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Free product subgroups between Chevalley groups $\mathrm{G}(\Phi, F)$ and $\mathrm{G}(\Phi, F[t])$ ☆

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ABSTRACT

We investigate subgroups of a Chevalley group $G = \mathrm{G}(\Phi, A)$ over a ring A, containing its elementary subgroup $E = \mathrm{E}(\Phi, F)$ over a subring $F \subseteq A$. Assume that the root system Φ is simply laced and $A = F[t]$ is a polynomial ring. We show that if G is of adjoint type, then there exists an element $g \in \mathrm{E}(\Phi, A)$ such that $\langle g, \mathrm{E}(\Phi, F)\rangle = \langle g\rangle * \mathrm{E}(\Phi, F)$, where $\langle X\rangle$ denotes the subgroup, generated by a set X, and $*$ stands for the free product.

It follows that under the above assumptions the lattice $\mathcal{L} = L(E, G)$ is not standard. Moreover, combining the above result with theorems of Nuzhin and the author one obtains a necessary and sufficient condition for $\mathcal{L}$ to be standard provided that A and F are fields of characteristic not 2 and $\Phi \neq G_2$.

Introduction

Let $\mathrm{G}(\Phi, \_) = \mathrm{G}_P(\Phi, \_)$ denote a Chevalley–Demazure group scheme with root system Φ and weight lattice P. Let A be a ring (all rings are assumed to be associative, commutative with a unit). Denote by $\mathrm{E}(\Phi, A) = \mathrm{E}_P(\Phi, A)$ the elementary subgroup of $\mathrm{G}(\Phi, A)$, i.e. the subgroup generated by all elementary root unipotent elements $x_\alpha(t)$, $\alpha \in \Phi$, $t \in A$. Let F be a subring of A. We study the lattice $\mathcal{L} = L(\mathrm{E}(\Phi, F), \mathrm{G}(\Phi, A))$ of subgroups of $\mathrm{G}(\Phi, A)$, containing $\mathrm{E}(\Phi, F)$. Denote by $N(R)$ the normalizer of $\mathrm{E}(\Phi, R)$ in $\mathrm{G}(\Phi, A)$.

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Definition. The lattice $\mathcal{L}$ is called *standard* if it breaks into disjoint union of intervals $\mathcal{L}_R = L(\mathrm{E}(\Phi, R), N(R))$ over all subrings R of A.

For example, for an algebraic field extension A/F the lattice $\mathcal{L}$ is known to be standard by the result of Nuzhin [8].

Assume that the root system Φ is *simply laced*, i.e. $\Phi = A_l, D_l, E_6, E_7$, or E_8 (in other words, all roots in Φ have the same length). Let $A = F[t]$ be the polynomial ring over a field F. We prove that in this case the lattice $\mathcal{L}$ is not standard, and this is a first published example of this kind for $\Phi \neq A_l$. With this end we construct a free product subgroup in $\mathcal{L}$. Denote by G the quotient of $\mathrm{G}(\Phi, A)$ by its center (so that G is a subgroup of $\mathrm{G}_{\mathrm{ad}}(\Phi, A)$) and let E be the image of $\mathrm{E}(\Phi, F)$ in G.

Theorem A. *There exists an element $g \in \mathrm{E}_{\mathrm{ad}}(\Phi, F[t])$ such that*

$$\langle g, E\rangle = \langle g\rangle * E,$$

where $\langle X\rangle$ denotes the subgroup, generated by a set X, and $*$ stands for the free product.

Combining the above result with theorems of Nuzhin [8] and the author [13] one obtains a necessary and sufficient condition for $\mathcal{L}$ to be standard provided A and F are fields of characteristic not 2 and $\Phi \neq G_2$.

Corollary. *Suppose that F is a field,* $\operatorname{char} F \neq 2$*, A is an F-algebra, and $\Phi \neq G_2$. The lattice $\mathcal{L}$ is not standard if and only if Φ is simply laced and A is not algebraic over F.*

In characteristic 2 the notion of standardness is more complicated, therefore the result for $\Phi = B_l, C_l, F_4$ is not written yet. This is the work in progress. The case $\Phi = G_2$ is not clear at all.

Now we need some notation.

Notation. Let G be a group. For two elements $x, y \in G$ we write $[x, y] = x^{-1}y^{-1}xy$ for their commutator and $x^y = y^{-1}xy$ for the element, conjugate to x by y. If Z is a subset of G, then $\langle Z\rangle$ denotes the subgroup generated by Z. For subgroups $X, Y \subseteq G$ we let X^Y denote the normal closure of X in $\langle X \cup Y\rangle$ whereas $[X, Y]$ stands for their mutual commutator subgroup.

To display relations between Theorem A and the above corollary we recall some group theoretic notions from [2]. Note that standardness of the lattice $\mathcal{L}$ is equivalent to the following statement: For any $H \in \mathcal{L}$ we have $\mathrm{E}(\Phi, F)^{\mathrm{E}(\Phi,F)^H} = \mathrm{E}(\Phi, F)^H = \mathrm{E}(\Phi, P)$ for some subring P.

Definition. (See Z.I. Borevich [2].) A subgroup D of a group G is called *polynormal* if $D^{D^H} = D^H$ for all subgroups H lying between D and G.

