



j -Multiplicity and depth of associated graded modules

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ABSTRACT

Let R be a Noetherian local ring. We define the minimal j -multiplicity and almost minimal j -multiplicity of an arbitrary R -ideal on any finite R -module. For any ideal I with minimal j -multiplicity or almost minimal j -multiplicity on a Cohen–Macaulay module M , we prove that under some residual conditions, the associated graded module $\mathrm{gr}_I(M)$ is Cohen–Macaulay or almost Cohen–Macaulay, respectively. Our work generalizes the results for minimal multiplicity and almost minimal multiplicity obtained by Sally, Rossi, Valla, Wang, Huckaba, Elias, Corso, Polini, and Vaz Pinto.

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1. Introduction

In this paper we investigate the behavior of the depth of the associated graded ring $\mathrm{gr}_I(R)$ of an ideal I in a Noetherian local ring (R, \mathfrak{m}) in terms of conditions on the j -multiplicity of I . The associated graded ring of I is an algebraic construction whose projective scheme represents the exceptional fiber of the blowup of a variety along a subvariety. Its arithmetical properties, like its depth, provide useful information, for instance, on the cohomology groups of the blowup. For an \mathfrak{m} -primary ideal I , the interplay between the Hilbert polynomial of I , more precisely its Hilbert coefficients, and the depth of the associated graded ring has been widely investigated. This line of study has its roots in the pioneering work of Sally. The idea is that extremal values of the Hilbert coefficients, most

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notably of the multiplicity of I , yield high depth of the associated graded ring and, conversely, good depth properties encode information about all the Hilbert coefficients, such as their positivity. The problem arises when one considers ideals which are not \mathfrak{m} -primary, because their Hilbert functions are not defined, thus there is no numerical information on Hilbert coefficients available to study the Cohen–Macaulayness of $\text{gr}_I(R)$. To remedy the lack of this tool, in this paper we propose to use the notion of j -multiplicity. The j -multiplicity was developed as a generalization of the Hilbert–Samuel multiplicity to arbitrary ideals. It was first introduced by Achilles and Manaresi in 1993 and, since then, it has been frequently used by both algebraists and geometers as an invariant to deal with improper intersections and varieties with non-isolated singularities [2].

In this introduction we will only discuss the case of associated graded rings, although in the rest of the paper we will treat associated graded modules.

Let I be an \mathfrak{m} -primary ideal. The Hilbert–Samuel function of I is the numerical function $H_I(n)$ that measures the growth of the length $\lambda(R/I^n)$ of the powers of I for all $n \geq 1$. For n sufficiently large, the Hilbert–Samuel function is a polynomial function in n of degree d , the dimension of R . This is the Hilbert–Samuel polynomial of I , whose coefficients $e_i(I)$, dubbed the Hilbert coefficients of I , are uniquely determined by I . It is well known that the normalized leading coefficient $e_0(I)$, the multiplicity of I , detects integral dependence of \mathfrak{m} -primary ideals. The integral closure of I , for instance, can be characterized as the largest ideal containing I with the same multiplicity e_0 , when the ring is equidimensional and universally catenary. Flennner and Manaresi were the first to use the j -multiplicity to generalize this fundamental theorem of Rees to arbitrary ideals [6].

In 1967 Abhyankar proved that the multiplicity $e(R) = e_0(\mathfrak{m})$ of a d -dimensional Cohen–Macaulay local ring is bounded below by $\mu(\mathfrak{m}) - d + 1$, where $\mu(\mathfrak{m})$ is the embedding dimension of R [1]. Rings for which $e_0 = \mu(\mathfrak{m}) - d + 1$ have since then been called rings of minimal multiplicity. In the case of minimal multiplicity, Sally had shown in [24] that the associated graded ring of \mathfrak{m} is always Cohen–Macaulay. Even if the multiplicity is almost minimal, the associated graded ring is Gorenstein provided the ambient ring is Gorenstein [25]. Unfortunately, for arbitrary Cohen–Macaulay local rings of almost minimal multiplicity the Cohen–Macaulayness of $\text{gr}_{\mathfrak{m}}(R)$ fails to hold, the exceptions being Cohen–Macaulay local rings of maximal type [26]. Based on this result, Sally conjectured that if the multiplicity of R is almost minimal then the depth of the associated graded ring is almost maximal, i.e., it is at least $d - 1$. This conjecture was proved later by Rossi and Valla [21], and independently by Wang [31]. In recent years there have been many generalizations of these results to \mathfrak{m} -primary ideals and modules with I -adic filtrations, where I is an ideal of definition, a condition that is required to define the Hilbert–Samuel multiplicity (see for example [22,10,4,5,20,17,23]).

In this paper, we investigate the depth of the associated graded ring of an arbitrary ideal using the j -multiplicity introduced by Achilles and Manaresi [2] and further studied in [8,6,7,16]. Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d > 0$ and I an R -ideal. One can assign generalized Hilbert coefficients $j_i(F)$ to every ideal filtration $F = \{F_j\}_{j \geq 0}$ whose Rees algebra is finite over $R[t] = \bigoplus_{j \geq 0} I^j t^j$ in the following way: let $A := \bigoplus_{j \geq 0} F_j/F_{j+1}$ be the associated graded ring of F , and denote by $\Gamma_{\mathfrak{m}}(A) = H_{\mathfrak{m}}^0(A)$ the ideal of A of elements supported on \mathfrak{m} . Since $\Gamma_{\mathfrak{m}}(A)$ is annihilated by a large power of \mathfrak{m} , it is a finite graded module over $\text{gr}_I(R) \otimes R/\mathfrak{m}^t$ for some t , hence its Hilbert polynomial is well defined:

$$P(j) = \sum_{i=0}^{d-1} (-1)^i j_i(F) \binom{j+d-i-1}{d-i-1}.$$

We call $P(j)$ the *generalized Hilbert polynomial* and $j_i(F)$, $0 \leq i \leq d-1$, the generalized Hilbert coefficients of F . The generalized Hilbert coefficients of the ideal filtration $\{I^j\}_{j \geq 0}$ will simply be denoted by $j_i(I)$. Notice that $j_0(I)$ coincides with the j -multiplicity defined by Achilles and Manaresi in [2]. Furthermore in the \mathfrak{m} -primary case $j_i(I) = e_i(I)$, so our definition recovers the standard one.

In Section 2, we prove a lower bound for the j -multiplicity of any ideal I . The definition of ideals with minimal j -multiplicity is thus immediate. In Section 3, under certain residual conditions, we prove that for any ideal with minimal j -multiplicity, the associated graded ring is Cohen–Macaulay. Furthermore if the ambient ring is Gorenstein, then the associated graded ring is Gorenstein as well,

which generalizes completely Sally's results. Finally, in Section 4, we deal with ideals having almost minimal j -multiplicity. We prove that, under the same residual conditions, the associated graded ring is almost Cohen–Macaulay. The technical novelty is a powerful combination of the methods used in the \mathfrak{m} -primary case with tools proper to residual intersection theory. This result can be viewed as a positive answer to Sally's conjecture for non \mathfrak{m} -primary ideals.

2. Minimal j -multiplicity

In this section we first prove a lower bound for the j -multiplicity; this bound leads to a notion of minimal j -multiplicity.

We start by fixing the notation that will be used throughout the paper. We first recall the definition of j -multiplicity according to [2] and [16].

Let (R, \mathfrak{m}) be a Noetherian local ring, I an arbitrary R -ideal, and M a finite R -module of dimension d . The I -adic filtration of M is a collection of submodules $\{I^j M\}_{j \geq 0}$. Let $G := \text{gr}_I(R) = \bigoplus_{j=0}^{\infty} I^j/I^{j+1}$ be the associated graded ring of I and $T := \text{gr}_I(M) = \bigoplus_{j=0}^{\infty} I^j M/I^{j+1} M$ the associated graded module of the filtration $\{I^j M\}_{j \geq 0}$. Notice that T is a finite graded module over the graded ring G . In general the homogeneous components of T may not have finite length, thus we consider the T -submodule of elements supported on \mathfrak{m} , $W := \Gamma_{\mathfrak{m}}(T) = 0 :_T \mathfrak{m}^{\infty} = \bigoplus_{j=0}^{\infty} \Gamma_{\mathfrak{m}}(I^j M/I^{j+1} M)$. Since W is annihilated by a large power of \mathfrak{m} , it is a finite graded module over $\text{gr}_I(R) \otimes R/\mathfrak{m}^t$ for some t , hence its Hilbert polynomial $P(j)$ is well defined. Notice that $\dim_G W \leq \dim_G T = d$, thus $P(j)$ has degree at most $d - 1$. The j -multiplicity of I on M is the normalized leading coefficient of $P(j)$ in degree $d - 1$, i.e.,

$$j(I, M) = (d - 1)! \lim_{t \rightarrow \infty} \frac{\lambda(\Gamma_{\mathfrak{m}}(I^t M/I^{t+1} M))}{t^{d-1}}.$$

Recall that the Krull dimension of the special fiber module $T/\mathfrak{m}T$ is called the *analytic spread* of I on M and is denoted by $\ell(I, M)$. In general, $\dim_G W \leq \ell(I, M) \leq d$ and equalities hold if and only if $\ell(I, M) = d$. Therefore $j(I, M) \neq 0$ if and only if $\ell(I, M) = d$ [16, 2.1].

If M/IM has finite length, the ideal I is said to be an *ideal of definition* on M . In this case each homogeneous component of T has finite length, thus $W = T$ and the j -multiplicity coincides with the usual multiplicity.

An element $x \in I$ is said to be a *superficial element* for I on M if there exists a non-negative integer c such that for every $j \geq c$,

$$(I^{j+1}M :_M x) \cap I^c M = I^j M.$$

A sequence of elements x_1, \dots, x_s in I is a *superficial sequence* for I on M if x_i is superficial for I on $M/(x_1, \dots, x_{i-1})M$ for $i = 1, \dots, s$. This notion, originally introduced by Zariski and Samuel, plays a significant role in the study of Hilbert functions because it allows to reduce the problems to lower dimensional ones. Notice that if M has positive depth then every superficial element is regular on M . If I is not nilpotent on M , i.e., $I^j M \neq 0$ for $j \geq 0$, then a superficial element has always order one, i.e., $x \in I/I^2$. Thus, in this case, the definition of superficial elements coincides with the definition of homogeneous filter-regular elements used in the study of j -multiplicity (see [2, 32, 16] for instance). More precisely, an element x is superficial for I on M if and only if x^* , the image of x in I/I^2 , is *filter-regular* for G_+ on T .

