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Cohomology of twisted tensor products ☆

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ABSTRACT

It is well known that the cohomology of a tensor product is essentially the tensor product of the cohomologies. We look at twisted tensor products, and investigate to which extent this is still true. We give an explicit description of the Ext-algebra of the tensor product of two modules, and under certain additional conditions, describe an essential part of the Hochschild cohomology ring of a twisted tensor product. As an application, we characterize precisely when the cohomology groups over a quantum complete intersection are finitely generated over the Hochschild cohomology ring. Moreover, both for quantum complete intersections and in related cases we obtain a lower bound for the representation dimension of the algebra.

1. Introduction

Given a field k and two k-algebras Λ and Γ, one may look at their tensor product $\Lambda \otimes_k \Gamma$. This is an algebra where multiplication is done componentwise. In other words, we use the multiplications in Λ and Γ, and define elements of Λ and Γ to commute with one another. Given a Λ-module M and a Γ-module N, it is well known that

$$\operatorname{Ext}^*_{\Lambda\otimes_k\Gamma}(M \otimes_k N, M \otimes_k N) = \operatorname{Ext}^*_\Lambda(M, M) \mathbin{\overline{\otimes}_k} \operatorname{Ext}^*_\Gamma(N, N),$$

$$\operatorname{HH}^*(\Lambda \otimes_k \Gamma) = \operatorname{HH}^*(\Lambda) \mathbin{\overline{\otimes}_k} \operatorname{HH}^*(\Gamma),$$

where $\overline{\otimes}$ is the usual tensor product, but with elements of odd degree anticommuting (and where HH^* denotes the Hochschild cohomology ring).

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In this paper we shall study graded algebras and twisted tensor products. That is, for two graded algebras Λ and Γ, we give their tensor product an algebra structure by defining elements from Λ and Γ to commute up to certain scalars, depending on the degrees of the elements. We denote these twisted tensor products by $\Lambda \otimes_k^t \Gamma$. Examples of algebras obtained in this way are quantum exterior algebras (see [5,6,10]), and, more generally, quantum complete intersections (see [4,7,8]).

The first main result of this paper (Theorem 3.7) shows that the first formula above holds for twisted tensor products. More precisely, we may make the identification

$$\operatorname{Ext}^*_{\Lambda\otimes_k^t\Gamma}\big(M \otimes_k^t N, M \otimes_k^t N\big) = \operatorname{Ext}^*_\Lambda(M, M) \otimes_k^{\tilde{t}} \operatorname{Ext}^*_\Gamma(N, N),$$

where the twist on the right-hand side is the combination of the twist we started with and the sign which already occurred in the classical case. This formula allows us to give an explicit description of the Ext-algebra of the simple module over a quantum complete intersection in Theorem 5.3. As for the second formula above, we shall see (Remark 5.4) that in general it does not carry over to twisted tensor products. However, in Theorem 4.7 we show that the Hochschild cohomology ring of a twisted tensor product contains a subalgebra, which is the twisted tensor product of corresponding subalgebras of the Hochschild cohomology rings of the factors. Under certain additional conditions, we show that these subalgebras are big enough to contain all the information on finite generation and complexity (Corollary 4.8). When finite generation holds, we may use these subalgebras and a result from [6] to find a lower bound for the representation dimension of the twisted tensor product.

In the final section we apply these results to quantum complete intersections. In particular, we show that the cohomology groups of such an algebra are all finitely generated over the Hochschild cohomology ring if and only if all the commutator parameters are roots of unity (Theorem 5.5). This allows us to give a lower bound for the representation dimension of these algebras (Corollary 5.6), thus generalizing the result of [8].

2. Notation

Throughout this paper, we fix a field k. All algebras considered are assumed to be associative k-algebras.

2.1. Definition. Let A be an abelian group. An *A-graded* algebra is an algebra Λ together with a decomposition $\Lambda = \bigoplus_{a\in A} \Lambda_a$ as k-vector spaces, such that $\Lambda_a \cdot \Lambda_{a'} \subseteq \Lambda_{a+a'}$. A module M over such a graded algebra Λ is a *graded module* if it has a decomposition $M = \bigoplus_{a\in A} M_a$ as k-vector spaces, such that $\Lambda_a \cdot M_{a'} \subseteq M_{a+a'}$. We denote the category of finitely generated graded Λ-modules by $\Lambda\text{-mod}_{\mathrm{gr}}$.

Let Λ, A and M be as above. We denote the degree of homogeneous elements $\lambda \in \Lambda$ and $m \in M$ by $|\lambda|$ and $|m|$, respectively. For an element $a \in A$ we denote by $M\langle a\rangle$ the shift of M having the same Λ-module structure as M, but with $M\langle a\rangle_{a'} = M_{a'-a}$. Now let M' be another graded Λ-module. To distinguish between graded and ungraded morphisms, we denote the set of all Λ-morphisms from M to M' by $\operatorname{Hom}_\Lambda(M, M')$, and the set of degree preserving morphisms by $\operatorname{grHom}_\Lambda(M, M')$. With this notation we obtain a decomposition

$$\operatorname{Hom}_\Lambda\big(M, M'\big) = \bigoplus_{a\in A} \operatorname{grHom}_\Lambda\big(M, M'\langle a\rangle\big).$$

Setting $\operatorname{Hom}_\Lambda(M, M')_a = \operatorname{grHom}_\Lambda(M, M'\langle a\rangle)$ turns $\operatorname{End}_\Lambda(M)$ and $\operatorname{End}_\Lambda(M')$ into A-graded algebras, and $\operatorname{Hom}(M, M')$ into a graded $\operatorname{End}_\Lambda(M)$–$\operatorname{End}_\Lambda(M')$ bimodule. Since M has a graded projective resolution $\mathbb{P}$, we can also define $\operatorname{Ext}^{i,a}_\Lambda(M, M') \overset{\text{def}}{=} H^i(\operatorname{grHom}(\mathbb{P}, M'\langle a\rangle))$. It follows that $\operatorname{Ext}^*_\Lambda(M, M)$ and $\operatorname{Ext}^*_\Lambda(M', M')$ are $(\mathbb{Z} \oplus A)$-graded algebras, and that $\operatorname{Ext}^*_\Lambda(M, M')$ is a graded $\operatorname{Ext}^*_\Lambda(M, M)$–$\operatorname{Ext}^*_\Lambda(M', M')$ bimodule.

