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On the Jacobson radical of constants of derivations ☆

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ABSTRACT

Let R be a semiprimitive algebra, d its algebraic derivation and $R^d = \ker d$ the subalgebra of constants of d. It is proved that the Jacobson radical $J(R^d)$ of R^d is nilpotent. It is also shown that the following properties are equivalent: R^d is semilocal; R is semisimple Artinian; R^d is left and right Artinian.

Introduction

In [13, Theorem 2.2] Haïly et al. proved that a semiprimitive algebra containing an algebraic element, whose centralizer is semiperfect, has to be Artinian. Using this result they proved that a semiprimitive complex Banach algebra containing an element whose centralizer is algebraic has to be finite-dimensional, answering affirmatively a question

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raised in [3]. Observe that the centralizer $C_R(a)$ of an element $a \in R$ is equal to the kernel of the inner derivation induced by a. Therefore, in terms of derivations, the above result says that if the constants of a certain algebraic derivation of a semiprimitive algebra are semiperfect, then the algebra is Artinian. On the other hand it is proved in [11, Theorem 4.7] that if d is an algebraic derivation of a semiprime algebra, then R^d is left Artinian if and only if R is semisimple Artinian. Thus comparing it with [13, Theorem 2.2] we obtain that the centralizer $C_R(a)$ of an algebraic element of a semiprimitive algebra R is semiperfect if and only if $C_R(a)$ is Artinian. In particular, in this situation the Jacobson radical $J(C_R(a))$ is nilpotent. Let us remind also that if R is semiprime and d algebraic, then the prime radical $P(R^d)$ is nilpotent (see [11, Theorem 3.4]). Therefore it is reasonable to ask whether the Jacobson radical of constants of an algebraic derivation of a semiprimitive algebra has to be nilpotent. In the first main result of this paper we answer this question affirmatively and prove

Theorem A. *Suppose that either R is a semiprimitive ring with a nilpotent derivation ($d^n = 0$) or R is a semiprimitive algebra over a field K with a derivation d satisfying the identity*

$$\alpha_0 d^n + \alpha_1 d^{n+1} + \cdots + \alpha_k d^{n+k} = 0,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k \in K$ and $\alpha_0 \neq 0$. Then the Jacobson radical $J(R^d)$ is nilpotent. More precisely, $J(R^d)^\gamma = 0$, where $\gamma \leqslant 2^n - 1$.

Observe that, under assumptions of the above theorem, if R^d is semilocal, then R^d is semiperfect. In the second main result of this paper we generalize Haïly et al. result on semiperfect centralizers [13, Theorem 2.2] to the following form

Theorem B. *Suppose that either R is a semiprimitive ring with a nilpotent derivation d or R is a semiprimitive K-algebra with a K-linear algebraic derivation d. The following conditions are equivalent*:

(1) *R^d is semilocal.*
(2) *R is semisimple Artinian.*
(3) *R^d is left and right Artinian.*

We can now introduce some of the notation and terminology that will be used throughout this paper. R will denote associative rings or associative algebras over a field K. Although some of the rings and algebras in this paper may not have an identity, when examining whether an algebra is Artinian, we will assume that the algebra has an identity. By a derivation of R we mean an additive map $d\colon R \to R$ such that $d(ab) = d(a)b + ad(b)$ for $a, b \in R$. The kernel $\ker d$ we denote by R^d and call constants of d. For an element $a \in R$, ad_a stands for the inner derivation induced by a, i.e., $\mathrm{ad}_a(r) = ar - ra$ for $r \in R$.

An ideal I of R is called a d-ideal provided $d(I) \subseteq I$. If $d^n = 0$ for some positive integer n, then d is said to be a nilpotent derivation. If A is a subset of R, then the smallest integer $n = n(A)$ such that $d^n(x) = 0$ for all $x \in A$ is called the index of nilpotency of d on A. We say that an ideal I of a ring R has a prime characteristic p, if $pI = 0$ for some prime integer $p > 0$.

For subsets A, B of R we let $\operatorname{ann}^r_A B = \{a \in A \mid Ba = 0\}$ to denote the right annihilator of B in A. Similarly $\operatorname{ann}^\ell_A B$ will denote the left annihilator of B in A.

Given a ring R denote its Jacobson (prime) radical by $J(R)$ (resp. $P(R)$). The Jacobson radical has a useful description in terms of elements. Namely an element $a \in R$ is said to be quasi-invertible if $1 - a$ is invertible in R. A subset of R is quasi-invertible if each its element is quasi-invertible. It is well known (cf. [14, Proposition 2.5.4]) that the Jacobson radical $J(R)$ is a quasi-invertible ideal of R which contains every quasi-invertible left ideal.