For an ideal $J \subseteq I$, one says that J is a *reduction* of I on M if $J I^t M = I^{t+1} M$ for some non-negative integer t . A *minimal reduction* is a reduction which is minimal with respect to inclusion. Minimal reductions always exist and, if R has infinite residue field, the minimal number of generators of any minimal reduction J of I on M equals the analytic spread $\ell(I, M)$. Furthermore, a minimal generating set of J can be chosen to be a superficial sequence for I on M [27, 3.1]. The least integer t with $J I^t M = I^{t+1} M$ is called the *reduction number* of I on M with respect to J and denoted by

$r_J(I, M)$. One then defines the *reduction number* $r(I, M)$ of I on M to be the least $r_J(I, M)$, where J varies over all minimal reductions of I on M .

Let $I = (a_1, \dots, a_n)$ and write $x_i = \sum_{j=1}^n \lambda_{ij} a_j$ for $1 \leq i \leq s$ and $(\lambda_{ij}) \in R^{sn}$. The elements x_1, \dots, x_s form a *sequence of general elements* in I (equivalently x_1, \dots, x_s are *general* in I) if there exists a dense open subset U of k^{sn} such that the image $(\bar{\lambda}_{ij}) \in U$. In this case we call the (λ_{ij}) *general elements* in R^{sn} . When $s = 1$ we say that $x = x_1$ is *general* in I . Observe that the notion of general elements of I is more restrictive than the one of *sequentially general* elements. The latter means that for every i with $1 \leq i \leq s$ and every fixed x_1, \dots, x_{i-1} the element x_i is general in I .

The notion of general elements is a fundamental tool for our study. They always exist if the ambient ring has infinite residue field. Furthermore they form a superficial sequence for I on M [32, 2.5]; they generate a minimal reduction whose reduction number $r_J(I, M)$ coincides with the reduction number $r(I, M)$ of I on M if $s = \ell(I, M)$ (see [28, 2.2] and [13, 8.6.6]); and they form a *super-reduction* in the sense of [2] whenever $s = \ell(I, M) = d = \dim_R M$ (see [32, 2.5]). One can compute the j -multiplicity using general elements and obtain a lower bound from it as the next proposition shows.

Proposition 2.1. *Let (R, \mathfrak{m}) be a Noetherian local ring with infinite residue field. Let M be a finite R -module and I an R -ideal with analytic spread $\ell(I, M) = d = \dim_R M$. Then for general elements x_1, \dots, x_{d-1} in I , we have*

(a) *the j -multiplicity of I on M is*

$$j(I, M) = e(I, \bar{M}) = \lambda(\bar{M}/x_d \bar{M}),$$

where $\bar{M} := M/((x_1, \dots, x_{d-1})M :_M I^\infty)$ and $x_d \in I$ is a parameter on \bar{M} .

(b) $j(I, M) \geq \lambda(I\bar{M}/I^2\bar{M})$.

Proof. By [32, 2.5], there exist general elements x_1, \dots, x_{d-1} in I which form part of a super-reduction in the sense of [2]. Now proceed as in the proof of [2, 3.8] or [16, 3.6] to obtain the desired formula of part (a). For part (b), as \bar{M} is a one-dimensional Cohen–Macaulay module and I is an ideal of definition on \bar{M} , we have $j(I, M) = e(I, \bar{M}) = \lambda(I\bar{M}/I^2\bar{M}) + \lambda(I^2\bar{M}/x_d I\bar{M}) \geq \lambda(I\bar{M}/I^2\bar{M})$, where the second equality follows from [23, Corollary 2.1]. \square

Once we show that $\lambda(I\bar{M}/I^2\bar{M})$ and $\lambda(I^2\bar{M}/x_d I\bar{M})$ do not depend on the choice of the general elements x_1, \dots, x_d in I , we will obtain the desired lower bound for the j -multiplicity.

Lemma 2.2. *Let (R, \mathfrak{m}) be a Noetherian local ring with infinite residue field k . Let $R' = R[z_1, \dots, z_t]$ be a polynomial ring over R . Let $M' \supseteq N'$ be two finite R' -modules with $\lambda_{R'}(M'_{\mathfrak{m}_{R'}}/N'_{\mathfrak{m}_{R'}}) = s$. If $(\bar{A}) = (\lambda_1, \dots, \lambda_t)$ is a vector in R^t , write (\bar{A}) for its image in k^t and $\pi(\cdot)$ for the evaluation map sending z_i to λ_i . Then there exists a dense open subset U of k^t such that $\lambda_R(\pi(M')/\pi(N')) \leq s$ whenever $(\bar{A}) \in U$.*

Proof. If $s = \infty$ then we are done. So we may assume $s < \infty$ and let $M' = M'_0 \supset M'_1 \supset \dots \supset M'_s = N'$ be a filtration such that $(M'_{l-1}/M'_l)_{\mathfrak{m}_{R'}} \cong k(z_1, \dots, z_t)$ for $1 \leq l \leq s$. We use induction on s to prove the lemma. When $s = 0$, i.e., $\lambda_{R'}(M'_{\mathfrak{m}_{R'}}/N'_{\mathfrak{m}_{R'}}) = 0$, there exists a polynomial $f \in R' \setminus \mathfrak{m}_{R'}$ such that $fM' \subseteq N'$. Let \bar{f} be the image of f in $k[z_1, \dots, z_t]$ and notice that $\bar{f} \neq 0$. Thus $U = D(\bar{f})$ is a dense open subset of k^t . If $(\bar{A}) \in U$ then $f(\bar{A})$ is a unit in R . Thus $f(\bar{A})\pi(M') = \pi(fM') \subseteq \pi(N')$ implies $\pi(M') \subseteq \pi(N')$. Now assume the lemma holds for $s - 1$, i.e., there exists a dense open subset $U_1 \subseteq k^t$ such that $\lambda_R(\pi(M')/\pi(M'_{s-1})) \leq s - 1$ whenever $(\bar{A}) \in U_1$. Since $(M'_{s-1}/M'_s)_{\mathfrak{m}_{R'}} \cong k(z_1, \dots, z_t)$, there exists $b' \in M'_{s-1}$ and a polynomial $f \in R' \setminus \mathfrak{m}_{R'}$ so that $fM'_{s-1} \subseteq R'b' + M'_s$ and $fmb' \in M'_s$. Also notice that the image \bar{f} of f in $k[z_1, \dots, z_t]$ is not zero and $U_2 = D(\bar{f})$ is a dense open subset of k^t . Let $U = U_1 \cap U_2$. Whenever $(\bar{A}) \in U$, $f(\bar{A})$ is a unit in R . Thus $\pi(M'_{s-1}) \subseteq R\pi(b') + \pi(M'_s)$ and $\mathfrak{m}\pi(b') \in \pi(M'_s)$. Therefore $\lambda_R(\pi(M'_{s-1})/\pi(M'_s)) \leq 1$ and thus we obtain $\lambda_R(\pi(M')/\pi(N')) = \lambda_R(\pi(M')/\pi(M'_{s-1})) + \lambda_R(\pi(M'_{s-1})/\pi(M'_s)) \leq s$ for every $(\bar{A}) \in U$. \square

Lemma 2.3. Let (R, \mathfrak{m}) be a Noetherian local ring with infinite residue field. Let M be a finite R -module and I an R -ideal with analytic spread $\ell(I, M) = d = \dim_R M$. For x_1, \dots, x_d general elements in I , write $\overline{M} = M/((x_1, \dots, x_{d-1})M :_M I^\infty)$, then the lengths $\lambda(I\overline{M}/I^2\overline{M})$ and $\lambda(I^2\overline{M}/x_d I\overline{M})$ are independent of x_1, \dots, x_d .

Proof. Let a_1, \dots, a_n be a set of generators of I and $Z = (z_{ij})$ be $d \times n$ variables. Write $R' = R[Z]$, $x'_i = \sum_{j=1}^n z_{ij}a_j$ for $1 \leq i \leq d$, and $M' = M \otimes_R R'$. Let $\overline{M'} = M'/((x'_1, \dots, x'_{d-1})M' :_{M'} IR'^\infty)$, by Proposition 2.1(a) and the proof of Proposition 2.1(b) (see also [2, 3.8], [16, 3.6] and [23, Corollary 2.1]) we have

$$j(I, M) = j(IR'_{\mathfrak{m}R'}, M'_{\mathfrak{m}R'}) = \lambda(I\overline{M'}_{\mathfrak{m}R'}/I^2\overline{M'}_{\mathfrak{m}R'}) + \lambda(I^2\overline{M'}_{\mathfrak{m}R'}/x'_d I\overline{M'}_{\mathfrak{m}R'}).$$

For general elements $(\Lambda) = (\lambda_{ij}) \in R'^{dn}$, write $\pi(\cdot)$ for the evaluation map sending z_{ij} to λ_{ij} . Observe $\pi(IM') = IM$, $\pi(I^2M') = I^2M$, $\pi(x'_d IM') = x_d IM$, and clearly

$$\pi((x'_1, \dots, x'_{d-1})M' :_{M'} IR'^\infty) \subseteq (x_1, \dots, x_{d-1})M :_M I^\infty.$$

Putting this together with Lemma 2.2, we obtain

$$\begin{aligned} \lambda(I\overline{M'}_{\mathfrak{m}R'}/I^2\overline{M'}_{\mathfrak{m}R'}) &= \lambda(IM'_{\mathfrak{m}R'}/[(x'_1, \dots, x'_{d-1})M'_{\mathfrak{m}R'} :_{IM'_{\mathfrak{m}R'}} (IR'_{\mathfrak{m}R'})^\infty + I^2M'_{\mathfrak{m}R'}]) \\ &\geq \lambda(IM/[\pi((x'_1, \dots, x'_{d-1})M' :_{IM'} IR'^\infty) + I^2M]) \\ &\geq \lambda(IM/[(x_1, \dots, x_{d-1})M :_{IM} I^\infty + I^2M]) = \lambda(I\overline{M}/I^2\overline{M}). \end{aligned}$$

In the same way we have

$$\begin{aligned} \lambda(I^2\overline{M'}_{\mathfrak{m}R'}/x'_d I\overline{M'}_{\mathfrak{m}R'}) &= \lambda(I^2M'_{\mathfrak{m}R'}/[(x'_1, \dots, x'_{d-1})M'_{\mathfrak{m}R'} :_{I^2M'_{\mathfrak{m}R'}} (IR'_{\mathfrak{m}R'})^\infty + x'_d IM'_{\mathfrak{m}R'}]) \\ &\geq \lambda(I^2M/[(x_1, \dots, x_{d-1})M :_{I^2M} I^\infty + x_d IM]) = \lambda(I^2\overline{M}/x_d I\overline{M}). \end{aligned}$$