Our main objects of study in this paper are twisted tensor products of two graded algebras, a concept we now define.

2.2. Definition/Construction. Let A and B be abelian groups, let Λ be an A-graded algebra and Γ a B-graded algebra. Let $t : A \otimes_{\mathbb{Z}} B \longrightarrow k^{\times}$ be a homomorphism of abelian groups, where $k^{\times}$ denotes the multiplicative group of nonzero elements in k. We write $t^{\langle a|b\rangle} = t(a \otimes b)$, and, by abuse of notation, also $t^{\langle \lambda|\gamma\rangle} = t^{\langle |\lambda||\gamma|\rangle}$ for homogeneous elements $\lambda \in \Lambda$ and $\gamma \in \Gamma$. The (t-)*twisted tensor product* of Λ and Γ is the algebra $\Lambda \otimes_k^t \Gamma$ defined by

$$\Lambda \otimes_k^t \Gamma = \Lambda \otimes_k \Gamma \quad \text{as } k\text{-vector spaces},$$

$$(\lambda \otimes \gamma) \cdot^t (\lambda' \otimes \gamma') \stackrel{\text{def}}{=} t^{\langle \lambda'|\gamma\rangle} \lambda\lambda' \otimes \gamma\gamma',$$

where $\lambda, \lambda' \in \Lambda$ and $\gamma, \gamma' \in \Gamma$ are homogeneous elements.

A straightforward calculation shows that this is indeed a well defined algebra. By defining $(\Lambda \otimes_k^t \Gamma)_{a,b}$ to be $\Lambda_a \otimes_k \Gamma_b$, this algebra becomes $(A \oplus B)$-graded.

We now define what it means for an algebra to have finitely generated cohomology.

2.3. Definition. Let Λ be an algebra. A commutative *ring of cohomology operators* is a commutative $\mathbb{Z}$-graded k-algebra H together with graded k-algebra morphisms $\phi_M : H \longrightarrow \operatorname{Ext}^*_\Lambda(M, M)$, for every $M \in \Lambda$-mod, such that for every pair $M, M' \in \Lambda$-mod the induced H-module structures on $\operatorname{Ext}^*_\Lambda(M, M')$ via ϕ_M and $\phi_{M'}$ coincide. If A is an abelian group and Λ is A-graded, then we require that H be a $(\mathbb{Z} \oplus A)$-graded algebra, and that the maps ϕ_M are morphisms of $(\mathbb{Z} \oplus A)$-graded algebras.

The main example of such a ring H is the even Hochschild cohomology ring of an algebra. Namely, by [18] the Hochschild cohomology ring is graded commutative, so its even part is commutative. Note that whenever an algebra is graded, then so is its Hochschild cohomology ring, and its even part is a commutative ring of graded cohomology operators.

2.4. Definition. An algebra Λ satisfies the *finite generation hypothesis* **Fg** if it has a commutative ring of operators H which is Noetherian and of finite type (i.e. $\dim_k H_i < \infty$ for all i), and such that for any $M, M' \in \Lambda$-mod the H-module $\operatorname{Ext}^*_\Lambda(M, M')$ is finitely generated.

Group algebras of finite groups and finite-dimensional complete intersections are examples of algebras satisfying **Fg** (cf. [11] and [3]). For more general finite-dimensional algebras, this concept was first studied in [9], using the Hochschild cohomology ring. We end this section with some remarks concerning finite generation of cohomology.

2.5. Remarks. (i) Assume that Λ is a finite-dimensional algebra, and let H be a commutative Noetherian ring of cohomology operators. Then all $\operatorname{Ext}^*_\Lambda(M, M')$ are finitely generated over H if and only if $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{Rad}\Lambda, \Lambda/\operatorname{Rad}\Lambda)$ is. This follows from an induction argument on the length of M and M'.

(ii) By [16, Proposition 5.7] the following are equivalent for an algebra Λ.

(1) Λ satisfies **Fg** with respect to its even Hochschild cohomology ring $\mathrm{HH}^{2*}(\Lambda)$,
(2) Λ satisfies **Fg** with respect to some subalgebra of its even Hochschild cohomology ring.

3. Tensor products of graded modules

Throughout this section, we fix two abelian groups A and B, together with an A-graded algebra Λ and a B-graded algebra Γ. Moreover, we fix a homomorphism $t : A \otimes_{\mathbb{Z}} B \longrightarrow k^{\times}$ of abelian groups. Given a graded Λ-module and a graded Γ-module, we construct a $\Lambda \otimes_k^t \Gamma$-module, and study homomorphisms and extensions between such modules.

3.1. Definition/Construction. Given modules $M \in \Lambda\text{-mod}_{\mathrm{gr}}$ and $N \in \Gamma\text{-mod}_{\mathrm{gr}}$, the tensor product $M \otimes_k N$ becomes a graded $\Lambda \otimes_k^t \Gamma$-module by defining

$$(\lambda \otimes \gamma) \cdot (m \otimes n) \stackrel{\text{def}}{=} t^{\langle m|\gamma \rangle} \lambda m \otimes \gamma n.$$

We denote this module by $M \otimes_k^t N$, its grading is given by $(M \otimes_k^t N)_{a,b} = M_a \otimes_k N_b$.

We now prove some elementary results on these tensor products, the first of which shows that the tensor product of two shifted modules is the shifted tensor product.