Recall that R is said to be semiprimitive (resp. semiprime) if $J(R) = 0$ (resp. $P(R) = 0$). The ring R is semilocal if $R/J(R)$ is semisimple Artinian. If in addition $J(R)$ is idempotent lifting, then R is called semiperfect.

1. The Jacobson radical of constants

In this section we prove that the Jacobson radical of constants of an algebraic derivation of a semiprimitive algebra is nilpotent. To this end, first we consider a special case where the derivation is nilpotent.

For a semiprime ring R let $\mathfrak{F}_R$ denote the set of all twosided ideals with zero annihilator. Equivalently, $\mathfrak{F}_R$ consists of those ideals of R which are essential as left (or right) ideals. For an ideal $S \in \mathfrak{F}_R$ the ring $\operatorname{End}(S_R)$ or right R-module endomorphisms of S is semiprime and $R \hookrightarrow \operatorname{End}(S_R)$ via the map $r \mapsto r_\ell$, where r_ℓ is the left multiplication by r acting on S. Each derivation $d: R \to R$ leaving S invariant has a unique extension to a derivation $\bar{d}: \operatorname{End}(S_R) \to \operatorname{End}(S_R)$; where $\bar{d}(f)(s) = d(f(s)) - f(d(s))$ for $f \in \operatorname{End}(S_R)$ and $s \in S$. A derivation d is said to be E-inner if there is a d-ideal $S \in \mathfrak{F}_R$ and $x \in \operatorname{End}(S_R)$ such that $\bar{d} = \operatorname{ad}_x$. It is clear that every E-inner derivation is X-inner, i.e., has an inner extension to the Martindale ring of quotients.

We will make use of the following description of nilpotent derivations

Theorem 1.1. *(See [9, Corollary 8].) If d is a nilpotent derivation of a semiprime ring R ($d^n = 0$) and if for every d-ideal I of R of a prime characteristic p, p does not divide the index of nilpotency of d on I, then d is E-inner. Furthermore, there exists an ideal $S \in \mathfrak{F}_R$ such that $d = \operatorname{ad}_x$, where x is an element of $\operatorname{End}(S_R)$ satisfying $x^{[(n+1)/2]} = 0$.*

In light of Theorem 1.1 the assumptions of the next result seem to be natural.

Theorem 1.2. *Let d be a nilpotent derivation of a semiprimitive ring R. Suppose that d is E-inner and $\bar{d} = \mathrm{ad}_x$, where $x \in \mathrm{End}(S_R)$, $S \in \mathfrak{F}_R$ and $x^m = 0$. Then $J(R^d)^\gamma = 0$, where $\gamma \leqslant 2^m - 1$.*

Proof. Let $R\langle x\rangle$ be the subring of $\mathrm{End}(S_R)$ generated by $R \cup \{x\}$. It is clear that $R\langle x\rangle = R + Rx + \cdots + Rx^{m-1}$, and the relation $xr = rx + d(r)$ is fulfilled in $R\langle x\rangle$. Notice that $J(R\langle x\rangle) = 0$. Since $Sx \cup xS \subset S$, it is clear that $SJ(R\langle x\rangle)$ is a quasi-invertible ideal of $R\langle x\rangle$ contained in S. Thus $SJ(R\langle x\rangle)$ is a quasi-invertible ideal of R and consequently $SJ(R\langle x\rangle) = 0$. Since $\mathrm{ann}_{\mathrm{End}(S_R)}(S) = \mathrm{ann}_R(S) = 0$, one obtains $J(R\langle x\rangle) = 0$.

Let $J = J(R^d)$. We will show that for $k = 0, 1, \ldots, m$ there exists an integer $f(k) \geqslant 0$ such that

$$x^{m-k} J^{f(k)} = 0 \quad \text{and} \quad f(k+1) \leqslant 2f(k) + 1.$$

For $k = 0$ it is enough to put $f(0) = 0$. Suppose that $k \geqslant 0$ and $x^{m-k} J^{f(k)} = 0$. For any simple left $R\langle x\rangle$-module M consider the left R^d-module $M_1 = x^{m-k-1} J^{f(k)} M$. We will prove that if M_1 is nonzero and not simple, then any proper R^d-submodule N of M_1 is annihilated by $SJ^{f(k)}$, i.e., $SJ^{f(k)} N = 0$. To this end, suppose that $SJ^{f(k)} N \neq 0$. Take an element $x^{m-k-1} a m_1 \in M_1 \setminus N$, where $a \in J^{f(k)}$ and an element $x^{m-k-1} b m_2 \in N$, where $b \in J^{f(k)}$ such that for some $c \in J^{f(k)}$