A. Bak called this description "*a sandwich classification theorem*" for subgroups between D and G.

In [12] the author proved that existence of a free product subgroup as in Theorem A ensures that $\mathrm{E}(\Phi, F)$ is not polynormal in $\mathrm{G}(\Phi, F[t])$. Further, if a pair of groups $D \leqslant H$ projects onto the pair $\mathrm{E}(\Phi, F) \leqslant \mathrm{G}(\Phi, F[t])$, then D is not polynormal in H. In particular, if a pair of rings $R \subseteq A$ projects onto the pair $F \subseteq F[x]$, then $\mathrm{E}(\Phi, R)$ is not polynormal in $\mathrm{G}(\Phi, A)$. Obviously, the same holds if the ring A is bigger.

In view of this, we call a ring extension $R \subseteq A$ *quasi-transcendental* if there exists a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & A \\ F \arrow[r] & F[x] & \end{tikzcd}$$

where F is a field, the horizontal arrows are injective, and the vertical ones are surjective. Otherwise the extension is called *quasi-algebraic*. Using the above observations we get the following result.

Corollary. *If a ring extension $R \subseteq A$ is quasi-transcendental and Φ is a simply laced root system, then $\mathrm{E}(\Phi, R)$ is not polynormal in $\mathrm{G}(\Phi, A)$.*

If F is a finitely generated algebra over a field or over $\mathbb{Z}$ and A is a domain, then the condition of being quasi-algebraic has especially elegant reformulation (see [12, Theorem B]), and therefore, we can rewrite the above corollary as follows.

Theorem B. *Let F be a finitely generated algebra over a field or over $\mathbb{Z}$ and let A be an F-algebra. Suppose that A is a domain and Φ is a simply laced root system. If the lattice of subgroups between $\mathrm{E}(\Phi, F)$ and $\mathrm{G}(\Phi, A)$ is polynormal, then one of the following holds*:

(1) *A is an integral extension of F; or*
(2) $\dim F \leqslant 1$ *and A is contained in the algebraic closure of the field of fractions of F.*

There is some evidence that the converse to the theorem above holds. Namely, the standard description of the lattice of subgroups between $\mathrm{E}(\Phi, F)$ and $\mathrm{G}(\Phi, A)$ is established by R.A. Schmidt in [10] for the case where $\Phi = A_l$ and A is the field of fractions of a Dedekind domain. Using the technique developed in [17] and the notion of ideal stable rank of a module in spirit of [11] we hope to extend the result by Schmidt to all Chevalley groups of rank $\geqslant 3$ (at present the standard description for all Chevalley groups over the field of fractions is known by Nuzhin and Yakushevich [9] only for the case where F is a Euclidean domain). The most complicated part is to extend the result by Nuzhin [8] to integral *ring* extensions. Once this done, one can try to combine the two above results to prove the converse to Theorem B.

The situation with Chevalley groups corresponding to not simply laced root systems is quite different. Recently the author has proved that if $\Phi \neq G_2$, then the lattice of subgroups between $\mathrm{E}(\Phi, F)$ and $\mathrm{G}(\Phi, A)$ is standard for an *arbitrary* pair of rings $F \subseteq A$ provided 2 is invertible in F. Technically, this is motivated by the following results by Golubchik, Mikhalev [4], Tomanov [15], Gordeev [5] and Nesterov and Stepanov [7] on so-called *identities with constants*.

Let G be a group. Denote by $\mathbf{F}(y_1, \ldots, y_k)$ the free group on symbols $y_1, \ldots, y_k$. Let $S(y_1, \ldots, y_k)$ be a nontrivial element in the free product $G * \mathbf{F}(y_1, \ldots, y_k)$. The equation $S(y_1, \ldots, y_k) = 1$ is called an identity with constants in G. Given $g_1, \ldots, g_k \in G$ the notation $S(g_1, \ldots, g_k)$ is self-explanatory. One says that an identity S with constants holds in G if for any elements $g_1, \ldots, g_k \in G$ the equality $S(g_1, \ldots, g_k) = 1$ is satisfied.

Theorem C *(Tomanov, Gordeev). Let $G = \mathrm{G}(\Phi, K)/\,\mathrm{Center}(\mathrm{G}(\Phi, K))$, where K is an infinite field. Suppose that the root system Φ is simply laced. Then there is no valid identity with constants in G, i.e. for any $S(y_1, \ldots, y_k)$ there are elements $g_1, \ldots, g_k \in G$ such that $S(g_1, \ldots, g_k) \neq 1$.*

In contrary, when Φ is not simply laced a certain identity with constants holds in G. To formulate the identity we need the concept of small elements from [15] and [5]. Let K be an algebraically closed field. Let $g \in \mathrm{G}(\Phi, K)$ be a semisimple element. Then it belongs to a torus T and the roots, being characters of torus, map T to K. A noncentral *semisimple element* is called *small* if it goes to the identity under all long roots. It is known that such elements exist if Φ is not simply laced provided $2 \neq 0$ for $\Phi = B_l, C_l, F_4$ and $3 \neq 0$ for $\Phi = G_2$. In the case of the bad characteristic (i.e. the conditions above do not hold) we say that a short root element is *a small unipotent element*. If R is a ring, then an element $g \in \mathrm{G}(\Phi, R)$ is called small if for any homomorphism φ from R to a closed field K the element $\mathrm{G}(\Phi, \varphi)(g)$ is small.

