that all simple modules of $K\mathcal{H}$ are realizable as modules of $\mathcal{H}$ over A. Therefore all characters $\chi \in \mathrm{Irr}(K\mathcal{H})$ take values in A when restricted to $\mathcal{H}$, such that we can safely consider characters in $\mathrm{Irr}(K\mathcal{H})$ as A-linear forms on $\mathcal{H}$ whenever convenient.

Hidden in the proof of the previous theorem are proofs for the following statements, which explain parts of the above theorem in more detail:

4.3. Theorem *(Left and right cells, equal cell modules, see [6, Proof of 1.4]). Let W be of type A_{n-1}. If Λ is a left cell, then $\Lambda^{-1} = \{x^{-1} \mid x \in \Lambda\}$ is a right cell. For $x, y \in \Lambda$ with $x \neq y$, we have $x^{-1} \not\sim_L y^{-1}$, such that we have:*

$$\text{If } x \sim_L y \text{ and } x \sim_R y, \text{ then } x = y.$$

If Ω is a two-sided cell, then the number of left cells in Ω is equal to the number of elements in each left cell in Ω and to the number of elements in each right cell in Ω and to the number of right cells in Ω. The intersection between any left cell in Ω and any right cell in Ω contains exactly one element. Moreover, if Λ_1 and Λ_2 are two left cells in the same two-sided cell, then there is a bijective mapping $\varphi : \Lambda_1 \to \Lambda_2$, such that

$$\tau(D_{x^{-1}} \cdot h \cdot C_z) = \tau(D_{\varphi(x)^{-1}} \cdot h \cdot C_{\varphi(z)}) \quad \text{for all } x, z \in \Lambda_1,$$

i.e. the two matrix representations with respect to the two bases

$$\left\{C_z + \mathrm{LC}^{(\Sigma_1)} \mid z \in \Lambda_1\right\} \quad \text{and} \quad \left\{C_z + \mathrm{LC}^{(\Sigma_2)} \mid z \in \Lambda_2\right\}$$

are in fact equal (here Σ_i is the union of all left cells $<_L$ than Λ_i for $i = 1, 2$).

The mapping φ can be described explicitly: For $x \in \Lambda_1$, the element $\varphi(x) \in \Lambda_2$ is the unique element of Λ_2 lying in the same right cell than x.

Comment on the proofs. The combinatorics regarding left and right cells follow from the results about the "generalized τ-invariant" cited there, which is the Robinson–Schensted correspondence in this type A case. The statements about the number of elements and the number of cells follow from the statements in Theorem 4.2 and the statement about equal matrix representations are proved in the proof of [6, Theorem 1.4] disguised as two W-graphs being isomorphic. See [8, Kapitel VI] for a more detailed exposition. □

In the rest of this section, we present a subset of the Conjectures **P1** to **P15** from [2, Chapter 14], under the assumption of which we will prove some results later on.

4.4. Conjectures *(Lusztig).*

P4. *If $z' \leqslant_{LR} z$ then $\mathbf{a}(z') \geqslant \mathbf{a}(z)$. Hence, if $z' \sim_{LR} z$, then $\mathbf{a}(z') = \mathbf{a}(z)$.*
P5. *If $d \in \mathcal{D}$, $y \in W$, and $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.*
P6. *If $d \in \mathcal{D}$, then $d^2 = 1$.*
P7. *For any $x, y, z \in W$, we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.*
P9. *If $z' \leqslant_L z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ then $z' \sim_L z$.*
P10. *If $z' \leqslant_R z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ then $z' \sim_R z$.*
P11. *If $z' \leqslant_{LR} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ then $z' \sim_{LR} z$.*
P13. *Any left cell Λ of W contains a unique element $d \in \mathcal{D}$. We have $\gamma_{x^{-1},x,d} \neq 0$ for all $x \in \Lambda$.*
P14. *For any $z \in W$ we have $z \sim_{LR} z^{-1}$.*

These conjectures are all proved for important cases: If $L = \ell$ and W is a finite or affine Weyl group (the "split case," see [2, Chapter 15]), if W is dihedral (see [2, Chapter 17]), and if W is quasi-split (see [2, Chapter 16]).

The following two corollaries are immediate consequences of **P4**, **P9**, and **P10**:

4.5. Corollary *(Left cells and two-sided cells, see [7, (5.1.13)]). Assume that* **P4** *and* **P9** *hold. Then we have*:

- *If $y \leqslant_L w$ and $y \sim_{LR} w$ hold, then also $y \sim_L w$.*

4.6. Corollary *(Right cells and two-sided cells). Assume that* **P4** *and* **P10** *hold. Then we have*:

- *If $y \leqslant_R w$ and $y \sim_{LR} w$ hold, then also $y \sim_R w$.*

5. Wedderburn decomposition

In this section, we show connections between the Kazhdan–Lusztig basis and the Wedderburn decomposition of $K\mathcal{H}$. The best results are for the case that all cell modules are simple as $K\mathcal{H}$ modules, where we show, how to construct a Wedderburn basis explicitly in terms of the Kazhdan–Lusztig basis and its dual basis.