By Proposition 2.1(a) and the proof of Proposition 2.1(b), the j -multiplicity is given by the sum of $\lambda(I\overline{M}/I^2\overline{M})$ and $\lambda(I^2\overline{M}/x_d I\overline{M})$, thus

$$\begin{aligned} j(I, M) &= \lambda(I\overline{M}/I^2\overline{M}) + \lambda(I^2\overline{M}/x_d I\overline{M}) \\ &\leq \lambda(I\overline{M'}_{\mathfrak{m}R'}/I^2\overline{M'}_{\mathfrak{m}R'}) + \lambda(I^2\overline{M'}_{\mathfrak{m}R'}/x'_d I\overline{M'}_{\mathfrak{m}R'}) \\ &= j(I, M). \end{aligned}$$

In turn this forces the equalities

$$\begin{aligned} \lambda(I\overline{M}/I^2\overline{M}) &= \lambda(I\overline{M'}_{\mathfrak{m}R'}/I^2\overline{M'}_{\mathfrak{m}R'}), \\ \lambda(I^2\overline{M}/x_d I\overline{M}) &= \lambda(I^2\overline{M'}_{\mathfrak{m}R'}/x'_d I\overline{M'}_{\mathfrak{m}R'}), \end{aligned}$$

and therefore shows the independence of these lengths from the general elements x_1, \dots, x_d . \square

Because of Proposition 2.1 and Lemma 2.3, we can now give the definition of *minimal j -multiplicity* of I on M which is the analogue of minimal multiplicity [23].

Definition 2.4. Let M be a finite module of dimension d over a Noetherian local ring R and I an R -ideal with analytic spread $\ell(I, M) = d$. We say that I has *minimal j -multiplicity* on M if $j(I, M) = \lambda(I\overline{M}/I^2\overline{M})$, where $\overline{M} = M/((x_1, \dots, x_{d-1})M :_M I^\infty)$ and x_1, \dots, x_{d-1} are general in I .

Notice that I has minimal j -multiplicity on M if and only if $\lambda(I^2\bar{M}/x_d I\bar{M})$ is zero, or equivalently, if and only if x_d generates a reduction of I on \bar{M} with reduction number one. The next lemma shows that for an ideal the assumption of having minimal j -multiplicity on M is quite strict. Indeed, if I has minimal j -multiplicity on M then the Hilbert function of I on \bar{M} is rigid, i.e., the value of the multiplicity determines the Hilbert function. We remark that results of this kind are really surprising since the multiplicity is just one of the Hilbert coefficients and, in turn, the Hilbert coefficients give only partial information on the Hilbert polynomial which gives only asymptotic information on the Hilbert function.

Corollary 2.5. *Let R , I , M and \bar{M} be as in Definition 2.4, then the j -multiplicity of I on M and on $I^t M$ coincides for every $t \geq 0$, i.e., $j(I, M) = j(I, I^t M)$ for every $t \geq 0$. Furthermore, if I has minimal j -multiplicity on M then $j(I, M) = \lambda(I^t \bar{M}/I^{t+1} \bar{M})$ for every $t \geq 1$.*

Proof. Observe that $\ell(I, M) = \dim T/\mathfrak{m}T = \dim T_{\geq t}/\mathfrak{m}T_{\geq t} = \ell(I, I^t M)$. For every $t \geq 0$,

$$\begin{aligned} j(I, I^t M) &= (d-1)! \lim_{s \rightarrow \infty} \frac{\lambda(\Gamma_{\mathfrak{m}}(I^{s+t} M/I^{s+t+1} M))}{s^{d-1}} \\ &= (d-1)! \lim_{s \rightarrow \infty} \frac{j(I, M)(s+t)^{d-1}}{(d-1)!s^{d-1}} = j(I, M). \end{aligned}$$

For the second assertion assume that I has minimal j -multiplicity on M , or equivalently, $I^2 \bar{M} = x_d I \bar{M}$. This gives $I^{t+2} \bar{M} = x_d I^{t+1} \bar{M}$ for every $t \geq 0$. In turn, this implies that x_d generates a reduction of I on $I^t \bar{M}$ with reduction number one, thus I has minimal j -multiplicity on $I^t M$ and therefore $j(I, I^t M) = \lambda(I I^t \bar{M}/I^2 I^t \bar{M}) = \lambda(I^{t+1} \bar{M}/I^{t+2} \bar{M})$. Hence $j(I, M) = j(I, I^t M) = \lambda(I^{t+1} \bar{M}/I^{t+2} \bar{M})$ for every $t \geq 0$. \square

3. Cohen–Macaulayness of the associated graded module

In this section we show that the associated graded module of any filtration with minimal j -multiplicity is Cohen–Macaulay, if the ideal has some residual properties. We start by describing the residual assumptions that are needed to prove the main theorem.

Let M be a finite faithful module of dimension d over a Noetherian local ring R . Let I be an R -ideal. The ideal I is said to satisfy the condition G_s on M if for every $\mathfrak{p} \in \text{Supp}_R(M/IM)$ with $\dim R_{\mathfrak{p}} = t < s$ the ideal I is generated by t elements on $M_{\mathfrak{p}}$, i.e., $IM_{\mathfrak{p}} = (x_1, \dots, x_t)M_{\mathfrak{p}}$ for some x_1, \dots, x_t in I . Write $H = (x_1, \dots, x_t)M :_M I$. If $IM_{\mathfrak{p}} = (x_1, \dots, x_t)M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq t-1$, then H is said to be a t -residual intersection of I on M . Now let H be a t -residual intersection of I on M . If in addition $IM_{\mathfrak{p}} = (x_1, \dots, x_t)M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Supp}_R(M/IM)$ with $\dim R_{\mathfrak{p}} \leq t$, then H is said to be a geometric t -residual intersection of I on M . If M is not faithful, then we say that I satisfies the condition G_s on M if \bar{R} satisfies the condition G_s on M , where $\bar{R} = R/\text{Ann } M$. We say H is a t -residual intersection or geometric t -residual intersection of I on M if H is a t -residual intersection or geometric t -residual intersection of $I\bar{R}$ on M respectively.

The next two lemmas contain basic facts about residual intersections. The ideas are already presented in [29, 1.6 and 1.7]. The first lemma says that the condition G_s gives rise to residual intersections.

Lemma 3.1. *Let R be a catenary and equidimensional Noetherian local ring with infinite residue field. Let M be a finite R -module and I an R -ideal satisfying condition G_s on M . For general elements x_1, \dots, x_s of I on M , write $H_i = (x_1, \dots, x_i)M :_M I$ for $0 \leq i \leq s$, then:*

- (a) H_i is an i -residual intersection of I on M for $0 \leq i \leq s$.
- (b) H_i is a geometric i -residual intersection of I on M for $0 \leq i \leq s-1$.

Proof. Let a_1, \dots, a_n be a set of generators of I on M , i.e., $IM = (a_1, \dots, a_n)M$, and $Z = (z_{ij})$ be $s \times n$ variables. Write $R' = R[Z]_{\mathfrak{m}R[Z]}$ and $M' = M \otimes_R R'$, where \mathfrak{m} is the maximal ideal of R . For $1 \leq i \leq s$, let $x'_i = \sum_{j=1}^n z_{ij}a_j$. We first claim that

(*) if $\mathfrak{p} \in \text{Supp}(M/IM)$ such that $IM_{\mathfrak{p}} = (y_1, \dots, y_l)M_{\mathfrak{p}}$ for some $y_1, \dots, y_l \in I$ where $1 \leq l \leq s$, then $IM'_{\mathfrak{p}} = (x'_1, \dots, x'_l)M'_{\mathfrak{p}}$.

To prove it, let X_1, \dots, X_n be variables over $R'_{\mathfrak{p}}/\mathfrak{p}R'_{\mathfrak{p}}$. Set $A' = M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}[X_1, \dots, X_n]$, $T'_{\mathfrak{p}} = \text{gr}_{IR'_{\mathfrak{p}}}(M'_{\mathfrak{p}})$, and $F' = T'_{\mathfrak{p}}/\mathfrak{p}T'_{\mathfrak{p}}$. Define the map $\varphi: A' \rightarrow F'$ by sending X_i to $a_i + \mathfrak{p}IR'_{\mathfrak{p}}$. For $1 \leq i \leq l$, write $y_i = \sum_{j=1}^n \lambda_{ij}a_j$, where $(\lambda_{ij}) \in R^{ln}$. Denote with $\bar{}$ the images of elements of R in $R'_{\mathfrak{p}}/\mathfrak{p}R'_{\mathfrak{p}}$. Observe that the preimages (under φ) of the y_i 's generate a vector space (over $M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}$) in $[A']_1$ of dimension n , i.e., $\dim_{M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}} \text{Span}\{b_i = \sum_{j=1}^n \bar{\lambda}_{ij}X_j\}_{1 \leq i \leq l}, [\ker(\varphi)]_1\} = n$. To show that $IM'_{\mathfrak{p}} = (x'_1, \dots, x'_l)M'_{\mathfrak{p}}$, it will be enough to show that the set $\{b'_i = \sum_{j=1}^n z_{ij}X_j\}_{1 \leq i \leq l}, [\ker(\varphi)]_1\}$ spans also an n -dimensional vector space (over $M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}$) in $[A']_1$, i.e., $\dim_{M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}} \text{Span}\{b'_i = \sum_{j=1}^n z_{ij}X_j\}_{1 \leq i \leq l}, [\ker(\varphi)]_1\} = n$. Indeed, define the map

$$\phi: A = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}[X_1, \dots, X_n] \rightarrow F = \text{gr}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})/(\mathfrak{p} \text{gr}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}))$$

by sending X_i to $a_i + \mathfrak{p}IR_{\mathfrak{p}}$. Since the extension from R to R' is flat, $\ker(\varphi) = \ker(\phi) \otimes_R R'$. Thus we can choose basis $b_i = \sum_{j=1}^n \bar{\lambda}_{ij}X_j$ in $[\ker(\phi)]_1$ where $\lambda_{ij} \in R$ and $l+1 \leq i \leq t$ such that

$$\dim_{M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}} \text{Span}(b_1, \dots, b_t) = n.$$

This forces $\dim_{M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}} \text{Span}(b'_1, \dots, b'_l, b_{l+1}, \dots, b_t) = n$ as well. If not, set

$$\alpha = \dim_{M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}} \text{Span}(b'_1, \dots, b'_l, b_{l+1}, \dots, b_t) < n.$$

Observe that each b'_i is a linear combination of X_j with coefficients polynomials in the variable z_{ij} of degree at most one, and let X be the $t \times n$ matrix obtained from these coefficients. Because $\alpha < n$ then all the $n \times n$ minors of X vanish. When we specialize z_{ij} to λ_{ij} for $1 \leq i \leq l$, then all the $n \times n$ minors of the $t \times n$ matrix obtained from the coefficients of $b_1, \dots, b_l, b_{l+1}, \dots, b_t$ as linear combinations of X_j vanish as well, which contradicts the fact that $\dim_{M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}} \text{Span}(b_1, \dots, b_t) = n$.