3.2. Lemma. *Given modules $M \in \Lambda\text{-mod}_{\mathrm{gr}}$ and $N \in \Gamma\text{-mod}_{\mathrm{gr}}$, the graded $\Lambda \otimes_k^t \Gamma$-modules $M\langle a \rangle \otimes_k^t N\langle b \rangle$ and $(M \otimes_k^t N)\langle a, b \rangle$ are isomorphic via the map*

$$\begin{gathered} M\langle a \rangle \otimes_k^t N\langle b \rangle \longrightarrow \big(M \otimes_k^t N\big)\langle a, b \rangle, \\ m \otimes n \longmapsto t^{\langle a|n \rangle} m \otimes n. \end{gathered}$$

Proof. The given map is clearly bijective, and it is straightforward to verify that it is a homomorphism. $\square$

The following lemma shows that the tensor product of projective modules is again projective. Given a graded algebra Δ, we denote by $\Delta\text{-proj}$ the category of finitely generated projective Δ-modules, and by $\Delta\text{-proj}_{\mathrm{gr}}$ the category of finitely generated graded projective Δ-modules.

3.3. Lemma. *Given modules $P \in \Lambda\text{-proj}_{\mathrm{gr}}$ and $Q \in \Gamma\text{-proj}_{\mathrm{gr}}$, the tensor product $P \otimes_k^t Q$ is a graded projective $\Lambda \otimes_k^t \Gamma$-module.*

Proof. By Lemma 3.2 we only have to consider the case $P = \Lambda$ and $Q = \Gamma$. In this case $P \otimes_k^t Q = \Lambda \otimes_k^t \Gamma$, so the lemma holds. $\square$

As the following result shows, the tensor product of morphism spaces is the morphism space of tensor products.

3.4. Lemma. *Given modules $M, M' \in \Lambda\text{-mod}_{\mathrm{gr}}$ and $N, N' \in \Gamma\text{-mod}_{\mathrm{gr}}$, the natural map*

$$\operatorname{grHom}_{\Lambda}(M, M') \otimes_k \operatorname{grHom}_{\Gamma}(N, N') \longrightarrow \operatorname{grHom}_{\Lambda \otimes_k^t \Gamma}\big(M \otimes_k^t N, M' \otimes_k^t N'\big)$$

is an isomorphism.

Proof. If $M = \Lambda\langle a \rangle$ and $N = \Gamma\langle b \rangle$ for some $a \in A$ and $b \in B$, then

$$\begin{aligned} \operatorname{grHom}_{\Lambda}\big(\Lambda\langle a \rangle, M'\big) \otimes_k \operatorname{grHom}_{\Gamma}\big(\Gamma\langle b \rangle, N'\big) &= M'_{-a} \otimes_k N'_{-b} \\ &= \big(M' \otimes_k^t N'\big)_{-a,-b} \\ &= \operatorname{grHom}\big(\big(\Lambda \otimes_k^t \Gamma\big)\langle a, b \rangle, M' \otimes_k^t N'\big) \\ &= \operatorname{grHom}\big(\big(\Lambda\langle a \rangle \otimes_k^t \Gamma\langle b \rangle\big), M' \otimes_k^t N'\big). \end{aligned}$$

Now note that both sides commute with cokernels in the M and N position. $\square$

Note that given degree a and b morphisms $\varphi : M \longrightarrow M'$ and $\psi : N \to N'$, we obtain a degree (a,b)-morphism

$$\varphi \otimes \psi : M \otimes N \longrightarrow M' \otimes N'$$

by composing the maps from Lemmas 3.4 and 3.2. Explicitly, the map is given by

$$(m \otimes n) \cdot (\varphi \otimes \psi) = t^{\langle \varphi | n \rangle} m \cdot \varphi \otimes n \cdot \psi$$

(we think of a module as a right module over its endomorphism ring). By applying this to the situation $M = M'$ and $N = N'$, we obtain the following result, showing that the endomorphism ring of a tensor product is the tensor product of the endomorphism rings.

3.5. Lemma. *Let $M \in \Lambda\text{-mod}_{\mathrm{gr}}$ and $N \in \Gamma\text{-mod}_{\mathrm{gr}}$. Then*

$$\operatorname{End}_{\Lambda \otimes_k^t \Gamma}\big(M \otimes_k^t N\big) = \operatorname{End}_\Lambda(M) \otimes_k^t \operatorname{End}_\Gamma(N).$$

As for projective resolutions, the behavior is also as expected. Namely, the following result shows that the tensor product of two projective resolutions is again a projective resolution.

3.6. Lemma. *Given modules $M \in \Lambda\text{-mod}_{\mathrm{gr}}$ and $N \in \Gamma\text{-mod}_{\mathrm{gr}}$ with graded projective resolutions*

$$\mathbb{P} : \cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$
$$\mathbb{Q} : \cdots \longrightarrow Q_i \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0,$$

the total complex of $\mathbb{P} \otimes_k^t \mathbb{Q}$ is a graded projective resolution of $M \otimes_k^t N$.

Proof. By Lemma 3.3 all the terms of the total complex $\operatorname{Tot}(\mathbb{P} \otimes_k^t \mathbb{Q})$ of $\mathbb{P} \otimes_k^t \mathbb{Q}$ are projective. Moreover, since k is a field $\operatorname{Tot}(\mathbb{P} \otimes_k^t \mathbb{Q})$ is exact. $\square$

We are now ready to prove the main result of this section. It shows that the Ext-algebra of a tensor product is the tensor product of the Ext-algebras.