We need one more piece of notation, namely the Schur elements.

5.1. Definition *(Schur elements).* Let Z be the element $\sum_{w\in W} T_w^\vee T_w \in \mathcal{H}$. For every irreducible character χ of $K\mathcal{H}$ we call the element

$$c_\chi := \frac{\chi(Z)}{\chi(T_{\mathrm{id}})^2} \in K$$

the *Schur element* of the character χ.

Note. Using that $\mathcal{H}$ is a symmetric algebra with symmetrizing trace τ one can define an averaging procedure to make a K-linear map from a $K\mathcal{H}$-module into itself into a $K\mathcal{H}$-endomorphism. The Schur elements play an important role in this as can be seen from [5, 7.2.1].

We need the following statement:

5.2. Theorem *(Semisimplicity and Schur elements, see [5, 7.2.6]). For the split semisimple algebra $K\mathcal{H}$ all Schur elements c_χ for all $\chi \in \mathrm{Irr}(K\mathcal{H})$ are non-zero and we have*:

$$\tau = \sum_{\chi \in \mathrm{Irr}(K\mathcal{H})} \frac{\chi}{c_\chi}$$

as A-linear maps.

We have now everything in place to proof our first main theorem:

5.3. Theorem *(Simple cell module, cf. [8, VI.(4.1)]). Let Λ be a left cell, such that the extension of scalars $K\mathrm{LC}^{(\Lambda)}$ of the left cell module $\mathrm{LC}^{(\Lambda)}$ is simple as a $K\mathcal{H}$ module. If χ denotes the corresponding irreducible character, then the elements*

$$\mathcal{B} := \left(c_\chi^{-1} C_x D_{y^{-1}}\right)_{x,y\in\Lambda}$$

are K-linearly independent and span the isotypic component of $K\mathcal{H}$ belonging to the character χ. Further, we have the relations

$$c_\chi^{-1} C_x D_{y^{-1}} \cdot c_\chi^{-1} C_{x'} D_{y'^{-1}} = \delta_{y,x'} \cdot c_\chi^{-1} C_x D_{y'^{-1}}$$

for all $x, y, x', y' \in \Lambda$. That is, $\mathcal{B}$ is a matrix unit for the isotypic component of $K\mathcal{H}$ corresponding to the simple module $K\mathrm{LC}^{(\Lambda)}$.

Proof. By [5, 7.2.7] we obtain a matrix unit for the isotypic component of $K\mathcal{H}$ corresponding to the simple module $K\mathrm{LC}^{(\Lambda)}$ by the elements

$$\frac{1}{c_\chi} \sum_{z\in W} \tau(D_{y^{-1}} \cdot C_z \cdot C_x) \cdot D_{z^{-1}} = \frac{1}{c_\chi} \sum_{z\in W} \tau(C_z \cdot C_x D_{y^{-1}}) \cdot D_{z^{-1}}$$

for $x, y \in \Lambda$. But this is equal to $c_\chi^{-1} \cdot C_x D_{y^{-1}}$, because $\{D_{z^{-1}}\}$ is the dual basis of $\{C_z\}$ with respect to τ (use Eq. (3)). $\square$

This result immediately yields explicit formulae for a Wedderburn decomposition of $K\mathcal{H}$ for the case that all cell modules are simple, which is for example true for type A by Theorem 4.2:

5.4. Corollary *(Wedderburn decomposition and basis). Assume that for all left cells Σ the extension of scalars $K\mathrm{LC}^{(\Sigma)}$ of the cell module is a simple $K\mathcal{H}$ module and that two such modules $K\mathrm{LC}^{(\Sigma)}$ and $K\mathrm{LC}^{(\Sigma')}$ are isomorphic if and only Σ and Σ' lie in the same two-sided cell.*

Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$ be left cells, such that from every two-sided cell exactly one left cell occurs, and let $\Lambda := \bigcup_{i=1}^r \Lambda_i$. We denote by $\chi_1, \chi_2, \ldots, \chi_r$ the irreducible characters of $\mathcal{H}$ afforded by the left cell modules $K\mathrm{LC}^{(\Lambda_1)}, \ldots, K\mathrm{LC}^{(\Lambda_r)}$ and with $c_{\chi_1}, \ldots, c_{\chi_r}$ the corresponding Schur elements. Let

$$\mathcal{B} := \left(c_{\chi_i}^{-1} \cdot C_x D_{y^{-1}} \in K\mathcal{H} \mid i \in \{1, 2, \ldots, r\} \text{ and } x, y \in \Lambda_i\right).$$

Then $\mathcal{B}$ is a K-basis of $K\mathcal{H}$, such that

$$\left(c_{\chi_i}^{-1} \cdot C_x D_{y^{-1}}\right) \cdot \left(c_{\chi_j}^{-1} \cdot C_{x'} D_{y'^{-1}}\right) = \delta_{i,j} \cdot \delta_{y,x'} \cdot \left(c_{\chi_i}^{-1} \cdot C_x D_{y'^{-1}}\right).$$

