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Liaison addition and the structure of a Gorenstein liaison class

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Abstract

We study the concept of liaison addition for codimension two subschemes of an arithmetically Gorenstein projective scheme. We show how it relates to liaison and biliaison classes of subschemes and use it to investigate the structure of Gorenstein liaison equivalence classes, extending the known theory for complete intersection liaison of codimension two subschemes. In particular, we show that on the non-singular quadric threefold in projective 4-space, every non-licci ACM curve can be obtained from a single line by successive liaison additions with lines and CI-biliaisons.

Keywords: Gorenstein liaison; CI-liaison; Liaison addition; Quadric hypersurface

1. Introduction

The Lazarsfeld–Rao property refers to a structure common to even liaison classes in codimension two under complete intersection liaison (cf. [1–3,7,12,15]). The goal of our work is to discover if there is an analogue of this property for Gorenstein liaison. In other words, we seek to find some structure for an even Gorenstein liaison class.

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At this point, more precisely, the Lazarsfeld–Rao theorem for complete intersection liaison is known only for subschemes of codimension two in an ambient projective scheme X, but X can be taken quite generally: the most general result to date is for X an integral projective scheme satisfying the condition S_3 of Serre and $H^1_*(\mathcal{O}_X) = 0$ [7]. In this case it says:

(1) If C is a codimension two subscheme of X (equidimensional without embedded components) that is not of minimal degree in its CI-biliaison equivalence class, then it admits a strictly descending biliaison.
(2) Any two subschemes C, C' of minimal degree in the same biliaison equivalence class can be joined by a sequence of elementary biliaisons of height 0.

Since nothing has been proved for subschemes of codimension three or higher in any ambient scheme, we will also stick to codimension two. We refer to [5] and [4] for definitions and basic results on CI-liaison, Gorenstein liaison, and Gorenstein biliaison on a projective scheme X. Our basic assumption throughout this paper is that $X \in \mathbb{P}^N$ is a normal arithmetically Gorenstein scheme, and that we deal with closed subschemes C that are equidimensional of codimension 2 without embedded points. Recall that a coherent sheaf $\mathcal{N}$ on X is called *extraverti* [7, Definition 2.9] if $H^1_*(\mathcal{N}^\vee) = 0$ and $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}_X) = 0$. A sheaf $\mathcal{F}$ is *dissocié* if $\mathcal{F} = \bigoplus \mathcal{O}_X(a_i)$ for some integers $r, a_1, \ldots, a_r$. An *$\mathcal{N}$-type resolution* of C [4, Definition 2.4] is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{N} \to I_C \to 0$$

with $\mathcal{L}$ dissocié and $\mathcal{N}$ extraverti. In addition if C is locally Cohen–Macaulay, then $\mathcal{N}$ is a locally Cohen–Macaulay sheaf on X. There are two (inequivalent) kinds of Gorenstein liaison that will concern us, so we phrase two questions.

Question 1. What is the structure of a Gorenstein biliaison equivalence class of codimension two subschemes of X (see [5] for Gorenstein biliaison)?

Question 2. What is the structure of an even Gorenstein liaison class of codimension two subschemes of X?

Since any CI-biliaison is also an even Gorenstein liaison and a Gorenstein biliaison, the first observation we can make is that each Gorenstein biliaison or even Gorenstein liaison class is a disjoint union of CI-biliaison classes, and within each of these, the "classical" Lazarsfeld–Rao property holds. So our problem is rather, how to get from one CI-biliaison class to another within the Gorenstein biliaison or even Gorenstein liaison class. And, while we are at it, do our constructions yield minimal elements of a CI-biliaison class if we start from a minimal element?

Interesting special cases are the following classes, all contained within the set of arithmetically Cohen–Macaulay subschemes:

$$\{\text{licci}\} \subseteq \{\text{gobilicci}\} \subseteq \{\text{glicci}\}.$$

Here we follow the commonly used acronyms for the liaison class (respectively Gorenstein biliaison class; respectively Gorenstein liaison class) of a complete intersection.

Another general observation is that in the case of Gorenstein biliaison, we have elementary Gorenstein biliaisons, which could play the role of elementary biliaisons in the traditional Lazarsfeld–Rao property. But in the case of even Gorenstein liaison, it may happen that the only Gorenstein biliaisons are CI-biliaisons, so we need some other operation to move from one class to another. We investigate liaison addition as a possibility in this case, and have some success. In particular, we give some additional justification to the name itself.

"Liaison Addition" was introduced by Schwartau in his thesis [17], where on page 1 he says: "Does there exist a geometric addition of curves in $\mathbb{P}^3$ corresponding to the direct sum of their liaison invariants? This is the *liaison addition* problem. ... We find that not only is there a way to add curves in $\mathbb{P}^3$, but that an explicit procedure is possible: that is, equations for the added curve may be written down from the equations of the curves being added. The addition procedure ... admits a purely intrinsic formulation reminiscent of liaison itself."

Since Schwartau considered the question only in $\mathbb{P}^3$, where Rao had shown that liaison reduces to a question about the "liaison invariants" (subsequently dubbed "Rao modules"), the name made perfect sense in his setting. Subsequently, liaison theory has exploded in the direction of Gorenstein liaison, thanks largely to [10] (see [13] for an extensive bibliography, albeit now quite outdated). Liaison addition has also been generalized substantially (see [2,3,6,13]); a treatment in the generality needed here can be found, for example, in [15]. The name has continued to make sense in the context of complete intersection liaison in codimension two (cf. [3, 15]). However, until now there has been no connection made between liaison addition and the more general notion of Gorenstein liaison.

In Section 2 of this paper we recall the construction and first properties of liaison addition. Our first main results of this paper are contained in Section 3, where we prove the following about the liaison addition C of given codimension two subschemes C_1 and C_2 with respect to forms $F_1 \in I_{C_2}$ and $F_2 \in I_{C_1}$.

- If C_2 is gobilicci then C_1 and C are in the same Gorenstein biliaison class on X.
- If C_2 is glicci then C_1 and C are in the same even Gorenstein liaison class on X.

It follows from Rao's theorem and the preparatory results on liaison addition that if C_2 is licci then C_1 and C are in the same CI even liaison class. However, it is only an existence result about a sequence of links. In Section 4 we make this more precise by showing that there is a very concrete sequence of links, preserving the liaison addition structure all along the way from C to C_1.

In Section 5 we begin the study of a Lazarsfeld–Rao-type structure theorem for a Gorenstein liaison class. Two specific situations that we will examine in some detail are the case of curves on the non-singular quadric hypersurface in $\mathbb{P}^4$ and on the singular quadric hypersurface with a single double point in $\mathbb{P}^4$. We hope that these cases will illustrate the type of phenomena one finds, and may suggest what kind of results one could hope for in more general situations. The special feature of these two examples is that in each case, the Rao module of a curve characterizes the even Gorenstein liaison (respectively G-biliaison) equivalence class of a curve, in analogy to the traditional Rao theorem, where the Rao module characterizes the CI-biliaison class of a curve in $\mathbb{P}^3$. (See [4, Theorem 6.2] for the first case and [5, Theorem 6.2] for the second.) Note that it is an open question whether the Rao module characterizes an even Gorenstein liaison class of curves in $\mathbb{P}^4$.