3.7. Theorem. *If $M, M' \in \Lambda\text{-mod}_{\mathrm{gr}}$ and $N, N' \in \Gamma\text{-mod}_{\mathrm{gr}}$ are modules, then*

$$\operatorname{Ext}^*_{\Lambda \otimes_k^t \Gamma}\big(M \otimes_k^t N, M \otimes_k^t N\big) = \operatorname{Ext}^*_\Lambda(M, M) \otimes_k^{\tilde{t}} \operatorname{Ext}^*_\Gamma(N, N),$$

with $\tilde{t}((i,a),(j,b)) = (-1)^{ij} t^{\langle a|b \rangle}$. Moreover

$$\operatorname{Ext}^*_{\Lambda \otimes_k^t \Gamma}\big(M \otimes_k^t N, M' \otimes_k^t N'\big) = \operatorname{Ext}^*_\Lambda(M, M') \otimes_k^{\tilde{t}} \operatorname{Ext}^*_\Gamma(N, N')$$

as $\operatorname{Ext}^*_\Lambda(M,M) \otimes_k^t \operatorname{Ext}^*_\Gamma(N,N)$–$\operatorname{Ext}^*_\Lambda(M',M') \otimes_k^t \operatorname{Ext}^*_\Gamma(N',N')$ bimodule.

Proof. Let $\mathbb{P}$ and $\mathbb{Q}$ be graded projective resolutions of M and N receptively. Then by Lemma 3.6 $\operatorname{Tot}(\mathbb{P} \otimes_k^t \mathbb{Q})$ is a projective resolution of $M \otimes_k^t N$, and therefore

$$\begin{aligned}\operatorname{Ext}^*_{\Lambda\otimes_k^t\Gamma}\big(M\otimes_k^t N, M'\otimes_k^t N'\big) &= H^*\big(\operatorname{Hom}_{\Lambda\otimes_k^t\Gamma}\big(\operatorname{Tot}\big(\mathbb{P}\otimes_k^t\mathbb{Q}\big), M'\otimes_k^t N'\big)\big)\\ &= H^*\big(\operatorname{Tot}\big(\operatorname{Hom}_{\Lambda\otimes_k^t\Gamma}\big(\mathbb{P}\otimes_k^t\mathbb{Q}\big), M'\otimes_k^t N'\big)\big)\\ &= H^*\big(\operatorname{Tot}\big(\operatorname{Hom}_\Lambda(\mathbb{P}, M')\otimes_k^t\operatorname{Hom}_\Gamma(\mathbb{Q}, N')\big)\big)\\ &= \operatorname{Ext}^*_\Lambda(M, M')\otimes_k^{\tilde{t}}\operatorname{Ext}^*_\Gamma(N, N').\end{aligned}$$

Here the third equality holds by Lemma 3.4, whereas the final one holds since k is a field. The multiplication is induced by the multiplication of morphisms in Lemma 3.5, with the additional signs needed because of the signs added when passing from the double complex to its total complex. $\square$

We end this section with the following result, which was shown in [17, Corollary 3.3 and Lemma 3.4] for untwisted tensor products (in which case we may forget about the grading). It will help us find upper bounds for the representation dimension of twisted tensor products. Given an algebra Δ, we denote by $\operatorname{gld}\Delta$ its global dimension.

3.8. Proposition. *Let $M \in \Lambda\text{-mod}_{\mathrm{gr}}$ and $N \in \Gamma\text{-mod}_{\mathrm{gr}}$ be graded modules, such that M generates and cogenerates $\Lambda\text{-mod}$, and such that N generates and cogenerates $\Gamma\text{-mod}$. Then $M\otimes_k^t N$ is a generator-cogenerator of $\Lambda\otimes_k^t\Gamma\text{-mod}$, and* $\operatorname{gld}\operatorname{End}_{\Lambda\otimes_k^t\Gamma}(M\otimes_k^t N) = \operatorname{gld}\operatorname{End}_\Lambda(M) + \operatorname{gld}\operatorname{End}_\Gamma(N)$.

4. Tensor products of bimodules

Throughout this section, we keep the notation from the last section. That is, we fix two abelian groups A and B, together with an A-graded algebra Λ and a B-graded algebra Γ. Moreover, we fix a homomorphism $t: A\otimes_{\mathbb{Z}} B \longrightarrow k^\times$ of abelian groups. Given an algebra Δ, we denote by Δ^{e} its enveloping algebra $\Delta\otimes_k\Delta^{\mathrm{op}}$. Note that if Δ is G-graded, where G is some abelian group, then so is Δ^{e}, and Δ is a graded Δ^{e}-module.

4.1. Definition/Construction. Given modules $X \in \Lambda^{\mathrm{e}}\text{-mod}_{\mathrm{gr}}$ and $Y \in \Gamma^{\mathrm{e}}\text{-mod}_{\mathrm{gr}}$, the tensor product $X\otimes_k Y$ becomes a graded $(\Lambda\otimes_k^t\Gamma)^{\mathrm{e}}$-module by defining

$$(\lambda\otimes\gamma)(x\otimes y)(\lambda'\otimes\gamma') \overset{\text{def}}{=} t^{\langle x|\gamma\rangle}t^{\langle\lambda'|y\rangle}t^{\langle\lambda'|\gamma\rangle}\lambda x\lambda'\otimes\gamma y\gamma'.$$

We denote this bimodule by $X\otimes_k^t Y$.

4.2. Remark. In general the graded $(\Lambda\otimes_k^t\Gamma)^{\mathrm{e}}$-modules $X\langle a\rangle\otimes_k^t Y\langle b\rangle$ and $(X\otimes_k^t Y)\langle a, b\rangle$ are not isomorphic. To see this, take $\Lambda = k[x]/(x^2)$ and $\Gamma = k$, both $\mathbb{Z}$-graded, and with x in degree one. Furthermore, choose a nonzero element $q \in k$, and define a homomorphism $t: x\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z} \to k^\times$ by $t^{\langle a|b\rangle} = q^{ab}$. If the $(\Lambda\otimes_k^t\Gamma)^{\mathrm{e}}$-modules $\Lambda\langle a\rangle\otimes_k^t\Gamma\langle b\rangle$ and $(\Lambda\otimes_k^t\Gamma)\langle a, b\rangle$ were isomorphic, then there would exist an isomorphism