Proof. This result follows immediately from Theorem 5.3. □

Remark. This shows that the elements $c_{\chi_i}^{-1} \cdot C_x D_{y^{-1}}$ form a Wedderburn basis of $K\mathcal{H}$, i.e.

$$K\mathcal{H} = K\mathcal{H}_1 \oplus K\mathcal{H}_2 \oplus \cdots \oplus K\mathcal{H}_r \quad \text{as } K\text{-algebras},$$

where $K\mathcal{H}_i := \langle c_{\chi_i}^{-1} \cdot C_x D_{y^{-1}} \mid x, y \in \Lambda_i \rangle_K$ is the isotypic component of $K\mathcal{H}$ corresponding to the irreducible character χ_i and the K-linear mapping from $K\mathcal{H}_i$ to the full matrix ring $M_{|\Lambda_i|}(K)$, mapping each $c_{\chi_i}^{-1} \cdot C_x D_{y^{-1}}$ to a matrix with exactly one 1 in position (x, y) and zeroes everywhere else, is an isomorphism of K-algebras.

These basis elements will also play a prominent role in the following sections about the Lusztig homomorphism.

For the case that the left cell modules to different left cells in the same two-sided cell yield the very same matrix representation (as is for example guaranteed for type A by Theorem 4.3) we even have the following corollary.

5.5. Corollary *(Different left cells in the same two-sided cell). Assume that Λ and Λ' are two left cells in the same two-sided cell, that $K\mathrm{LC}^{(\Lambda)}$ is a simple $K\mathcal{H}$-module and that there is a bijection $\varphi : \Lambda \to \Lambda'$ with*

$$\tau(D_{y^{-1}} \cdot h \cdot C_x) = \tau(D_{\varphi(y)^{-1}} \cdot h \cdot C_{\varphi(x)}) \quad \textit{for all } x, y \in \Lambda \textit{ and } h \in \mathcal{H}.$$

Then $C_x D_{y^{-1}} = C_{\varphi(x)} D_{\varphi(y)^{-1}}$ for all $x, y \in \Lambda$.

Proof. This follows directly from the formulae in the proof of Theorem 5.3. □

In the rest of this section we investigate, which of the nice results above can still be obtained in the case that the cell modules are not all simple. For the next few results, we still do not have to use Conjectures **P1** to **P15**.

5.6. Lemma. *If Λ is a left cell, then the element $s := \sum_{z \in \Lambda} C_z D_{z^{-1}}$ is central in $\mathcal{H}$.*

Proof. If χ_Λ is the character corresponding to the left cell module $\mathrm{LC}^{(\Lambda)}$, then we have for all elements $h, h' \in \mathcal{H}$ (note Eq. (4) in Section 3):

$$\begin{aligned}\tau(sh \cdot h') &= \sum_{z \in \Lambda} \tau(D_{z^{-1}} hh' C_z) = \chi_\Lambda(hh') = \chi_\Lambda(h'h) = \sum_{z \in \Lambda} \tau(D_{z^{-1}} h'h C_z) \\ &= \tau(hs \cdot h').\end{aligned}$$

Thus $sh = hs$ for all $h \in \mathcal{H}$, because τ is non-degenerate. □

5.7. Proposition. *Let Λ and Λ' be two left cells that do not lie in the same two-sided cell. Then the two left cell modules $K\mathrm{LC}^{(\Lambda)}$ and $K\mathrm{LC}^{(\Lambda')}$ have no common simple $K\mathcal{H}$-module as constituent.*

Proof. The elements $s := \sum_{z\in\Lambda} C_z D_{z^{-1}}$ and $s' := \sum_{z'\in\Lambda'} C_{z'} D_{z'^{-1}}$ are both central in $\mathcal{H}$ by Lemma 5.6 and we have

$$\chi_\Lambda(h) = \tau(s\cdot h) \quad \text{and} \quad \chi_{\Lambda'}(h) = \tau(s'\cdot h)$$

for all $h \in \mathcal{H}$ as in the proof of Lemma 5.6. Thus s is a linear combination of the central primitive idempotents in $K\mathcal{H}$ corresponding to the simple constituents occurring in $K\mathrm{LC}^{(\Lambda)}$, namely

$$s = \sum_{\chi\in\mathrm{Irr}(K\mathcal{H})} (\chi,\chi_\Lambda) c_\chi e_\chi,$$

where (χ,χ_Λ) stands for the number of simple constituents of $K\mathrm{LC}^{(\Lambda)}$ with character χ and e_χ is the central primitive idempotent in $K\mathcal{H}$ corresponding to the character χ. To proof this, calculate

$$\begin{aligned}\tau\Biggl(\sum_{\chi\in\mathrm{Irr}(K\mathcal{H})} (\chi,\chi_\Lambda) c_\chi e_\chi\cdot h\Biggr) &= \sum_{\chi\in\mathrm{Irr}(K\mathcal{H})} (\chi,\chi_\Lambda) c_\chi\cdot\tau(e_\chi\cdot h)\\ &= \sum_{\chi\in\mathrm{Irr}(K\mathcal{H})} (\chi,\chi_\Lambda) c_\chi\cdot\Biggl(\sum_{\chi'\in\mathrm{Irr}(K\mathcal{H})} c_{\chi'}^{-1}\chi'(e_\chi\cdot h)\Biggr)\\ &= \sum_{\chi\in\mathrm{Irr}(K\mathcal{H})} (\chi,\chi_\Lambda)\chi(h) = \chi_\Lambda(h)\end{aligned}$$

using Theorem 5.2 and the fact, that $\chi'(e_\chi\cdot h) = \delta_{\chi,\chi'}\cdot\chi(h)$ for all $h\in\mathcal{H}$ and $\chi,\chi'\in\mathrm{Irr}(K\mathcal{H})$. The analogous statement holds for s'.