$$Scx^{m-k-1} bm_2 = Sx^{m-k-1} cbm_2 \neq 0.$$

Observe that our assumption that $x^{m-k} J^{f(k)} = 0$ implies that $x^{m-k} a = cx^{m-k} = 0$, so for any $s \in S$ the element

$$T_{a,c}(s) = \sum_{i=0}^{m-k-1} x^i ascx^{m-k-i-1}$$

centralizes x. Thus $T_{a,c}(s) \in R^d$. Notice that

$$S \cdot R\langle x\rangle x^{m-k-1} cbm_2 = \sum_{i=0}^{m-1} SRx^i x^{m-k-1} cbm_2 = Sx^{m-k-1} cbm_2 \neq 0.$$

Since M is simple as a left $R\langle x\rangle$-module, we obtain that $M = Sx^{m-k-1} cbm_2$. In particular, there exists $s \in S$ such that $sx^{m-k-1} cbm_2 = m_1$. Observe that

$$\begin{aligned} x^{m-k-1} am_1 &= x^{m-k-1} a\bigl(sx^{m-k-1} cbm_2\bigr) = x^{m-k-1} ascx^{m-k-1} bm_2 \\ &= T_{a,c}(s) x^{m-k-1} bm_2 \in N, \end{aligned}$$

which contradicts the assumption that $x^{m-k-1} am_1 \in M_1 \setminus N$. Consequently, any proper submodule of M_1 is annihilated by $SJ^{f(k)}$. In the case when M_1 is not finitely generated as a left R^d-module, any its cyclic submodule is proper, so any cyclic submodule of M_1 is annihilated by $SJ^{f(k)}$. It implies that $0 = SJ^{f(k)}M_1 = Sx^{m-k-1}J^{2f(k)}M$. On the other hand if M_1 is finitely generated as a left R^d-module, then by the Nakayama Lemma $JM_1 \neq M_1$. Thus $SJ^{f(k)}JM_1 = 0$ and

$$Sx^{m-k-1}J^{2f(k)+1}M = 0.$$

The above holds for every simple $R\langle x\rangle$-module, so since $R\langle x\rangle$ is semiprimitive, we obtain

$$Sx^{m-k-1}J^{2f(k)+1} = 0.$$

The ideal S of R has zero annihilator in R and in $R\langle x\rangle$, so $x^{m-k-1}J^{2f(k)+1} = 0$. Now we can take the smallest integer $f(k+1) \leqslant 2f(k)+1$ satisfying $x^{m-k-1}J^{f(k+1)} = 0$.

In particular, for $k = m$ we have $J^{f(m)} = 0$. Since $f(0) = 0$ and $f(k+1) \leqslant 2f(k)+1$, it follows by induction that $f(m) \leqslant 2^m - 1$. $\square$

Now consider the case when $pR = 0$ for some prime number $p > 0$. In this situation it may happen that the derivation d is not E-inner. Instead of a suitable endomorphism ring we will make use of a construction of a minimum extension $R\{x\}$ of R in which a nilpotent derivation d of R becomes inner and adjoint to a nilpotent element. Namely, if d is a nilpotent derivation of a semiprime ring R consider the ring of differential polynomials $R[X;d]$ and its ideal (X^n), where $d^n = 0$. It is easy to see that $R \cap (X^n) \subset d^n(R)R = 0$. Thus we can take a minimal positive integer m such that $R \cap (X^m) = 0$ and an ideal M of $R[X;d]$ containing X^m and maximal with respect to the property $R \cap M = 0$. We denote the factor ring $R[X;d]/M$ by $R\{x\}$, where x is the coset of X. Clearly we have a natural embedding $R \hookrightarrow R\{x\}$, $d = (\mathrm{ad}_x)_{|R}$ and $x^m = 0$. Since nonzero ideals of $R\{x\}$ have nonzero intersections with R, the ring $R\{x\}$ has to be semiprime. The constructed ring $R\{x\}$ will be called the d-extension of R. This ring was extensively employed in the papers [9–12]. More information on the structure of d-extensions of prime rings can be found in Chuang and Lee paper [4]. If I is a d-ideal of R, by $I\{x\}$ we denote the ideal of $R\{x\}$ generated by I. We need the following lemma proved in [11]:

Lemma 1.3. *(See [11, Lemma 3.2].) Suppose that R is semiprime, $R\{x\}$ is a d-extension of R and $pR = 0$. Let S be a d-ideal of R and let k be an integer such that $\operatorname{ann}_S^{\ell} x^{p^{k-1}} = 0$. For a nonempty subset I of R^d, let j be a minimal integer with respect to the property*

$$x^j IS = SIx^j = 0.$$

Then p^k divides j.