2. Liaison addition

Throughout this note we denote by $R = k[x_0, \dots, x_n]$ the homogeneous coordinate ring of $\mathbb{P}^n$ where k is any infinite field. We begin with the definition.

Definition 2.1. Let C_1, C_2 be codimension 2 subschemes of $X \subset \mathbb{P}^n$. Let $F_1 \in I_{C_1,X}$ and $F_2 \in I_{C_2,X}$ be homogeneous elements of degree f_1 and f_2, respectively, such that $\{F_1, F_2\}$ is an S-regular sequence, where $S = R/I_X$. Then the subscheme $C \subset X$ defined by the ideal

$$I_{C,X} := F_2 \cdot I_{C_1,X} + F_1 \cdot I_{C_2,X}$$

is called the *liaison addition of* C_1 *and* C_2 *with respect to* F_1 *and* F_2.

We record some of its properties:

Lemma 2.2. *Assume X is an arithmetically Gorenstein scheme and that C_1 and C_2 are codimension 2 equidimensional subschemes. Then the ideal $J = F_2 \cdot I_{C_1,X} + F_1 \cdot I_{C_2,X}$ is a saturated ideal in the coordinate ring $S = R/I_X$ of X. Thus, it is the homogeneous ideal of a subscheme $C \subset X$ which has the following properties*:

(a) *The Hilbert function of C is for all integers j*:

$$h_C(j) = h_{C_1}(j - f_2) + h_{C_2}(j - f_1) + h_Y(j)$$

where Y is the complete intersection defined by (F_1, F_2).

(b) *Let*

$$0 \to \mathcal{L}_i \to \mathcal{N}_i \to \mathcal{I}_{C_i,X} \to 0$$

be an $\mathcal{N}$-type resolution of C_i, $i = 1, 2$, on X. Then C has the following $\mathcal{N}$-type resolution on X

$$0 \to \mathcal{O}_X(-f_1 - f_2) \oplus \mathcal{L}_1(-f_2) \oplus \mathcal{L}_2(-f_1) \to \mathcal{N}_1(-f_2) \oplus \mathcal{N}_2(-f_1) \to \mathcal{I}_{C,X} \to 0.$$

(c) *If* $\dim X = 3$ *with* $d = \deg X$ *and* $\omega_X = \mathcal{O}_X(e)$, *and* C_1, C_2 *are curves of degrees* d_1 *and* d_2 *and arithmetic genera* g_1, g_2, *respectively, then the degree of* C *is*

$$\deg C = d_1 + d_2 + d f_1 f_2$$

and its arithmetic genus is

$$g_C = g_1 + g_2 - 1 + d_1 f_2 + d_2 f_1 + \frac{1}{2} d f_1 f_2 (f_1 + f_2 + e).$$

Proof. Most of the claims are covered by [15, Proposition 4.1]. In any case, the proof given there shows that there is an exact sequence

$$0 \to S(-f_1 - f_2) \to I_{C_1,X}(-f_2) \oplus I_{C_2,X}(-f_1) \to I_{C,X} \to 0.$$

This implies (a) and (c). Claim (b) follows from (a) by noting that the arithmetic genus of the complete intersection Y is $g_Y = \frac{1}{2}df_1f_2(f_1 + f_2 + e) + 1$. $\square$

Corollary 2.3. *The CI-liaison class of C depends only on the CI-liaison classes of C_1, C_2 and the difference $f_1 - f_2$ of the degrees of the hypersurfaces F_1, F_2.*

Proof. Indeed, the $\mathcal{N}$-type resolution of C involves $\mathcal{N}_1(-f_2) \oplus \mathcal{N}_2(-f_1)$, whose stable equivalence class depends only on those of $\mathcal{N}_1, \mathcal{N}_2$ and the difference $f_1 - f_2$. Hence, the claim follows from Rao's theorem [16], [7, Corollary 3.14] that CI-biliaison classes are determined by the sheaf $\mathcal{N}$ appearing in the $\mathcal{N}$-type resolution, up to stable equivalence. $\square$

3. Properties of liaison addition

We hope to use liaison addition for elucidating the structure of a Gorenstein liaison class. To this end it is important to know under what conditions on C_2 the new subscheme C is in the same G-biliaison or even G-liaison class as C_1.

Proposition 3.1.

(a) *If C_2 is licci, then C_1 and C are in the same CI-biliaison class.*
(b) *If C and C_1 are in the same G-biliaison or even G-liaison class, then C_2 must be ACM.*

Proof. (a) If C_2 is licci, then it has an $\mathcal{N}$-type resolution with $\mathcal{N}_2$ dissocié. Hence C_1 and C have $\mathcal{N}$-type resolutions with stably equivalent $\mathcal{N}$.

(b) G-biliaison and even G-liaison preserve deficiency modules, up to twist, so the deficiency modules of C_2 must be all zero, i.e. C_2 is ACM. $\square$

Even though the above result has a very simple proof, it is based on deep theorems and the links are not given explicitly (see however Section 4).

We now weaken the assumption on C_2.

Theorem 3.2. *Let X be a normal arithmetically Gorenstein subscheme of $\mathbb{P}^N$ and let C_1, C_2 be locally Cohen–Macaulay codimension two subschemes of X. Let C be the liaison addition of C_1 and C_2 with respect to forms $F_1 \in I_{C_2}$, $F_2 \in I_{C_1}$.*

(a) *If C_2 is gobilicci then C_1 and C are in the same Gorenstein biliaison class on X.*
(b) *If C_2 is glicci then C_1 and C are in the same even Gorenstein liaison class on X.*

To prove Theorem 3.2 we will use the fact that an $\mathcal{N}$-type resolution of C is obtained essentially as a direct sum of $\mathcal{N}$-type resolutions of C_1 and C_2 (see Lemma 2.2(c) above). Then we use criteria from the papers [5] and [4] respectively characterizing subschemes in the same Gorenstein biliaison (respectively liaison) class to prove the results.

First we need an alternative form of [5, Theorem 3.1] characterizing Gorenstein biliaison equivalence classes.