We now consider the element $ss' = s's$:

$$\sum_{z\in\Lambda}\sum_{z'\in\Lambda'} C_z D_{z^{-1}} C_{z'} D_{z'^{-1}} = \Biggl(\sum_{z\in\Lambda} C_z D_{z^{-1}}\Biggr)\cdot\Biggl(\sum_{z'\in\Lambda'} C_{z'} D_{z'^{-1}}\Biggr) = ss' \tag{5}$$

$$= s's = \sum_{z\in\Lambda}\sum_{z'\in\Lambda'} C_{z'} D_{z'^{-1}} C_z D_{z^{-1}}. \tag{6}$$

Assume that this sum is non-zero.

There must be at least one pair $(z_1, z_1') \in \Lambda\times\Lambda'$ with $D_{z_1^{-1}} C_{z_1'} \neq 0$ (see Eq. (5)) and at least one pair $(z_2, z_2') \in \Lambda\times\Lambda'$ with $D_{z_2'^{-1}} C_{z_2} \neq 0$ (see Eq. (6)). But this means $z_1 \leqslant_R z_1' \sim_L z_2' \leqslant_R z_2 \sim_L z_1$ by Lemma 4.1 and thus contradicts the hypothesis that Λ and Λ' are not in the same two-sided cell. Therefore we have shown that $ss' = 0 = s's$.

Thus, we have

$$0 = \sum_{\chi\in\mathrm{Irr}(K\mathcal{H})} (\chi,\chi_\Lambda)\cdot(\chi,\chi_{\Lambda'})\cdot c_\chi^2 e_\chi$$

which proves the proposition, because all Schur elements are non-zero. □

Comment. This and the fact that every simple $K\mathcal{H}$-module occurs in at least one left cell module shows that the two-sided cells induce a partition of the set of isomorphism classes of simple $K\mathcal{H}$-modules. Therefore, we speak of "the two-sided cell a character $\chi \in \mathrm{Irr}(K\mathcal{H})$ belongs to" in the sequel.

5.8. Proposition. *Assume that $x \sim_L y \nsim_{LR} z \sim_L w$ holds for $x, y, z, w \in W$. Then*

$$C_x D_{y^{-1}} \cdot C_z D_{w^{-1}} = 0.$$

Proof. Denote the left cell in which x and y lie by Λ and the two-sided cell in which Λ lies by Σ.

We consider the action of $C_x D_{y^{-1}}$ on the left cell module $\mathrm{LC}^{(\Lambda')}$ for some left cell Λ' outside of Σ. The representing matrix for $C_x D_{y^{-1}}$ is

$$\bigl(\tau(D_{w'^{-1}} \cdot C_x D_{y^{-1}} \cdot C_{z'})\bigr)_{z', w' \in \Lambda'}.$$

By Lemma 4.1 $D_{w'^{-1}} \cdot C_x = 0$ unless $w' \leqslant_R x$ and $D_{y^{-1}} \cdot C_{z'} = 0$ unless $y \leqslant_R z'$. But in that case we would have $w' \leqslant_R x \sim_L y \leqslant_R z' \sim_L w'$ and this would mean that Λ' is in Σ, contrary to our assumption.

Thus we have shown that $C_x D_{y^{-1}}$ acts as zero on all left cell modules of left cells Λ' outside of Σ. This means that $C_x D_{y^{-1}}$ lies in the direct sum of those isotypic components of irreducible characters that belong to Σ.

The same arguments hold for $C_z D_{w^{-1}}$ and for the two-sided cell in which z and w lie. Therefore the product must be zero. □

5.9. Definition/Proposition *(Ideals $\mathcal{L}_y$ and $\mathcal{R}_x$).* Let Λ be a left cell and $x, y \in \Lambda$.

Then the A-module $\mathcal{L}_y := \langle C_z D_{y^{-1}} \mid z \sim_L y \rangle_A$ is a left ideal in $\mathcal{H}$, and $\mathcal{L}_y$ is isomorphic as left $\mathcal{H}$-module to $\mathrm{LC}^{(\Lambda)}$ via the A-linear map

$$C_z + \mathrm{LC}^{(\Lambda')} \mapsto C_z D_{y^{-1}},$$

where Λ' is the union of left cells that are $<_L \Lambda$.