Remark 1.4. Observe that in the above lemma $SIx^{m-j} = 0$ if and only if $x^{m-j}IS = 0$. Indeed, it is clear for $j = 0$. If $j \geqslant 0$ and $SIx^{m-j} = 0$, then $SIx^{m-i} = 0$ for $i \leqslant j$, so by induction one can assume that $x^{m-j+1}IS = 0$. Then $x^{m-j}IS$ is a right ideal of $R\{x\}$

and $(x^{m-j}IS)^2 = x^{m-j}I(Sx^{m-j}I)S = x^{m-j}I(SIx^{m-j})S = 0$. Thus by semiprimeness of $R\{x\}$ it follows that $x^{m-j}IS = 0$.

Now let $S_1, S_2, S_3, \dots$ be a chain of d-ideals of R defined as follows:

- $S_1 = \operatorname{ann}^{\ell}_R x^{p^{i_1}} \cdot R$, where $i_1 = \min\{i \mid i \geqslant 0 \text{ and } \operatorname{ann}^{\ell}_R x^{p^i} \neq 0\}$;
- if $S_1, \dots, S_l$ are defined then consider $A_l = \operatorname{ann}^{\ell}_R(S_1 \oplus \dots \oplus S_l)$ and put $S_{l+1} = \operatorname{ann}^{\ell}_{A_l} x^{p^{i_{l+1}}} \cdot R$, where $i_{l+1} = \min\{i \mid \operatorname{ann}^{\ell}_{A_l} x^{p^i} \neq 0\}$.

It follows from the definition that $i_1 < i_2 < \cdots$ and since $x^m = 0$, the construction stabilizes.

Lemma 1.5. *If R is semiprime and $pR = 0$, then R contains a collection of d-ideals $S_1, \dots, S_t$ and there exists a sequence $0 \leqslant i_1 < \dots < i_t$ such that*

(1) $S = S_1 \oplus \dots \oplus S_t \in \mathfrak{F}_R$;
(2) $\operatorname{ann}^{\ell}_{S_l} x^{p^{i_l}-1} = 0$ *for* $l = 1, \dots, t$;
(3) $x^{p^{i_l}} S_l \subseteq S_l$ *and* $S_l x^{p^{i_l}} \subseteq S_l$ *for* $l = 1, \dots, t$.

Proof. (1) follows directly from the construction.

For (2) suppose that $L = \operatorname{ann}^{\ell}_{S_l} x^{p^{i_l}-1} \neq 0$. Then L is a d-stable left ideal contained in S_l. Clearly $A_{l-1}L^d \subset L \subset S_l \subset A_{l-1}$. Since $\operatorname{ann}^{\ell}_{A_{l-1}} x^{p^{i_l-1}} = 0$ and $A_{l-1}L^d x^{p^{i_l}-1} = 0$, by Lemma 1.3 the number $j \leqslant p^{i_l}-1$ minimal with respect to the property $A_{l-1}L^d x^j = 0$ is divided by p^{i_l}, what is impossible.

For (3) notice that $x^{p^{i_l}} S_l = x^{p^{i_l}}(\operatorname{ann}^{\ell}_{A_{l-1}} x^{p^{i_l}} \cdot R) \subset d^{p^{i_l}}(\operatorname{ann}^{\ell}_{A_{l-1}} x^{p^{i_l}})R \subset S_l$ and $S_l x^{p^{i_l}} = (\operatorname{ann}^{\ell}_{A_{l-1}} x^{p^{i_l}} \cdot R)x^{p^{i_l}} \subset (\operatorname{ann}^{\ell}_{A_{l-1}} x^{p^{i_l}}) \cdot d^{p^{i_l}}(R) \subset S_l$. $\square$

The above implies immediately that for any integer $k > 0$ if $k \equiv j \pmod{p^{i_l}}$, where $0 \leqslant j < p^{i_l}$, then $S_l x^k \subset S_l x^j$. Thus the ideal of $R\{x\}$ generated by S_l has the form $S_l\{x\} = S_l + S_l x + \dots + S_l x^{p^{i_l}-1}$. Since $\operatorname{ann}^{\ell}_{S_l} x^{p^{i_l}-1} = 0$, we have a decomposition into a direct sum of left R-modules

$$S_l\{x\} = S_l \oplus S_l x \oplus \dots \oplus S_l x^{p^{i_l}-1}.$$

Since $xr = rx + d(r)$ for $r \in R$, it can be easily checked that $S_l\{x\}$ has a decomposition into a direct sum of right R-modules $S_l \oplus xS_l \oplus \dots \oplus x^{p^{i_l}-1}S_l$. Therefore we can write

$$S_l\{x\} = S_l \oplus S_l x \oplus \dots \oplus S_l x^{p^{i_l}-1} = S_l \oplus xS_l \oplus \dots \oplus x^{p^{i_l}-1}S_l.$$