Proposition 3.3. *Let X be a normal projective arithmetically Gorenstein scheme, and let C_1, C_2 be codimension two subschemes without embedded components in X. Then C_1 and C_2 are in the same G-biliaison equivalence class if and only if they have $\mathcal{N}$-type resolutions*

$$0 \to \mathcal{L}_1 \to \mathcal{N}_1 \to \mathcal{I}_{C_1}(a_1) \to 0,$$
$$0 \to \mathcal{L}_2 \to \mathcal{N}_2 \to \mathcal{I}_{C_2}(a_2) \to 0$$

and there exists an extraverti sheaf $\mathcal{F}$ and exact sequences (*with the same $\mathcal{F}$ on the left!*)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{N}_1 \arrow[r] & \mathcal{E}_1^{\vee\sigma\vee} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{N}_2 \arrow[r] & \mathcal{E}_2^{\vee\sigma\vee} \arrow[r] & 0, \end{tikzcd}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are layered ACM sheaves (*see* [5] *for definition*) *of the same rank, and the rank 1 factors of the layerings of $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic, up to twist, in some order.* (*Here* $^\vee$ *represents dual, and* $^\sigma$ *represents the syzygy sheaf, see* [4].)

Proof. This is obtained by rewriting the result of [5, Theorem 3.1] in terms of the $\mathcal{N}$-type resolutions. Given a sequence

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{I}_C(a) \to 0$$

as in the statement of [5, Theorem 3.1], where $\mathcal{E}$ is an ACM sheaf and $\mathcal{N}$ is just assumed to be coherent (note this is neither an $\mathcal{E}$-type nor an $\mathcal{N}$-type resolution, in spite of the notation!), we proceed as follows.

First take a sequence

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{N} \to 0$$

where $\mathcal{N}'$ is extraverti, which exists by [9, §2]. Then letting $\mathcal{E}'$ be the kernel of $\mathcal{N}' \to \mathcal{I}_C(a)$, we get a new sequence

$$0 \to \mathcal{E}' \to \mathcal{N}' \to \mathcal{I}_C(a) \to 0$$

as above, where now $\mathcal{N}'$ is extraverti. In other words, dropping primes, we may assume that the original $\mathcal{N}$ was extraverti.

Next take the syzygies of $\mathcal{E}^\vee$

$$0 \to \mathcal{E}^{\vee\sigma} \to \mathcal{L} \to \mathcal{E}^\vee \to 0$$

with $\mathcal{L}$ dissocié. Dualize to obtain

$$0 \to \mathcal{E} \to \mathcal{L}^\vee \to \mathcal{E}^{\vee\sigma\vee} \to 0.$$

Now we create a push-out diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{I}_C(a) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{L}^\vee \arrow[r] \arrow[d] & \mathcal{N}' \arrow[r] \arrow[d] & \mathcal{I}_C(a) \arrow[r] & 0. \\
 & \mathcal{E}^{\vee\sigma\vee} \arrow[r, equal] \arrow[d] & \mathcal{E}^{\vee\sigma\vee} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The middle row is then an $\mathcal{N}$-type resolution of C.

To prove the proposition, first let C_1, C_2 be in the same Gorenstein biliaison class. Then there are sequences

$$0 \to \mathcal{E}_1 \to \mathcal{N} \to \mathcal{I}_{C_1}(a_1) \to 0,$$
$$0 \to \mathcal{E}_2 \to \mathcal{N} \to \mathcal{I}_{C_2}(a_2) \to 0$$

as in [5, Theorem 3.1] with the same sheaf $\mathcal{N}$ in the middle. As above, we may assume $\mathcal{N}$ is extraverti. Then performing the push-out construction for $\mathcal{E}_1$ and $\mathcal{E}_2$ as above, we get $\mathcal{N}$-type resolutions for C_1 and C_2 with sheaves $\mathcal{N}_1$ and $\mathcal{N}_2$ (as the $\mathcal{N}'$ above) and exact sequences as desired with $\mathcal{F}$ taken as the $\mathcal{N}$ above.

Conversely, given $\mathcal{N}$-type resolutions $\mathcal{N}_1$ and $\mathcal{N}_2$ for C_1 and C_2 respectively, and given the sheaf $\mathcal{F}$ relating the two as above, for each one create a diagram (dropping subscripts)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{R} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{E}^{\vee\sigma\vee} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{E}^{\vee\sigma\vee} \arrow[r] & 0 \\
 & \mathcal{I}_C(a) \arrow[r, equal] \arrow[d] & \mathcal{I}_C(a) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

defining $\mathcal{R}$ as the kernel of $\mathcal{L} \to \mathcal{E}^{\vee\sigma\vee} \to 0$. Then $\mathcal{R}$ becomes the syzygy sheaf of $\mathcal{E}^{\vee\sigma\vee}$ up to a dissocié and this is just $\mathcal{E}$ up to a dissocié [4, Proposition 4.1(b)]. So the left-hand columns of these two diagrams give the sequences of [5, Theorem 3.1] with $\mathcal{R}, \mathcal{F}$ in place of $\mathcal{E}, \mathcal{N}$. Hence C_1, C_2 are in the same Gorenstein biliaison class. $\square$

Corollary 3.4. *C is gobilicci if and only if it has an $\mathcal{N}$-type resolution whose $\mathcal{N}$ belongs to an exact sequence*

$$0 \to \mathcal{E}' \to \mathcal{N} \to \mathcal{E}^{\vee\sigma\vee} \to 0$$

where $\mathcal{E}$ and $\mathcal{E}'$ are layered ACM sheaves with the same rank 1 factors up to twist and order.

Proof. In the proposition we can take $C_1 = C$ and C_2 to be a complete intersection. Then $\mathcal{N}_2$ is dissocié, so $\mathcal{F}$ becomes just $\mathcal{E}_2$ up to a dissocié and we get the desired result.

Conversely, given this sequence for $\mathcal{N}$, consider the syzygy sequence for $\mathcal{E}'^{\vee}$, compare its dual

$$0 \to \mathcal{E}' \to \mathcal{L}^{\vee} \to \mathcal{E}'^{\vee\sigma\vee} \to 0,$$

and apply Proposition 3.3 to see that C is gobilicci. $\square$

Proof of Theorem 3.2(a). Let C_1 and C_2 have $\mathcal{N}$-type resolutions with sheaves $\mathcal{N}_1, \mathcal{N}_2$. Assuming that C_2 is gobilicci, $\mathcal{N}_2$ admits a sequence

$$0 \to \mathcal{E}' \to \mathcal{N}_2 \to \mathcal{E}^{\vee\sigma\vee} \to 0$$

with $\mathcal{E}, \mathcal{E}'$ as above.

Then by Lemma 2.2 above, C has an $\mathcal{N}$-type resolution with $\mathcal{N} = \mathcal{N}_1(-f_1) \oplus \mathcal{N}_2(-f_2)$. To show that C_1 and C are in the same Gorenstein biliaison class, we apply Proposition 3.3. For simplicity, we drop the twists from the notation.