Theorem D *(Golubchik, Mikhalev, Gordeev, Nesterov, Stepanov). Let $h \neq 1$ be a small element. Let $u \neq 1$ be a long root unipotent element. Then the following identity with constants holds*: $[u, [u, u^{h^y}]] = 1$. *Moreover, if $\Phi = B_l, C_l, F_4$, then the identity $[u, u^{h^y}] = 1$ holds.*

Theorem C provides another motivation for investigation of free product subgroups in $\mathrm{G}(\Phi, F[t])$. Let A be the affine algebra of $\mathrm{G}(\Phi, F)$ and let g denote the "generic element" of $\mathrm{G}(\Phi, A)$. Then the theorem shows in particular (with $k = 1$) that the subgroup in $\mathrm{G}(\Phi, A)/\mathrm{Center}(\mathrm{G}(\Phi, F))$ generated by $\mathrm{G}(\Phi, F)/\mathrm{Center}(\mathrm{G}(\Phi, F))$ and the cyclic subgroup $\langle g \rangle$ is their free product. From this point of view Theorem A is a strengthening of this corollary.

1. Opposite transvections

In this section we develop some tools for the proof of Theorem A.

Notation. Let Φ be a root system and let R be a ring. For a root $\alpha \in \Phi$ and an element $\xi \in R$ we denote by $x_\alpha(\xi)$ the corresponding elementary root unipotent element and by $X_\alpha(R) = \{x_\alpha(\xi) \mid \xi \in R\}$ the root subgroup. For a given order on Φ $B(R)$ denotes the standard Borel subgroup of $\mathrm{G}(\Phi, R)$. Of course, B and X_α are affine group schemes. However, we shall write B and X_α instead of $B(R)$ and $X_\alpha(R)$ when this cannot lead to confusion and R is uniquely specified by context.

Let K be a field and $G = \mathrm{G}(\Phi, K)$. Let α be a long root and $a, b \in G$. Consider two subgroups X_α^a and X_α^b. It is easy to prove (see [16]) that there exists $c \in G$ such that $(X_\alpha^a)^c = X_\alpha$ and $(X_\alpha^b)^c = X_\beta$ for some long root $\beta \in \Phi$. Recall that the subgroups X_α^a and X_α^b are called *opposite* if $\beta = -\alpha$. In this case, any two elements $x_\alpha(\xi)^a$ and $x_\alpha(\zeta)^b$ also will be called opposite.

In the sequel we use the following reformulation of the latter condition.

Lemma 1.1. *Let α be a long root and $a, b \in G$. Choose a split maximal torus in G and an ordering on Φ such that α is a maximal root. Then the following conditions are equivalent.*

(1) *X_α^a and X_α^b are opposite.*
(2) *If $ba^{-1} \in BwB$ is a Bruhat decomposition of ba^{-1}, then $w(\alpha) = -\alpha$.*
(3) *For any $x \in X_\alpha^a \setminus \{1\}$ and $y \in X_\alpha^b \setminus \{1\}$ the commutator $[x, y]$ does not commute with x.*

Proof. Obviously, X_α^a and X_α^b are opposite if and only if X_α and $X_\alpha^{ba^{-1}}$ are opposite. Let $ba^{-1} = b_1 w b_2$ be a Bruhat decomposition of ba^{-1}. Since α is the maximal root, B normalizes X_α. Therefore, $X_\alpha^{ba^{-1}} = X_\alpha^{wb_2}$. Conjugating by b_2 we see that X_α and $X_\alpha^{ba^{-1}}$ are opposite if and only if X_α and X_α^w are opposite which is equivalent to saying that $w(\alpha) = -\alpha$. Thus, (1) $\iff$ (2).

Clearly condition (3) is stable under conjugation. Therefore, we may assume that $x \in X_\alpha$ and $y \in X_\beta$ for some long root $\beta \in \Phi$. If $\beta = -\alpha$, then X_α and X_β generates the subgroup isomorphic to $SL_2(K)$ or $\mathrm{PGL}_2(K)$. Moreover, this isomorphism sends x and y to the elementary transvections $t_{12}(\xi)$ and $t_{21}(\zeta)$, respectively. It is clear that $[t_{12}(\xi), t_{21}(\zeta)]$ does not commute with $t_{12}(\xi)$, thus, $[x, y]$ does not commute with x. Conversely, if $\beta \neq -\alpha$, then either $\alpha + \beta$ is not a root, hence $[x, y] = 1$, or $\gamma = \alpha + \beta \in \Phi$ but $\alpha + \gamma \notin \Phi$ in which case $[x, y] \in X_\gamma$ and X_α commutes with X_γ. $\square$

From now on we always assume that Φ **is a simply laced** root system. Let F be a field and $F[t]$ the polynomial ring over F. In the sequel we use the notion of degree of an element from a Chevalley group over $F[t]$ or over its quotient field $F(t)$.