As a result the ideal of $R\{x\}$ generated by S has the form

$$S\{x\} = \bigoplus_{l=1}^{t} \bigl(S_l \oplus S_l x \oplus \dots \oplus S_l x^{p^{i_l}-1}\bigr). \tag{1}$$

From the construction it follows immediately that $S\{x\} \in \mathfrak{F}_{R\{x\}}$. We are now ready to prove

Lemma 1.6. *If $pR = 0$ and $J(R) = 0$, then $J(R\{x\}) = 0$.*

Proof. Notice that $S_l\{x\}J(R\{x\})$ is a quasi-invertible ideal contained in $S_l\{x\}$, for $l = 1, \dots, t$. Suppose $J(R\{x\}) \neq 0$. Then $S_l\{x\}J(R\{x\}) \neq 0$ for some l. Hence $J(S_l\{x\}) \neq 0$. Since every nonzero ideal of $R\{x\}$ has a nonzero intersection with R, $J(S_l\{x\}) \cap R$ must be nonzero. We claim that $J(S_l\{x\}) \cap S_l \neq 0$. Indeed, if $a = a_0 + a_1x + \dots + a_kx^k \in R$, where $a_j \in S_l$, then S_la must be nonzero. Otherwise, the decomposition $S_l\{x\} = S_l \oplus S_lx \oplus \dots \oplus S_lx^{p^{i_l}-1}$ implies that $S_la_j = 0$ for all j. Thus $S_l \cap \operatorname{ann}^r_R S_l \neq 0$, what is impossible in a semiprime ring. Consequently, $J(S_l\{x\}) \cap S_l \neq 0$. Again using the decomposition of $S_l\{x\}$ into a direct sum of left R-modules, we obtain that $J(S_l\{x\}) \cap S_l$ is a quasi-invertible ideal of S_l. Thus $J(S_l) \neq 0$. Therefore also $J(R) \neq 0$. $\square$

Remark 1.7. In the general case, one can consider the set π_n of prime numbers $p \leqslant n$ such that $R_p = \{r \in R \mid pr = 0\} \neq 0$. It is clear that each R_p is a d-ideal of R. Let $A = \operatorname{ann}^\ell_R(\bigoplus_{p\in\pi_n} R_p)$. Then A is a d-ideal and $A \oplus \bigoplus_{p\in\pi_n} R_p \in \mathfrak{F}_R$. According to Theorem 1.1, d acts on A as an E-inner derivation adjoint to a nilpotent element from $\operatorname{End}(B_A)$, where $B \in \mathfrak{F}_A$. Indeed, from the construction of $R\{x\}$ and the proof of Theorem 1.1 it follows that $B = \operatorname{ann}^\ell_A x \cdot A$ satisfies the required property. Furthermore, for any $p \in \pi_n$ there exists a d-ideal S_p of R, such that $S_p \subseteq R_p$ and $S_p\{x\}$ has the form (1). Now it is easy to see that Lemma 1.6 holds for every semiprimitive ring (without restrictions on characteristic and with the same proof).

We are now able to prove that the Jacobson radical of constants is nilpotent in the case when R is a semiprimitive ring of a prime characteristic.

Theorem 1.8. *Let d be a nilpotent derivation of a semiprimitive ring R such that $pR = 0$, where p is a prime number. Then $J(R^d)^\gamma = 0$, where $\gamma \leqslant 2^n - 1$ and n is the nilpotency index of d.*

Proof. Consider the ring $R\{x\} = R + Rx + \dots + Rx^{m-1}$, d-ideals $S_1, \dots, S_t$ of R and a sequence $0 \leqslant i_1 < \dots < i_t$ such that

(a) $S = S_1 \oplus \dots \oplus S_t \in \mathfrak{F}_R$;
(b) $\operatorname{ann}^\ell_{S_l} x^{p^{i_l}-1} = 0$ for $l = 1, \dots, t$;
(c) $x^{p^{i_l}} S_l \subseteq S_l$ and $S_lx^{p^{i_l}} \subseteq S_l$ for $l = 1, \dots, t$.