Let

$$0 \to \mathcal{E}' \to \mathcal{L} \to \mathcal{E}'^{\vee\sigma\vee} \to 0$$

be the dual syzygy sequence for $\mathcal{E}'$. Then $\mathcal{L}$ is dissocié, so C_1 also has an $\mathcal{N}$-type resolution with $\mathcal{N}_1 \oplus \mathcal{L}$ in the middle. We take $\mathcal{F} = \mathcal{N}_1 \oplus \mathcal{E}'$ and use the sequences

$$0 \to \mathcal{F} \to \mathcal{N}_1 \oplus \mathcal{L} \to \mathcal{E}'^{\vee\sigma\vee} \to 0,$$
$$0 \to \mathcal{F} \to \mathcal{N}_1 \oplus \mathcal{N}_2 \to \mathcal{E}^{\vee\sigma\vee} \to 0$$

which show by Proposition 3.3 that C_1 and C are in the same Gorenstein biliaison class. $\square$

Proof of Theorem 3.2(b). This time we use results from [4]. If C_2 is glicci then it has an $\mathcal{N}$-type resolution whose sheaf $\mathcal{N}_2 \cong \mathcal{N}_0 \oplus \mathcal{M}_0$ with $\mathcal{N}_0$ double-layered and $\mathcal{M}_0$ dissocié [4, Corollary 5.3].

Let C_1 have an $\mathcal{N}$-type resolution $\mathcal{N}_1$. Then C has an $\mathcal{N}$-type resolution with $\mathcal{N}_1 \oplus \mathcal{N}_2$ (dropping twists). We apply [4, Proposition 5.1]. Since $\mathcal{N}_0$ is double-layered, we take the filtration given in the definition [4, Definition 4.4], and insert $\mathcal{N}_1$ and $\mathcal{M}_0$ in the middle, to satisfy the criterion of [4, Proposition 5.1] and show that C_1 and C are in the same Gorenstein liaison class. It is an even Gorenstein liaison class because the sheaf in the middle of the filtration is $\mathcal{N}_1$ and not $\mathcal{N}_1^{\sigma\vee}$. □

4. Licci subschemes

It is interesting to see that the links needed in Proposition 3.1 can be described in a very concrete way. We begin with the following preliminary tool.

Proposition 4.1. *Let X be a normal arithmetically Gorenstein subscheme of $\mathbb{P}^n$ and let C_1 and C_2 be codimension two subschemes of X. Choose $B, F \in I_{C_1}$ and $A, G \in I_{C_2}$ such that AF and BG form a regular sequence. We make the following codimension two subschemes*:

(1) *C_1' is the residual to C_1 in the complete intersection (F, B)*;
(2) *C_2' is the residual to C_2 in the complete intersection (G, A)*;
(3) *C is the liaison addition subscheme defined by $I_C = G \cdot I_{C_1} + F \cdot I_{C_2}$*;
(4) *C' is the liaison addition subscheme defined by $I_{C'} = A \cdot I_{C_1'} + B \cdot I_{C_2'}$.*

Then C and C' are directly linked by the complete intersection (AF, BG).

Proof. By hypothesis we have

$$(F, B) : I_{C_1} = I_{C_1'},$$
$$(G, A) : I_{C_2} = I_{C_2'}.$$

We have to show that

$$(AF, BG) : (G \cdot I_{C_1} + F \cdot I_{C_2}) = A \cdot I_{C_1'} + B \cdot I_{C_2'}. \tag{4.1}$$

We proceed in two steps. First we show the inclusion $\supseteq$. To this end, let $H \in I_{C_1'}$. By construction we have $H \cdot I_{C_1} \subset (F, B)$. Then it follows that

$$AH \cdot (G \cdot I_{C_1} + F \cdot I_{C_2}) \subset (AF, BG)$$

since

$$AHG \cdot I_{C_1} \subset (AGF, AGB) \subset (AF, BG)$$

and clearly

$$AHF \cdot I_{C_2} \subset (AF, BG).$$

Now let $K \in I_{C_2'}$. In a completely analogous way we have

$$BK \cdot (G \cdot I_{C_1} + F \cdot I_{C_2}) \subset (AF, BG).$$

We thus have shown the inclusion $\supseteq$ of (4.1).

Now, all ideals under consideration are the saturated ideals of codimension two subschemes of X. Hence the equality of (4.1) will be established if we can show that both sides define schemes of the same degree. Let $f = \deg F$, $g = \deg G$, $a = \deg A$, $b = \deg B$, and $d = \deg X$. Then

$$\deg C = \deg C_1 + \deg C_2 + dfg,$$
$$\deg C' = \deg C_1' + \deg C_2' + dab$$

and we know $\deg C_1 + \deg C_1' = dbf$ and $\deg C_2 + \deg C_2' = dag$, so

$$\begin{aligned}\deg C + \deg C' &= d(bf + ag + fg + ab)\\ &= d(a + f)(b + g),\end{aligned}$$

which completes the proof. □

In the following corollary, the fact that C is evenly CI-linked to C_1 follows immediately from Rao's theorem, as noted above, since the $\mathcal{N}$-type resolutions give bundles that are stably equivalent. The content of this corollary is that we can follow the links in such a precise way.

Corollary 4.2. *Let X be a normal arithmetically Gorenstein subscheme of $\mathbb{P}^n$ and let C_1, C_2 be codimension two subschemes of X. Assume that C_2 is licci, with r minimal generators. Choose $F \in I_{C_1}$ and $G \in I_{C_2}$ such that F and G form a regular sequence, and let C be the liaison addition subscheme defined by the saturated ideal $I_C = G \cdot I_{C_1} + F \cdot I_{C_2}$. Then C is CI-linked to C_1 in an even number of steps. Furthermore, there is a sequence of subschemes*

$$C, Z_{r-1}, Y_{r-1}, Z_{r-2}, Y_{r-2}, \dots, Z_2, Y_2, Z_1, Y_1, D, C_1$$

where

(a) *any two consecutive subschemes in the sequence are directly linked (by complete intersections that we will specify);*
(b) *for $i \geqslant 2$, each Y_i is obtained as the liaison addition of C_1 with a licci subscheme with i minimal generators;*
(c) *Each Z_i is obtained as the liaison addition of a (fixed) subscheme directly linked to C_1 with a licci subscheme;*
(d) *Y_1 is a basic double link of C_1, i.e. the liaison addition of C_1 with the trivial subscheme.*

Proof. By Rao's theorem, a codimension two licci subscheme Y of X has a minimal free R/I_X-resolution of the form

$$0 \to L_2 \to L_1 \to I_Y \to 0.$$

By a standard trick due to Gaeta (in modern language this is shown via mapping cones—cf. [13]), we have the following possibilities for linking Y. Suppose that $F_1, F_2 \in I_Y$ are a regular sequence, linking Y to a residual subscheme Y' of X.

(1) If F_1 and F_2 are both minimal generators of I_Y then neither is a minimal generator of $I_{Y'}$. In this case $I_{Y'}$ has one less minimal generator than does I_Y.

(2) If F_1 is a minimal generator of I_Y but F_2 is not, then F_1 is again a minimal generator of $I_{Y'}$, but F_2 is not. In this case $I_{Y'}$ has the same number of minimal generators as does I_Y.
(3) If neither F_1 nor F_2 are minimal generators of I_Y then both are minimal generators of $I_{Y'}$. In this case $I_{Y'}$ has one more minimal generator than does I_Y.