Definition 1.2. First, define the degree $\deg \frac{p}{q}$ of a rational function $\frac{p}{q}$ as $\deg p - \deg q$. Let $\rho : \mathrm{G}(\Phi, F(t)) \to \mathrm{GL}_n(F(t))$ be a representation of $\mathrm{G}(\Phi, F(t))$ and $g \in \mathrm{G}(\Phi, F(t))$. Then we define the degree $\deg(g)$ of an element $g \in \mathrm{G}(\Phi, F(t))$ (with respect to ρ) to be the maximal degree of the entries of the matrix $\rho(g)$.

The following lemma is a key step in the proof of the main theorem.

Lemma 1.3. *Let $\alpha \in \Phi$ be a root. There exists an element $g \in \mathrm{G}(\Phi, F[t])$ such that for any noncentral element a in $\mathrm{G}(\Phi, F)$ the root subgroups X_α and $X_\alpha^{gag^{-1}}$ are opposite.*

Proof. For $g \in \mathrm{G}(\Phi, F[t])$ denote by $C(g)$ the set of all elements $a \in \mathrm{G}(\Phi, F)$ such that

$$\big[\big[X_\alpha, X_\alpha^{gag^{-1}}\big], X_\alpha\big] = \{1\}. \tag{1}$$

Clearly, $C(g)$ is Zariski closed in $\mathrm{G}(\Phi, F)$. Now we construct a sequence of elements $g_0 = 1, g_1, \dots, g_k, \dots$ in $\mathrm{G}(\Phi, F[t])$ such that $C(g_0) \supsetneqq C(g_1) \supsetneqq \cdots \supsetneqq C(g_k) \supsetneqq \cdots$ is a proper chain of Zariski closed subsets in $\mathrm{G}(\Phi, F)$ and their intersection equals to the center of $\mathrm{G}(\Phi, F)$. Since $\mathrm{G}(\Phi, F)$ is a Noetherian variety, this chain terminates in some $C(g_n) = \mathrm{Center}(\mathrm{G}(\Phi, F))$.

Suppose that we have obtained $g_0, \dots, g_k$ satisfying the above conditions, except that $C(g_k) \neq \mathrm{Center}(\mathrm{G}(\Phi, F))$. It suffices to prove that we can construct the next element of the chain. Clearly, we can choose $M \in \mathbb{N}$ such that

$$\big[\big[X_\alpha, X_\alpha^{g_k a g_k^{-1}}\big], X_\alpha\big] \not\equiv \{1\} \mod t^M F[t]$$

for any $a \notin C(g_k)$. Indeed, if $m \geqslant \deg(g), \deg(g^{-1})$, then for any $a \in \mathrm{G}(\Phi, F)$ the degree of

$$\big[\big[x_\alpha(1), x_\alpha(1)^{gag^{-1}}\big], x_\alpha(1)\big]$$

is at most $16m$. Therefore, it equals 1 if and only if it is equivalent to 1 modulo $t^M F[t]$ where $M > 16m$.

Now, take $a \in C(g_k)$. Obviously, the set X of all $g \in \mathrm{G}(\Phi, F(t))$ such that

$$\big[\big[X_\alpha, X_\alpha^{(g_k g)a(g_k g)^{-1}}\big], X_\alpha\big] \neq \{1\}$$

is open in $\mathrm{G}(\Phi, F(t))$. By Gordeev's Theorem C, this set is nonempty. Note that the congruence subgroup $\mathrm{G}(\Phi, F[t], t^M F[t])$ is Zariski dense in $\mathrm{G}(\Phi, F(t))$ (because $\mathrm{G}(\Phi, F[t], t^M F[t]) \cap X_\alpha$ is dense in X_α as an infinite subset in $\mathbb{A}^1$ over a field). Therefore, it has nonempty intersection with X. Take $g \in X \cap \mathrm{G}(\Phi, F[t], t^M F[t])$ and put $g_{k+1} = g_k g$. Since $g_{k+1} \equiv g_k \mod t^M F[t]$, by the choice of M we have $C(g_{k+1}) \subseteq C(g_k)$. But $a \in C(g_k) \setminus C(g_{k+1})$, therefore, $C(g_{k+1}) \neq C(g_k)$, which completes the proof. $\square$

2. Proof of the main theorem

For the proof of the main theorem we need to evaluate degrees of certain elements of a Chevalley group and its Lie algebra. Let ρ be a faithfull representation of $\mathrm{G}(\Phi, \_)$. For an element $g \in \mathrm{G}(\Phi, F(t))$ we define the *negative degree* $\mathrm{n.deg}\, g$ as a supremum of $-\deg(\rho(g)_{ij})$, where $1 \leqslant i, j \leqslant n$.

First we consider Bruhat decomposition in $\mathrm{G}(\Phi, F(t))$. Fix a split maximal torus T and the ordering on Φ. Denote by U the unipotent radical of the standard Borel subgroup B and by N the (scheme theoretic) normalizer of T. As usually, $W = W(\Phi)$ denotes the Weyl group. Let R be a commutative ring. For $w \in W$ denote by $U_w(R)$ the subgroup of $U(R)$ generated by root subgroups $X_\alpha(R)$ for all roots $\alpha \in \Phi^+$ such that $w(\alpha) \in \Phi^-$. Then, $B(R)wB(R) = U_w(R)wB(R)$ and $a = b'\dot{w}b''$ is called a reduced Bruhat decomposition of an element $a \in \mathrm{G}(\Phi, R)$ if $b' \in U_w(R)$, $b'' \in B$ and $\dot{w} \in N(R)$ is a preimage of w.

