Journal of Algebra 323 (2010) 2943–2959

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Computing the Stanley depth ☆

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ARTICLE INFO

Article history:
Received 25 November 2009
Available online 16 December 2009
Communicated by Steven Dale Cutkosky

Keywords:
Monomial ideals
Stanley decompositions
Stanley depth

ABSTRACT

Let Q and Q' be two monomial primary ideals of a polynomial algebra S over a field. We give an upper bound for the Stanley depth of $S/(Q \cap Q')$ which is reached if Q, Q' are irreducible. Also we show that Stanley's Conjecture holds for $Q_1 \cap Q_2$, $S/(Q_1 \cap Q_2 \cap Q_3)$, $(Q_i)_i$ being some irreducible monomial ideals of S.

Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring over K in n variables and M a finitely generated multigraded (i.e. $\mathbb{Z}^n$-graded) S-module. Given $z \in M$ a homogeneous element in M and $Z \subseteq \{x_1, \ldots, x_n\}$, let $zK[Z] \subset M$ be the linear K-subspace of all elements of the form zf, $f \in K[Z]$. This subspace is called Stanley space of dimension $|Z|$, if $zK[Z]$ is a free $K[Z]$-module. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces $\mathcal{D}$: $M = \bigoplus_{i=1}^r z_i K[Z_i]$. Set $\operatorname{sdepth} \mathcal{D} = \min\{|Z_i|:\ i = 1, \ldots, r\}$. The number

$$\operatorname{sdepth}(M) := \max\bigl\{\operatorname{sdepth}(\mathcal{D}):\ \mathcal{D} \text{ is a Stanley decomposition of } M\bigr\}$$

is called the Stanley depth of M. This is a combinatorial invariant which has some common properties with the homological invariant depth. Stanley conjectured (see [17]) that $\operatorname{sdepth} M \geqslant \operatorname{depth} M$, but this conjecture is still open for a long time in spite of some results obtained mainly for $n \leqslant 5$ (see [1,16,8,

☆ The authors would like to express their gratitude to ASSMS of GC University Lahore for creating a very appropriate atmosphere for research work. This research is partially supported by HEC Pakistan. The first author was mainly supported by CNCSIS Grant ID-PCE no. 51/2007.

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doi:10.1016/j.jalgebra.2009.11.025

2,12,13]). An algorithm to compute the Stanley depth is given in [9] and was used here to find several examples. Very important in our computations were the results from [3,6,15].

Let Q, Q' be two monomial primary ideals such that $\dim S/(Q+Q')=0$. Then

$$\operatorname{sdepth} S/\big(Q\cap Q'\big)\leqslant \max\left\{\min\left\{\dim\big(S/Q'\big),\left\lceil\frac{\dim(S/Q)}{2}\right\rceil\right\},\min\left\{\dim(S/Q),\left\lceil\frac{\dim(S/Q')}{2}\right\rceil\right\}\right\},$$

and the bound is reached when Q, Q' are non-zero irreducible monomial ideals (see Proposition 2.2, or more general in Corollary 2.4), $\lceil\frac{a}{2}\rceil$ being the smallest integer $\geqslant a/2$, $a\in\mathbf{Q}$.

Let Q_1, Q_2, Q_3 be three non-zero irreducible monomial ideals of S. If $\dim S/(Q_1+Q_2)=0$ then

$$\operatorname{sdepth}(Q_1\cap Q_2)\geqslant\left\lceil\frac{\dim(S/Q_1)}{2}\right\rceil+\left\lceil\frac{\dim(S/Q_2)}{2}\right\rceil$$

(see Lemma 4.3, or more general in Theorem 4.5). In this case, our bound is better than the bound given by [10] and [11] (see Remark 4.2). Using these results we show that $\operatorname{sdepth}(Q_1\cap Q_2)\geqslant\operatorname{depth}(Q_1\cap Q_2)$, and

$$\operatorname{sdepth} S/(Q_1\cap Q_2\cap Q_3)\geqslant\operatorname{depth} S/(Q_1\cap Q_2\cap Q_3),$$

that is Stanley's Conjecture holds for $Q_1\cap Q_2$ and $S/(Q_1\cap Q_2\cap Q_3)$ (see Theorems 5.6, 5.9).

1. A lower bound for Stanley's depth of some cycle modules

We start with few simple lemmas which we include for the completeness of our paper.

Lemma 1.1. *Let Q be a monomial primary ideal in $S=K[x_1,\dots,x_n]$. Suppose that $\sqrt{Q}=(x_1,\dots,x_r)$ where $1\leqslant r\leqslant n$, Then there exists a Stanley decomposition*

$$S/Q=\bigoplus uK[x_{r+1},\dots,x_n],$$

where the sum runs on monomials $u\in K[x_1,\dots,x_r]\setminus(Q\cap K[x_1,\dots,x_r])$.

Proof. Given $u,v\in K[x_1,\dots,x_r]\setminus(Q\cap K[x_1,\dots,x_r])$ and $h,g\in K[x_{r+1},\dots,x_n]$ with $uh=vg$ then we get $u=v$, $g=h$. Thus the given sum is direct. Note that there exist just a finite number of monomials in $K[x_1,\dots,x_r]\setminus(Q\cap K[x_1,\dots,x_r])$. Let $0\neq\alpha\in(S\setminus Q)$ be a monomial. Then $\alpha=uf$, where $f\in K[x_{r+1},\dots,x_n]$ and $u\in K[x_1,\dots,x_r]$. Since $\alpha\notin Q$ we have $u\notin Q$. Thus $S/Q\subset\bigoplus uK[x_{r+1},\dots,x_n]$, the other inclusion being trivial. □

Lemma 1.2. *Let Q be a monomial primary ideal in $S=K[x_1,\dots,x_n]$. Then* $\operatorname{sdepth} S/Q=\dim S/Q=\operatorname{depth} S/Q$.

Proof. Let $\dim S/Q=n-r$ for some $0\leqslant r\leqslant n$. We have $\dim S/Q\geqslant\operatorname{sdepth} S/Q$ by [1, Theorem 2.4]. Renumbering variables we may suppose that $\sqrt{Q}=(x_1,\dots,x_r)$. Using the above lemma we get the converse inequality. As S/Q is Cohen Macaulay it follows $\dim S/Q=\operatorname{depth} S/Q$, which is enough. □

Lemma 1.3. *Let I, J be two monomial ideals of $S=K[x_1,\dots,x_n]$. Then*

$$\begin{aligned}\operatorname{sdepth}\big(S/(I\cap J)\big)\geqslant\max\big\{&\min\big\{\operatorname{sdepth}(S/I),\operatorname{sdepth}\big(I/(I\cap J)\big)\big\},\\&\min\big\{\operatorname{sdepth}(S/J),\operatorname{sdepth}\big(J/(I\cap J)\big)\big\}\big\}.\end{aligned}$$

Proof. Consider the following exact sequence of S-modules:

$$0 \to I/(I \cap J) \to S/(I \cap J) \to S/I \to 0.$$

By [14, Lemma 2.2], we have

$$\operatorname{sdepth}\bigl(S/(I \cap J)\bigr) \geqslant \min\bigl\{\operatorname{sdepth}(S/I), \operatorname{sdepth}\bigl(I/(I \cap J)\bigr)\bigr\}. \tag{1}$$

Similarly, we get

$$\operatorname{sdepth}\bigl(S/(I \cap J)\bigr) \geqslant \min\bigl\{\operatorname{sdepth}(S/J), \operatorname{sdepth}\bigl(J/(I \cap J)\bigr)\bigr\}. \tag{2}$$

The proof ends using (1) and (2). □

Proposition 1.4. *Let Q, Q' be two monomial primary ideals in $S = K[x_1, \dots, x_n]$ with different associated prime ideals. Suppose that $\sqrt{Q} = (x_1, \dots, x_t)$, $\sqrt{Q'} = (x_{r+1}, \dots, x_n)$ for some integers t, r with $0 \leqslant r \leqslant t \leqslant n$. Then*

$$\begin{aligned}
&\operatorname{sdepth}\bigl(S/\bigl(Q \cap Q'\bigr)\bigr) \\
&\quad \geqslant \max\Bigl\{\min_v\bigl\{r, \operatorname{sdepth}\bigl(Q' \cap K[x_{t+1}, \dots, x_n]\bigr), \operatorname{sdepth}\bigl(\bigl(Q' : v\bigr) \cap K[x_{t+1}, \dots, x_n]\bigr)\bigr\}, \\
&\qquad \min_w\bigl\{n - t, \operatorname{sdepth}\bigl(Q \cap K[x_1, \dots, x_r]\bigr), \operatorname{sdepth}\bigl((Q : w) \cap K[x_1, \dots, x_r]\bigr)\bigr\}\Bigr\},
\end{aligned}$$

where v, w run in the set of monomials containing only variables from $\{x_{r+1}, \dots, x_t\}$, $w \notin Q$, $v \notin Q'$.