Now, in our situation, we will show that C can be linked in two steps to a codimension two subscheme C'' that is the liaison addition of C_1 and a licci subscheme C_2'' whose ideal has one fewer minimal generator than does I_{C_2}. The result will then follow by induction.

Choose $B \in I_{C_1}$ and $A \in I_{C_2}$ such that A and B form a regular sequence, and such that furthermore A is a minimal generator of I_{C_2}. As before, let C_1 be directly linked to C_1' by (F, B) and let C_2 be directly linked to C_2' by (G, A). Then by Proposition 4.1, C is directly linked via (AF, BG) to the liaison addition subscheme C' corresponding to the ideal $A \cdot I_{C_1'} + B \cdot I_{C_2'}$.

Now replace the data (C_1, C_2, F, G, B, A) by the data $(C_1', C_2', B, A, F, A')$ where A' is a minimal generator of $I_{C_2'}$. We get that C_2' is directly linked by (A, A') to a subscheme C_2'', C_1' is directly linked by (B, F) back to C_1, and C' is directly linked by $(A'B, FA)$ to a subscheme C'' defined by the liaison addition $I_{C''} = A' \cdot I_{C_1} + F \cdot I_{C_2''}$.

Now we consider the number of minimal generators of I_{C_2} and $I_{C_2''}$. Since G may or may not have been a minimal generator of I_{C_2}, while A was a minimal generator, we have two possibilities.

If G was a minimal generator of I_{C_2} then $I_{C_2'}$ has one fewer minimal generator than does I_{C_2}, but then $I_{C_2''}$ has the same number of minimal generators as $I_{C_2'}$, which is one less than I_{C_2}.

If G was not a minimal generator of I_{C_2} then $I_{C_2'}$ has the same number of minimal generators as I_{C_2}, but then A and A' are both minimal generators of $I_{C_2'}$, so $I_{C_2''}$ has one fewer minimal generator. Note that neither A nor A' are minimal generators of $I_{C_2''}$.

By induction, we arrive in an even number of steps to the liaison addition of C_1 and a complete intersection, C_2. As we have seen above, but using the notation of Proposition 4.1, the polynomial $G \in I_{C_2}$ used in the liaison addition is not one of the minimal generators of I_{C_2}. One more link as we did above results in the subscheme C' consisting of the liaison addition of C_1' with another complete intersection, but this time the polynomial $A \in I_{C_2'}$ used in the liaison addition *is* a minimal generator of $I_{C_2'}$. We will write $I_{C_2'} = (A, A')$, where A and A' have no common factor.

So at this stage we are considering the subscheme C' which is a liaison addition of the form

$$A \cdot I_{C_1'} + B \cdot (A', A).$$

This is linked by the complete intersection $(A'B, FA)$ to the subscheme C'' defined by $A' \cdot I_{C_1} + (F)$, since C_2' is linked by (A, A') to the trivial ideal R/I_X. But C' is precisely a basic double link ideal, which is linked in two steps to C_1. $\square$

Remark 4.3. In the proof of Corollary 4.2, we used in a heavy way the theory of liaison for codimension two licci ideals. We wonder about the following questions.

(1) Since Proposition 4.1 did not assume that C_2 was licci, it should have additional applications. If we start with *any* liaison addition of subschemes C_1 and C_2, can we explicitly link it in a finite number of steps to a suitable liaison addition of a *minimal* element in the even liaison class of C_1 and a minimal element in the even liaison class of C_2?

(2) Liaison addition and basic double linkage have been developed for higher codimension [2, 6,15] in a way that is very similar to the codimension two picture. Can Proposition 4.1 and Corollary 4.2 also be extended to higher codimension?

5. Curves on quadric threefolds in $\mathbb{P}^4$—toward a Lazarsfeld–Rao-type structure

There is a beautiful structure theorem for an even liaison class of codimension two subschemes. It was discovered for curves in $\mathbb{P}^3$ by Martin-Deschamps and Perrin [12] and for codimension two subschemes in $\mathbb{P}^n$ by Ballico, Bolondi and Migliore [1], based on a conjecture of Harris and a special case proved by Lazarsfeld and Rao. It has been extended to codimension two subschemes of arithmetically Gorenstein varieties in [3] and in a more general way in [15] and in [7], but always for even CI-liaison. It was pointed out in [13] that extending this property to Gorenstein liaison will be difficult. Here, we study the question in two special cases.

Non-singular quadrics

Let X be a non-singular quadric 3-fold in $\mathbb{P}^4$. The only surfaces on X are complete intersections, so for curves in X, Gorenstein biliaison is equivalent to CI-biliaison, which is also equivalent to even CI-liaison. The even CI-liaison class of a curve C is determined by a triple (M, P, α), where $M = H^1_*(\mathcal{I}_C)$ is the Rao module of C, P is a maximal Cohen–Macaulay module over the homogeneous coordinate ring of X, say $S = H^0_*(\mathcal{O}_X) = R/Q$ (where Q is the defining polynomial of X), and $\alpha : P^\vee \to M^* \to 0$ is a surjective map of graded S-modules [7, Corollary 4.3]. This triple is determined up to isomorphism and shift of degrees for M, up to stable equivalence and (the same) shift of degrees for P, and compatible maps α.

On the other hand, the even G-liaison class of a curve on X is determined by the Rao module alone (up to shift)—cf. [4, Theorem 6.2].

For each CI-biliaison equivalence class we have the traditional Lazarsfeld–Rao theorem [7, Theorem 3.4]. Each even Gorenstein liaison class is a union of CI-biliaison classes, so we can ask what kind of structure the even Gorenstein liaison class can have. That is, how are the CI subclasses related?

We will show that for ACM curves, which form one even Gorenstein liaison class since their Rao modules are zero, we can obtain all the non-licci curves by a combination of CI-biliaisons and liaison additions with a line, starting from a line. On the other hand, for an even Gorenstein liaison class of curves with non-zero Rao module, it is not possible to obtain them all from a single one, or even from minimal ones, by CI-biliaisons and liaison additions with ACM curves (see Remark 5.4 and Example 5.5).

Theorem 5.1. *Every CI-biliaison class of non-licci ACM curves on the non-singular quadric 3-fold X contains a minimal curve C that can be obtained by liaison addition with a line from a minimal curve of lower degree in another such class, unless C is already a line.*

The proof requires some preparation. Let $L \subset X$ be a line. Let $\mathcal{E}_0$ be the locally free sheaf defined by the minimal $\mathcal{N}$-type resolution of L:

$$0 \to \mathcal{O}_X \to \mathcal{E}_0 \to \mathcal{I}_{L,X}(1) \to 0.$$

Then, according to [4], each non-licci CI-liaison class of ACM curves on X corresponds via its $\mathcal{N}$-type resolution to the stable equivalence class of one of the sheaves

$$\mathcal{N}_{0,a_2,\dots,a_r} := \mathcal{E}_0 \oplus \mathcal{E}_0(-a_2) \oplus \mathcal{E}_0(-a_2-a_3) \oplus \cdots \oplus \mathcal{E}_0(-a_2-\cdots-a_r)$$

where $a_2,\dots,a_r \geqslant 0$ are integers. Note that $\mathcal{N}_0 = \mathcal{E}_0$.