Further, the A-module $\mathcal{R}_x := \langle C_x D_{z^{-1}} \mid z \sim_L x \rangle_A$ is a right ideal in $\mathcal{H}$ and as a right module isomorphic to the dual module $\hom_A(\mathrm{LC}^{(\Lambda)}, A)$ of the left cell module $\mathrm{LC}^{(\Lambda)}$ via the A-linear map

$$\bigl(C_z + \mathrm{LC}^{(\Lambda')}\bigr)^{\vee} \mapsto C_x D_{z^{-1}},$$

where $((C_z + \mathrm{LC}^{(\Lambda')})^{\vee})_{z \in \Lambda}$ stands for the dual basis of the basis $(C_z + \mathrm{LC}^{(\Lambda')})_{z \in \Lambda}$.

Proof. For arbitrary $h \in \mathcal{H}$ we have (use Eq. (3))

$$h \cdot C_z = \sum_{x \in W} \tau(h \cdot C_z D_{x^{-1}}) \cdot C_x = \sum_{x \leqslant_L z} \tau(h \cdot C_z D_{x^{-1}}) \cdot C_x.$$

Initially, x runs through all of W, but we may restrict the sum to $x \leqslant_L z$ because otherwise $C_z D_{x^{-1}} = 0$ by Lemma 4.1, thereby proving the second equality. We multiply this equation from the right by $D_{y^{-1}}$ and get

$$h \cdot C_z D_{y^{-1}} = \sum_{x \leqslant_L z} \tau(h \cdot C_z D_{x^{-1}}) \cdot C_x D_{y^{-1}} = \sum_{x \sim_L y} \tau(h \cdot C_z D_{x^{-1}}) \cdot C_x D_{y^{-1}}.$$

The term $C_x D_{y^{-1}}$ is equal to 0 unless we have $y \leqslant_L x$ again by Lemma 4.1. But in that case, we have $y \leqslant_L x \leqslant_L z \sim_L y$ and thus $x \sim_L y$. This shows the right equality and thus that $\mathcal{L}_y$ is indeed a left ideal in $\mathcal{H}$.

As a left $\mathcal{H}$-module the left ideal $\mathcal{L}_y$ is isomorphic to $\mathrm{LC}^{(\Lambda)}$ via the A-linear map that maps $C_z + \mathrm{LC}^{(\Lambda')}$ to $C_z D_{y^{-1}}$ for all $z \in \Lambda$, because the entries of the representing matrices of h with respect to these bases are in both cases $\tau(D_{x^{-1}} \cdot h \cdot C_z)$ for $x, z \in \Lambda$.

The proofs for the right handed version are completely analogous. □

Remark. An analogous result can be obtained for the right cell modules by using elements of the form $D_{y^{-1}} C_x$ with $x \sim_R y$.

2. Remark. This result specializes for an irreducible left cell module to the fact that in a full matrix ring the set of matrices having zeroes everywhere except in a single column form a left ideal isomorphic to the natural left module and that the set of matrices having zeroes everywhere except in a single row form a right ideal isomorphic to the natural right module. Compare to Theorem 5.3.

5.10. Theorem *(Generation of $K\mathcal{H}$). The algebra $K\mathcal{H}$ is generated as a K-vectorspace by the set $\{C_x D_{y^{-1}} \mid x \sim_L y\}$.*

Proof. The element $Z := \sum_{w \in W} C_w D_{w^{-1}}$ is central in $\mathcal{H}$ as sum of central elements (see Lemma 5.6), and has the property that

$$\rho(h) = \tau(Z \cdot h) = \sum_{w \in W} \tau(D_{w^{-1}} \cdot h \cdot C_w)$$

for all $h \in K\mathcal{H}$ holds, where ρ is the regular character. Thus Z is equal to

$$Z = \sum_{\chi \in \mathrm{Irr}(K\mathcal{H})} \chi(T_{\mathrm{id}}) \cdot c_\chi \cdot e_\chi,$$

where c_χ is the Schur element and e_χ is the central primitive idempotent belonging to the character χ (remember Theorem 5.2 and the proof of Proposition 5.7). In particular Z is invertible in $K\mathcal{H}$.

Therefore, we can write every element $h \in \mathcal{H}$ as

$$h = h \cdot Z^{-1} \cdot Z = \left(h \cdot Z^{-1}\right) \cdot \sum_{w \in W} C_w D_{w^{-1}}.$$

Because the element $C_w D_{w^{-1}}$ lies in the left ideal $\mathcal{L}_w$, which is spanned as an A-module by the elements $C_x D_{w^{-1}}$ with $x \sim_L w$, the theorem is proved. □

Remark. Compare to Theorem 6.4, which shows (under the assumption of Conjectures **P1** to **P15**) that the right regular $K\mathcal{H}$-module $K\mathcal{H}_{K\mathcal{H}}$ is isomorphic to the direct sum $\bigoplus_{d\in\mathcal{D}} K\mathcal{R}_d$, because the set $\{C_d D_{z^{-1}} \mid z \in W \text{ and } d \in \mathcal{D} \text{ with } d \sim_L z\}$ is a K-basis of $K\mathcal{H}$.