Lemma 2.1. *Given $m \in \mathbb{N}$ there exists $M \in \mathbb{N}$ satisfying the following: for any element $a \in \mathrm{G}(\Phi, F[t])$ of degree at most m there are $b', b'' \in U(F(t))$ and $\dot{w} \in N(F(t))$ such that $a = b'\dot{w}b''$ and* $\deg(b'), \deg(b''), \deg(\dot{w}), \mathrm{n.deg}(\dot{w}), \deg(b'^{-1}), \deg(b''^{-1}), \deg(\dot{w}^{-1}), \mathrm{n.deg}(\dot{w}^{-1}) \leqslant M$.

Proof. Let $\rho : G \to \mathrm{GL}_n$ be a faithfull representation of G. If $a = b'\dot{w}b''$ is a reduced Bruhat decomposition of a, then the matrix coefficients of $\rho(b')$, $\rho(b'')$ and $\rho(\dot{w})$ are rational functions on matrix coefficient of $\rho(a)$, hence the result follows.

A rigorous proof uses formalism of affine schemes. In this proof T, B, U, U_w and N denote the corresponding group subschemes of $G = \mathrm{G}(\Phi, \_)$. For each $w \in W$ fix a preimage $\tilde{w}$ of w in $N(\mathbb{Z})$. Let $\varphi : U_w \times B \to G$ be a morphism of affine schemes given by $\varphi_R(u, b) = u\tilde{w}b$, where R is a ring, $u \in U_w(R)$, $b \in B(R)$, and $\tilde{w}$ is identified with its canonical image in $N(R)$. The image C_w of φ is called a Bruhat cell. It is known that C_w is an open subscheme of a closed subscheme of G and that φ induces an isomorphism of schemes $U_w \times B \to C_w$ (see [3, Exp. XXII, 5.7.3]). As affine schemes $B \cong T \times U$, thus we obtain an isomorphism $\psi : U_w \times T \times U \to C_w$.

Let M_n denote the full matrix ring. Then M_n is an affine scheme with affine algebra $\mathbb{Z}[M_n] = \mathbb{Z}[h_{ij} \mid 1 \leqslant i, j \leqslant n]$. The representation ρ induces a morphism $\sigma : C_w \to M_n$ and a ring homomorphism $\sigma^* : \mathbb{Z}[M_n] \to \mathbb{Z}[C_w]$. Put $g_{ij} = \sigma^*(h_{ij})$. Since ρ is faithfull, the map σ is a composition of open and closed immersions. Therefore, the image of C_w is an open subscheme of a closed subscheme of M_n ("closed of an open" and "open of a closed" is the same thing for affine schemes, see, e.g., [6, Ch. 3, Exercise 2.3]). It follows that $\mathbb{Z}[C_w]$ is a localization of $\sigma^*(\mathbb{Z}[M_n]) = \mathbb{Z}[g_{ij}]$.

Consider a commutative diagram

$$\begin{tikzcd} U_w \times T \times U \arrow[r, "\tau"] \arrow[d, "\psi"'] & M_n \times M_n \times M_n \arrow[d, "\theta"] \\ C_w \arrow[r, "\sigma"] & M_n \end{tikzcd}$$

where the horizontal arrows are induced by ρ and $\theta_R(b, c, d) = b\tilde{w}cd$ for any ring R and $b, c, d \in M_n(R)$. Let $a \in G(F[t]) \leqslant G(F(t))$ be an element of degree $\leqslant m$. The group $G(F(t))$ is a disjoint union of Bruhat cells $C_w(F(t))$ (see [1, 2.11]). If $w \in W$ is such that $a \in C_w(F(t))$, then $a = b'\tilde{w}sb''$, where $b' \in U_w(F(t))$, $s \in T(F(t))$, and $b'' \in U(F(t))$. By definition an R-rational point of a $\mathbb{Z}$-scheme X is a ring homomorphism $\mathbb{Z}[X] \to R$. Thus, we obtain the following commutative diagram.

$$\begin{tikzcd} \mathbb{Z}[h_{ij}] = \mathbb{Z}[M_n] \arrow[r, "\sigma^*"] \arrow[d, "\theta^*"'] & \mathbb{Z}[C_w] \arrow[r, "a"] \arrow[d, "\psi^*"'] & F(t) \\ \mathbb{Z}[x_{ij}, y_{ij}, z_{ij}] = \mathbb{Z}[M_n] \otimes \mathbb{Z}[M_n] \otimes \mathbb{Z}[M_n] \arrow[r, "\tau^*"] & \mathbb{Z}[U_w] \otimes \mathbb{Z}[T] \otimes \mathbb{Z}[U] \arrow[r, "b' \otimes s \otimes b''"] & F(t) \otimes F(t) \otimes F(t) \arrow[u, "mult"'] \end{tikzcd}$$

where *mult* is the multiplication homomorphism. Note that

$$\rho(b')_{ij} = b' \circ \rho^*(x_{ij}) = mult \circ (b' \otimes s \otimes b'') \circ \tau^*(x_{ij}) = a \circ (\psi^*)^{-1} \circ \tau^*(x_{ij}).$$

Similarly, $\rho(s)_{ij} = a \circ (\psi^*)^{-1} \circ \tau^*(y_{ij})$, $\rho(b'')_{ij} = a \circ (\psi^*)^{-1} \circ \tau^*(z_{ij})$, and $\rho(a)_{ij} = a \circ \sigma^*(h_{ij}) = a(g_{ij})$.