Proof. If Q, or Q' is zero then the inequality holds trivially. If $r = 0$ then $Q \cap K[x_1, \dots, x_r] = Q \cap K = 0$, and the inequality is clear. A similar case is $t = n$. Thus we may suppose $1 \leqslant r \leqslant t < n$. Applying Lemma 1.3 it is enough to show that

$$\operatorname{sdepth}\bigl(Q'/\bigl(Q \cap Q'\bigr)\bigr) \geqslant \min\bigl\{\operatorname{sdepth}\bigl(Q' \cap K[x_{t+1}, \dots x_n]\bigr), \operatorname{sdepth}\bigl(\bigl(Q' : v\bigr) \cap K[x_{t+1}, \dots x_n]\bigr)\bigr\},$$

where v is a monomial of $K[x_{r+1}, \dots, x_n] \setminus (Q \cap Q')$. We have a canonical injective map

$$Q'/\bigl(Q \cap Q'\bigr) \to S/Q.$$

By Lemma 1.1 we get

$$Q'/\bigl(Q \cap Q'\bigr) = Q' \cap \left(\bigoplus uK[x_{t+1}, \dots, x_n]\right) = \bigoplus\bigl(Q' \cap uK[x_{t+1}, \dots, x_n]\bigr),$$

where u runs in the monomials of $K[x_1, \dots, x_t] \setminus Q$. Here

$$Q' \cap uK[x_{t+1}, \dots, x_n] = u\bigl(Q' \cap K[x_{t+1}, \dots, x_n]\bigr) \quad \text{if } u \in K[x_1, \dots, x_r]$$

and

$$Q' \cap uK[x_{t+1}, \dots, x_n] = u\bigl(\bigl(Q' : u\bigr) \cap K[x_{t+1}, \dots, x_n]\bigr) \quad \text{if } u \notin K[x_1, \dots, x_r].$$

If $u \in Q'$ then $Q' : u = S$. We have

$$Q'/(Q \cap Q') = \Bigl(\bigoplus u\bigl(Q' \cap K[x_{t+1}, \dots, x_n]\bigr)\Bigr) \oplus \Bigl(\bigoplus zK[x_{t+1}, \dots, x_n]\Bigr) \oplus \Bigl(\bigoplus uv\bigl((Q' : v) \cap K[x_{t+1}, \dots, x_n]\bigr)\Bigr),$$

where the sum runs for all monomials $u \in (K[x_1, \dots, x_r] \setminus Q)$, $z \in Q' \setminus Q$ and $v \in K[x_{r+1}, \dots, x_t]$, $v \notin Q' \cup Q$. Now it is enough to apply [14, Lemma 2.2] to get the above inequality. $\square$

Theorem 1.5. *Let Q and Q' be two irreducible monomial ideals of S. Then*

$$\operatorname{sdepth}_S S/(Q \cap Q') \geqslant \max\Bigl\{\min\Bigl\{\dim(S/Q'), \Bigl\lceil \frac{\dim(S/Q) + \dim(S/(Q + Q'))}{2} \Bigr\rceil\Bigr\}, \min\Bigl\{\dim(S/Q), \Bigl\lceil \frac{\dim(S/Q') + \dim(S/(Q + Q'))}{2} \Bigr\rceil\Bigr\}\Bigr\}.$$

Proof. If the associated prime ideals of Q, Q' are the same then the above inequality says that $\operatorname{sdepth}_S S/(Q \cap Q') \geqslant \dim S/Q$, which follows from Lemma 1.2. Thus we may suppose that the associated prime ideals of Q, Q' are different. We may suppose that Q is generated in variables $\{x_1, \dots, x_t\}$ and Q' is generated in variables $\{x_{r+1}, \dots, x_p\}$ for some integers $0 \leqslant r \leqslant t \leqslant p \leqslant n$. Since $\dim(S/Q) = n - t$, $\dim(S/Q') = n - p + r$ and $\dim(S/(Q + Q')) = n - p$ we get

$$n - t - \Bigl\lfloor \frac{p - t}{2} \Bigr\rfloor = \Bigl\lceil \frac{(n - t) + (n - p)}{2} \Bigr\rceil = \Bigl\lceil \frac{\dim(S/Q) + \dim(S/(Q + Q'))}{2} \Bigr\rceil,$$

$\lfloor \frac{a}{2} \rfloor$ being the biggest integer $\leqslant a/2$, $a \in \mathbf{Q}$. Similarly, we have

$$n - p + r - \Bigl\lfloor \frac{r}{2} \Bigr\rfloor = \Bigl\lceil \frac{\dim(S/Q') + \dim(S/(Q + Q'))}{2} \Bigr\rceil.$$

On the other hand by [6], and [15, Theorem 2.4] $\operatorname{sdepth}(Q' \cap K[x_{t+1}, \dots, x_n]) = n - t - \lfloor \frac{p-t}{2} \rfloor$ and $\operatorname{sdepth}(Q \cap K[x_1, \dots, x_r, x_{p+1}, \dots, x_n]) = n - p + r - \lfloor \frac{r}{2} \rfloor$. In fact, the quoted result says in particular that sdepth of each irreducible ideal L depends only on the number of variables of the ring and the number of variables generating L (a description of irreducible monomial ideals is given in [18]). Since $(Q' : v) \cap K[x_{t+1}, \dots, x_n]$ is still an irreducible ideal generated by the same variables as Q' we conclude that

$$\operatorname{sdepth}\bigl((Q' : v) \cap K[x_{t+1}, \dots, x_n]\bigr) = \operatorname{sdepth}\bigl(Q' \cap K[x_{t+1}, \dots, x_n]\bigr),$$

$v \notin Q'$ being any monomial. Similarly,

$$\operatorname{sdepth}\bigl((Q : w) \cap K[x_1, \dots, x_r, x_{p+1}, \dots, x_n]\bigr) = \operatorname{sdepth}\bigl(Q \cap K[x_1, \dots, x_r, x_{p+1}, \dots, x_n]\bigr).$$

It follows that our inequality holds if $p = n$ by Proposition 1.4.

Set $S' = K[x_1, \dots, x_p]$, $q = Q \cap S'$, $q' = Q' \cap S'$. As above (case $p = n$) we get

$$\operatorname{sdepth}_{S'} S'/(q \cap q') \geqslant \max\Bigl\{\min\Bigl\{\dim(S'/q'), \Bigl\lceil \frac{\dim(S'/q)}{2} \Bigr\rceil\Bigr\}, \min\Bigl\{\dim(S'/q), \Bigl\lceil \frac{\dim(S'/q')}{2} \Bigr\rceil\Bigr\}\Bigr\} = \max\Bigl\{\min\Bigl\{r, \Bigl\lceil \frac{p - t}{2} \Bigr\rceil\Bigr\}, \min\Bigl\{p - t, \Bigl\lceil \frac{r}{2} \Bigr\rceil\Bigr\}\Bigr\}.$$

Using [9, Lemma 3.6], we have

$$\operatorname{sdepth}_S\big(S/(Q\cap Q')\big)=\operatorname{sdepth}_S\big(S/(q\cap q')S\big)=n-p+\operatorname{sdepth}_{S'}\big(S'/(q\cap q')\big).$$