Write $H'_i = (x'_1, \dots, x'_i)M' :_{M'} IR'$ for $0 \leq i \leq s$. We claim that:

- (a') H'_i is an i -residual intersection of IR' on M' for $0 \leq i \leq s$.
- (b') H'_i is a geometric i -residual intersection of IR' on M' for $0 \leq i \leq s-1$.

Now by factoring out $\text{Ann } M$, we may assume M is faithful. For part (a'), let $1 \leq i \leq s$ and $\mathfrak{p}' \in \text{Spec}(R')$ with $\text{ht } \mathfrak{p}' \leq i-1$. Then $\mathfrak{p}' = \mathfrak{p}R'$ for some $\mathfrak{p} \in \text{Spec}(R)$ with $\text{ht } \mathfrak{p} \leq i-1$. If $\mathfrak{p} \notin \text{Supp}(M/IM)$, then $IM'_{\mathfrak{p}'} = M'_{\mathfrak{p}'} = (x'_1, \dots, x'_i)M'_{\mathfrak{p}'}$, where the last equality holds because $x'_iR'_{\mathfrak{p}'} = R'_{\mathfrak{p}'}$. Therefore we may assume $\mathfrak{p} \in \text{Supp}(M/IM)$. Since I satisfies condition G_s on M and $\mathfrak{p} \in \text{Spec}(R)$ with $\text{ht } \mathfrak{p} \leq i-1 < s$, we have $IM_{\mathfrak{p}} = (y_1, \dots, y_{i-1})M_{\mathfrak{p}}$ for some $y_1, \dots, y_{i-1} \in I$. By (*), this implies $IM'_{\mathfrak{p}} = (x'_1, \dots, x'_{i-1})M'_{\mathfrak{p}}$. Part (b') follows by employing the same argument.

Finally since for $0 \leq i \leq s-1$, H'_i is a geometric i -residual intersection of IR' on M' and H'_s is an s -residual intersection of IR' on M' , we have $\text{ht}((x'_1, \dots, x'_i)M' :_{R'} IM') \geq i$ for $0 \leq i \leq s$ and $\text{ht}((x'_1, \dots, x'_i)M' :_{R'} IM' + IM' :_{R'} M') \geq i+1$ for $0 \leq i \leq s-1$. Let $k = R/\mathfrak{m}$, $(\Lambda) = (\lambda_{ij}) \in R^{sn}$ and $(\bar{\Lambda})$ be the image of (Λ) in k^{sn} . Write $\pi(\cdot)$ for the evaluation map sending z_{ij} to λ_{ij} . By [9, 3.1] for all i , there exists a dense open subset U of k^{sn} such that $\text{ht}(\pi((x'_1, \dots, x'_i)M' :_{R'} IM')) \geq i$ and $\text{ht}(\pi((x'_1, \dots, x'_i)M' :_{R'} IM' + IM' :_{R'} M')) \geq i+1$ whenever $(\bar{\Lambda}) \in U$. Since $\pi((x'_1, \dots, x'_i)M' :_{R'} IM') \subseteq (x_1, \dots, x_i)M :_R IM$ and $\pi(IM' :_{R'} M') \subseteq IM :_R M$, then H_i is also a geometric i -residual intersection of I on M for $0 \leq i \leq s-1$ and H_s is an s -residual intersection of I on M . \square

Assume that M is Cohen–Macaulay. The ideal I is said to have the *Artin–Nagata property* AN_t^- on M if for every $0 \leq i \leq t$ and every geometric i -residual intersection H of I on M , the module M/H is Cohen–Macaulay. In the next lemma we exhibit some basic facts about Artin–Nagata properties that will be useful in the proof of Theorem 3.8.

Lemma 3.2. *Let M be a Cohen–Macaulay module of dimension d over a Noetherian local ring R with infinite residue field. Let I be an R -ideal with $\ell(I, M) = s$ satisfying G_s and AN_{s-1}^- on M . For general elements x_1, \dots, x_s of I on M , write $H_i = (x_1, \dots, x_i)M :_M I$ for $0 \leq i \leq s$, then:*

- (a) x_{i+1} is regular on M/H_i and $H_i = (x_1, \dots, x_i)M :_M x_{i+1}$ if $0 \leq i \leq s-1$.
- (b) M/H_i is unmixed of dimension $d-i$.
- (c) $\text{depth } M/(x_1, \dots, x_i)M = d-i$.
- (d) $\text{depth}(M/(x_1, \dots, x_i)IM) \geq \min\{d-i, \text{depth}(M/IM)\}$.
- (e) $(x_1, \dots, x_i)M = H_i \cap IM$ if $0 \leq i \leq s-1$.
- (f) If $\text{depth}(M/IM) \geq d-s+1$ then

$$(x_1, \dots, x_{s-1})M :_{I^2M} I^\infty = (x_1, \dots, x_{s-1})M :_{I^2M} I = (x_1, \dots, x_{s-1})M :_{I^2M} x_s = (x_1, \dots, x_{s-1})IM.$$

- (g) Let $\bar{M} = M/H_0$, then I satisfies G_s and AN_{s-1}^- on \bar{M} .

Proof. By Lemma 3.1, for general elements x_1, \dots, x_s of I on M , the module H_i is a geometric i -residual intersection of I on M for $0 \leq i \leq s-1$ and H_s is an s -residual intersection of I on M . Thus parts (a), (b), (c), (e) and (g) follow as in the proofs of [29, 1.7] and [14, 2.3, 2.4]. To prove (d), we use induction on i . First when $i=0$, clearly $\text{depth } M \geq \min\{d, \text{depth}(M/IM)\}$. Assume $\text{depth}(M/(x_1, \dots, x_i)IM) \geq \min\{d-i, \text{depth}(M/IM)\}$ for some $0 \leq i < s$. By (a) and (e), $(x_1, \dots, x_i)M :_M x_{i+1} \cap IM = (x_1, \dots, x_i)M$. For $i+1$, we have

$$\begin{aligned} (x_1, \dots, x_i)IM \cap x_{i+1}IM &= x_{i+1}[(x_1, \dots, x_i)IM :_M x_{i+1}] \cap IM \\ &\subseteq x_{i+1}[(x_1, \dots, x_i)M :_M x_{i+1}] \cap IM \\ &= x_{i+1}(x_1, \dots, x_i)M \subseteq (x_1, \dots, x_i)IM \cap x_{i+1}IM. \end{aligned}$$

Thus we obtain an exact sequence:

$$0 \rightarrow x_{i+1}(x_1, \dots, x_i)M \rightarrow (x_1, \dots, x_i)IM \oplus x_{i+1}IM \rightarrow (x_1, \dots, x_{i+1})IM \rightarrow 0.$$

The element x_{i+1} is regular on IM and therefore on $(x_1, \dots, x_i)M$ because $(0_M :_M x_{i+1}) \cap (x_1, \dots, x_i)M \subseteq (0_M :_M x_{i+1}) \cap IM = 0_M$, thus $x_{i+1}(x_1, \dots, x_i)M \cong (x_1, \dots, x_i)M$ and $x_{i+1}IM \cong IM$. Therefore $\text{depth}(x_{i+1}(x_1, \dots, x_i)M) \geq \min\{d, d-i+1\}$ and $\text{depth}(x_{i+1}IM) \geq \min\{d, \text{depth}(M/IM)+1\}$. By induction hypothesis $\text{depth}((x_1, \dots, x_i)IM) \geq \min\{d-i+1, \text{depth}(M/IM)+1\}$, thus the above exact sequence yields $\text{depth}((x_1, \dots, x_{i+1})IM) \geq \min\{d-i, \text{depth}(M/IM)+1\}$. We finally conclude that $\text{depth}(M/(x_1, \dots, x_{i+1})IM) \geq \min\{d-i-1, \text{depth}(M/IM)\}$.

To show assertion (f), since

$$(x_1, \dots, x_{s-1})IM \subseteq (x_1, \dots, x_{s-1})M :_{I^2M} I \subseteq (x_1, \dots, x_{s-1})M :_{I^2M} I^\infty,$$

it is enough to check the equality locally at every prime ideal $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{s-1})IM)$. By (d), $\text{depth}(M/(x_1, \dots, x_{s-1})IM) \geq d-s+1$. Thus for every $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{s-1})IM)$, $\text{ht } \mathfrak{p} \leq s-1$ and hence either $IM_{\mathfrak{p}} = M_{\mathfrak{p}}$ or $IM_{\mathfrak{p}} = (x_1, \dots, x_{s-1})M_{\mathfrak{p}}$ since H_{s-1} is a geometric $s-1$ -residual intersection of I on M . Therefore, if $IM_{\mathfrak{p}} = (x_1, \dots, x_{s-1})M_{\mathfrak{p}}$ then

$$[(x_1, \dots, x_{s-1})M :_{I^2M} I^\infty]_{\mathfrak{p}} = [(x_1, \dots, x_{s-1})M :_{I^2M} I]_{\mathfrak{p}} = (x_1, \dots, x_{s-1})IM_{\mathfrak{p}} = I^2M_{\mathfrak{p}}.$$

Otherwise by (e), we immediately obtain $(x_1, \dots, x_{s-1})M_{\mathfrak{p}} = (H_{s-1})_{\mathfrak{p}}$, and the latter coincides with the module $[(x_1, \dots, x_{s-1})M :_{I^2M} I]_{\mathfrak{p}} = [(x_1, \dots, x_{s-1})M :_{I^2M} I^{\infty}]_{\mathfrak{p}}$.

The full assertion now follows from part (a). \square

The following lemma shows that the presence of G_d along with the Artin–Nagata condition AN_{d-2}^- is actually sufficient to obtain AN_d^- .