We have already shown that any element of $\mathbb{Z}[C_w]$ is a rational function on the elements g_{ij}. Let k_w be an integer greater than degrees of numerators and denominators of $(\psi^*)^{-1} \circ \tau^*(x_{ij})$, $(\psi^*)^{-1} \circ \tau^*(y_{ij})$, and $(\psi^*)^{-1} \circ \tau^*(z_{ij})$, viewed as rational functions on g_{ij}'s. Put $M_1 = nm \cdot \max_{w \in W} k_w$. Since $\deg \rho(a)_{ij} \leqslant m$, the degrees of numerators and denominators of $\rho(b')_{ij}$, $\rho(s)_{ij}$, and $\rho(b'')_{ij}$ are not greater than $m \cdot \max_{w \in W} k_w$. On the other hand, $a = b'\dot{w}b''$ where $\dot{w} = \tilde{w}s$. Since $\tilde{w}$ comes from $N(\mathbb{Z})$, degrees of numerators and denominators of elements $\rho(\dot{w})_{ij}$ are not greater than M_1.

The elements of a matrix are rational functions on the elements of its inverse. Since the degrees of numerators and denominators of b', $\dot{w}$, and b are bounded by M_1, there exists an integer M, satisfying the condition of the lemma. □

Notation. Let $\alpha_1, \dots, \alpha_r$ be the simple roots. We consider the adjoint representation of $\mathrm{G}(\Phi, F(t))$ on its Lie algebra $\mathrm{Lie}(\Phi, F(t))$ with a Chevalley base consisting of root elements e_α and elements $h_i = h_{\alpha_i}$ from the Cartan subalgebra $H = H(\Phi, F(t))$. For $g \in \mathrm{G}(\Phi, F(t))$ and $u \in \mathrm{Lie}(\Phi, F(t))$ we write u^g to denote the (right) action of g on u.

For an element $u = \sum_{\beta \in \Phi} \lambda_\beta e_\beta + \sum_{i=1}^r \mu_i h_i \in \mathrm{Lie}(\Phi, F(t))$ we write $\deg(u)$ to denote the maximum of the degrees of the coefficients λ_β and μ_i. For a given root α we use the following notation:

$$\deg_\alpha(u) = \deg \lambda_\alpha \quad \text{and} \quad \widehat{\deg}_\alpha(u) = \max_{\beta \neq \alpha}(\deg \lambda_\beta, \deg \mu_i).$$

Lemma 2.2. *Let $\alpha \in \Phi$ be the maximal root. Put $d = b\dot{w}x_\alpha(mt^N)$, where $b \in U(F(t))$, $\dot{w} \in N(F(t))$, w is its image in W, and $m, N \in \mathbb{Z}$. Suppose that $w(\alpha) = -\alpha$. For $u \in \mathrm{Lie}(\Phi, F(t))$ we have:*

(1) $\deg(u^d) \leqslant \deg(u) + 2N + \deg(b) + \deg(b^{-1}) + \deg(\dot{w}) + \deg(\dot{w}^{-1})$;
(2) $\widehat{\deg}_\alpha(u^d) \leqslant \deg(u) + N + \deg(b) + \deg(b^{-1}) + \deg(\dot{w}) + \deg(\dot{w}^{-1})$;
(3) $\deg_\alpha(e_\alpha^d) \geqslant 2N - \mathrm{n.deg}(\dot{w}) - \mathrm{n.deg}(\dot{w}^{-1})$;
(4) $\deg_\alpha(e_\alpha^d) - \widehat{\deg}_\alpha(e_\alpha^d) = N$.

Proof. For the proof we need the following calculations in Lie algebras $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

$$e_{-\alpha}^{x_\alpha(mt^N)} = -m^2t^{2N}e_\alpha + t^N h + e_{-\alpha}, \quad \text{for some } h \in H,\ \deg(h) = 0,$$

$$e_\gamma^{x_\alpha(mt^N)} = e_\gamma \pm mt^N e_{\alpha+\gamma}, \quad \text{if } \alpha + \gamma \in \Phi,$$

$$e_\gamma^{x_\alpha(mt^N)} = e_\alpha, \quad \text{if } \alpha + \gamma \notin \Phi \cup \{0\},$$

$$h_i^{x_\alpha(mt^N)} = h_i + kt^N e_\alpha, \quad \text{for some } k \in \mathbb{Z}.$$

Clearly, $\deg(u^{b\dot{w}}) \leqslant \deg(u) + \deg(b) + \deg(\dot{w}) + \deg(b^{-1}) + \deg(\dot{w}^{-1})$. On the other hand, the above formulas show that $\deg(f^{x_\alpha(mt^N)}) \leqslant \deg(f) + 2N$ and $\widehat{\deg}_\alpha(f^{x_\alpha(mt^N)}) \leqslant \deg(f) + N$ for any $f \in \mathrm{Lie}(\Phi, F(t))$, which proves the first and the second assertions of the lemma.