Using liaison addition we first construct curves that we will then show to be minimal in their CI-biliaison classes.

Lemma 5.2. *For each $a_2,\dots,a_r \geqslant 0$ there is a curve $C_{0,a_2,\dots,a_r}$ with $\mathcal{N}$-type resolution*

$$\begin{aligned}0 &\to \big(\mathcal{O}_X \oplus \mathcal{O}_X^2(-a_2) \oplus \cdots \oplus \mathcal{O}_X^2(-a_2-\cdots-a_r)\big)(-r)\\ &\to \mathcal{N}_{0,a_2,\dots,a_r}(-r) \to \mathcal{I}_{C_{0,a_2,\dots,a_r},X} \to 0\end{aligned}$$

and with $\mathcal{E}$-type resolution:

$$\begin{aligned}0 &\to \mathcal{N}_{0,a_2,\dots,a_r}(-r-1)\\ &\to \big(\mathcal{O}_X^3 \oplus \mathcal{O}_X^2(-a_2) \oplus \cdots \oplus \mathcal{O}_X^2(-a_2-\cdots-a_r)\big)(-r) \to \mathcal{I}_{C_{0,a_2,\dots,a_r},X} \to 0.\end{aligned}$$

Furthermore, for $r \geqslant 2$ the curve $C_{0,a_2,\dots,a_r}$ can be obtained by liaison addition with a line from $C_{0,a_3,\dots,a_r}$.

Proof. We use induction on r. For $r=1$, we take C_0 to be a line L. Its $\mathcal{N}$- and $\mathcal{E}$-type resolutions are well known. If $r \geqslant 2$, then the $\mathcal{E}$-type resolution of $C_{0,a_3,\dots,a_r}$ shows that its homogeneous ideal contains an element F_2 of degree $r-1$. Let $F_1 \in I_{L,X}$ be an element of degree $a_2+1 \geqslant 1$, such that $\{F_1, F_2\}$ is a regular sequence. Then Lemma 2.2 shows that the curve $C_{0,a_2,\dots,a_r}$ defined by the liaison addition ideal $F_2 \cdot I_L + F_1 \cdot I_{C_{0,a_3,\dots,a_r}}$ has the required $\mathcal{N}$-type resolution. The $\mathcal{E}$-type resolution of the curve $C_{0,a_2,\dots,a_r}$ can be obtained using the syzygy sequence

$$0 \to \mathcal{E}_0(-1) \to \mathcal{O}_X^4 \to \mathcal{E}_0 \to 0$$

and the method of converting an $\mathcal{N}$-type resolution to an $\mathcal{E}$-type resolution described in [4, Proposition 4.3(a)] (see also [14]).

This proves the lemma, since all these curves have been constructed by liaison addition with a line, starting with $C_0 = L$ which is a line. $\square$

We now show that the constructed curves are minimal in their CI-biliaison classes.

Lemma 5.3. *Each curve $C_{0,a_2,\dots,a_r}$ described in Lemma 5.2 is minimal in its CI-biliaison class.*

Proof. For this lemma we will change the above notation and rewrite the $\mathcal{N}$-type resolution of $C = C_{0,a_2,\dots,a_r}$ as

$$0 \to \mathcal{O}_X^{2r_1-1} \oplus \mathcal{O}_X^{2r_2}(-b_2) \oplus \cdots \oplus \mathcal{O}_X(-b_m)^{2r_m} \to \mathcal{N} \to \mathcal{I}_{C,X}(r) \to 0$$

where $\mathcal{N} = \mathcal{E}_0^{r_1} \oplus \mathcal{E}_0(-b_2)^{r_2} \oplus \cdots \oplus \mathcal{E}_0(-b_m)^{r_m}$ is of total rank $r = 2\sum r_i$ and $0 = b_1 < b_2 < \cdots < b_m$. To prove that C is minimal, it is enough to show that if

$$0 \to \bigoplus_{i=1}^{2r-1} \mathcal{O}_X(-c_i) \to \mathcal{N} \to \mathcal{I}_{D,X}(s) \to 0$$

is the $\mathcal{N}$-type resolution of any other curve D on X, then

$$\sum_{i=1}^{2r-1} c_i \geqslant 2\sum_{j=1}^{m} r_j b_j.$$

We may assume that $c_1 \leqslant \cdots \leqslant c_{2r-1}$. It follows from the existence of D that for any subset J of $\{1, \ldots, 2r-1\}$, the cokernel of the map

$$\alpha_J : \bigoplus_{i \in J} \mathcal{O}_X(-c_i) \to \mathcal{N}$$

is torsion free. Let $J := \{c_i \mid c_i < b_2\}$. Then the image of α_J lands inside the sheaf $\mathcal{E}_0^{r_1}$, so the rank of α_J must be less than $2r_1$. This means that for all $i \geqslant 2r_1$, we have $c_i \geqslant b_2$.

Next, let $J := \{c_i \mid c_i < b_3\}$. Then the image of α_J lands inside the sheaf $\mathcal{E}_0^{r_1} \oplus \mathcal{E}_0(-b_2)^{r_2}$, and the same argument shows that, for all $i \geqslant 2(r_1 + r_2)$, we must have $c_i \geqslant b_3$.

Continuing in this fashion, the inequalities on the c_i's and b_j's show that $\sum c_i \geqslant 2\sum r_j b_j$, as required. □

The above theorem follows now easily.

Proof of Theorem 5.1. Each non-licci CI-biliaison class of ACM curves on X corresponds to one of the sheaves $\mathcal{N}_{0,a_2,\ldots,a_r}$ of rank $\geqslant 2$. Thus, the theorem follows by combining Lemmas 5.2 and 5.3 and the Lazarsfeld–Rao property for CI-biliaison classes. □

Remark 5.4. In the case of the traditional Lazarsfeld–Rao property, Bolondi and Migliore [3, Corollary 4.10] have shown that one can get from a minimal scheme in a codimension two CI-liaison class to any other (up to flat deformation) by liaison addition with a licci scheme. In general, the analogous result is not true in even Gorenstein liaison classes.