From now on, we will assume Conjectures **P1** to **P15** if necessary. Then also the relations for a matrix unit can be generalized to the case of reducible cell modules:

5.11. Proposition *(Generalized matrix unit relations). Assume that* **P4**, **P9**, *and* **P10** *hold. Let Σ be a two-sided cell, $x, y, z, w \in \Sigma$ and $x \sim_L y$ and $z \sim_L w$. Then we have*

$$C_x D_{y^{-1}} \cdot C_z D_{w^{-1}} = \begin{cases} 0 & \text{for } y \not\sim_R z, \\ \sum_{u\in\mathcal{S}(w,x)} \tau(C_x D_{y^{-1}} C_z D_{u^{-1}}) C_u D_{w^{-1}} & \text{for } y \sim_R z, \end{cases}$$
$$= \begin{cases} 0 & \text{for } y \not\sim_R z, \\ \sum_{u\in\mathcal{S}(x,w)} \tau(C_u D_{y^{-1}} C_z D_{w^{-1}}) C_x D_{u^{-1}} & \text{for } y \sim_R z, \end{cases}$$

where $\mathcal{S}(x, y)$ stands for the set $\{u \in W \mid u \sim_L x$ and $u \sim_R y\}$. (Note that this means that $\mathcal{S}(x, y)$ is the intersection of the left cell of x with the right cell of y.)

Proof. If $y \not\leqslant_R z$, the product $D_{y^{-1}} \cdot C_z$ is zero by Lemma 4.1. If, on the other hand, we have $y \leqslant_R z$, then by Corollary 4.6 we have $y \sim_R z$, because all elements lie in the two-sided cell Σ. In this case, we get

$$C_x D_{y^{-1}} C_z = \sum_{u\in W} a_u C_u = \sum_{\substack{u\leqslant_L z\\ u\leqslant_R x}} a_u C_u,$$

where $a_u = \tau(C_x D_{y^{-1}} C_z D_{u^{-1}}) \in A$. Thus, we have $a_u = 0$ for $u \not\leqslant_L z$ or $u \not\leqslant_R x$, so we only have to sum over those u with $u \leqslant_L z$ and $u \leqslant_R x$ as indicated in the sum. We now multiply this equation by $D_{w^{-1}}$ from the right:

$$C_x D_{y^{-1}} C_z D_{w^{-1}} = \sum_{\substack{u\leqslant_L z\\ u\leqslant_R x}} a_u C_u D_{w^{-1}},$$

where all summands vanish for which $u \not\geqslant_L w$. Thus, because of $z \sim_L w$, we only have to sum over the left cell of w. However, since all elements u in this left cell are in Σ, and we conclude $u \sim_R x$ from $u \leqslant_R x$ by Corollary 4.6, we finally reach the summation over $\mathcal{S}(w, x)$ as stated in the statement of the proposition.

The other equality follows completely analogously by writing $D_{y^{-1}} C_z D_{w^{-1}}$ as linear combination of the $D_{u^{-1}}$, multiplying the result by C_x from the left, and analogous reasoning using Corollary 4.5. □

6. The asymptotic algebra $\mathcal{J}$ and the Lusztig homomorphism ϕ

In this section we briefly recall the definitions for the asymptotic algebra $\mathcal{J}$ and the Lusztig homomorphism from $\mathcal{H}$ to $\mathcal{J}$, which were first introduced in [1]. Then we derive preimages of the elements t_z in the standard basis of $\mathcal{J}$, which leads to a new interpretation of the Lusztig homomorphism. Finally, the results are made even more explicit for the case that the left cell modules are simple $K\mathcal{H}$-modules like for example in type A.

We assume from now on that **P1** to **P15** hold. Note that we assume that W is finite throughout, therefore we automatically have that W is tame in the sense of [2, 1.1.11], which allows us to use the results of [2, Chapter 18].

6.1. Definition/Proposition *(Lusztig's asymptotic algebra, see [2, Chapter 18]).* Let J be the free abelian group with basis $\{t_w \mid w \in W\}$. It becomes an associative $\mathbb{Z}$-algebra by using the $\gamma_{x,y,z^{-1}}$ as structure constants, i.e. if we set

$$t_x \cdot t_y := \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z \quad \text{for all } x, y \in W.$$

Its unit element is $\sum_{d \in \mathcal{D}} n_d t_d$. We denote the associative A-algebra $A \otimes_{\mathbb{Z}} J$ by $\mathcal{J}$, it is called *Lusztig's asymptotic algebra*.

Proof. See [2, 18.3]. □

The asymptotic algebra turns out to be closely related to $\mathcal{H}$, as is shown by the following statement:

6.2. Definition/Proposition *(Lusztig homomorphism, see [2, 18.9]).* As in [2, 3.5] we denote by $h \mapsto h^\dagger$ the unique algebra involution of $\mathcal{H}$ such that $T_s^\dagger = -T_s^{-1}$ for all $s \in S$.