Since α is the maximal root, e_α commutes with U. Since $w(\alpha) = -\alpha$, we have $e_\alpha^{\dot{w}} = pe_{-\alpha}$, where $p \in F(t)$ and $\deg p \geqslant -\mathrm{n.deg}(\dot{w}) - \mathrm{n.deg}(\dot{w}^{-1})$. Now, items (3) and (4) follow from the first of the displayed formulas. □

Now, we are ready to prove the main result of the article.

Proof of Theorem A. Since $\mathrm{G}(\Phi, F[t])/\mathrm{Center}(\mathrm{G}(\Phi, F[t]))$ is a subgroup of the adjoint group $\mathrm{G_{ad}}(\Phi, F[t])$ containing $\mathrm{E_{ad}}(\Phi, F[t])$, we may assume that G is of adjoint type. Let $\rho : \mathrm{G}(\Phi, F(t)) \to \mathrm{GL}_n(F(t))$ be the adjoint representation. Since ρ is faithfull we may identify the elements of $\mathrm{G}(\Phi, F(t))$ with their images under ρ. Choose a split maximal torus T, and an ordering on Φ. Let

$\alpha \in \Phi$ be the maximal root. By Lemma 1.3 we can find $a \in \mathrm{G}(\Phi, F[t])$ such that for any $c \in \mathrm{G}(\Phi, F)$ the root subgroups X_α^a and X_α^{ac} are opposite.

Let $m \geqslant \deg(a), \deg(a^{-1})$. Obviously, the degree of the element aca^{-1} is at most $2m$ for any $c \in \mathrm{G}(\Phi, F)$. Let $b'\dot{w}b''$ be the reduced Bruhat decomposition of aca^{-1} with b', b'' in the unipotent radical $U = U(F(t))$ of the Borel subgroup. By Lemma 2.1 there exists $M \in \mathbb{N}$ such that

$$\sup_{c \in \mathrm{G}(\Phi,F)} \bigl(\deg\bigl(b'\bigr), \deg\bigl(b'^{-1}\bigr), \deg\bigl(b''\bigr), \deg\bigl(b''^{-1}\bigr), \deg(\dot{w}), \deg\bigl(\dot{w}^{-1}\bigr), \mathrm{n.deg}(\dot{w}), \mathrm{n.deg}\bigl(\dot{w}^{-1}\bigr)\bigr) = M.$$

Take an integer $N > 8M$ and put $g = x_\alpha(t^N)^a$. By the main result of [14] the elementary subgroup is normal, therefore $g \in \mathrm{E}(\Phi, F[t])$.

Consider the product $z = c_1 g^{m_1} c_2 \cdots g^{m_l} c_{l+1}$, where $c_i \in \mathrm{G}(\Phi, F)$, $c_i \neq 1$ for all $i = 2, \dots, l$, and each m_i is not divisible by $\operatorname{char} F$ (otherwise, $g^{m_i} = 1$). We have to prove that $z \neq 1$. Suppose that z is a product of minimal length such that $z = 1$. Conjugating the equation $z = 1$ by c_{l+1} we may assume that $c_{l+1} = 1$ and $c_1 \neq 1$ (otherwise the product z can be shortened by conjugation by g^{m_l}). Now, conjugating z by a^{-1} we get

$$aza^{-1} = \bigl(ac_1a^{-1}\bigr)x_\alpha\bigl(m_1t^N\bigr)\bigl(ac_2a^{-1}\bigr)x_\alpha\bigl(m_2t^N\bigr)\cdots\bigl(ac_la^{-1}\bigr)x_\alpha\bigl(m_lt^N\bigr) = 1.$$

Let $ac_ka^{-1} = b'_k w_k b''_k$ be the reduced Bruhat decomposition of ac_ka^{-1} with $b'_k, b''_k \in U$. Since X_α commutes with U, we can rewrite the equation above in the form

$$b_1w_1x_\alpha\bigl(m_1t^N\bigr)b_2w_2x_\alpha\bigl(m_2t^N\bigr)\cdots b_lw_lx_\alpha\bigl(m_lt^N\bigr) = \bigl(b''_l\bigr)^{-1} \qquad (\star)$$

where $b_k = b''_{k-1}b'_k$ for $k > 1$ and $b_1 = b'_1$. Recall that by the choice of a we have $w_k(\alpha) = -\alpha$ and $\deg(b_k), \deg(b_k^{-1}) \leqslant 2M$.

Since $e_\alpha^U = e_\alpha$ and $(b''_l)^{-1} \in U$, the right hand side of $(\star)$ acts trivially on e_α. Consider the action of the left hand side of $(\star)$ on e_α. Let

$$u_j = e_\alpha^{y_j}, \quad \text{where } y_j = b_1w_1x_\alpha\bigl(m_1t^N\bigr)\cdots b_jw_jx_\alpha\bigl(m_jt^N\bigr).$$

We prove by induction on j that $\deg_\alpha(u_j) \geqslant \widehat{\deg}_\alpha(u_j) + N$. Base of induction coincides with item (4) of Lemma 2.2.