Lemma 3.3. (See [29, 1.9].) *Let M be a Cohen–Macaulay module of dimension d over a Noetherian local ring R and I an R -ideal satisfying G_d and AN_{d-2}^- on M . Then I satisfies AN_d^- on M .*

Proof. The claim follows from Lemma 3.2(b). \square

Now we show that, as in the \mathfrak{m} -primary case, minimal multiplicity yields reduction number at most one.

Theorem 3.4. *Let M be a Cohen–Macaulay module of dimension d over a Noetherian local ring R and let I be an R -ideal with $\ell(I, M) = d$. Assume $\text{depth}(M/IM) \geq \min\{\dim(M/IM), 1\}$ and I satisfies G_d and AN_{d-2}^- on M . If I has minimal j -multiplicity on M then $r(I, M) \leq 1$.*

Proof. By adjoining variables to R and localizing, we may assume that the residue field is infinite. If $\dim M/IM = 0$ then the assertion follows from [23, Theorem 2.9]. Now assume $\dim M/IM > 0$. For general elements x_1, \dots, x_d in I , let $\bar{M} = M/((x_1, \dots, x_{d-1})M :_M I^{\infty})$. By Proposition 2.1 and Definition 2.4 the j -multiplicity can be computed using x_1, \dots, x_d thus $j(I, M) = \lambda(I\bar{M}/I^2\bar{M})$. From Definition 2.4 (see also the proof of Corollary 2.5), we obtain $I^2M = x_dIM + (x_1, \dots, x_{d-1})M :_{I^2M} I^{\infty}$. By Lemma 3.2(f)

$$(x_1, \dots, x_{d-1})M :_{I^2M} I^{\infty} = (x_1, \dots, x_{d-1})IM$$

thus we conclude at once that the reduction number of I on M with respect to (x_1, \dots, x_d) is at most one. Now [28, 2.2] or [13, 8.6.6] imply that $r(I, M) \leq 1$. \square

Our main application is the case when $M = R$. We obtain that ideals with residual intersection properties and minimal j -multiplicity have Cohen–Macaulay associated graded rings.

Corollary 3.5. *Let R be a Cohen–Macaulay local ring of dimension d . Let I be an R -ideal with $\ell(I) = d$. Assume $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$ and I satisfies G_d and AN_{d-2}^- . If I has minimal j -multiplicity then the associated graded ring $\text{gr}_I(R)$ is Cohen–Macaulay.*

Proof. The assertion follows from Theorem 3.4 and [14, 3.1]. \square

Let R be a Cohen–Macaulay local ring. Notice that the ‘residual intersection assumptions’: $\ell(I) = d$, $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$, I satisfies G_d and AN_{d-2}^- are all vacuous in the 0-dimensional case and the condition on the minimal j -multiplicity becomes the usual assumption of minimal multiplicity found in the original work of Sally. If the ambient ring is Gorenstein she was able to prove that the associated graded ring is Gorenstein as well. We recover this result in Corollary 3.6.

We remark that the Artin–Nagata property AN_{d-2}^- holds if $d \leq \text{ht } I + 1$, or if I satisfies G_d and the sliding depth conditions $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq d - \text{ht } I + 1$, a linear weakening of the Cohen–Macaulay property for consecutive powers of I (see [14, 2.1]). The depth inequalities are satisfied by strongly Cohen–Macaulay ideals with G_d , i.e., ideals whose Koszul homology modules are Cohen–Macaulay (see [29, 2.10]). Examples of ideals satisfying the latter condition are quite common – Cohen–Macaulay almost complete intersections, Cohen–Macaulay ideals generated by $2 + \text{ht } I$ elements, perfect ideals of codimension two, perfect Gorenstein ideals of codimension three, and, more generally, any ideal in the linkage class of a complete intersection, namely, licci ideals (see [12, 1.11]).

Corollary 3.6. Let R be a Gorenstein local ring of dimension d . Let I be an R -ideal with $\ell(I) = d$. Assume $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq d - \text{ht } I + 1$ and I satisfies G_d . If I has minimal j -multiplicity then the associated graded ring $\text{gr}_I(R)$ is Gorenstein.

Proof. The assertion follows from Theorem 3.4 and [14, 2.1, 5.3]. \square

Corollary 3.7. Let R be a Cohen–Macaulay local ring of dimension d . Let I be a strongly Cohen–Macaulay R -ideal with $\ell(I) = d$. If I has G_d and minimal j -multiplicity then the associated graded ring $\text{gr}_I(R)$ is Cohen–Macaulay. In addition if R is Gorenstein then $\text{gr}_I(R)$ is Gorenstein.

Proof. The first assertion follows from Theorem 3.4 and [14, 3.2], and the second from [29, 2.10] and Corollary 3.6. \square

In the next theorem we demonstrate that $r(I, M) \leq 1$ implies the Cohen–Macaulayness of the associated graded module $\text{gr}_I(M)$.

Theorem 3.8. Let M be a Cohen–Macaulay module of dimension d over a Noetherian local ring R and let I be an R -ideal with $\ell(I, M) = d$. Assume that I satisfies G_d and AN_{d-2}^- on M . If $r(I, M) \leq 1$ then the associated graded module $\text{gr}_I(M)$ is Cohen–Macaulay.

Proof. By adjoining variables to R and localizing, we may assume that the residue field is infinite. Set $g = \text{grade}(I, M)$. Let x_1, \dots, x_d be general elements in I and x_1^*, \dots, x_d^* their initial forms in $\text{gr}_I(R)$. First we show that x_1^*, \dots, x_g^* form a $\text{gr}_I(M)$ -regular sequence. By [30, 2.6], we only need to show $(x_1, \dots, x_g)M \cap I^j M = (x_1, \dots, x_g)I^{j-1}M$ for every $j \geq 1$.

We use induction on j to prove $(x_1, \dots, x_i)M \cap I^j M = (x_1, \dots, x_i)I^{j-1}M$ for every $j \geq 1$ and $0 \leq i \leq d$. This is clear if $j = 1$. So we assume $j \geq 2$ and the equality holds for $j - 1$. Set $J = (x_1, \dots, x_d)$. Since the reduction number of I on M is at most one, thus by [28, 2.2] or [13, 8.6.6] we have $I^2 M = JIM$. Therefore $JM \cap I^j M = JI^{j-1}M$. Now we use descending induction on i and assume $(x_1, \dots, x_{i+1})M \cap I^j M = (x_1, \dots, x_{i+1})I^{j-1}M$. Then

$$\begin{aligned} (x_1, \dots, x_i)M \cap I^j M &= (x_1, \dots, x_i)M \cap (x_1, \dots, x_{i+1})I^{j-1}M \quad \text{by induction on } i \\ &= (x_1, \dots, x_i)M \cap ((x_1, \dots, x_i)I^{j-1}M + x_{i+1}I^{j-1}M) \\ &= (x_1, \dots, x_i)I^{j-1}M + (x_1, \dots, x_i)M \cap x_{i+1}I^{j-1}M \\ &= (x_1, \dots, x_i)I^{j-1}M + x_{i+1}[(x_1, \dots, x_i)M :_M x_{i+1}] \cap I^{j-1}M \\ &= (x_1, \dots, x_i)I^{j-1}M + x_{i+1}[(x_1, \dots, x_i)M \cap I^{j-1}M] \quad \text{by Lemma 3.2(a) and (e)} \\ &= (x_1, \dots, x_i)I^{j-1}M + x_{i+1}(x_1, \dots, x_i)I^{j-2}M \quad \text{by induction on } j \\ &\subseteq (x_1, \dots, x_i)I^{j-1}M. \end{aligned}$$

Set $\delta(I, M) = d - g$. Now we prove that the associated graded module $\text{gr}_I(M)$ is Cohen–Macaulay by induction on δ . If $\delta = 0$, the assertion follows because x_1^*, \dots, x_g^* form a $\text{gr}_I(M)$ -regular sequence. Thus we may assume $\delta(I, M) \geq 1$ and the theorem holds for smaller values of $\delta(I, M)$. In particular, $d \geq g + 1$. Again since x_1^*, \dots, x_g^* form a $\text{gr}_I(M)$ -regular sequence, we may factor out x_1, \dots, x_g to assume $g = 0$. Now $d = \delta(I, M) \geq 1$. Set $H_0 = 0_M :_M I$ and $\overline{M} = M/H_0$. Then \overline{M} is Cohen–Macaulay since I satisfies AN_d^- on M (see Lemma 3.3). By Lemma 3.2(e), (a) and (g) and Lemma 3.3, $IM \cap H_0 = 0$, $\text{grade}(I, \overline{M}) \geq 1$, I still satisfies G_d and AN_d^- on \overline{M} . Furthermore $\dim M = \dim \overline{M} = d$ and $IM \cap H_0 = 0$ imply $\ell(I, \overline{M}) = \ell(I, M) = d$. Again by $IM \cap H_0 = 0$, there is a graded exact sequence

$$0 \rightarrow H_0 \rightarrow \text{gr}_I(M) \rightarrow \text{gr}_I(\overline{M}) \rightarrow 0. \quad (1)$$

Notice that $r(I, \overline{M}) \leq 1$. Since $\delta(I, \overline{M}) = d - \text{grade}(I, \overline{M}) < d = \delta(I, M)$, by induction hypothesis $\text{depth}(\text{gr}_I(\overline{M})) \geq d$. Observe $\text{depth}(H_0) \geq d$ since \overline{M} is Cohen–Macaulay. The Cohen–Macaulayness of $\text{gr}_I(M)$ follows at once by the short exact sequence (1). \square

Now we are ready to prove the main theorem.

Theorem 3.9. *Let M be a Cohen–Macaulay module of dimension d over a Noetherian local ring R and let I be an R -ideal with $\ell(I, M) = d$. Assume $\text{depth}(M/IM) \geq \min\{\dim(M/IM), 1\}$ and I satisfies G_d and AN_{d-2}^- on M . If I has minimal j -multiplicity on M then the associated graded module $\text{gr}_I(M)$ is Cohen–Macaulay.*

Proof. By Theorem 3.4, the reduction number of I on M is at most one. Now the assertion follows from Theorem 3.8. \square

Next we give an example of an ideal with minimal j -multiplicity that does not satisfy G_d and whose associated graded ring is not Cohen–Macaulay (see [3]). This demonstrates that the assumptions of Theorem 3.9 are necessary.

Example 3.10. Let $R = k[[x, y, z]]/(x^3 - x^2y)$ and $J = (xy^t, z)$ for any $t \geq 0$. Notice that R is a two-dimensional local Cohen–Macaulay ring, that $\ell(J) = 2$, thus J has reduction number zero, in particular it has minimal j -multiplicity. However, J does not satisfy G_2 . Using Macaulay2 [15] one sees that $\text{gr}_J(R)$ is not Cohen–Macaulay.