Example 5.5. To see that a method similar to that used for ACM curves cannot work with non-ACM classes of curves on X, consider the even Gorenstein liaison class of curves with Rao module $M = k$. Among these there is one CI-biliaison class corresponding to a triple (M, P, α), where P is a non-free maximal Cohen–Macaulay module over S, and $\alpha : P^\vee \to M^* \to 0$. There is another CI-biliaison class corresponding to the triple (M, S, β), where we think of S as the free rank 1 S-module and $\beta : S \to k \to 0$ the natural map [7, Corollary 4.3]. Since liaison addition acts as direct sum on Rao modules and on $\mathcal{N}$-type resolutions, one sees easily that it also acts as direct sums on the triples (M, P, α). An ACM curve will have triple $(0, Q, \sigma)$. Adding this to the first curve above will give a triple $(M, P \oplus Q, \alpha)$, where α acts by 0 on the Q factor. Similarly, adding to the second will give $(M, S \oplus Q, \beta)$, with β acting as 0 on the Q-factor. So it is clear that no combination of liaison additions with ACM curves will ever connect these two types of curves with Rao module k, because in one case k is covered by a non-free maximal Cohen–Macaulay module and in the other case by a free maximal Cohen–Macaulay module.

Singular quadrics with one double point

Let X be the singular quadric threefold in $\mathbb{P}^4$ having just one double point. In this case two curves are in the same Gorenstein biliaison equivalence class if and only if their Rao modules are isomorphic up to shift [5, Theorem 6.2]. It follows that Gorenstein biliaison is the same as even Gorenstein liaison in this case. Indeed, we know in general that any Gorenstein biliaison is an even Gorenstein liaison. Conversely, if two curves are in the same even Gorenstein liaison class, then their Rao modules are isomorphic up to shift, and so by the theorem above they are equivalent for Gorenstein biliaison.

Thus, having the operation of Gorenstein biliaison available, one might hope, as in the case of the traditional Lazarsfeld–Rao theorem, that in any Gorenstein biliaison equivalence class, every curve could be obtained by a finite sequence of ascending Gorenstein biliaisons from one, or a small number of "minimal" curves in the class. The following example shows that this is too ambitious.

Example 5.6. For ACM curves on X, the natural choice for minimal curves would be a line or a conic. There are three types of lines: those contained in a D-plane, those contained in an E-plane, or those passing through the double point of X, where D and E refer to the rulings over a general hyperplane section Q of X (which is a smooth quadric surface), and we think of X as a cone over Q. These lines have $\mathcal{N}$-type resolution using $\mathcal{E}_1, \mathcal{E}_1'$ or $\mathcal{I}_D \oplus \mathcal{I}_E$ respectively, in the notation of [5, Theorem 6.2]. However, we will exhibit here an infinite sequence of ACM curves that do not admit any descending Gorenstein biliaisons.

Take two D-planes in X, say D_1 and D_2. They meet only at the singular point P of X. Take curves $C_1 \subseteq D_1$, $C_2 \subseteq D_2$ of degrees d and e, respectively, each passing simply through the common point P of the two planes. Consider the exact sequence

$$0 \to I_{C_1} \cap I_{C_2} \to I_{C_1} \oplus I_{C_2} \to I_{C_1} + I_{C_2} \to 0.$$

Note that $I_{C_1} + I_{C_2} = I_P$, and that C_1 and C_2 are both ACM, being plane curves. Then sheafifying this sequence and taking cohomology, it is immediate to see that $C = C_1 \cup C_2$ is an ACM curve of degree $d + e$. Assuming that $e \geqslant 2$ and $d \geqslant e + 2$, we will show that C does not admit any descending Gorenstein biliaison in X.

So suppose that C admits a descending Gorenstein biliaison on an ACM surface Y in X. We distinguish four cases.

Case 1. $Y = D_1 \cup D_2 \cup F$ for some other surface F. This is impossible, because there is nothing in F to subtract a hyperplane section from.

Case 2. $Y = D_1 \cup F$, where F is some other surface containing C_2, but not containing D_2. Letting (a, b) be the bidegree of F on X, i.e. $F \sim aD + bE$ on X, we see that $b \geqslant e$ by intersecting F with D_2. On the other hand, since C_2 passes only simply through P, we must have $a > 0$. Then the degree of F is $a + b > e$, and we cannot subtract a hyperplane section of F from C_2.

Case 3. $Y = F \cup D_2$. This is similar to Case 2.

Case 4. Y does not contain either D_1 or D_2. Then Y has bidegree (a, b) with $b \geqslant d$, and since Y is an ACM surface on X, we must have $|a - b| \leqslant 1$. Hence

$$\deg Y = a + b \geqslant (b - 1) + b \geqslant (d - 1) + d > e + d,$$

because of our hypotheses on d and e. Hence we cannot bilink down on Y.

Remark 5.7. As in the case of the non-singular quadric threefold, one could ask whether every ACM curve can be obtained by a succession of complete intersection bilinks and liaison additions, for example with a line. On the non-singular quadric threefold X, there is only one non-trivial indecomposable ACM sheaf, up to twist, namely $\mathcal{E}_0$, and it corresponds to a line. Thus any ACM sheaf can be obtained by adding direct sums of twists of this one to a dissocié sheaf and this explains why the method shown above works in this case.

On the singular quadric threefold, there are two infinite sequence of indecomposable ACM sheaves, $\mathcal{E}_\ell$ and $\mathcal{E}'_\ell$ for $\ell = 1, 2, \ldots$ [5, proof of Theorem 6.2], so in order to formulate an analogous result, one would have to allow (at least) liaison additions with plane curves of all degrees in both D-planes and E-planes, either passing or not passing through the singular point P.

Example 5.8. Now we consider curves with Rao module k. In $\mathbb{P}^4$ the curves with minimal leftward shift of k, namely k in degree 0, have been classified [11], [8, Proposition 4.1]. They exist in any degree $d \geqslant 2$, and the general such curve is the disjoint union of a plane curve of degree $d - 1$ and a line not meeting the plane of the first curve. Curves of this kind for every $d \geqslant 2$ can be found on the singular quadric threefold X, so we take these as the minimal curves. One might hope that curves whose Rao module has a shift into positive degrees of the module k could be obtained by ascending Gorenstein biliaison from these minimal curves. We show this is not the case by exhibiting some non-singular curves of degree 5 and genus 0 on X that have Rao module k in degree 1 and do not admit any descending Gorenstein biliaison on X.

We begin by recalling some basic facts about degree 5 and genus 0 curves C in $\mathbb{P}^4$ [8, Example 4.3]. Such curves can be obtained by generic projection from the rational normal curve in $\mathbb{P}^5$. As long as the curve is non-degenerate (i.e. not contained in any $\mathbb{P}^3$), we find from Riemann–Roch that $h^0(\mathcal{I}_C(2)) = 4$. Taking two general quadric hypersurfaces containing C, the curve C will be contained in their intersection Y, a degree 4 Del Pezzo surface, which will be non-singular provided C is general. Conversely, on the Del Pezzo surface Y, we can find smooth curves in the divisor classes $(2; 1, 0^4)$, $(3; 2, 1^2, 0^2)$, $(4; 2^3, 1, 0)$, and $(5; 3, 2^3, 1)$ (and their permutations), which we will denote by C_1, C_2, C_3, C_4, respectively. Here we use the standard notation for divisor classes on Y [8, Notation 3.3]. There is no difference between these curves as curves in $\mathbb{P}^4$. However, as curves on Y they are distinguished by their divisor classes on Y.