Let $j > 1$. Put $d = b_jw_jx_\alpha(m_jt^N)$. Let $u_{j-1} = pe_\alpha + u$, where u belongs to the span of h_i and e_β for all $i = 1, \dots, r$ and $\beta \in \Phi \setminus \{\alpha\}$. By induction hypothesis $\deg p \geqslant N + \deg(u)$. Items (3) and (1) of Lemma 2.2 imply that

$$\deg_\alpha\bigl(pe_\alpha^d\bigr) \geqslant \deg p + 2N - \mathrm{n.deg}\, w_j - \mathrm{n.deg}\, w_j^{-1} \geqslant \deg(u) + 3N - 2M \geqslant \deg\bigl(u^d\bigr).$$

Therefore, $\deg_\alpha(u_j) = \deg_\alpha(pe_\alpha^d) \geqslant \deg p + 2N - \mathrm{n.deg}(w_j) - \mathrm{n.deg}(w_j^{-1})$. By item (4) of Lemma 2.2, $\widehat{\deg}_\alpha(pe_\alpha^d) = \deg_\alpha(u_j) - N$. Finally, by item (1) of Lemma 2.2,

$$\begin{aligned}
\widehat{\deg}_\alpha\bigl(u^d\bigr) &\leqslant \deg(u) + 2N + \deg(b_j) + \deg\bigl(b_j^{-1}\bigr) + \deg(w_j) + \deg\bigl(w_j^{-1}\bigr) \\
&\leqslant \deg p + N + \deg(b_j) + \deg\bigl(b_j^{-1}\bigr) + \deg(w_j) + \deg\bigl(w_j^{-1}\bigr) \\
&\leqslant \deg p + 2N - \mathrm{n.deg}(w_j) - \mathrm{n.deg}\bigl(w_j^{-1}\bigr) \leqslant \deg_\alpha(u_j).
\end{aligned}$$

This competes the proof that $\deg_\alpha(u_j) \geqslant \widehat{\deg}_\alpha(u_j) + N$ and shows that Eq. $(\star)$ is impossible, i.e. $z \neq 1$. $\square$

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References

[1] A. Borel, J. Tits, Groupes réductifs, Publ. Math. Inst. Hautes Etudes Sci. 27 (1965) 55–151.
[2] Z.I. Borevich, O.N. Macedonskaja, On the lattice of subgroups, Zap. Nauchn. Sem. LOMI 103 (1980) 13–19 (in Russian).
[3] M. Demazure, A. Grothendieck, Schémas en groupes, Lecture Notes in Math., vol. 151–153, Springer-Verlag, Berlin, Heidelberg, New York, 1970.
[4] I.Z. Golubchik, A.V. Mikhalev, Generalized group identities is classical groups, J. Soviet Math. 27 (4) (1984) 2902–2918.
[5] N.L. Gordeev, Freedom in conjugacy classes of simple algebraic groups and identities with constants, St. Petersburg Math. J. 9 (4) (1998) 709–723.
[6] Q. Liu, Algebraic Geometry and Arithmetic Curves, Oxford Univ. Press, USA, 2002.
[7] V.V. Nesterov, A.V. Stepanov, Identity with constant in a Chevalley group of type F_4, St. Petersburg Math. J. (2009).
[8] Ya.N. Nuzhin, Groups contained between groups of Lie type over different fields, Algebra Logic 22 (1983) 378–389.
[9] Ya.N. Nuzhin, A.V. Yakushevich, Intermediate subgroups of Chevalley groups over a field of fractions of a ring of principal ideals, Algebra Logic 39 (3) (2000) 199–206.
[10] R.A. Shmidt, Subgroups of the general linear group over the field of quotients of a Dedekind ring, J. Soviet Math. 19 (1982) 1052–1059.
[11] A.V. Stepanov, Description of subgroups of the general linear group over a ring with the use of stability conditions, in: Rings and Linear Groups, Kubansky State University, Krasnodar, 1988, pp. 82–91 (in Russian).
[12] A.V. Stepanov, Nonstandard subgroups between $\mathrm{E}_n(R)$ and $\mathrm{GL}_n(A)$, Algebra Colloq. 10 (3) (2004) 321–334.
[13] A.V. Stepanov, Subring subgroups in Chevalley groups with doubly laced root systems, preprint: http://alexei.stepanov.spb.ru/papers/positive.pd, 2009.
[14] G. Taddei, Normalité des groupes élémentaire dans les groupes de Chevalley sur un anneau, Contemp. Math. 55 (1986) 693–710.
[15] G.M. Tomanov, Generalized group identities is linear groups, Math. USSR-Sb. 51 (1) (1985) 33–46.
[16] N.A. Vavilov, On the geometry of long root subgroups in Chevalley groups, Vestnik Leningrad Univ. Math. Mech. Astronom. 21 (1) (1988) 5–10.
[17] Dengyin Wang, Shangzhi Li, Overgroups of $L(K)$ in $L(F)$, Algebra Colloq. 5 (4) (1998) 417–424.