4. Almost minimal j -multiplicity

We start by giving the definition of almost minimal j -multiplicity, which is the analogue of almost minimal multiplicity [23].

Definition 4.1. Let M be a finite module of dimension d over a Noetherian local ring R and I an R -ideal with analytic spread $\ell(I, M) = d$. We say that I has *almost minimal j -multiplicity* on M if $j(I, M) = \lambda(I\overline{M}/I^2\overline{M}) + 1$, where $\overline{M} = M/((x_1, \dots, x_{d-1})M :_M I^\infty)$ and x_1, \dots, x_{d-1} are general in I .

Notice that by Lemma 2.3 the definition of almost minimal j -multiplicity is independent on the general sequence chosen in I .

Remark 4.2. If I has almost minimal j -multiplicity on M then

$$\lambda(I^2M/[x_dIM + (x_1, \dots, x_{d-1})M :_{I^2M} I^\infty]) = 1$$

for any general sequence x_1, \dots, x_d in I .

Notation and discussion 4.3. Let M be a finite module over a Noetherian local ring R and I an R -ideal. For every $j \geq 1$, let $\widetilde{I^j}M := \bigcup_{t \geq 1} (I^{j+t}M :_M I^t)$ be the Ratliff–Rush filtration of I on M (see [19,18,23]). If $\text{depth}_I M > 0$, by [23, Lemma 3.1], there exists an integer n_0 such that $\widetilde{I^j}M = I^jM$ for every $j \geq n_0$. In particular, for every reduction \widetilde{J} of I on M and every $j \geq n_0$, we obtain $\widetilde{I^{j+1}}M = \widetilde{J\widetilde{I^j}M} + I^{j+1}M$. Thus the module $N := \bigoplus_{j \geq 0} (I^{j+1}M/J\widetilde{I^j}M + I^{j+1}M)$ has finitely many non-zero components. Since each component is a finite R -module, N itself is finite as an R -module; we denote with $q = \mu(N)$ its minimal number of generators.

The next result is the key step in relating the reduction number of an I -adic filtration to invariants of the Ratliff–Rush filtration of I on M . The idea originated in the work of Rossi and Valla (see [21,20,23]). For clarity of exposition we state here the version for modules and ideals that are not necessarily \mathfrak{m} -primary.

Theorem 4.4. Use the notation of 4.3 and assume $\text{depth}_I M > 0$. Let J be an ideal generated by d general elements in I . Then

$$I^q \subseteq JI^{q-1} + (I^{q+j}M :_R \tilde{J}^j M)$$

for every positive integer j .

Proof. The proof is the same as the proof of [23, Theorem 4.1]. \square

Corollary 4.5. Use the notation of 4.3 and assume $\text{depth}_I M > 0$. Let J be an ideal generated by d general elements in I . Then $r(I, M) \leq t + q$, where $t = \min\{j \mid I^{j+1}M \subseteq \tilde{J}^j M\}$.

Proof. The proof is the same as the proof of [23, Corollaries 4.1 and 4.2]. \square

Lemma 4.6. Let M be a Cohen–Macaulay module of dimension d over a Noetherian local ring R . Let I be an R -ideal with $\ell(I, M) = d$. Assume that I satisfies G_d and AN_{d-2}^- on M and let J be an ideal generated by d general elements in I . Then the lengths $\lambda(I^j IM / JI^{j-1} IM)$, $\lambda(I^j IM / JI^{j-1} IM)$ and $\lambda(\tilde{J}^j IM / I^j IM)$ are finite for all $j \geq 1$.

Proof. Clearly $\lambda(IM/JM) < \infty$ and $\lambda(I^j IM / JI^{j-1} IM) < \infty$ for every $j \geq 1$, since IM and JM are the same on the punctured spectrum. To show the remaining assertions, observe that J has analytic spread d and satisfies G_d and AN_{d-2}^- on M as well, for instance by Lemma 3.1 and [29, 3.1]. Since $r(J, M) = 0$, by Theorem 3.8 the associated graded module $\text{gr}_J(M)$ is Cohen–Macaulay. In particular, J is Ratliff–Rush closed on M , i.e., $\tilde{J}^j M = J^j M$ for all $j \geq 1$. Thus on the punctured spectrum I is Ratliff–Rush closed on M as well, in particular $\lambda(I^j M / J^j M) < \infty$ for every $j \geq 1$. \square

Theorem 4.7. Let M be a Cohen–Macaulay module of dimension 2 over a Noetherian local ring R with infinite residue field. Let I be an R -ideal with $\ell(I, M) = 2$. Assume $\text{depth}(M/IM) \geq \min\{\dim(M/IM), 1\}$ and I satisfies G_2 and AN_0^- on M . Let x_1 be a general element in I . If I has almost minimal j -multiplicity on M then

- (a) x_1^* is regular on $\text{gr}_I(M)_+$;
- (b) $\text{depth}(\text{gr}_I(M)) \geq 1$.

Proof. If $\dim M/IM = 0$ then both claims follow from [23, Theorem 4.4]. Thus we may assume $\text{depth}(M/IM) > 0$.

Since I has almost minimal j -multiplicity on M , by Remark 4.2, for general elements x_1, x_2 in I , $\lambda(I^2 M / [x_2 IM + x_1 M :_{I^2 M} I^\infty]) = 1$. Set $J = (x_1, x_2)$. By Lemma 3.2(f), we have $x_1 M :_{I^2 M} I^\infty = x_1 IM$, thus $\lambda(I^2 M / JIM) = 1$. Hence $I^2 M = abR + JIM$ for some $a \in I, b \in IM$ with $ab \notin JIM$. For $j \geq 2$, the multiplication by a gives a surjective map from $I^j M / JI^{j-1} M$ to $I^{j+1} M / JI^j M$. Thus $\lambda(I^j M / JI^{j-1} M) \leq 1$ for every $j \geq 2$.

Notice that x_1 is regular on IM since $(0 :_M x_1) \cap IM = 0$ (Lemma 3.2(e)). Thus to prove that x_1^* is regular on $\text{gr}_I(M)_+ = \text{gr}_I(IM)$ we only need to show $x_1 IM \cap I^j IM = x_1 I^{j-1} IM$ for every $j \geq 1$ by [30, 2.6] (see also [23, Lemma 1.1]). This is clear if $j = 1$; hence we can assume $j \geq 2$. Let $\overline{}$ denote images in $\overline{IM} = IM/x_1 M$ and set $s = r_J(I, \overline{IM})$. We claim that it is enough to show $r_J(I, IM) = s$. Indeed, if $1 \leq j \leq s$ then $JI^{j-1} IM + (x_1 M \cap I^j IM) = JI^{j-1} IM$. This follows from the following easy inequality of lengths

$$\begin{aligned} 0 &< \lambda(I^j IM / JI^{j-1} IM + (x_1 M \cap I^j IM)) \\ &= \lambda(I^j IM / JI^{j-1} IM) - \lambda(JI^{j-1} IM + (x_1 M \cap I^j IM) / JI^{j-1} IM) \\ &= 1 - \lambda(JI^{j-1} IM + (x_1 M \cap I^j IM) / JI^{j-1} IM). \end{aligned}$$

On the other hand, if $j \geq s+1 = r_J(I, IM) + 1$, then $I^j IM = JI^{j-1} IM$. Thus we have for all $j \geq 1$

$$x_1 IM \cap I^j IM = x_1 IM \cap JI^{j-1} IM. \quad (2)$$

Now we proceed by induction on $j \geq 2$ as in the proof of Theorem 3.8:

$$\begin{aligned} x_1 IM \cap I^j IM &= x_1 IM \cap JI^{j-1} IM \quad \text{by (2)} \\ &= x_1 IM \cap (x_1 I^{j-1} IM + x_2 I^{j-1} IM) \\ &= x_1 I^{j-1} IM + (x_1 IM \cap x_2 I^{j-1} IM) \\ &= x_1 I^{j-1} IM + x_2 [(x_1 IM :_{IM} x_2) \cap I^{j-1} IM] \\ &= x_1 I^{j-1} IM + x_2 [x_1 IM \cap I^{j-1} IM]. \end{aligned}$$

The last equality follows because $x_1 IM \cap I^{j-1} IM \subseteq (x_1 IM :_{IM} x_2) \cap I^{j-1} IM \subseteq (x_1 M :_{I^2 M} x_2) \cap I^{j-1} IM = x_1 IM \cap I^{j-1} IM$ because $j \geq 2$ and by Lemma 3.2(f). Now using induction on j , we obtain

$$= x_1 I^{j-1} IM + x_2 [x_1 I^{j-2} IM] = x_1 I^{j-1} IM.$$

To complete the proof of (a), we still need to show that $r_J(I, IM) = s$. For this purpose we will use the Ratliff–Rush filtration $\widetilde{I^j IM}$ as it is done for ideals of definition (see [23, Theorem 4.2]). As noticed earlier x_1 is regular on IM . Thus, for instance by [23, Lemma 3.1], there exists an integer n_0 such that $I^j IM = \widetilde{I^j IM}$ for $j \geq n_0$, and

$$\widetilde{I^{j+1} IM} :_{IM} x_1 = \widetilde{I^j IM} \quad \text{for every } j \geq 0. \quad (3)$$

As before, let $\overline{IM} = IM/x_1 M$ and $\widetilde{}$ denote images in \overline{IM} . There are two filtrations:

$$\overline{IM}: \quad \overline{IM} \supseteq \overline{IIM} = \overline{I^2 M} \supseteq \dots \supseteq \overline{I^{j-1} IM} = \overline{I^j IM} \supseteq \dots$$

and

$$\widetilde{IM}: \quad \overline{IM} \supseteq \widetilde{IIM} := \widetilde{IIM} \supseteq \dots \supseteq \widetilde{I^{j-1} IM} := \widetilde{I^{j-1} IM} \supseteq \dots.$$

Notice that \overline{IM} is an I -adic filtration and \widetilde{IM} is a good I -filtration on \overline{IM} (see [23, p. 9] for the definition of good filtration). Furthermore, I is an ideal of definition on \overline{IM} , i.e., $\lambda_R(\overline{IM}/I\overline{IM}) < \infty$. Indeed, $(x_1 M :_M x_2) \cap IM = x_1 M$ (see Lemma 3.2(e)) which forces $x_2 \in I$ to be regular on \overline{IM} , in turns this yields $\lambda_R(\overline{IM}/I\overline{IM}) \leq \lambda_R(\overline{IM}/x_2 \overline{IM}) < \infty$. Thus we are in the context of the filtrations treated in [23]. Since $I^{j-1} \overline{IM} = \widetilde{I^{j-1} IM}$ for $j \geq n_0$, the associated graded modules $\text{gr}_{\overline{IM}}(\overline{IM})$ and $\text{gr}_{\widetilde{IM}}(\overline{IM})$ have the same Hilbert coefficients e_0 and e_1 . Again, because there exists an element in I which is regular on \overline{IM} , by [23, Lemmas 2.1 and 2.2] we have

$$s = \sum_{j \geq 0} \lambda(I^{j+1} \overline{IM}/x_2 I^j \overline{IM}) = e_1(\overline{IM}) = e_1(\widetilde{IM}) = \sum_{j \geq 0} \lambda(\widetilde{I^{j+1} IM}/x_2 \widetilde{I^j IM}). \quad (4)$$

Observe that the first equality holds because $\lambda(I^{j+1} \overline{IM}/x_2 I^j \overline{IM}) = 1$ for all $0 \leq j \leq s-1$ and $\lambda(I^{j+1} \overline{IM}/x_2 I^j \overline{IM}) = 0$ for $j \geq s = r_J(I, \overline{IM})$.