It follows that

$$\begin{aligned}\operatorname{sdepth}_S\big(S/(Q\cap Q')\big)&\geqslant n-p+\max\left\{\min\left\{r,\left\lceil\frac{p-t}{2}\right\rceil\right\},\min\left\{p-t,\left\lceil\frac{r}{2}\right\rceil\right\}\right\}\\&=\max\left\{\min\left\{n-p+r,n-p+\left\lceil\frac{p-t}{2}\right\rceil\right\},\min\left\{n-t,n-p+\left\lceil\frac{r}{2}\right\rceil\right\}\right\}\\&=\max\left\{\min\left\{n-p+r,n-t-\left\lfloor\frac{p-t}{2}\right\rfloor\right\},\min\left\{n-t,n-p+r-\left\lfloor\frac{r}{2}\right\rfloor\right\}\right\},\end{aligned}$$

which is enough. □

2. An upper bound for Stanley's depth of some cycle modules

Let Q, Q' be two monomial primary ideals of S. Suppose that Q is generated in variables $\{x_1,\dots,x_t\}$ and Q' is generated in variables $\{x_{r+1},\dots,x_n\}$ for some integers $1\leqslant r\leqslant t<n$. Thus the prime ideals associated to $Q\cap Q'$ have dimension $\geqslant 1$ and it follows $\operatorname{depth}(S/(Q\cap Q'))\geqslant 1$. Then $\operatorname{sdepth}(S/(Q\cap Q'))\geqslant 1$ by [5, Corollary 1.6], or [7, Theorem 1.4]. Let $\mathcal{D}$: $S/(Q\cap Q')=\bigoplus_{i=1}^s u_iK[Z_i]$ be a Stanley decomposition of $S/(Q\cap Q')$ with $\operatorname{sdepth}\mathcal{D}=\operatorname{sdepth}(S/(Q\cap Q'))$. Thus $|Z_i|\geqslant 1$ for all i. Renumbering (u_i,Z_i) we may suppose that $1\in u_1K[Z_1]$, so $u_1=1$. Note that Z_i cannot have mixed variables from $\{x_1,\dots,x_r\}$ and $\{x_{t+1},\dots,x_n\}$ because otherwise $u_iK[Z_i]$ will be not a free $K[Z_i]$-module. As $|Z_1|\geqslant 1$ we may have either $Z_1\subset\{x_1,\dots,x_r\}$ or $Z_1\subset\{x_{t+1},\dots,x_n\}$.

Lemma 2.1. *Suppose $Z_1\subset\{x_1,\dots,x_r\}$. Then* $\operatorname{sdepth}(\mathcal{D})\leqslant\min\{r,\lceil\frac{n-t}{2}\rceil\}$.

Proof. Clearly $\operatorname{sdepth}(\mathcal{D})\leqslant|Z_1|\leqslant r$. Let $a\in\mathbb{N}$ be such that $x_i^a\in Q'$ for all $t<i\leqslant n$. Let $T=K[y_{t+1},\dots,y_n]$ and $\varphi:T\to S$ be the K-morphism given by $y_i\to x_i^a$. The composition map $\psi:T\to S\to S/(Q\cap Q')$ is injective. Note also that we may consider $Q'\cap K[x_{t+1},\dots,x_n]\subset S/(Q\cap Q')$ since $Q\cap K[x_{t+1},\dots,x_n]=0$. We have

$$(y_{t+1},\dots,y_n)=\psi^{-1}\big(Q'\cap K[x_{t+1},\dots,x_n]\big)=\bigoplus\psi^{-1}\big(u_jK[Z_j]\cap Q'\cap K[x_{t+1},\dots,x_n]\big).$$

If $u_jK[Z_j]\cap Q'\cap K[x_{t+1},\dots,x_n]\neq 0$ then $u_j\in K[x_{t+1},\dots,x_n]$. Also we have $Z_j\subset\{x_{t+1},\dots,x_n\}$, otherwise $u_jK[Z_j]$ is not free over $K[Z_j]$. Moreover, if $\psi^{-1}(u_jK[Z_j]\cap Q'\cap K[x_{t+1},\dots,x_n])\neq 0$ then $u_j=x_{t+1}^{b_{t+1}}\dots x_n^{b_n}$, $b_i\in\mathbb{N}$ is such that if $x_i\notin Z_j$, $t<i\leqslant n$, then $a\mid b_i$, let us say $b_i=ac_i$ for some $c_i\in\mathbb{N}$. Denote $c_i=\lceil\frac{b_i}{a}\rceil$ when $x_i\in Z_j$. We get

$$\psi^{-1}\big(u_jK[Z_j]\cap Q'\cap K[x_{t+1},\dots,x_n]\big)=y_{t+1}^{c_{t+1}}\dots y_n^{c_n}K[V_j],$$

where $V_j=\{y_i\colon\ t<i\leqslant n, x_i\in Z_j\}$. Thus $\psi^{-1}(u_jK[Z_j]\cap Q'\cap K[x_{t+1},\dots,x_n])$ is a Stanley space of T and so $\mathcal{D}$ induces a Stanley decomposition $\mathcal{D}'$ of $(y_{t+1},\dots,y_n)$ such that $\operatorname{sdepth}(\mathcal{D})\leqslant\operatorname{sdepth}(\mathcal{D}')\leqslant\operatorname{sdepth}(y_{t+1},\dots,y_n)$ because $|Z_j|=|V_j|$. Consequently $\operatorname{sdepth}(\mathcal{D})\leqslant\lceil\frac{n-t}{2}\rceil$ by [3] and so $\operatorname{sdepth}(\mathcal{D})\leqslant\min\{r,\lceil\frac{n-t}{2}\rceil\}$.

Note also that if $t=n$, or $r=0$ then the same proof works; so $\operatorname{sdepth}S/(Q\cap Q')=0$, which is clear because $\operatorname{depth}S/(Q\cap Q')=0$ (see [5, Corollary 1.6]). □

Proposition 2.2. *Let Q, Q' be two non-zero monomial primary ideals of S with different associated prime ideals. Suppose that* $\dim(S/(Q+Q'))=0$. *Then*

$$\begin{aligned}&\operatorname{sdepth}_S\big(S/\big(Q\cap Q'\big)\big)\\&\quad\leqslant\max\left\{\min\left\{\dim\big(S/Q'\big),\left\lceil\frac{\dim(S/Q)}{2}\right\rceil\right\},\min\left\{\dim(S/Q),\left\lceil\frac{\dim(S/Q')}{2}\right\rceil\right\}\right\}.\end{aligned}$$

Proof. If one of Q, Q' is of dimension zero then $\operatorname{depth}(S/(Q\cap Q'))=0$ and so by [5, Corollary 1.6] (or [7, Theorem 1.4]) $\operatorname{sdepth}(S/(Q\cap Q'))=0$, that is the inequality holds trivially. Thus we may suppose after renumbering of variables that Q is generated in variables $\{x_1,\dots,x_t\}$ and Q' is generated in variables $\{x_{r+1},\dots,x_p\}$ for some integers t,r,p with $1\leqslant r\leqslant t<p\leqslant n$, or $0\leqslant r<t\leqslant n$. By hypothesis we have $p=n$. Let $\mathcal{D}$ be the Stanley decomposition of $S/(Q\cap Q')$ such that $\operatorname{sdepth}(\mathcal{D})=\operatorname{sdepth}(S/(Q\cap Q'))$. Let Z_1 be defined as in Lemma 2.1, that is $K[Z_1]$ is the Stanley space corresponding to 1. If $Z_1\subset\{x_1,\dots,x_r\}$ then by Lemma 2.1,

$$\operatorname{sdepth}(\mathcal{D})\leqslant\min\left\{r,\left\lceil\frac{n-t}{2}\right\rceil\right\}=\min\left\{\dim\big(S/Q'\big),\left\lceil\frac{\dim(S/Q)}{2}\right\rceil\right\}.$$

If $Z_1\subset\{x_{t+1},\dots,x_n\}$ we get analogously

$$\operatorname{sdepth}(\mathcal{D})\leqslant\min\left\{n-t,\left\lceil\frac{r}{2}\right\rceil\right\}=\min\left\{\dim(S/Q),\left\lceil\frac{\dim(S/Q')}{2}\right\rceil\right\},$$

which shows our inequality. □

Theorem 2.3. *Let Q and Q' be two non-zero monomial primary ideals of S with different associated prime ideals. Then*

$$\begin{aligned}\operatorname{sdepth}_S S/\big(Q\cap Q'\big)\leqslant\max\Bigg\{&\min\left\{\dim\big(S/Q'\big),\left\lceil\frac{\dim(S/Q)+\dim(S/(Q+Q'))}{2}\right\rceil\right\},\\&\min\left\{\dim(S/Q),\left\lceil\frac{\dim(S/Q'+\dim(S/(Q+Q')))}{2}\right\rceil\right\}\Bigg\}.\end{aligned}$$

Proof. As in the proof of Proposition 2.2 we may suppose that Q is generated in variables $\{x_1,\dots,x_t\}$ and Q' is generated in variables $\{x_{r+1},\dots,x_p\}$ for some integers $1\leqslant r\leqslant t<p\leqslant n$, or $0\leqslant r<t\leqslant n$ but now we have not in general $p=n$. Set $S'=K[x_1,\dots,x_p]$, $q=Q\cap S'$, $q'=Q'\cap S'$. Using Proposition 2.2 we get

$$\operatorname{sdepth}_S\big(S/\big(q\cap q'\big)\big)\leqslant\max\left\{\min\left\{\dim\big(S/q'\big),\left\lceil\frac{\dim(S/q)}{2}\right\rceil\right\},\min\left\{\dim(S/q),\left\lceil\frac{\dim(S/q')}{2}\right\rceil\right\}\right\}.$$