Let $J = J(R^d)$. We will prove that for $l = 1, \dots, t$

$$S_lJ^{\gamma_l} = 0 \quad \text{for some } \gamma_l \leqslant 2^{[m/p^{i_l}]} - 1. \tag{2}$$

The case $i_1 = 0$ is covered by Theorem 1.2. Thus it suffices to show that if A is a nonzero d-ideal of R and $k \geqslant 1$ is an integer such that $\operatorname{ann}^{\ell}_A x^{p^{k-1}} = 0$ and $Ax^{p^k} \cup x^{p^k} A \subseteq A$, then

$$Ax^{(q-j)p^k} J^{f(j)} = 0, \tag{3}$$

for $j = 0, 1, \dots, q = [m/p^k]$, where $f(0) = 0$ and $f(j+1) \leqslant 2f(j) + 1$.

Since $Ax^m = 0$, Lemma 1.3 (for $S = A$ and $I = \{1\}$) forces that $Ax^{qp^k} = 0$; so one can put $f(0) = 0$ (We mean here $J^0 = R^d$). Suppose that $j \geqslant 0$ and

$$x^{(q-j)p^k} J^{f(j)} = 0.$$

Let M be a simple $R\{x\}$-module and $M_1 = x^{(q-j)p^k-1} J^{f(j)} M$. Then M_1 is a left R^d-module. Similarly as in Theorem 1.2 we prove that for any proper R^d-submodule $N \subset M_1$, $AJ^{f(j)}N = 0$. If $AJ^{f(j)}N \neq 0$, one can take $a, b, c \in J^{f(j)}$ and $m_1, m_2 \in M$ such that

$$x^{(q-j)p^k-1} am_1 \in M_1 \setminus N, \qquad x^{(q-j)p^k-1} bm_2 \in N$$

and

$$Acx^{(q-j)p^k-1} bm_2 = Ax^{(q-j)p^k-1} cbm_2 \neq 0.$$

For $s \in A$ let

$$T_{a,c}(s) = \sum_{i=0}^{(q-j)p^k-1} ax^{(q-j)p^k-1-i} s x^i c.$$

It is clear that $T_{a,c}(s)$ centralizes x and from computations in [11, Theorem 3.3], it follows that

$$T_{a,c}(s) = \sum_{i=1}^{q-j} (-1)^i \binom{q-j}{i} ax^{(q-j-i)p^k} d^{ip^k-1}(s) c \in R.$$

Thus $T_{a,c}(s) \in R^d$ for any $s \in A$. Since M is simple as a left $R\{x\}$-module, one obtains $AR\{x\}x^{(q-j)p^k-1}cbm_2 = Ax^{(q-j)p^k-1}cbm_2 = M$. Hence there exists $s \in A$ such that $sx^{(q-j)p^k-1}cbm_2 = m_1$. Now it is clear that

$$\begin{aligned} x^{(q-j)p^k-1} am_1 &= x^{(q-j)p^k-1} a\bigl(sx^{(q-j)p^k-1}cbm_2\bigr) \\ &= x^{(q-j)p^k-1} ascx^{(q-j)p^k-1} bm_2 = T_{a,c}(s) x^{(q-j)p^k-1} bm_2 \in N, \end{aligned}$$

a contradiction with assumption that $x^{(q-j)p^k-1}am_1 \notin N$. Therefore $AJ^{f(j)}N = 0$ for any proper submodule $N \subset M_1$ and hence

$$Ax^{(q-j)p^k-1}J^{2f(j)+1}M = 0,$$

for any simple $R\{x\}$-module M. Since $J(R\{x\}) = 0$, we obtain

$$Ax^{(q-j)p^k-1}J^{2f(j)+1} = AJ^{2f(j)+1}x^{(q-j)p^k-1} = 0.$$

Now Lemma 1.3 forces that $AJ^{2f(j)+1}x^{(q-j-1)p^k} = 0$. This finishes the proof of (3). Since $S \in \mathfrak{F}_R$ and by (2) $\gamma_l \leqslant 2^{[m/p^{i_l}]} - 1 \leqslant 2^{[m/p^{i_1}]} - 1$, we obtain

$$SJ^{2^{[m/p^{i_1}]}-1} = \sum_{l=1}^{t} S_l J^{2^{[m/p^{i_1}]}-1} = 0.$$

Therefore $J(R^d)^\gamma = 0$, for some $\gamma \leqslant 2^{[m/p^{i_1}]} - 1 \leqslant 2^m - 1$. $\square$

We can now use Theorems 1.1, 1.2 and 1.8 to obtain Theorem A.