We prove that $\lambda(I^{j+1} \overline{IM}/x_2 \widetilde{I^j IM}) = \lambda(I^{j+1} IM/J\widetilde{I^j IM})$ for every $j \geq 0$. Since

$$\widetilde{I^{j+1} IM}/x_2 \widetilde{I^j IM} \cong \widetilde{I^{j+1} IM}/(x_1 M \cap \widetilde{I^{j+1} IM} + x_2 \widetilde{I^j IM}),$$

we just need to show $x_1 M \cap \widetilde{I^{j+1}}IM = x_1 \widetilde{I^j}IM$. We first prove $x_1 M \cap \widetilde{I}IM = x_1 IM$. Since $x_1 M \cap \widetilde{I}IM \supseteq x_1 IM$, it suffices to show the equality locally at every associated prime ideal of $M/x_1 IM$. By Lemma 3.2(d), every $\mathfrak{p} \in \text{Ass}(M/x_1 IM)$ is not maximal. By the proof of Lemma 4.6, $x_1 M_{\mathfrak{p}} = \widetilde{I}M_{\mathfrak{p}} = IM_{\mathfrak{p}}$ thus $x_1 M_{\mathfrak{p}} \cap \widetilde{I}IM_{\mathfrak{p}} = \widetilde{I}IM_{\mathfrak{p}} = x_1 IM_{\mathfrak{p}}$. Therefore $x_1 M \cap \widetilde{I}IM = x_1 IM$. Now for any $j \geq 1$, $x_1 M \cap \widetilde{I^{j+1}}IM = x_1 IM \cap \widetilde{I^{j+1}}IM = x_1 (\widetilde{I^{j+1}}IM :_{IM} x_1) = x_1 \widetilde{I^j}IM$, where the last equality holds by (3).

Now (4) gives us

$$\sum_{j \geq 0} \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM) = s, \quad (5)$$

and

$$p = \inf\{j \mid \widetilde{I^j}IM = \widetilde{I^{j+1}}IM\} \leq s.$$

Let $t = \inf\{j \mid I^{j+1}IM \subseteq \widetilde{I^j}IM\}$. Observe that $t \leq p \leq s$ because $I^{p+1}IM \subseteq \widetilde{I^{p+1}}IM = \widetilde{I^p}IM$. Let l be a positive integer such that for all $0 \leq j \leq l$ we have $I^{j+1}IM \cap \widetilde{I}IM = \widetilde{I^j}IM$. If $t \leq l$, then $r_J(I, IM) \leq t \leq s$ and we are done. So we can assume that $t > l$ and we have:

$$l < t \leq p \leq s.$$

By Corollary 4.5, the reduction number is bounded above by $t + q$ where

$$q \leq \sum_{j \geq 0} \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM). \quad (6)$$

To prove that $r_J(I, IM) = s$, it will be enough to show that $\sum_{j \geq 0} \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM) \leq s - t$. Observe that for $0 \leq j \leq l$, we can relate $\lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM)$ to the difference of the length of the factors of the filtrations \mathbb{N} and \mathbb{M} . Indeed, for $0 \leq j \leq l$ we have $\widetilde{I^j}IM \cap I^{j+1}IM = \widetilde{I^j}IM$ and therefore we obtain the following family of short exact sequences:

$$0 \rightarrow \widetilde{I^j}IM / \widetilde{I^j}IM \rightarrow \widetilde{I^{j+1}}IM / I^{j+1}IM \rightarrow \widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM \rightarrow 0,$$

from which we obtain:

$$\begin{aligned} \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM) &= \lambda(\widetilde{I^{j+1}}IM / I^{j+1}IM) - \lambda(\widetilde{I^j}IM / \widetilde{I^j}IM) \\ &= \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM) - \lambda(I^{j+1}IM / \widetilde{I^j}IM) \quad \text{for } 0 \leq j \leq l. \end{aligned}$$

For this we conclude that

$$\lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM) = \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM) - 1, \quad 0 \leq j \leq l, \quad (7)$$

$$\lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM) \leq \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM) - 1, \quad l+1 \leq j \leq t-1, \quad (8)$$

$$\lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM) = \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM), \quad j \geq t. \quad (9)$$

Now by means of (5), (6), (7) and (8) we obtain

$$q \leq \sum_{j \geq 0} \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM + I^{j+1}IM) \leq \sum_{j \geq 0} \lambda(\widetilde{I^{j+1}}IM / \widetilde{I^j}IM) - t = s - t.$$

This concludes the proof of (a) since $r_J(I, IM) \leq t + q \leq t + s - t = s$ (see Corollary 4.5).

Finally part (b) follows from (a). Indeed, by assumption $\text{depth}(M/IM) > 0$ and by the exact sequence:

$$0 \rightarrow M/IM \rightarrow \text{gr}_I(M) \rightarrow \text{gr}_I(M)_+ \rightarrow 0,$$

we conclude $\text{depth}(\text{gr}_I(M)) \geq \min\{\text{depth } M/IM, \text{depth}(\text{gr}_I(M)_+)\} \geq 1$. \square

The following lemma is a generalization of the Sally machine to non \mathfrak{m} -primary ideals. It is inspired by [11, 2.2].

Lemma 4.8. *Let M be a finite module over a Noetherian local ring R , I an R -ideal, and $T = \text{gr}_I(M)$ the associated graded module. Let x_1 be a superficial element in I and set $\bar{T} = \text{gr}_I(\bar{M})$, where $\bar{M} = M/x_1 M$. Assume there exists $x_2 \in I$ such that either x_2^* is regular on \bar{T} or x_2^* is regular on \bar{T}_+ and $I^{j+t}M :_M x_1^t = I^{j+t} :_{IM} x_1^t$ for every $j, t \geq 1$. Then x_1^* is regular on T .*

Proof. If x_2^* is regular on \bar{T} then the assertion follows from the proof of [11, 2.2]. Otherwise let us assume that x_2^* is regular on \bar{T}_+ and $I^{j+t}M :_M x_1^t = I^{j+t} :_{IM} x_1^t$ for every $j, t \geq 1$. We need to show

$$I^{j+1}M :_M x_1 = I^j M$$

for every $j \geq 0$. This is clear if $j = 0$. So assume $j \geq 1$. Since x_2^* is regular on \bar{T}_+ , we have

$$I^{j+\delta}M :_{IM} x_2^\delta \subseteq I^j M + x_1 M \quad (10)$$

for every $j, \delta \geq 1$. Since x_1 is superficial in I , for every $t \geq 1$ there exists $c_t \geq 1$ such that for every $j \geq c_t$ we have

$$(I^{j+t}M :_M x_1^t) \cap I^{c_t}M = I^j M.$$

Hence if $\delta \geq c_t$ we obtain

$$x_2^\delta (I^{j+t}M :_M x_1^t) \subseteq [I^{\delta+j+t}M :_M x_1^t] \cap I^{c_t}M = I^{\delta+j}M,$$

which yields

$$I^{j+t}M :_M x_1^t \subseteq I^{j+\delta}M :_M x_2^\delta. \quad (11)$$

Therefore

$$\begin{aligned} I^{j+t}M :_M x_1^t &= I^{j+t}M :_{IM} x_1^t \quad \text{by assumption} \\ &\subseteq I^{j+\delta}M :_{IM} x_2^\delta \quad \text{by (11)} \\ &\subseteq I^j M + x_1 M \quad \text{by (10)}. \end{aligned}$$

Hence for every $t \geq 1$ we have

$$I^{j+t}M :_M x_1^t = I^j M + x_1 M \cap [I^{j+t}M :_M x_1^t] = I^j M + x_1 [I^{j+t}M :_M x_1^{t+1}],$$

which yields

$$I^{j+t}M :_M x_1^t = I^j M + x_1 [I^{j+t}M :_M x_1^{t+1}]. \quad (12)$$

Finally applying (12) repeatedly we obtain

$$I^{j+1}M :_M x_1 = I^jM + x_1(I^{j+1}M :_M x_1^2) = I^jM + x_1I^{j-1}M + \cdots + x_1^jM = I^jM,$$

as asserted. \square

In the next lemma we show that the residual properties on I assure us the second condition on Lemma 4.8.

Lemma 4.9. *Let M be a Cohen–Macaulay module of dimension d over a Noetherian local ring R with infinite residue field. Let I be an R -ideal with $\ell(I, M) = d \geq 2$. Assume $\text{depth}(M/IM) \geq 1$, $\text{grade}(I, M) \geq 1$, and I satisfies G_d and AN_{d-2}^- on M . Then for a general element $x_1 \in I$,*

$$I^{j+t}M :_M x_1^t = I^{j+t} :_{IM} x_1^t$$

for every $j, t \geq 1$.

Proof. We choose general elements x_1, \dots, x_d such that $J = (x_1, \dots, x_d)$ form a reduction of I . Observe x_1 is regular on M . We first claim that it is enough to show

$$x_1^t M \cap I^{j+t}M = x_1^t IM \cap I^{j+t}M. \quad (13)$$

Indeed,

$$\begin{aligned} x_1^t(I^{j+t}M :_M x_1^t) &= x_1^t M \cap I^{j+t}M \\ &= x_1^t IM \cap I^{j+t}M \\ &= x_1^t(I^{j+t} :_{IM} x_1^t) \end{aligned}$$

which gives us the desired equality as x_1^t is regular on M .