By [9, Lemma 3.6] we have

$$\operatorname{sdepth}_S\big(S/\big(Q\cap Q'\big)\big)=\operatorname{sdepth}_S\big(S/\big(q\cap q'\big)S\big)=n-p+\operatorname{sdepth}_{S'}\big(S'/\big(q\cap q'\big)\big).$$

As in the proof of Theorem 1.5, it follows that

$$\begin{aligned}\operatorname{sdepth}_S\big(S/\big(Q\cap Q'\big)\big) &\leqslant n-p+\max\left\{\min\left\{r,\left\lceil\frac{p-t}{2}\right\rceil\right\},\min\left\{p-t,\left\lceil\frac{r}{2}\right\rceil\right\}\right\}\\ &=\max\left\{\min\left\{n-p+r,n-t-\left\lfloor\frac{p-t}{2}\right\rfloor\right\},\min\left\{n-t,n-p+r-\left\lfloor\frac{r}{2}\right\rfloor\right\}\right\},\end{aligned}$$

which is enough. □

Corollary 2.4. *Let Q and Q' be two non-zero monomial irreducible ideals of S with different associated prime ideals. Then*

$$\begin{aligned}\operatorname{sdepth}_S S/(Q\cap Q') = \max\Bigg\{&\min\left\{\dim\big(S/Q'\big),\left\lceil\frac{\dim(S/Q)+\dim(S/(Q+Q'))}{2}\right\rceil\right\},\\ &\min\left\{\dim(S/Q),\left\lceil\frac{\dim(S/Q')+\dim(S/(Q+Q'))}{2}\right\rceil\right\}\Bigg\}.\end{aligned}$$

For the proof apply Theorem 1.5 and Theorem 2.3.

Corollary 2.5. *Let P and P' be two different non-zero monomial prime ideals of S, which are not included one in the other. Then*

$$\begin{aligned}\operatorname{sdepth}_S S/\big(P\cap P'\big) = \max\Bigg\{&\min\left\{\dim\big(S/P'\big),\left\lceil\frac{\dim(S/P)+\dim(S/(P+P'))}{2}\right\rceil\right\},\\ &\min\left\{\dim(S/P),\left\lceil\frac{\dim(S/P')+\dim(S/(P+P'))}{2}\right\rceil\right\}\Bigg\}.\end{aligned}$$

Proof. For the proof apply Corollary 2.4. □

Corollary 2.6. *Let $\triangle$ be a simplicial complex in n vertices with only two different facets F, F'. Then*

$$\operatorname{sdepth} K[\triangle] = \max\left\{\min\left\{\left|F'\right|,\left\lceil\frac{|F|+|F\cap F'|}{2}\right\rceil\right\},\min\left\{|F|,\left\lceil\frac{|F'|+|F\cap F'|}{2}\right\rceil\right\}\right\}.$$

3. An illustration

Let $S=K[x_1,\dots,x_6]$, $Q=(x_1^2,x_2^2,x_3^2,x_4^2,x_1x_2x_4,x_1x_3x_4)$, $Q'=(x_4^2,x_5,x_6)$. By our Theorem 2.3 we get

$$\operatorname{sdepth} S/\big(Q\cap Q'\big)\leqslant\max\left\{\min\left\{3,\left\lceil\frac{2}{2}\right\rceil\right\},\min\left\{2,\left\lceil\frac{3}{2}\right\rceil\right\}\right\}=\max\{1,2\}=2.$$

On the other hand, we claim that $I=((Q:w)\cap K[x_1,x_2,x_3])=(x_1^2,x_2^2,x_3^2,x_1x_2,x_1x_3)$ for $w=x_4$ and $\operatorname{sdepth} I=1<2=\operatorname{sdepth}(Q\cap K[x_1,x_2,x_3])$. Thus our Proposition 1.4 gives

$$\operatorname{sdepth} S/\big(Q\cap Q'\big)\geqslant\max\left\{\min\left\{3,\left\lceil\frac{2}{2}\right\rceil\right\},\min\left\{2,\left\lceil\frac{3}{2}\right\rceil,1\right\}\right\}=1.$$

In this section, we will show that $\operatorname{sdepth}(S/(Q\cap Q'))=1$.

First we prove our claim. Suppose that there exists a Stanley decomposition $\mathcal{D}$ of I with $\operatorname{sdepth}\mathcal{D}\geqslant 2$. Among the Stanley spaces of $\mathcal{D}$ we have five important $x_1^2K[Z_1]$, $x_2^2K[Z_2]$, $x_3^2K[Z_3]$,

$x_1x_2K[Z_4]$, $x_1x_3K[Z_5]$ for some subsets $Z_i \subset \{x_1,x_2,x_3\}$ with $|Z_i| \geqslant 2$. If $Z_4 = \{x_1,x_2,x_3\}$ and Z_5 contains x_2 then the last two Stanley spaces will have a non-zero intersection and if Z_1 contains x_2 then the first and the fourth Stanley space will have non-zero intersection. Now if $x_2 \notin Z_5$ and $x_2 \notin Z_1$ then the first and the last space will intersect. Suppose that $Z_4 = \{x_1,x_2\}$. Then $x_2 \notin Z_1$ (resp. $x_1 \notin Z_2$) because otherwise the intersection of $x_1x_2K[Z_4]$ with the first Stanley space (resp. the second one) will be again non-zero. As $|Z_1|, |Z_2| \geqslant 2$ we get $Z_1 = \{x_1,x_3\}$, $Z_2 = \{x_2,x_3\}$. But $x_1 \notin Z_3$ because otherwise the first and the third Stanley space will contain $x_1^2x_3^2$, which is impossible. Similarly, $x_2 \notin Z_3$, which contradicts $|Z_3| \geqslant 2$. The case $Z_5 = \{x_1,x_3\}$ gives a similar contradiction.

Now suppose that $Z_4 = \{x_1,x_3\}$. If $Z_5 \supset \{x_1,x_2\}$ we see that the intersection of the last two Stanley spaces from the above five, contains $x_1^2x_2x_3$ and if $Z_5 = \{x_2,x_3\}$ we see that the intersection of the same Stanley spaces contains $x_1x_2x_3$. Contradiction (we saw that $Z_5 \neq \{x_1,x_3\}$)! Hence $\operatorname{sdepth}\mathcal{D} \leqslant 1$ and so $\operatorname{sdepth} I = 1$ using [5].

Next we show that $\operatorname{sdepth} S/(Q \cap Q') = 1$. Suppose that $\mathcal{D}'$ is a Stanley decomposition of $S/(Q \cap Q')$ such that $\operatorname{sdepth} S/(Q \cap Q') = 2$. We claim that $\mathcal{D}'$ has the form

$$S/\big(Q \cap Q'\big) = \Big(\bigoplus vK[x_5,x_6]\Big) \oplus \left(\bigoplus_{i=1}^{s} u_iK[Z_i]\right)$$

for some monomials $v \in (K[x_1,\dots,x_4] \setminus Q)$, $u_i \in (Q \cap K[x_1,\dots,x_4])$ and $Z_i \subset \{x_1,x_2,x_3\}$. Indeed, let $v \in (K[x_1,\dots,x_4] \setminus Q)$. Then vx_5, vx_6 belong to some Stanley spaces of $\mathcal{D}'$, let us say $uK[Z]$, $u'K[Z']$. The presence of x_5 in u or Z implies that Z does not contain any x_i, $1 \leqslant i \leqslant 3$, otherwise $uK[Z]$ will be not free over $K[Z]$. Thus $Z \subset \{x_5,x_6\}$. As $|Z| \geqslant 2$ we get $Z = \{x_5,x_6\}$ and similarly $Z' = \{x_5,x_6\}$. Thus $vx_5x_6 \in (uK[Z] \cap u'K[Z'])$ and it follows that $u = u'$, $Z = Z'$ because the sum in $\mathcal{D}'$ is direct. It follows that $u|vx_5$, $u|vx_6$ and so $u|v$, that is $v = uf$, f being a monomial in x_5, x_6. As $v \in K[x_1,\dots,x_4]$ we get $f = 1$ and so $u = v$.

A monomial $w \in (Q \setminus Q')$ is not a multiple of x_5, x_6, because otherwise $w \in Q'$. Suppose w belongs to a Stanley space $uK[Z]$ of $\mathcal{D}'$. If $u \in (K[x_1,\dots,x_4] \setminus Q)$ then as above $\mathcal{D}'$ has also a Stanley space $uK[x_5,x_6]$ and both spaces contains u. This is false since the sum is direct. Thus $u \in (Q \cap K[x_1,\dots,x_4])$, which shows our claim.