Proof of Theorem A. First suppose that R is a semiprimitive ring, $d^n = 0$ and consider d-extension $R \subset R\{x\}$. According to Remark 1.7, for any $p \in \pi_n$ let $B \subseteq A$, $S_p \subseteq R_p$ be d-ideals of R such that $d_{|A} = \mathrm{ad}_x$, $xB \cup Bx \subseteq B$, and $S_p\{x\}$ is an ideal of $R_p\{x\}$ having decomposition (1).

By Theorem 1.8, applied to the extension $R_p \subset R_p\{x\} \subset R\{x\}$, we obtain $S_pJ(R^d)^\gamma = 0$ for $\gamma = 2^n - 1$. Since from the proof of Theorem 1.2 it follows that $BJ(R^d)^\gamma = 0$, we get that

$$\left(B \oplus \bigoplus_{p\in\pi_n} S_p\right) J\left(R^d\right)^\gamma = 0.$$

However, $B \oplus \bigoplus_{p\in\pi_n} S_p$ has zero right annihilator in R, so $J(R^d)^\gamma = 0$.

If in addition R is a semiprimitive algebra over a field K and d is an algebraic derivation of R, then d acts on the 0-eigenspace $R_0 = \bigcup_{j\geqslant 1} \ker d^j$ as a nilpotent derivation and clearly $R^d = R_0^d \subset R_0$. Furthermore, R_0 is semiprimitive. Indeed, it is easy to see that if K has a prime characteristic p, then there exists a finite separable extension of fields $K \subset F$, such that F contains all eigenvalues of d^{p^k}, for some integer $p^k \geqslant n$. Clearly the kernel of $\delta = d^{p^k}$ coincides with the zero eigenspace of d, so $R_0 \otimes_K F = (R \otimes_K F)^{\delta\otimes 1}$ can be viewed as the identity component of the algebra $R \otimes_K F$ graded by a finite subgroup of F^+ generated by eigenvalues of δ. Since $J(R \otimes_K F) = J(R) \otimes_K F$ (cf. [14, Theorem 2.5.36]) and $J(R_e) = J(R) \cap R_e$ for a ring $R = \oplus \sum_{g\in G} R_g$ graded by a finite group G (cf. [7, Theorem 4.4] or [8, Corollary 2c]) it follows that the algebras $R \otimes_K F$, $R_0 \otimes_K F$, and consequently also R_0 are semiprimitive. If K has characteristic zero we can apply a similar argument. Namely, it suffices to consider a finite field extension $K \subset F$ containing all eigenvalues of d. Then the algebra $R \otimes_K F$ is graded by the subgroup G of the additive group F^+ generated by the eigenvalues of d. Furthermore $R \otimes_K F$ can be viewed as an algebra graded by a finite abelian group (being a homomorphic image of G), with the same set of homogeneous components (cf. [1, Lemma 2.2]). $\square$

2. Rings with semilocal constants of algebraic derivations

In this section we prove that if the subalgebra of constants R^d of an algebraic derivation d of a semiprimitive algebra R is semilocal, then R must be semisimple Artinian. An important role will be played by the following

Theorem 2.1. *Suppose that d is either a nilpotent derivation of a semiprime ring or an algebraic derivation of a semiprime algebra over a field. Then*

(a) [10, Theorem 4] *$I^d = I \cap R^d$ is nonnilpotent for any nonzero d-stable ideal I of R.*
(b) [11, Theorem 3.4] *the prime radical $P(R^d)$ of R^d is nilpotent. More precisely, if $d^n = 0$, then $P(R^d)^\gamma = 0$, where $\gamma \leqslant 2^n - 1$.*

We should point out that from [5] it follows even more; namely $I^d = I \cap R^d$ is not nil of bounded index.

The next lemma is well known. We include a short proof using known properties of group graded rings (cf. [6]).

Lemma 2.2. *Assume that $J(R) = 0$ and $1 \in R$ can be written as $e_1 + e_2 + \cdots + e_n$ where e_i's are orthogonal idempotents. If every e_iRe_i is left Artinian, then R is left Artinian*

Proof. First consider the case $n = 2$. Notice that

$$R = R_0 \oplus R_1, \quad \text{where } R_0 = e_1Re_1 + e_2Re_2 \text{ and } R_1 = e_1Re_2 + e_2Re_1.$$

It is easy to see that $R_0R_0 \cup R_1R_1 \subseteq R_0$, $R_1R_0 \cup R_0R_1 \subseteq R_1$. Thus R is a semiprimitive $\mathbb{Z}_2$-graded ring with the semisimple Artinian identity component R_0. Hence R must be left Artinian by [6]. The result follows by induction on n. $\square$

Recall that a ring R is said to be d-simple, if R does not contain nonzero proper d-ideals.