$$\big(\Lambda\otimes_k^t\Gamma\big)\langle a, b\rangle \xrightarrow{f} \Lambda\langle a\rangle\otimes_k^t\Gamma\langle b\rangle$$

such that

$$1\otimes 1 \mapsto \alpha(1\otimes 1) + \beta(x\otimes 1),$$

where α and β are scalars with $\alpha \neq 0$. But then

$$\begin{aligned} q^b\alpha(x\otimes 1) &= f(1\otimes 1)\cdot(x\otimes 1)\\ &= f(x\otimes 1)\\ &= (x\otimes 1)\cdot f(1\otimes 1)\\ &= \alpha(x\otimes 1)\end{aligned}$$

in $\Lambda\langle a\rangle \otimes_k^t \Gamma\langle b\rangle$, hence when $q^b \neq 1$ the modules cannot be isomorphic.

The following results are analogues of Lemmas 3.3, 3.4 and 3.6. We prove only the first result, as the proofs of the other two results are more or less the same as those of Lemmas 3.4 and 3.6.

4.3. Lemma. *Given modules $X \in \Lambda^{\mathrm{e}}\text{-proj}_{\mathrm{gr}}$ and $Y \in \Gamma^{\mathrm{e}}\text{-proj}_{\mathrm{gr}}$, the tensor product $X \otimes_k^t Y$ is a graded projective $(\Lambda \otimes_k^t \Gamma)^{\mathrm{e}}$-module.*

Proof. It suffices to show that $\Lambda^{\mathrm{e}}\langle a\rangle \otimes_k^t \Gamma^{\mathrm{e}}\langle b\rangle$ is graded projective for any $a \in A$ and $b \in B$. This can be seen by noting that the map

$$\begin{aligned} \big(\Lambda \otimes_k^t \Gamma\big)^{\mathrm{e}}\langle a, b\rangle &\longrightarrow \Lambda^{\mathrm{e}}\langle a\rangle \otimes_k^t \Gamma^{\mathrm{e}}\langle b\rangle,\\ (l\otimes g)\otimes(l'\otimes g') &\longmapsto t^{\langle l'|g\rangle} t^{\langle a|g\rangle} t^{\langle l'|b\rangle}(l\otimes l')\otimes(g\otimes g')\end{aligned}$$

is an isomorphism of graded $(\Lambda \otimes_k^t \Gamma)^{\mathrm{e}}$-modules. □

4.4. Lemma. *Given modules $X, X' \in \Lambda^{\mathrm{e}}\text{-mod}_{\mathrm{gr}}$ and $Y, Y' \in \Gamma^{\mathrm{e}}\text{-mod}_{\mathrm{gr}}$, the natural map*

$$\operatorname{grHom}_{\Lambda^{\mathrm{e}}}(X, X') \otimes_k \operatorname{grHom}_{\Gamma^{\mathrm{e}}}(Y, Y') \longrightarrow \operatorname{grHom}_{(\Lambda\otimes_k^t\Gamma)^{\mathrm{e}}}\big(X \otimes_k^t Y, X' \otimes_k^t Y'\big)$$

is an isomorphism.

4.5. Lemma. *Given modules $X \in \Lambda^{\mathrm{e}}\text{-mod}_{\mathrm{gr}}$ and $Y \in \Gamma^{\mathrm{e}}\text{-mod}_{\mathrm{gr}}$ with graded projective bimodule resolutions*

$$\begin{aligned} \mathbb{P}: \cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0,\\ \mathbb{Q}: \cdots \longrightarrow Q_i \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow Y \longrightarrow 0,\end{aligned}$$

the total complex of $\mathbb{P} \otimes_k^t \mathbb{Q}$ is a graded projective bimodule resolution of $X \otimes_k^t Y$.

Now note that for a fixed $b \in B$ the map t induces a morphism $t^{\langle -|b\rangle} : A \longrightarrow k^\times$ (and similarly for a fixed $a \in A$). With this notation, we make the following observation.

4.6. Lemma. *Let $a' \in \bigcap_{b\in B} \operatorname{Ker} t^{\langle -|b\rangle} \leqslant A$ and $b' \in \bigcap_{a\in A} \operatorname{Ker} t^{\langle a|-\rangle} \leqslant B$. Then the map*

$$\begin{aligned} \Lambda\langle a'\rangle \otimes_k^t \Gamma\langle b'\rangle &\longrightarrow \big(\Lambda \otimes_k^t \Gamma\big)\langle a, b\rangle,\\ \lambda\otimes\gamma &\longmapsto \lambda\otimes\gamma\end{aligned}$$

is an isomorphism of graded $(\Lambda \otimes_k^t \Gamma)^{\mathrm{e}}$-modules.

Using the above notation, we now prove the main result of this section. It shows that Hochschild cohomology commutes with twisted tensor products, provided we only consider the graded parts corresponding to the subgroups $\bigcap_{b\in B} \operatorname{Ker} t^{\langle -|b\rangle} \leqslant A$ and $\bigcap_{a\in A} \operatorname{Ker} t^{\langle a|-\rangle} \leqslant B$.

4.7. Theorem. *Let $A' = \bigcap_{b \in B} \operatorname{Ker} t^{\langle -|b\rangle} \leqslant A$ and $B' = \bigcap_{a \in A} \operatorname{Ker} t^{\langle a|-\rangle} \leqslant B$. Then there is an isomorphism*

$$\mathrm{HH}^{*,A'}(\Lambda) \otimes_k^{(-1)^{**}} \mathrm{HH}^{*,B'}(\Gamma) \longrightarrow \mathrm{HH}^{*,A' \oplus B'}\bigl(\Lambda \otimes_k^t \Gamma\bigr),$$

where $(-1)^{**}$ denotes the morphism mapping $((i,a'),(j,b'))$ to $(-1)^{ij}$.