Now we prove (13). Since $x_1^t M \cap I^{j+t}M = x_1^t M \cap I^{1+t}M \cap I^{j+t}M$, we just need to show $x_1^t M \cap I^{1+t}M = x_1^t IM$. By the residual intersection properties, $\text{depth}(M/x_1 IM) \geq 1$. Hence $\text{depth}(x_1^t IM) = \text{depth}(x_1 IM) \geq 2$, therefore

$$\text{depth}(M/x_1^t IM) \geq 1.$$

Hence for every $\mathfrak{p} \in \text{Ass}(M/x_1^t IM)$, \mathfrak{p} is not maximal, either $IM_{\mathfrak{p}} = M_{\mathfrak{p}}$ or $IM_{\mathfrak{p}} = (x_1, \dots, x_{d-1})M_{\mathfrak{p}}$. To show $x_1^t M_{\mathfrak{p}} \cap I^{1+t}M_{\mathfrak{p}} = x_1^t IM_{\mathfrak{p}}$, we just need to show x_1^t is regular on $\text{gr}_I(M)_{\mathfrak{p}}$. By [30, 2.6] it will be enough to prove $x_1 M_{\mathfrak{p}} \cap I^j M_{\mathfrak{p}} = x_1 I^{j-1} M_{\mathfrak{p}}$ for every $j \geq 1$. This is clear if $IM_{\mathfrak{p}} = M_{\mathfrak{p}}$. So assume $IM_{\mathfrak{p}} = (x_1, \dots, x_{d-1})M_{\mathfrak{p}}$. We use increasing induction on j and descending induction on i to show $(x_1, \dots, x_i)M_{\mathfrak{p}} \cap I^j M_{\mathfrak{p}} = (x_1, \dots, x_i)I^{j-1} M_{\mathfrak{p}}$ for $1 \leq i \leq d-1$. This is clear if $j = 1$ for all i and if $i = d-1$ for all j . Now assume $j \geq 2$, $i \leq d-2$, further assume

$$(x_1, \dots, x_{i+1})M_{\mathfrak{p}} \cap I^j M_{\mathfrak{p}} = (x_1, \dots, x_{i+1})I^{j-1} M_{\mathfrak{p}} \quad \text{and} \quad (14)$$

$$(x_1, \dots, x_i)M_{\mathfrak{p}} \cap I^{j-1} M_{\mathfrak{p}} = (x_1, \dots, x_i)I^{j-2} M_{\mathfrak{p}}. \quad (15)$$

Then using the residual properties of I we have

$$\begin{aligned}
(x_1, \dots, x_i)M_{\mathfrak{p}} \cap I^j M_{\mathfrak{p}} &= (x_1, \dots, x_i)M_{\mathfrak{p}} \cap (x_1, \dots, x_{i+1})I^{j-1}M_{\mathfrak{p}} \quad \text{by (14)} \\
&= (x_1, \dots, x_i)M_{\mathfrak{p}} \cap ((x_1, \dots, x_i)I^{j-1}M_{\mathfrak{p}} + x_{i+1}I^{j-1}M_{\mathfrak{p}}) \\
&= (x_1, \dots, x_i)I^{j-1}M_{\mathfrak{p}} + [(x_1, \dots, x_i)M_{\mathfrak{p}} \cap x_{i+1}I^{j-1}M_{\mathfrak{p}}] \\
&= (x_1, \dots, x_i)I^{j-1}M_{\mathfrak{p}} + x_{i+1}[(x_1, \dots, x_i)M_{\mathfrak{p}} :_{M_{\mathfrak{p}}} x_{i+1}] \cap I^{j-1}M_{\mathfrak{p}} \\
&= (x_1, \dots, x_i)I^{j-1}M_{\mathfrak{p}} + x_{i+1}[(x_1, \dots, x_i)M_{\mathfrak{p}} \cap I^{j-1}M_{\mathfrak{p}}] \\
&= (x_1, \dots, x_i)I^{j-1}M_{\mathfrak{p}} + x_{i+1}(x_1, \dots, x_i)I^{j-2}M_{\mathfrak{p}} \quad \text{by (15)} \\
&= (x_1, \dots, x_i)I^{j-1}M_{\mathfrak{p}},
\end{aligned}$$

as claimed. \square

We are now ready to prove our main theorem:

Theorem 4.10. *Let M be a Cohen–Macaulay module of dimension d over a Noetherian local ring R . Let I be an R -ideal with $\ell(I, M) = d$. Assume $\text{depth}(M/IM) \geq \min\{\dim(M/IM), 1\}$ and I satisfies G_d and AN_{d-2}^- on M . If I has almost minimal j -multiplicity on M then*

- (a) *for a general $x_1 \in I$, x_1^* is regular on $\text{gr}_I(M)_+$.*
- (b) *$\text{depth}(\text{gr}_I(M)) \geq d - 1$.*

Proof. As in the proof of Theorem 3.4, we may assume that the residue field of R is infinite. We prove the theorem by induction on d . The case $d = 2$ being proven in Theorem 4.7. Let $d \geq 3$ and assume the theorem holds for $d - 1$. We first reduce to the case $\text{grade}(I, M) \geq 1$. If $\text{grade}(I, M) = 0$, let $H_0 = 0 :_M I$. As in the proof of Theorem 3.8, all assumptions still hold for the module M/H_0 . Furthermore $IM/H_0 = IM/(H_0 \cap IM) = IM$, $\text{grade}(I, M/H_0) \geq 1$ and again as in the proof of Theorem 3.8, $\text{depth}(\text{gr}_I(M)) \geq \text{depth}(\text{gr}_I(M/H_0))$. So we are reduced to the case where the ideal I has at least one M -regular element. Thus if x_1 is a general element in I then x_1 is regular on M .

If $\dim M/IM = 0$ then the assertion follows from [23, Theorem 4.4]. Thus we may assume $\dim M/IM > 0$. Let $\bar{}$ denote images in $\bar{M} = M/x_1 M$. Observe that \bar{M} is a Cohen–Macaulay module of dimension $d - 1$ and $\ell(I, \bar{M}) = d - 1$. Also I satisfies G_{d-1} and AN_{d-3}^- on \bar{M} by Lemma 3.2. Furthermore, observe $\bar{M}/I\bar{M} \cong M/IM$ thus $\text{depth}(\bar{M}/I\bar{M}) = \text{depth}(M/IM) \geq \min\{\dim M/IM, 1\} = \{\dim \bar{M}/I\bar{M}, 1\}$. Clearly I has almost minimal j -multiplicity on \bar{M} . By induction hypothesis, for a general $x_2 \in I$, x_2^* is regular on $\text{gr}_I(\bar{M})_+$, and $\text{depth}(\text{gr}_I(\bar{M})) \geq d - 2$.

Now Lemmas 4.8 and 4.9 imply that x_1^* is regular on $\text{gr}_I(M)$. Since $\text{depth}(\text{gr}_I(\bar{M})) \geq d - 2$ and x_1^* is regular on $\text{gr}_I(M)$, we have $\text{depth}(\text{gr}_I(M)) \geq d - 1$. \square

Again our main application is the case when $M = R$. We obtain that ideals with residual intersection properties and almost minimal j -multiplicity have associated graded rings almost Cohen–Macaulay.

Corollary 4.11. *Let R be a Cohen–Macaulay local ring of dimension d . Let I be an R -ideal with $\ell(I) = d$. Assume $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$ and I satisfies G_d and AN_{d-2}^- . If I has almost minimal j -multiplicity then $\text{depth}(\text{gr}_I(R)) \geq d - 1$.*

As noticed in Section 3 before Corollary 3.6, the assumptions on Corollary 4.11 are all vacuous in the 0-dimensional case and the condition on the almost minimal j -multiplicity becomes the usual assumption of almost minimal multiplicity found in Sally's conjecture and in the work of Rossi and Valla and Wang (see [26,21,31,10,4,20,5]). Thus Corollary 4.11 can be viewed as a positive answer to Sally's conjecture for arbitrary ideals.

Again the conclusion of Corollary 4.11 will hold true for I which is generically a complete intersection with $\text{ht } I = d - 1$ or for strongly Cohen–Macaulay ideals satisfying G_d , in particular, for Cohen–Macaulay almost complete intersections, Cohen–Macaulay ideals generated by $2 + \text{ht } I$ elements, perfect ideals of codimension two, perfect Gorenstein ideals of codimension three, and, more generally licci ideals.

We will finish our paper with the following three examples.

Example 4.12. Let S be a 3-dimensional Cohen–Macaulay local ring and x, y, z a system of parameters for S . We set $R = S/(x^2 - yz)S$ and $I = (x, y)R$. Observe that R is a Cohen–Macaulay local ring of dimension $d = 2$. By [16, 4.2], I is a Cohen–Macaulay ideal of height $1 = d - 1$ which is generically a complete intersection and $\ell(I) = 2$. In particular, I has reduction number zero and therefore I has minimal j -multiplicity. By Theorem 3.9 the associated graded ring $\text{gr}_I(R)$ is Cohen–Macaulay, and if S is Gorenstein then $\text{gr}_I(R)$ is Gorenstein as well.

Example 4.13. Let $R = k[[x, y, z, v, w]]$ be a power series ring over an infinite field k and $I = I_2(A)$, where

$$A = \begin{pmatrix} x & y & z & v \\ y & z & v & w \end{pmatrix}.$$

Observe that I is a perfect ideal of grade 3 and is a complete intersection on the punctured spectrum with analytic spread $\ell(I) = 5$. Hence I satisfies G_5 and AN_5^- . By Macaulay2 [15], $\lambda(I^2/JI) = 0$, where J is a general minimal reduction of I . Hence I has minimal j -multiplicity. By Theorem 3.9, the associated graded ring $G = \text{gr}_I(R)$ is Cohen–Macaulay.

Example 4.14. Let $R = k[[x, y, z, v]]$ be a power series ring over an infinite field k and $I = I_2(B)$, where

$$B = \begin{pmatrix} x & y & z & v \\ v & x & y & z \end{pmatrix}.$$

Observe that I is a perfect ideal of grade 3 and is a generically a complete intersection with analytic spread $\ell(I) = 4$. Hence I satisfies G_4 and AN_4^- . Let J be a general minimal reduction of I , using Macaulay2 [15], one computes $\lambda(I^2/JI) = 1$. Hence I has almost minimal j -multiplicity. By Corollary 4.11 the associated graded ring $G = \text{gr}_I(R)$ has depth at least 3. Indeed, G is Cohen–Macaulay.

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