The A-linear mapping $\phi : \mathcal{H} \to \mathcal{J}$ with

$$\phi\left(C_x^\dagger\right) := \sum_{\substack{d \in \mathcal{D},\, z \in W \\ a(d)=a(z)}} h_{x,d,z} \cdot \hat{n}_z \cdot t_z = \sum_{\substack{d \in \mathcal{D},\, z \in W \\ z \sim_L d}} h_{x,d,z} \cdot \hat{n}_z \cdot t_z \quad \text{for all } x \in W,$$

where $\hat{n}_z := n_{d'}$ for the unique (use **P13**) $d' \in \mathcal{D}$ with $d' \sim_L z^{-1}$, is a homomorphism of A-algebras. It is called the *Lusztig homomorphism*. We denote the extension $\mathrm{id}_K \otimes_A \phi : K\mathcal{H} \to K\mathcal{J}$ to the extension of scalars with K also by ϕ. It is an isomorphism of K-algebras.

Comment on proofs. See [2, 18.8 to 18.12]. The fact that $\mathrm{id}_K \otimes_A \phi$ is an isomorphism follows from [2, 18.12.(b)]. The second equals sign in the formula above follows from the fact that $h_{x,d,z} = \tau(C_x C_d D_{z^{-1}}) = 0$ unless $z \leqslant_L d$ (use Lemma 4.1) and that $z \sim_L d$ is equivalent to $z \leqslant_L d$ and $\mathbf{a}(d) = \mathbf{a}(z)$ by **P9** and **P4**. □

6.3. Corollary *(Lusztig homomorphism for a general element). Let $h \in \mathcal{H}$ be arbitrary. Then we have for the image $\phi(h^\dagger)$ of $h^\dagger$ under the Lusztig homomorphism*:

$$\phi\left(h^\dagger\right) = \sum_{\substack{d \in \mathcal{D},\, z \in W \\ z \sim_L d}} \tau(h \cdot C_d D_{z^{-1}}) \cdot \hat{n}_z \cdot t_z.$$

Proof. Write $h = \sum_{u \in W} \alpha_u C_u$ and use $h_{u,d,z} = \tau(C_u C_d D_{z^{-1}})$ and the fact that τ and ϕ are A-linear. □

We can now find the preimages of the elements t_z under the Lusztig homomorphism:

6.4. Theorem *(Preimages of the t_z under the Lusztig homomorphism). The set*

$$\mathcal{B}^\vee := \{C_d D_{z^{-1}} \mid z \in W \text{ and } d \in \mathcal{D} \text{ with } d \sim_L z\}$$

is K-linearly independent in $K\mathcal{H}$. Let $\{h_y \in K\mathcal{H} \mid y \in W\}$ be the dual basis of $\mathcal{B}^\vee$ in $K\mathcal{H}$ with respect to τ in the sense that $\tau(h_y \cdot C_d D_{z^{-1}}) = \delta_{y,z}$ for $z \in W$ and $d \in \mathcal{D}$ with $d \sim_L z$. Then $\phi(h_z^\dagger) = \hat{n}_z t_z$ for all $z \in W$.

Proof. We derive these results in the opposite order than in the theorem:

The map $\phi : K\mathcal{H} \to K\mathcal{J}$ is an isomorphism of K-algebras. Therefore, every element $\hat{n}_z t_z \in \mathcal{J}$ has a unique preimage, which we denote by $h_z^\dagger$. Because the set $\{\hat{n}_z t_z \mid z \in W\}$ is a K-basis of $\mathcal{J}$ and $h \mapsto h^\dagger$ is a K-linear involution, the set $\{h_z \mid z \in W\}$ is a K-basis of $K\mathcal{H}$. By Corollary 6.3 we have

$$\phi\big(h_y^\dagger\big) = \sum_{\substack{d \in \mathcal{D},\, z \in W \\ z \sim_L d}} \tau(h_y \cdot C_d D_{z^{-1}}) \cdot \hat{n}_z \cdot t_z = \hat{n}_y \cdot t_y.$$

Thus, each h_y fulfills the relation $\tau(h_y \cdot C_d D_{z^{-1}}) = \delta_{y,z}$ for all $z \in W$ and $d \in \mathcal{D}$ with $d \sim_L z$. Therefore, the set $\mathcal{B}^\vee$ is the dual basis of $\{h_y \mid y \in W\}$ with respect to τ and in particular K-linearly independent. □

6.5. Corollary *(Direct sum decomposition of $K\mathcal{H}_{K\mathcal{H}}$). The right regular module $K\mathcal{H}_{K\mathcal{H}}$ is isomorphic as a right $K\mathcal{H}$-module to the direct sum*

$$\bigoplus_{d \in \mathcal{D}} \mathcal{R}_d = \bigoplus_{d \in \mathcal{D}} \langle C_d D_{z^{-1}} \mid z \sim_L d \rangle_K$$

of right ideals. Each $\mathcal{R}_d$ is isomorphic to the dual module $\hom_K(K\mathrm{LC}^{(\Lambda)}, K)$ *of the left cell module $K\mathrm{LC}^{(\Lambda)}$ (where Λ is the left cell with $d \in \Lambda$ and Λ' is the union of left cells that are $<_L$ than Λ) via the K-linear map*

$$\big(C_z + \mathrm{LC}^{(\Lambda')}\big)^\vee \mapsto C_d D_{z^{-1}},$$

where $((C_z + \mathrm{LC}^{(\Lambda')})^\vee)_{z \in \Lambda}$ stands for the dual basis of the basis $(C_z + \mathrm{LC}^{(\Lambda')})_{d \in \Lambda}$.