Hence $\mathcal{D}'$ induces two Stanley decompositions $S/Q = \bigoplus_{v \in (K[x_1,\dots,x_4] \setminus Q)} vK[x_5,x_6]$, $Q/(Q \cap Q') = \bigoplus_{i=1}^{s} u_iK[Z_i]$, where $u_i \in (Q \cap K[x_1,\dots,x_4])$ and $Z_i \subset \{x_1,x_2,x_3\}$. Then we get the following Stanley decompositions

$$Q \cap K[x_1,\dots,x_3] = \bigoplus_{i=1,\, u_i \notin (x_4)}^{s} u_iK[Z_i], \qquad I = \bigoplus_{i=1,\, x_4|u_i}^{s} (u_i/x_4)K[Z_i].$$

As $2 \leqslant \min_i |Z_i|$ we get $\operatorname{sdepth} I \geqslant 2$. Contradiction!

4. A lower bound for Stanley's depth of some ideals

Let Q, Q' be two non-zero irreducible monomial ideals of S such that $\sqrt{Q} = (x_1,\dots,x_t)$, $\sqrt{Q'} = (x_{r+1},\dots,x_p)$ for some integers r,t,p with $1 \leqslant r \leqslant t < p \leqslant n$, or $0 = r < t < p \leqslant n$, or $1 \leqslant r \leqslant t = p \leqslant n$.

Lemma 4.1. *Suppose that $p = n$, $t = r$. Then*

$$\operatorname{sdepth}\big(Q \cap Q'\big) \geqslant \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{n-r}{2} \right\rceil \geqslant n/2.$$

Proof. It follows $1 \leqslant r < p$. Let $f \in Q \cap K[x_1,\dots,x_r]$, $g \in Q' \cap K[x_{r+1},\dots,x_n]$ and $\mathcal{M}(T)$ be the monomials from an ideal T. The correspondence $(f,g) \to fg$ defines a map $\varphi : \mathcal{M}(Q \cap K[x_1,\dots,x_r]) \times$

$\mathcal{M}(Q' \cap K[x_{r+1},\dots,x_n]) \to \mathcal{M}(Q \cap Q')$, which is injective. If w is a monomial of $Q \cap Q'$, let us say $w = fg$ for some monomials $f \in K[x_1,\dots,x_r]$, $g \in K[x_{r+1},\dots,x_n]$ then $fg \in Q$ and so $f \in Q$ because the variables x_i, $i > r$ are regular on S/Q. Similarly, $g \in Q'$ and so $w = \varphi((f,g))$, that is φ is surjective. Let $\mathcal{D}$ be a Stanley decomposition of $Q \cap K[x_1,\dots,x_r]$,

$$\mathcal{D}:\quad Q \cap K[x_1,\dots,x_r] = \bigoplus_{i=1}^{s} u_i K[Z_i]$$

with $\operatorname{sdepth}\mathcal{D} = \operatorname{sdepth}(Q \cap K[x_1,\dots,x_r])$ and $\mathcal{D}'$ a Stanley decomposition of $Q' \cap K[x_{r+1},\dots,x_n]$,

$$\mathcal{D}':\quad Q' \cap K[x_{r+1},\dots,x_n] = \bigoplus_{j=1}^{e} v_j K[T_j]$$

with $\operatorname{sdepth}\mathcal{D}' = \operatorname{sdepth}(Q' \cap K[x_{r+1},\dots,x_n])$. They induce a Stanley decomposition

$$\mathcal{D}'':\quad Q \cap Q' = \bigoplus_{j=1}^{e}\bigoplus_{i=1}^{s} u_i v_j K[Z_i \cup T_j]$$

because of the bijection φ. Thus

$$\begin{aligned}
\operatorname{sdepth}\bigl(Q \cap Q'\bigr) &\geqslant \operatorname{sdepth}\mathcal{D}'' = \min_{i,j}\bigl(|Z_i| + |T_j|\bigr) \geqslant \min_i |Z_i| + \min_j |T_j| \\
&= \operatorname{sdepth}\mathcal{D} + \operatorname{sdepth}\mathcal{D}' \\
&= \operatorname{sdepth}\bigl(Q \cap K[x_1,\dots,x_r]\bigr) + \operatorname{sdepth}\bigl(Q' \cap K[x_{r+1},\dots,x_n]\bigr) \\
&= \left(r - \left\lfloor \frac{r}{2} \right\rfloor\right) + \left(n - r - \left\lfloor \frac{n-r}{2} \right\rfloor\right) = \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{n-r}{2} \right\rceil \geqslant n/2. \qquad \square
\end{aligned}$$

Remark 4.2. Suppose that $n = 8$, $r = 1$. Then by the above lemma we get $\operatorname{sdepth}(Q \cap Q') \geqslant \lceil \frac{1}{2} \rceil + \lceil \frac{7}{2} \rceil = 5$. Since $|G(Q \cap Q')| = 7$ we get by [10,11] the same lower bound $\operatorname{sdepth}(Q \cap Q') \geqslant 8 - \lfloor \frac{7}{2} \rfloor = 5$. If $n = 8$, $r = 2$ then by [10,11] we have $\operatorname{sdepth}(Q \cap Q') \geqslant 8 - \lfloor \frac{12}{2} \rfloor = 2$ but our previous lemma gives $\operatorname{sdepth}(Q \cap Q') \geqslant \lceil \frac{2}{2} \rceil + \lceil \frac{6}{2} \rceil = 4$.

Lemma 4.3. *Suppose that $p = n$. Then*

$$\operatorname{sdepth}\bigl(Q \cap Q'\bigr) \geqslant \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{n-t}{2} \right\rceil.$$

Proof. We show that

$$\begin{aligned}
Q \cap Q' = {} & \bigl(Q \cap Q' \cap K[x_{r+1},\dots,x_t]\bigr)S \\
& \oplus \left(\bigoplus_{w} w\bigl(\bigl(\bigl(Q \cap Q'\bigr) : w\bigr) \cap K[x_1,\dots,x_r,x_{t+1},\dots,x_n]\bigr)\right),
\end{aligned}$$

where w runs in the monomials of $K[x_{r+1},\dots,x_t] \setminus (Q \cap Q')$. Indeed, a monomial h of S has the form $h = fg$ for some monomials $f \in K[x_{r+1},\dots,x_t]$, $g \in K[x_1,\dots,x_r,x_{t+1},\dots,x_n]$. Since Q, Q' are irreducible we see that $h \in Q \cap Q'$ either when f is a multiple of a minimal generator of $Q \cap Q' \cap K[x_{r+1}, \dots, x_t]$, or $f \notin (Q \cap Q' \cap K[x_{r+1}, \dots, x_t])$ and then

$$h \in f\big(\big(\big(Q \cap Q'\big) : f\big) \cap K[x_1, \dots, x_r, x_{t+1}, \dots, x_n]\big).$$

Let $\mathcal{D}$ be a Stanley decomposition of $(Q \cap Q' \cap K[x_{r+1}, \dots, x_t])S$,

$$\mathcal{D}\colon \quad \big(Q \cap Q' \cap K[x_{r+1}, \dots, x_t]\big)S = \bigoplus_{i=1}^{s} u_i K[Z_i]$$

with $\operatorname{sdepth} \mathcal{D} = \operatorname{sdepth}(Q \cap Q' \cap K[x_{r+1}, \dots, x_t])S$ and for all $w \in (K[x_{r+1}, \dots, x_t] \setminus (Q \cap Q'))$, let $\mathcal{D}_w$ be a Stanley decomposition of $((Q \cap Q') : w) \cap K[x_1, \dots, x_r, x_{t+1}, \dots, x_n]$,

$$\mathcal{D}_w\colon \quad \big(\big(Q \cap Q'\big) : w\big) \cap K[x_1, \dots, x_r, x_{t+1}, \dots, x_n] = \bigoplus_{w} \bigoplus_{j} v_{wj} K[T_{wj}]$$

with $\operatorname{sdepth} \mathcal{D}_w = \operatorname{sdepth}(((Q \cap Q') : w) \cap K[x_1, \dots, x_r, x_{t+1}, \dots, x_n])$. Since $K[x_{r+1}, \dots, x_t] \setminus (Q \cap Q')$ contains just a finite set of monomials we get a Stanley decomposition of $Q \cap Q'$,

$$\mathcal{D}'\colon \quad Q \cap Q' = \left(\bigoplus_{i=1}^{s} u_i K[Z_i]\right) \oplus \left(\bigoplus_{w} \bigoplus_{j} w v_{wj} K[T_{wj}]\right),$$

where w runs in the monomials of $K[x_{r+1}, \dots, x_t] \setminus (Q \cap Q')$. Then

$$\begin{aligned}
\operatorname{sdepth} \mathcal{D}' &= \min_{w} \{\operatorname{sdepth} \mathcal{D}, \operatorname{sdepth} \mathcal{D}_w\} \\
&= \min_{w} \big\{\operatorname{sdepth}\big(Q \cap Q' \cap K[x_{r+1}, \dots, x_t]\big)S, \\
&\quad \operatorname{sdepth}\big(\big(\big(Q \cap Q'\big) : w\big) \cap K[x_1, \dots, x_r, x_{t+1}, \dots, x_n]\big)\big\}.
\end{aligned}$$