Proof. Let $\mathbb{P}$ and $\mathbb{Q}$ be graded bimodule projective resolutions of Λ and Γ, respectively. Given $a \in A$ and $b \in B$, the same arguments as in the proof of Theorem 3.7 give

$$\begin{aligned}\mathrm{HH}^{*,a,b}\bigl(\Lambda \otimes_k^t \Gamma\bigr) &= H^*\bigl(\operatorname{grHom}_{(\Lambda \otimes_k^t \Gamma)^{\mathrm{e}}}\bigl(\operatorname{Tot}\bigl(\mathbb{P} \otimes_k^t \mathbb{Q}, \bigl(\Lambda \otimes_k^t \Gamma\bigr)\langle a,b\rangle\bigr)\bigr)\bigr) \\ &= H^*\bigl(\operatorname{Tot}\bigl(\operatorname{Hom}_{(\Lambda \otimes_k^t \Gamma)^{\mathrm{e}}}\bigl(\mathbb{P} \otimes_k^t \mathbb{Q}, \bigl(\Lambda \otimes_k^t \Gamma\bigr)\langle a,b\rangle\bigr)\bigr)\bigr)\end{aligned}$$

and

$$\begin{aligned}&H^*\bigl(\operatorname{Tot}\bigl(\operatorname{Hom}_{(\Lambda \otimes_k^t \Gamma)^{\mathrm{e}}}\bigl(\mathbb{P} \otimes_k^t \mathbb{Q}, \Lambda\langle a\rangle \otimes_k^t \Gamma\langle b\rangle\bigr)\bigr)\bigr) \\ &\quad = H^*\bigl(\operatorname{Tot}\bigl(\operatorname{Hom}_{\Lambda^{\mathrm{e}}}\bigl(\mathbb{P}, \Lambda\langle a\rangle\bigr) \otimes_k^t \operatorname{Hom}_{\Gamma^{\mathrm{e}}}(\mathbb{Q}, \Gamma)\bigr)\bigr) \\ &\quad = \mathrm{HH}^{*,a}(\Lambda) \otimes_k^{\tilde{t}} \mathrm{HH}^{*,b}(\Gamma),\end{aligned}$$

where $\tilde{t} = (-1)^{**} \cdot t$ as in Theorem 3.7. Now if $a \in A'$ and $b \in B'$, then from Lemma 4.6 we see that we may identify

$$H^*\bigl(\operatorname{Tot}\bigl(\operatorname{Hom}_{(\Lambda \otimes_k^t \Gamma)^{\mathrm{e}}}\bigl(\mathbb{P} \otimes_k^t \mathbb{Q}, \bigl(\Lambda \otimes_k^t \Gamma\bigr)\langle a,b\rangle\bigr)\bigr)\bigr)$$

with

$$H^*\bigl(\operatorname{Tot}\bigl(\operatorname{Hom}_{(\Lambda \otimes_k^t \Gamma)^{\mathrm{e}}}\bigl(\mathbb{P} \otimes_k^t \mathbb{Q}, \Lambda\langle a\rangle \otimes_k^t \Gamma\langle b\rangle\bigr)\bigr)\bigr).$$

Finally, note that $\mathrm{HH}^{*,A'}(\Lambda) \otimes_k^{\tilde{t}} \mathrm{HH}^{*,B'}(\Gamma) = \mathrm{HH}^{*,A'}(\Lambda) \otimes_k^{(-1)^{**}} \mathrm{HH}^{*,B'}(\Gamma)$, since all degrees occurring are in the kernel of t. $\square$

We end this section with the following corollary to Theorem 4.7. It shows that, given certain conditions, if Λ and Γ satisfy **Fg**, then so does $\Lambda \otimes_k^t \Gamma$.

4.8. Corollary. *With the same notation as in Theorem* 4.7, *assume Λ and Γ satisfy* **Fg** *with respect to their even Hochschild cohomolgy rings $\mathrm{HH}^{2*}(\Lambda)$ and $\mathrm{HH}^{2*}(\Gamma)$. Moreover, suppose $[A : A']$ and $[B : B']$ are finite, and that $\Lambda/\operatorname{Rad}\Lambda$ and $\Gamma/\operatorname{Rad}\Gamma$ are separable over k. Then $\Lambda \otimes_k^t \Gamma$ satisfies* **Fg** *with respect its even Hochschild cohomolgy ring $\mathrm{HH}^{2*}(\Lambda \otimes_k^t \Gamma)$.*

Proof. Since $[A : A']$ is finite, the algebra $\mathrm{HH}^{2*,A}(\Lambda)$ is a finitely generated module over $\mathrm{HH}^{2*,A'}(\Lambda)$. Therefore, since Λ satisfies **Fg**, we see that $\operatorname{Ext}_\Lambda^*(\Lambda/\operatorname{Rad}\Lambda, \Lambda/\operatorname{Rad}\Lambda)$ is finitely generated over $\mathrm{HH}^{2*,A'}(\Lambda)$. The same arguments apply to Γ, hence

$$\operatorname{Ext}_\Lambda^*(\Lambda/\operatorname{Rad}\Lambda, \Lambda/\operatorname{Rad}\Lambda) \otimes_k^t \operatorname{Ext}_\Gamma^*(\Gamma/\operatorname{Rad}\Gamma, \Gamma/\operatorname{Rad}\Gamma)$$

is finitely generated over $\mathrm{HH}^{2*,A'}(\Lambda) \otimes_k \mathrm{HH}^{2*,B'}(\Gamma)$. Then by Theorems 3.7 and 4.7, we see that

$$\operatorname{Ext}_{\Lambda \otimes_k^t \Gamma}^*\bigl(\Lambda/\operatorname{Rad}\Lambda \otimes_k^t \Gamma/\operatorname{Rad}\Gamma, \Lambda/\operatorname{Rad}\Lambda \otimes_k^t \Gamma/\operatorname{Rad}\Gamma\bigr)$$