Next we observe that each general non-singular, non-degenerate curve C in $\mathbb{P}^4$ of degree 5 and genus 0 has a unique trisecant. Indeed, any trisecant of C must lie in every quadric hypersurface containing C and therefore on the Del Pezzo surface Y. Then, checking each of the lines on Y, one finds exactly one trisecant for each C_i on Y.

We can also find non-singular degree 5 genus 0 curves C_5 on the rational cubic scroll S in the divisor class $(4; 3)$, and this is the only possibility. This curve C_5 meets a fiber $F = (1; 1)$ of the ruling in one point, and it meets the exceptional curve $E = (0; 1)$ three times. Thus, the ruling determines an isomorphism of C_5 to the projective line E. This isomorphism is uniquely determined by the three intersection points of C_5 with E (which are necessarily distinct), and we can recover the surface S as the closure of the union of lines joining corresponding points on C_5 and E. This shows that C_5 lies on a unique cubic scroll. Since we saw earlier that each general degree 5 genus 0 curve in $\mathbb{P}^4$ has a unique trisecant, it follows that every such curve C is contained in this way in a cubic scroll.

On the cubic scroll, the divisor class $C - H = (2; 2)$ contains a union of two fibers of the rulings. The passage from $C - H$ to C is a Gorenstein biliaison in $\mathbb{P}^4$, hence C has Rao module k in degree 1, and as a curve in $\mathbb{P}^4$ it does admit a descending Gorenstein biliaison [8, Example 4.3].

Now we consider non-singular degree 5 genus 0 curves C on the singular quadric threefold X in $\mathbb{P}^4$. There are cubic scrolls S in X, having bidegrees $(2, 1)$ and $(1, 2)$. If C is in S, then its projection $\pi(C)$ from the singular point P of X to a general hyperplane section Q, which is a non-singular quadric surface in $\mathbb{P}^3$, is isomorphic to C because the projection maps S to Q birationally, blowing up the point P and blowing down the ruling through P, which meets C just once. Hence $\pi(C)$ is a curve of bidegree $(1, 4)$ or $(4, 1)$ on Q.

There are also Del Pezzo surfaces Y on X, having bidegree $(2, 2)$. The intersections of the two families of planes on X with Y are conics adding to a hyperplane section of Y, and without loss of generality we can take these to be $\Gamma = (1; 1, 0^4)$ and $\Gamma' = (2; 0, 1^4)$. For any curve C on X, the two intersection numbers $C.\Gamma, C.\Gamma'$ will give the bidegree of the projection $\pi(C)$ of C onto the hyperplane section Q. Thus we find that $\pi(C_1)$ and $\pi(C_2)$ have bidegree $(4, 1)$ or $(1, 4)$, whereas $\pi(C_3)$ and $\pi(C_4)$ have bidegree $(3, 2)$ or $(2, 3)$. Since we saw above that a $(5, 0)$ curve on a cubic scroll must project onto a curve with bidegree $(1, 4)$ or $(4, 1)$ on Q, the curves C_3 and C_4 cannot be contained in any cubic scroll on X.

Now, finally, we show that neither C_3 nor C_4 admit any descending biliaison on X. Indeed, let C be one of these two curves and suppose that C is contained in an ACM surface T in X and that $C - H$ is effective on T. Then $\deg(C - H) = 5 - \deg T$ must be at least 2, since any curve of degree 1 is ACM. So $\deg T \leqslant 3$. The degree cannot be 2, since C is not contained in a hyperplane. We conclude that $\deg T = 3$. However, the only irreducible surfaces of degree 3 in X are the cubic scroll, which does not contain C, and the cone over a twisted cubic, which contains no non-singular curves of degree 5 and genus 0. Hence a descending Gorenstein biliaison of C is not possible on X.

Note of course that either curve C_3 or C_4 is contained in a unique cubic scroll in $\mathbb{P}^4$, but this argument shows that in this case the cubic scroll lies outside of X, and intersects X only in the curve C_i $(i = 3, 4)$ together with its trisecant.

We close this section by wondering if the concept of liaison addition can be extended. This could potentially be useful to address some of the problems we encountered above.

Remark 5.9. Let C_1, C_2 be two codimension 2 subschemes of some projective scheme $X \subset \mathbb{P}^n$ with Rao modules M_1 and M_2. Then the liaison addition with respect to hypersurfaces of degrees d_1, d_2 is a curve with Rao module $M_1(-d_2) \oplus M_2(-d_1)$. Assume now that N is any graded module corresponding to an extension

$$0 \to M_2(-d_1) \to N \to M_1(-d_2) \to 0.$$

Is it then possible to construct a curve C starting directly from C_1 and C_2 such that the Rao module of C is isomorphic to N? If the answer is affirmative, this could possibly provide a natural extension of liaison addition and it would be justified to call the curve C a *liaison extension* of C_1 and C_2.

6. Conclusion

Our motivation for the work in this paper was to investigate Questions 1 and 2 from the introduction: What is the structure of a Gorenstein biliaison class or an even Gorenstein liaison class of codimension 2 subschemes of an arithmetically Gorenstein subscheme? To address this question we employ the idea of liaison addition. Liaison addition has been used to investigate CI-liaison classes. Here we show that it can also be used to study Gorenstein liaison classes.

Our two main test cases have been the non-singular quadric 3-fold and the singular quadric 3-fold with one double point in $\mathbb{P}^4$, because in each case we have available explicit descriptions of CI-biliaison classes as well as of Gorenstein biliaison and even liaison classes of curves.

We found a satisfactory answer for ACM curves on the non-singular quadric 3-fold where the non-licci ACM curves can all be obtained starting from a single line by successive liaison additions with a line and CI-biliaisons. One could perhaps get an analogous result for ACM curves on the singular quadric 3-fold, using liaison additions with plane curves of all degrees in both families. This gives some hope for liaison addition as a key operation in explaining the structure of an even Gorenstein liaison class. However, simple examples show that this alone is not sufficient to deal with the case of non-ACM curves. Thus, we wonder if liaison addition can be extended to liaison extension.

On the singular quadric 3-fold we have available the method of Gorenstein biliaison, and one might hope to reach any curve by ascending Gorenstein biliaison from a suitable class of "minimal" curves. For curves with non-zero Rao module, the natural definition would be those whose Rao module has the left-most shift. But examples show that with this definition there are non-minimal curves with no descending biliaisons. For ACM curves, we have found infinitely many classes of curves with no descending biliaisons, so there does not appear to be a suitable class of "minimal" curves from which all others can be obtained by ascending biliaisons.

In summary, some new ideas will be necessary to give satisfying answers to Questions 1 and 2 in general.

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