Proof. The direct sum decomposition follows directly from Theorem 6.4 and the fact that $\mathcal{R}_d$ is a right ideal from Proposition 5.9. The isomorphism to the dual of the left cell module follows by extension of scalars from Proposition 5.9. □

6.6. Corollary *(A new interpretation of the Lusztig homomorphism). Let $\mathcal{B}^\vee$ be the K-basis*

$$\mathcal{B}^\vee := \{C_d D_{z^{-1}} \mid z \in W \text{ and } d \in \mathcal{D} \text{ with } d \sim_L z\}$$

of $K\mathcal{H}$ *and*

$$\mathcal{B} := \{h_z \mid z \in W\}$$

be the dual basis of $\mathcal{B}^\vee$ *with respect to* τ *in the sense of* $\tau(h_y \cdot C_d D_{z^{-1}}) = \delta_{y,z}$ *for all* $y, z \in W$ *and* $d \in \mathcal{D}$ *with* $d \sim_L z$.

Then the asymptotic algebra $\mathcal{J}$ *is, as an* A*-algebra, isomorphic to the* A*-algebra*

$$\langle \mathcal{B} \rangle_A^\dagger := \langle h_z \mid z \in W \rangle_A^\dagger$$

via the restriction of the Lusztig homomorphism ϕ *to* $\langle \mathcal{B} \rangle_A^\dagger$.

Thus, the Lusztig homomorphism $\phi : \mathcal{H} \to \mathcal{J}$ *can be interpreted as the inclusion of* $\mathcal{H}$ *into* $\langle \mathcal{B} \rangle_A^\dagger$.

Proof. We have $\langle \mathcal{B}^\vee \rangle_A \subseteq \mathcal{H}$ and by Theorem 6.4 the set $\mathcal{B}^\vee$ is a K-basis of $K\mathcal{H}$. Thus, because $\mathcal{H}$ is a symmetric algebra with symmetrizing trace form τ, the A-span $\langle \mathcal{B} \rangle_A$ of $\mathcal{B}$ contains $\mathcal{H}$. Therefore $\mathcal{H} = \mathcal{H}^\dagger$ is a subset of $\langle \mathcal{B}^\dagger \rangle_A = \langle \mathcal{B} \rangle_A^\dagger$. The result follows now directly from Theorem 6.4, because ϕ maps $\langle \mathcal{B} \rangle_A^\dagger$ bijectively onto $\mathcal{J}$. $\square$

Finally we can give even more explicit results for the special case, that the left cell modules are simple $K\mathcal{H}$-modules (like for example if W is of type A_{n-1} by Theorem 4.2):

6.7. Corollary *(Explicit interpretation for the case of simple cell modules). Let* Λ *be a left cell, such that* $K\mathrm{LC}^{(\Lambda)}$ *is a simple* $K\mathcal{H}$*-module.*

For $y \in \Lambda$ *let* $h_y := c_\chi^{-1} \cdot C_y D_{d'} \in K\mathcal{H}$, *where* $d' \in \mathcal{D}$ *is the unique element with* $d' \sim_L y$ *and* c_χ *is the Schur element to the character* χ *corresponding to the left cell module* $\mathrm{LC}^{(\Lambda)}$. *Then* $\phi(h_y^\dagger) = \hat{n}_y t_y$.

Proof. By Theorem 6.4 we only have to show that $\tau(h_y \cdot C_d D_{z^{-1}}) = \delta_{y,z}$ for all $z \in W$ and $d \in \mathcal{D}$ with $d \sim_L z$. Let $z \in W$ and $d \in \mathcal{D}$ with $d \sim_L z$. Then

$$\tau\left(c_\chi^{-1} \cdot C_y D_{d'} \cdot C_d D_{z^{-1}}\right) = \tau\left(c_\chi^{-1} \cdot D_{d'} \cdot C_d D_{z^{-1}} \cdot C_y\right)$$

is equal to zero unless $d' \leqslant_R d$ and $z \leqslant_R y$ by Lemma 4.1 (note $d' = d'^{-1}$ by **P6**). Now assume that case. Then we have $y \sim_L d' \leqslant_R d \sim_L z \leqslant_R y$ and thus that all four elements d', y, d, and z are in the same two-sided cell. By **P4** and **P10** it follows, that $d' \sim_R d$ and $z \sim_R y$. Using **P6** and **P13** we conclude that $d = d'$ and that all four elements are in Λ. Now we can apply Theorem 5.3 and get

$$\tau\left(c_\chi^{-1} \cdot C_y D_{d'} \cdot C_{d'} D_{z^{-1}}\right) = \tau(C_y D_{z^{-1}}) = \delta_{y,z}$$

proving all our claims. $\square$

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