But $((Q \cap Q') : w) \cap K[x_1, \dots, x_r, x_{t+1}, \dots, x_n]$ is still an intersection of two irreducible ideals and

$$\operatorname{sdepth}\big(\big(\big(Q \cap Q'\big) : w\big) \cap K[x_1, \dots, x_r, x_{t+1}, \dots, x_n]\big) \geqslant \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{n-t}{2} \right\rceil$$

by Lemma 4.1. We have $\operatorname{sdepth}(Q \cap Q' \cap K[x_{r+1}, \dots, x_t]) \geqslant 1$ and so

$$\operatorname{sdepth}\big(Q \cap Q' \cap K[x_{r+1}, \dots, x_t]\big)S \geqslant 1 + n - t + r$$

by [9, Lemma 3.6]. Thus

$$\operatorname{sdepth}\big(Q \cap Q'\big) \geqslant \operatorname{sdepth} \mathcal{D}' \geqslant \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{n-t}{2} \right\rceil.$$

Note that the proof goes even when $0 \leqslant r < t \leqslant n$ (anyway $\operatorname{sdepth} Q \cap Q' \geqslant 1$ if $n = t$, $r = 0$). $\square$

Lemma 4.4.

$$\operatorname{sdepth}\big(Q \cap Q'\big) \geqslant n - p + \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{p-t}{2} \right\rceil.$$

Proof. As usual we see that there are now $(n-p)$ free variables and it is enough to apply [9, Lemma 3.6] and Lemma 4.3. $\square$

Theorem 4.5. *Let Q and Q' be two non-zero irreducible monomial ideals of S. Then*

$$\begin{aligned}\operatorname{sdepth}_S(Q\cap Q') &\geqslant \dim(S/(Q+Q')) + \left\lceil \frac{\dim(S/Q')-\dim(S/(Q+Q'))}{2}\right\rceil \\ &\quad + \left\lceil \frac{\dim(S/Q)-\dim(S/(Q+Q'))}{2}\right\rceil \\ &\geqslant \left\lceil \frac{\dim(S/Q')+\dim(S/Q)}{2}\right\rceil.\end{aligned}$$

Proof. After renumbering of variables, we may suppose as above that $\sqrt{Q}=(x_1,\dots,x_t)$, $\sqrt{Q'}=(x_{r+1},\dots,x_p)$ for some integers r,t,p with $1\leqslant r\leqslant t<p\leqslant n$, or $0=r<t<p\leqslant n$, or $1\leqslant r\leqslant t=p\leqslant n$. If $n=p$, $r=0$ then $\sqrt{Q}\subset\sqrt{Q'}$ and the inequality is trivial. It is enough to apply Lemma 4.4 because $n-p=\dim(S/(Q+Q'))$, $r=\dim(S/Q')-\dim(S/(Q+Q'))$, $p-t=\dim(S/Q)-\dim(S/(Q+Q'))$. $\square$

Remark 4.6. If Q, Q' are non-zero irreducible monomial ideals of S with $\sqrt{Q}=\sqrt{Q'}$ then we have $\operatorname{sdepth}_S(Q\cap Q')\geqslant 1+\dim S/Q$.

Example 4.7. Let $S=K[x_1,x_2]$, $Q=(x_1)$, $Q'=(x_1^2,x_2)$. We have

$$\operatorname{sdepth}(Q\cap Q')\geqslant \left\lceil \frac{\dim(S/Q')+\dim(S/Q)}{2}\right\rceil = \left\lceil\frac{1+0}{2}\right\rceil = 1$$

by the above theorem. As $Q\cap Q'$ is not a principle ideal its Stanley depth is <2. Thus

$$\operatorname{sdepth}(Q\cap Q')=1.$$

Example 4.8. Let $S=K[x_1,x_2,x_3,x_4,x_5]$, $Q=(x_1,x_2,x_3^2)$, $Q'=(x_3,x_4,x_5)$. As $\dim(S/(Q+Q'))=0$, $\dim S/Q=2$ and $\dim S/Q'=2$ we get

$$\operatorname{sdepth}(Q\cap Q')\geqslant \left\lceil \frac{\dim(S/Q')+\dim(S/Q)}{2}\right\rceil = \left\lceil\frac{2+2}{2}\right\rceil = 2$$

by the above theorem. Note also that

$$\operatorname{sdepth}(Q\cap Q'\cap K[x_1,x_2,x_4,x_5]) = \operatorname{sdepth}(x_1x_4,x_1x_5,x_2x_4,x_2x_5)K[x_1,x_2,x_4,x_5]=3,$$

and

$$\begin{aligned}\operatorname{sdepth}(((Q\cap Q'):x_3)\cap K[x_1,x_2,x_4,x_5]) &= \operatorname{sdepth}((x_1,x_2)K[x_1,x_2,x_4,x_5]) \\ &= 4-\left\lfloor\frac{2}{2}\right\rfloor = 3,\end{aligned}$$

by [15]. But $\operatorname{sdepth}(Q\cap Q')\geqslant 3$ because of the following Stanley decomposition

$$\begin{aligned}Q \cap Q' = {} & x_1x_4K[x_1, x_4, x_5] \oplus x_1x_5K[x_1, x_2, x_5] \oplus x_2x_4K[x_1, x_2, x_4] \oplus x_2x_5K[x_2, x_4, x_5] \\ & \oplus x_3^2K[x_3, x_4, x_5] \oplus x_2x_3K[x_2, x_3, x_4] \oplus x_1x_3K[x_1, x_2, x_3] \oplus x_1x_3x_4K[x_1, x_2, x_4, x_5] \\ & \oplus x_1x_3x_5K[x_1, x_3, x_5] \oplus x_2x_3x_5K[x_2, x_3, x_4, x_5] \oplus x_1x_2x_4x_5K[x_1, x_2, x_4, x_5] \\ & \oplus x_1x_3^2x_4K[x_1, x_3, x_4, x_5] \oplus x_1x_2x_3x_5K[x_1, x_2, x_3, x_5] \oplus x_1x_2x_3^2x_4K[x_1, x_2, x_3, x_4, x_5].\end{aligned}$$

5. Applications

Let $I \subset S$ be a non-zero monomial ideal. A. Rauf presented in [14] the following:

Question 5.1. *Does it hold the inequality*

$$\operatorname{sdepth} I \geqslant 1 + \operatorname{sdepth} S/I?$$

The importance of this question is given by the following:

Proposition 5.2. *Suppose that Stanley's Conjecture holds for cyclic S-modules and the above question has a positive answer for all monomial ideals of S. Then Stanley's Conjecture holds for all monomial ideals of S.*

For the proof note that $\operatorname{sdepth} I \geqslant 1 + \operatorname{sdepth} S/I \geqslant 1 + \operatorname{depth} S/I = \operatorname{depth} I$.

Remark 5.3. In [12] it is proved that Stanley's Conjecture holds for all multigraded cycle modules over $S = K[x_1, \dots, x_5]$. If the above question has a positive answer then Stanley's Conjecture holds for all monomial ideals of S. Actually this is true for all square free monomial ideals of S as [13] shows.

We show that the above question holds for the intersection of two non-zero irreducible monomial ideals.