must be a finitely generated $\mathrm{HH}^{2*,A'\oplus B'}(\Lambda \otimes_k^t \Gamma)$-module. Finally, since $\Lambda/\operatorname{Rad}\Lambda$ and $\Gamma/\operatorname{Rad}\Gamma$ are separable over k, the equality

$$\Lambda/\operatorname{Rad}\Lambda \otimes_k^t \Gamma/\operatorname{Rad}\Gamma = \big(\Lambda \otimes_k^t \Gamma\big)/\operatorname{Rad}\big(\Lambda \otimes_k^t \Gamma\big)$$

holds. The claim now follows from Remarks 2.5. □

5. Quantum complete intersections

We now apply the cohomology theory of twisted tensor products to the class of finite-dimensional algebras known as *quantum complete intersections*. Throughout this section, fix integers $n \geqslant 1$ and $a_1, \dots, a_n \geqslant 2$, together with a nonzero element $q_{ij} \in k$ for every $1 \leqslant i < j \leqslant n$. We define the algebra Λ by

$$\Lambda \stackrel{\text{def}}{=} k\langle x_1, \dots, x_n\rangle/\big(x_i^{a_i}, x_j x_i - q_{ij} x_i x_j\big),$$

a codimension n quantum complete intersection in its most general form. This is a selfinjective algebra of dimension $\prod a_i$. We shall determine precisely when such an algebra satisfies **Fg**, and consequently obtain a lower bound for its representation dimension.

Note that Λ is $\mathbb{Z}^n$ graded by $|x_i| \stackrel{\text{def}}{=} (0, \dots, 1, \dots, 0)$, the ith unit vector. In particular, we use the $\mathbb{Z}$-grading $|x| = 1$ for the special case of a codimension one quantum complete intersection $k[x]/(x^a)$. The following observation allows us to study the cohomology inductively, starting with the well known case $k[x]/(x^a)$.

5.1. Lemma. *Let Λ' be the subalgebra of Λ generated by $x_1, \dots, x_{n-1}$. Then*

$$\Lambda = \Lambda' \otimes_k^t k[x_n]/\big(x_n^{a_n}\big),$$

where $t^{\langle d_1,\dots,d_{n-1}|d_n\rangle} \stackrel{\text{def}}{=} \prod_{i=1}^{n-1} q_{in}^{d_i d_n}$.

As for quantum complete intersections of codimension one, that is, truncated polynomial algebras, their cohomology is well known. We record this in the following lemma.

5.2. Lemma. *If $\Gamma = k[x]/(x^a)$, then*

(1) $\mathrm{HH}^{2*}(\Gamma) = k[x,z]/(x^a, a x^{a-1} z)$,

(2) $\operatorname{Ext}_\Gamma^*(k,k) = \begin{cases} k[y,z]/(y^2 = z) & \text{if } a = 2, \\ k[y,z]/(y^2) & \text{if } a \neq 2, \end{cases}$

with $|x| = 0$, $|y| = 1$ and $|z| = 2$. In particular, the algebra Γ satisfies **Fg** *with respect to its even Hochschild cohomology ring.*

Proof. The first part is [12, Theorem 3.2], the second part can be read off directly from the projective resolution. □

Using this lemma and Theorem 3.7, we obtain the following result on the Ext-algebra of the simple module of a quantum complete intersection.

5.3. Theorem. *The* Ext-*algebra of k is given by*

$$\operatorname{Ext}_\Lambda^*(k,k) = k\langle y_1, \dots, y_n, z_1, \dots, z_n\rangle/I,$$

where I is the ideal in $k\langle y_1,\dots,y_n,z_1,\dots,z_n\rangle$ defined by the relations

$$\begin{pmatrix} y_iz_i - z_iy_i & \\ y_jy_i + q_{ij}y_iy_j & i<j \\ y_jz_i - q_{ij}^{a_i}z_iy_j & i<j \\ z_jy_i - q_{ij}^{a_j}y_iz_j & i<j \\ z_jz_i - q_{ij}^{a_ia_j}z_iz_j & i<j \\ y_i^2 = z_i & a_i = 2 \\ y_i^2 & a_i \neq 2 \end{pmatrix}.$$

5.4. Remark. Lemma 5.2 shows that if Γ and Δ are arbitrary algebras, then the algebra $\mathrm{HH}^*(\Gamma) \otimes_k^{\tilde t} \mathrm{HH}^*(\Delta)$ does not in general embed into $\mathrm{HH}^*(\Gamma \otimes_k^t \Delta)$. Namely, the latter is always graded commutative, whereas $\mathrm{HH}^*(\Gamma) \otimes_k^{\tilde t} \mathrm{HH}^*(\Delta)$ need not be.

We are now ready to characterize precisely when a quantum complete intersection satisfies **Fg**.

5.5. Theorem. *The following are equivalent.*

(1) *Λ satisfies* **Fg**,
(2) *Λ satisfies* **Fg** *with respect to its even Hochschild cohomology ring* $\mathrm{HH}^{2*}(\Lambda)$,
(3) *all the commutators q_{ij} are roots of unity.*

Proof. The implication (2) $\Rightarrow$ (1) is obvious, and the implication (3) $\Rightarrow$ (2) follows from Corollary 4.8. To show (1) $\Rightarrow$ (3), we assume that (1) holds but not (3), so there are i and j such that q_{ij} is not a root of unity. By (1), the Ext-algebra of k is finitely generated as a module over its center, hence so is every quotient of this ring. By factoring out all y_k, z_k with $k \notin \{i,j\}$ and $\{y_k \mid k \in \{i,j\}$ and $y_k^2 = 0\}$, we obtain a ring of the form $k\langle r,s\rangle/(sr - qrs)$, where q is not a root of unity. The center of this ring is trivial, hence the ring cannot be finitely generated over its center, a contradiction. $\square$

As a corollary, we obtain a lower bound for the representation dimension of a quantum complete intersection. Recall that the representation dimension of a finite-dimensional algebra Δ is defined as

$$\operatorname{repdim} \Delta \stackrel{\text{def}}{=} \inf\bigl\{\operatorname{gld} \operatorname{End}_\Delta(M)\bigr\},$$

where the infimum is taken over all the finitely generated Δ-modules which generate and cogenerate Δ-mod.