Proposition 5.4. *Question* 5.1 *has a positive answer for intersections of two non-zero irreducible monomial ideals.*

Proof. First suppose that Q, Q' have different associated prime ideals. After renumbering of variables we may suppose as above that $\sqrt{Q} = (x_1, \dots, x_t)$, $\sqrt{Q'} = (x_{r+1}, \dots, x_p)$ for some integers r, t, p with $1 \leqslant r \leqslant t < p \leqslant n$, or $0 = r < t < p \leqslant n$, or $1 \leqslant r \leqslant t = p \leqslant n$. Then

$$\operatorname{sdepth}(Q \cap Q') \geqslant n - p + \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{p-t}{2} \right\rceil$$

by Lemma 4.4. Note that

$$\operatorname{sdepth}(S/(Q \cap Q')) = n - p + \max\left\{\min\left\{r, \left\lceil \frac{p-t}{2} \right\rceil\right\}, \min\left\{p - t, \left\lceil \frac{r}{2} \right\rceil\right\}\right\}$$

by Corollary 2.4. Thus

$$1 + \operatorname{sdepth}(S/(Q \cap Q')) \leqslant n - p + \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{p-t}{2} \right\rceil \leqslant \operatorname{sdepth}(Q \cap Q').$$

Finally, if Q, Q' have the same associated prime ideal then $\operatorname{sdepth}(Q \cap Q') \geqslant 1 + \dim S/Q$ by Remark 4.6 and so $\operatorname{sdepth}(Q \cap Q') \geqslant 1 + \operatorname{sdepth} S/(Q \cap Q')$. $\square$

Next we will show that Stanley's Conjecture holds for intersections of two primary monomial ideals. We start with a simple lemma.

Lemma 5.5. *Let Q, Q' be two primary ideals in $S = K[x_1, \ldots, x_n]$. Suppose $\sqrt{Q} = (x_1, \ldots, x_t)$ and $\sqrt{Q} = (x_{r+1}, \ldots, x_p)$ for integers $0 \leqslant r \leqslant t \leqslant p \leqslant n$. Then $\operatorname{sdepth}(S/(Q \cap Q')) \geqslant \operatorname{depth}(S/(Q \cap Q'))$, that is Stanley's Conjecture holds for $S/(Q \cap Q')$.*

Proof. If either $r = 0$, or $t = p$ then $\operatorname{depth} S/(Q \cap Q') \leqslant n - p \leqslant \operatorname{sdepth}(S/(Q \cap Q'))$ by [9, Lemma 3.6]. Now suppose that $r > 0$, $t < p$ and let $S' = K[x_1, \ldots, x_p]$ and $q = Q \cap S'$, $q' = Q' \cap S'$. Consider the following exact sequence of S'-modules

$$0 \to S'/\big(q \cap q'\big) \to S'/q \oplus S'/q' \to S'/\big(q + q'\big) \to 0.$$

By Lemma 1.2

$$\begin{aligned}
\operatorname{depth}\big(S'/q \oplus S'/q'\big) &= \min\big\{\operatorname{depth}\big(S'/q\big), \operatorname{depth}\big(S'/q'\big)\big\} \\
&= \min\big\{\dim\big(S'/q\big), \dim\big(S'/q'\big)\big\} \\
&= \min\{r, p - t\} \geqslant 1 > 0 \\
&= \operatorname{depth}\big(S'/\big(q + q'\big)\big).
\end{aligned}$$

Thus by Depth Lemma (see e.g. [4])

$$\operatorname{depth}\big(S'/q \cap q'\big) = \operatorname{depth}\big(S'/\big(q + q'\big)\big) + 1 = 1.$$

But $\operatorname{sdepth}(S'/(q \cap q')) \geqslant 1$ by [5, Corollary 1.6] and so

$$\begin{aligned}
\operatorname{sdepth}\big(S/\big(Q \cap Q'\big)\big) &= \operatorname{sdepth}\big(S'/\big(q \cap q'\big)\big) + n - p \geqslant 1 + n - p \\
&= n - p + \operatorname{depth}\big(S'/\big(q \cap q'\big)\big) \\
&= \operatorname{depth}\big(S/\big(Q \cap Q'\big)\big)
\end{aligned}$$

by [9, Lemma 3.6]. □

Theorem 5.6. *Let Q, Q' be two non-zero irreducible ideals of S. Then $\operatorname{sdepth}(Q \cap Q') \geqslant \operatorname{depth}(Q \cap Q')$, that is Stanley's Conjecture holds for $Q \cap Q'$.*

Proof. By Proposition 5.4, Question 5.1 has a positive answer, so by the proof of Proposition 5.2 it is enough to know that Stanley's Conjecture holds for $S/(Q \cap Q')$. This is given by the above lemma. □

Next we consider the cycle module given by an irredundant intersection of 3 irreducible ideals.

Lemma 5.7. *Let Q_1, Q_2, Q_3 be three non-zero irreducible monomial ideals of $S = K[x_1, \ldots, x_n]$. Then*

$$\begin{aligned}
&\operatorname{sdepth}\big((Q_2 \cap Q_3)/(Q_1 \cap Q_2 \cap Q_3)\big) \\
&\quad \geqslant \dim\big(S/(Q_1 + Q_2 + Q_3)\big) + \left\lceil \frac{\dim(S/(Q_1 + Q_2)) - \dim(S/(Q_1 + Q_2 + Q_3))}{2} \right\rceil
\end{aligned}$$

$$+\left\lceil\frac{\dim(S/(Q_1+Q_3))-\dim(S/(Q_1+Q_2+Q_3))}{2}\right\rceil$$
$$\geqslant\left\lceil\frac{\dim(S/(Q_1+Q_2))+\dim(S/(Q_1+Q_3))}{2}\right\rceil.$$

If $Q_3\subset Q_1+Q_2$ then

$$\operatorname{sdepth}\bigl((Q_2\cap Q_3)/(Q_1\cap Q_2\cap Q_3)\bigr)\geqslant\left\lceil\frac{\dim(S/Q_1)+\dim(S/(Q_1+Q_3))}{2}\right\rceil.$$

Proof. Renumbering the variables we may assume that $\sqrt{Q_1}=(x_1,\dots,x_t)$ and $\sqrt{Q_2+Q_3}=(x_{r+1},\dots,x_p)$, where $0\leqslant r\leqslant t<p\leqslant n$. If $t=p$ then $\sqrt{Q_1+Q_2}=\sqrt{Q_1+Q_3}$ and the inequality is trivial by [9, Lemma 3.6]. Let $S'=K[x_1,\dots,x_p]$ and $q_1=Q_1\cap S'$, $q_2=Q_2\cap S'$, $q_3=Q_3\cap S'$. We have a canonical injective map $(q_2\cap q_3)/(q_1\cap q_2\cap q_3)\to S'/q_1$. Now by Lemma 1.1, we have

$$S'/q_1=\bigoplus uK[x_{t+1},\dots,x_p]$$

and so

$$(q_2\cap q_3)/(q_1\cap q_2\cap q_3)=\bigoplus\bigl((q_2\cap q_3)\cap uK[x_{t+1},\dots,x_p]\bigr),$$

where u runs in the monomials of $K[x_1,\dots,x_t]\setminus(q_1\cap K[x_1,\dots,x_t])$. If $u\in K[x_1,\dots,x_r]$ then

$$(q_2\cap q_3)\cap uK[x_{t+1},\dots,x_p]=u\bigl(q_2\cap q_3\cap K[x_{t+1},\dots,x_p]\bigr)$$

and if $u\notin K[x_1,\dots,x_r]$ then

$$(q_2\cap q_3)\cap uK[x_{t+1},\dots,x_p]=u\bigl(\bigl((q_2\cap q_3):u\bigr)\cap K[x_{t+1},\dots,x_p]\bigr).$$

Since $(q_2\cap q_3):u$ is still an intersection of irreducible monomial ideals we get by Lemma 4.3 that

$$\operatorname{sdepth}\bigl(\bigl((q_2\cap q_3):u\bigr)\cap K[x_{t+1},\dots,x_p]\bigr)$$
$$\geqslant\left\lceil\frac{\dim K[x_{t+1},\dots,x_p]/q_2\cap K[x_{t+1},\dots,x_p]}{2}\right\rceil+\left\lceil\frac{\dim K[x_{t+1},\dots,x_p]/q_3\cap K[x_{t+1},\dots,x_p]}{2}\right\rceil.$$

Also we have

$$q_2/(q_1\cap q_2)=\bigoplus u\bigl(q_2\cap K[x_{t+1},\dots,x_p]\bigr),$$

and it follows

$$S'/(q_1+q_2)\cong\bigl(S'/q_1\bigr)/\bigl(q_2/(q_1\cap q_2)\bigr)=\bigoplus u\bigl(K[x_{t+1},\dots,x_p]/q_2\cap K[x_{t+1},\dots,x_p]\bigr).$$

Thus $\dim S'/(q_1+q_2)=\dim K[x_{t+1},\dots,x_p]/q_2\cap K[x_{t+1},\dots,x_p]$ and similarly

$$\dim S'/(q_1+q_3)=\dim K[x_{t+1},\dots,x_p]/q_3\cap K[x_{t+1},\dots,x_p].$$

Hence

$$\begin{aligned}\operatorname{sdepth}\big((q_2\cap q_3)/(q_1\cap q_2\cap q_3)\big) &\geqslant \left\lceil\frac{\dim(S'/(q_1+q_2))}{2}\right\rceil+\left\lceil\frac{\dim(S'/(q_1+q_3))}{2}\right\rceil\\ &=\left\lceil\frac{\dim(S/(Q_1+Q_2))-\dim(S/(Q_1+Q_2+Q_3))}{2}\right\rceil\\ &\quad+\left\lceil\frac{\dim(S/(Q_1+Q_3))-\dim(S/(Q_1+Q_2+Q_3))}{2}\right\rceil.\end{aligned}$$

If $Q_3\subset Q_1+Q_2$ then $(q_2\cap q_3)\cap K[x_{t+1},\dots,x_p]=q_3\cap K[x_{t+1},\dots,x_p]$ and so

$$\begin{aligned}\operatorname{sdepth}_{S'}(q_2\cap q_3)/(q_1\cap q_2\cap q_3) &\geqslant \operatorname{sdepth}\big((q_2\cap q_3)\cap K[x_{t+1},\dots,x_p]\big)\\ &=p-t-\left\lfloor\frac{ht(q_3\cap K[x_{t+1},\dots,x_p])}{2}\right\rfloor\\ &=\left\lceil\frac{p-t+\dim K[x_{t+1},\dots,x_p]/q_3\cap K[x_{t+1},\dots,x_p]}{2}\right\rceil\\ &=\left\lceil\frac{\dim(S'/q_1)+\dim(S'/(q_1+q_3))}{2}\right\rceil.\end{aligned}$$

Now it is enough to apply [9, Lemma 3.6]. □

Proposition 5.8. *Let Q_1, Q_2, Q_3 be three non-zero irreducible ideals of S and $R=S/Q_1\cap Q_2\cap Q_3$. Suppose that* $\dim S/(Q_1+Q_2+Q_3)=0$. *Then*

$$\begin{aligned}\operatorname{sdepth} R\geqslant \max\Bigg\{&\min\left\{\operatorname{sdepth} S/(Q_2\cap Q_3),\left\lceil\frac{\dim(S/(Q_1+Q_2))}{2}\right\rceil+\left\lceil\frac{\dim(S/(Q_1+Q_3))}{2}\right\rceil\right\},\\ &\min\left\{\operatorname{sdepth} S/(Q_1\cap Q_3),\left\lceil\frac{\dim(S/(Q_1+Q_2))}{2}\right\rceil+\left\lceil\frac{\dim(S/(Q_2+Q_3))}{2}\right\rceil\right\},\\ &\min\left\{\operatorname{sdepth} S/(Q_1\cap Q_2),\left\lceil\frac{\dim(S/(Q_3+Q_2))}{2}\right\rceil+\left\lceil\frac{\dim(S/(Q_1+Q_3))}{2}\right\rceil\right\}\Bigg\}.\end{aligned}$$

For the proof apply Lemma 1.3 and Lemma 5.7.

Theorem 5.9. *Let Q_1, Q_2, Q_3 be three non-zero irreducible ideals of S and $R=S/(Q_1\cap Q_2\cap Q_3)$. Then* $\operatorname{sdepth} R\geqslant \operatorname{depth} R$, *that is Stanley's Conjecture holds for R.*

Proof. Applying [9, Lemma 3.6] we may reduce the problem to the case when

$$\dim S/(Q_1+Q_2+Q_3)=0.$$

If one of the Q_i has dimension 0 then $\operatorname{depth} R=0$ and there exists nothing to show. Assume that all Q_i have dimension >0. If one of the Q_i has dimension 1 then $\operatorname{depth} R=1$ and by [5] (or [7]) we get $\operatorname{sdepth} R\geqslant 1=\operatorname{depth} R$. From now on we assume that all Q_i have dimension >1.

If Q_1+Q_2 has dimension 0 and $Q_3\not\subset Q_1+Q_2$ then from the exact sequence

$$0\to R\to S/Q_1\oplus S/Q_2\cap Q_3\to S/(Q_1+Q_2)\cap(Q_1+Q_3)\to 0,$$

we get $\operatorname{depth} R=1$ by Depth Lemma and we may apply [5] (or [7]) to get as above $\operatorname{sdepth} R\geqslant 1=\operatorname{depth} R$. If $Q_3\subset Q_1+Q_2$ then by Lemma 1.3, Theorem 5.6 and Lemma 5.7 we have

$$\begin{aligned}\operatorname{sdepth} R &\geqslant \min\left\{\operatorname{depth} S/(Q_2\cap Q_3), \left\lceil \frac{\dim(S/Q_1)+\dim(S/(Q_1+Q_3))}{2}\right\rceil\right\}\\ &\geqslant 1+\min\{\dim S/(Q_2+Q_3), \dim S/(Q_1+Q_3)\}\\ &= \operatorname{depth} R\end{aligned}$$

from the above exact sequence and a similar one. Thus we may suppose that Q_1+Q_2, Q_2+Q_3, Q_1+Q_3 have dimension $\geqslant 1$. Then from the exact sequence

$$0\to S/(Q_1+Q_2)\cap(Q_1+Q_3)\to S/(Q_1+Q_2)\oplus S/(Q_1+Q_3)\to S/(Q_1+Q_2+Q_3)\to 0$$

we get by Depth Lemma $\operatorname{depth} S/(Q_1+Q_2)\cap(Q_1+Q_3)=1$. Renumbering Q_i we may suppose that $\dim(Q_2+Q_3)\geqslant \max\{\dim(Q_1+Q_3), \dim(Q_2+Q_1)\}$. Using Proposition 5.8 we have

$$\operatorname{sdepth} R\geqslant \min\left\{\operatorname{sdepth} S/Q_2\cap Q_3, \left\lceil\frac{\dim(S/(Q_1+Q_2))}{2}\right\rceil+\left\lceil\frac{\dim(S/(Q_1+Q_3))}{2}\right\rceil\right\}.$$

We may suppose that $\operatorname{sdepth} R<\dim S/Q_i$ because otherwise $\operatorname{sdepth} R\geqslant \dim S/Q_i\geqslant \operatorname{depth} R$. Thus using Theorem 1.5 we get

$$\begin{aligned}\operatorname{sdepth} R\geqslant \min\Bigg\{&\left\lceil\frac{\dim S/Q_3+\dim S/(Q_2+Q_3)}{2}\right\rceil,\\ &\left\lceil\frac{\dim(S/(Q_1+Q_2))}{2}\right\rceil+\left\lceil\frac{\dim(S/(Q_1+Q_3))}{2}\right\rceil\Bigg\}.\end{aligned}$$

If $Q_1\not\subset\sqrt{Q_3}$ then $\dim S/Q_3>\dim S/(Q_1+Q_3)$ and we get

$$\dim S/Q_3+\dim S/(Q_2+Q_3)>\dim\big(S/(Q_1+Q_2)\big)+\dim\big(S/(Q_1+Q_3)\big)$$

because $\dim S/(Q_2+Q_3)$ is maxim by our choice. It follows that

$$\operatorname{sdepth} R\geqslant\left\lceil\frac{\dim(S/(Q_1+Q_2))}{2}\right\rceil+\left\lceil\frac{\dim(S/(Q_1+Q_3))}{2}\right\rceil\geqslant 2.$$

But from the first above exact sequence we get $\operatorname{depth} R=2$ with Depth Lemma, that is $\operatorname{sdepth} R\geqslant \operatorname{depth} R$.

If $Q_1\not\subset\sqrt{Q_2}$ we note that $\dim S/Q_2+\dim S/(Q_2+Q_3)>\dim(S/(Q_1+Q_2))+\dim(S/(Q_1+Q_3))$ and we proceed similarly as above with Q_2 instead Q_3. Note also that if $Q_1\subset\sqrt{Q_2}$ and $Q_1\subset\sqrt{Q_3}$ we get $\dim S/(Q_2+Q_3)\geqslant\dim S/(Q_2+Q_1)=\dim S/Q_2$, respectively $\dim S/(Q_2+Q_3)\geqslant \dim S/(Q_3+Q_1)=\dim S/Q_3$. Thus $Q_1\subset\sqrt{Q_3}=\sqrt{Q_2}$ and it follows $\operatorname{sdepth} R\geqslant\dim S/Q_2$, which is a contradiction. □

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