5.6. Corollary. *Define the integer $c \geqslant 0$ by*

$$c \stackrel{\text{def}}{=} \max\bigl\{\operatorname{card} I \bigm| I \subseteq \{1,\dots,n\} \text{ and } q_{ij} \text{ is a root of unity } \forall i,j \in I,\ i<j\bigr\}.$$

Then $\operatorname{repdim} \Lambda \geqslant c+1$. *In particular, if all the commutators q_{ij} are roots of unity, then* $\operatorname{repdim} \Lambda \geqslant n+1$.

In order to prove this result we need to recall some notions. Let Δ be an algebra, and let M be a finitely generated Δ-module with minimal projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

say. The *complexity* of M, denoted $\operatorname{cx} M$, is defined as

$$\operatorname{cx} M \stackrel{\text{def}}{=} \inf\bigl\{t \in \mathbb{N} \cup \{0\} \bigm| \exists r \in \mathbb{R} \text{ such that } \dim_k P_n \leqslant rn^{t-1} \text{ for } n \gg 0\bigr\}.$$

Now let V be a positively graded k-vector space of finite type, i.e. $\dim_k V_n < \infty$ for all n. The *rate of growth* of V, denoted $\gamma(V)$, is defined as

$$\gamma(V) \stackrel{\text{def}}{=} \inf\bigl\{t \in \mathbb{N} \cup \{0\} \bigm| \exists r \in \mathbb{R} \text{ such that } \dim_k V_n \leqslant rn^{t-1} \text{ for } n \gg 0\bigr\}.$$

It is well known that the complexity of a module M equals $\gamma(\operatorname{Ext}^*_\Delta(M, \Delta/\operatorname{Rad}\Delta))$. Now suppose Δ is selfinjective, and denote by $\Delta\text{-}\underline{\mathrm{mod}}$ the stable module category of Δ-mod, that is, the category obtained from Δ-mod by factoring out all morphisms which factor through a projective module. This is a triangulated category, and we denote by $\dim(\Delta\text{-}\underline{\mathrm{mod}})$ its dimension, as defined in [15].

Proof of Corollary 5.6. Choose a subset I of $\{1, \dots, n\}$ realizing the maximum in the definition of the integer c, and let Λ' be the subalgebra of Λ generated by the x_i with $i \in I$. By Theorem 5.5 the algebra Λ' satisfies **Fg**, and so from [6, Theorem 3.1] we see that $\dim(\Lambda'\text{-}\underline{\mathrm{mod}}) \geqslant \operatorname{cx}_{\Lambda'} k - 1$. Moreover, by Theorem 5.3 the complexity of k as a Λ'-module equals $\operatorname{card} I$, giving $\dim(\Lambda'\text{-}\underline{\mathrm{mod}}) \geqslant \operatorname{card} I - 1$.

The forgetful functor $\Lambda\text{-mod} \longrightarrow \Lambda'\text{-mod}$ is exact, dense, and maps projective Λ-modules to projective Λ'-modules. Therefore it induces a dense triangle functor $\Lambda\text{-}\underline{\mathrm{mod}} \longrightarrow \Lambda'\text{-}\underline{\mathrm{mod}}$, and so from [15, Lemma 3.4] we obtain the inequality $\dim(\Lambda\text{-}\underline{\mathrm{mod}}) \geqslant \dim(\Lambda'\text{-}\underline{\mathrm{mod}})$. Finally, by [14, Proposition 3.7] the inequality $\operatorname{repdim}\Lambda \geqslant \dim(\Lambda\text{-}\underline{\mathrm{mod}}) + 2$ holds, and the proof is complete. □

5.7. Remark. By [8, Theorem 3.2] the inequality $\operatorname{repdim}\Lambda \leqslant 2n$ always holds.

It was shown in [13] that the representation dimension of the truncated polynomial algebra $k[x, y]/(x^2, y^a)$ is three. Using their construction and exactly the same proof, one can show that the quantum complete intersection $\Gamma = k\langle x, y\rangle/(yx - qxy, x^2, y^a)$ has a generator-cogenerator M which is graded with $\operatorname{gld}\operatorname{End}_\Gamma(M) = 3$. Moreover, for a quantum exterior algebra Γ on n variables (that is, a codimension n quantum complete intersection where all the defining exponents are 2), the global dimension of the endomorphism ring of the graded generator-cogenerator $\bigoplus \Gamma/(\operatorname{Rad}\Gamma)^i$ is $n + 1$ (cf. [1]). Using this and Proposition 3.8, we obtain the following improvement of Remark 5.7.

5.8. Theorem. *If $h = \operatorname{card}\{i \mid a_i = 2\}$, then*

$$\operatorname{repdim}\Lambda \leqslant \begin{cases} 2n - h & \text{if } h \leqslant n/2, \\ 2n - h + 1 & \text{if } h > n/2. \end{cases}$$

Proof. In the first case decompose the algebra into h parts of the form $k\langle x, y\rangle/(yx - qxy, x^2, y^a)$, and $n - 2h$ parts of the form $k[x]/(x^a)$. Adding up the global dimensions of the endomorphism rings of the graded Auslander generators (which we may do by Proposition 3.8), we obtain $h \cdot 3 + (n - 2h) \cdot 2 = 2n - h$. In the second case, we decompose the algebra into $n - h$ parts of the form $k\langle x, y\rangle/(yx - qxy, x^2, y^a)$, and a quantum exterior algebra on $2h - n$ variables, and add up global dimensions as above. □

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