Lemma 2.3. *If R is semiprimitive and d-simple, then either $R\{x\} = R$ or there exists a positive integer $m = p^k$, where p is a prime number such that $pR = 0$ and*

$$R\{x\} = R \oplus Rx \oplus \cdots \oplus Rx^{m-1} = R \oplus xR \oplus \cdots \oplus x^{m-1}R.$$

In this case

$$R\{x\}^{\mathrm{ad}_x} = R^d \oplus R^d x \oplus \cdots \oplus R^d x^{m-1}$$

is a finite centralizing extension. Furthermore, if R^d is a local ring then $R\{x\}^{\mathrm{ad}_x}$ is also local. If $R\{x\}$ is left Artinian, then R is also left Artinian.

Proof. First part follows immediately form the decomposition (1). Since

$$\mathrm{ad}_x\left(\sum_{i=0}^{m-1} r_i x^i\right) = \sum_{i=0}^{m-1} \mathrm{ad}_x(r_i)x^i = \sum_{i=0}^{m-1} d(r_i)x^i,$$

it is clear that $\mathrm{ad}_x(\sum_{i=0}^{m-1} r_i x^i) = 0$ if and only if $r_i \in R^d$. Suppose that R^d is local with the Jacobson radical $J = J(R^d)$ (we know that J is nilpotent). Since x is nilpotent and central in $R\{x\}^{\mathrm{ad}_x}$, $\hat{J} = J \oplus R^d x \oplus \cdots \oplus R^d x^{m-1}$ is a nilpotent ideal of $R\{x\}^{\mathrm{ad}_x}$, and clearly $R\{x\}^{\mathrm{ad}_x}/\hat{J}$ is isomorphic to R^d/J. Therefore $R\{x\}^{\mathrm{ad}_x}$ is local.

Finally, $L \mapsto R\{x\}L = L \oplus xL \oplus \cdots \oplus x^{m-1}L$ is an order preserving, injective map from the lattice of left ideals of R into the lattice of left ideals of $R\{x\}$. Therefore if $R\{x\}$ is left Artinian, then R also must be left Artinian. □

We are now ready to prove the main result of this section. The above construction allows us to extend [13, Theorem 2.2] to nilpotent derivations with semilocal constants.

Theorem 2.4. *Let d be a nilpotent derivation of a semiprimitive ring R. If the ring of constants R^d is semilocal, then R is semisimple Artinian.*

Proof. Suppose that R^d is semilocal. Since $J(R^d)$ is nilpotent, the ring R^d is semiperfect. By [14, Proposition 2.7.20] there exist orthogonal idempotents $e_1, \ldots, e_l$ of R^d, such that $e_1 + \cdots + e_l = 1$ and each $e_i R^d e_i$ is a local ring. It is clear that $(e_i R e_i)^d = e_i (e_i R e_i)^d e_i$, so $(e_i R e_i)^d = e_i R^d e_i$.

First, we examine a special case where R is semiprimitive and R^d is local. In this situation, R has to be d-simple. Indeed, if I is a nonzero proper d-stable ideal of R, then $I^d = R^d \cap I$ is a proper ideal of a local ring R^d. By previous section the Jacobson radical of R^d is nilpotent. Thus $I^d = R^d \cap I$ is nilpotent, but this is impossible in light of Theorem 2.1(a). Consequently, R must be d-simple. Notice that if $R\{x\} = R$, then d is inner and hence R has to be simple. On the other hand, if the second case of Lemma 2.3 is satisfied, then since $R\{x\}^{\mathrm{ad}_x}$ is local, the ring $R\{x\}$ is ad_x-simple and consequently simple. Recapitulating, we may assume that R is simple, d is inner, $d = \mathrm{ad}_x$, where x is a nilpotent element of R.

Suppose $x^m = 0$ and $x^{m-1} \neq 0$. Let M be a simple left R-module. Since R is a simple ring with 1, M is faithful, that is $\mathrm{ann}_R M = 0$. Consider the division ring $\Delta = (\mathrm{End}_R M)^{\mathrm{op}}$; i.e., multiplication in Δ is the composition of maps in the reverse order. It is clear that the map $\lambda_x: M \to M$, given by $\lambda_x(m) = xm$ is Δ-linear, i.e., λ_x is a nilpotent endomorphism of the right vector space M_Δ. We claim that $\dim_\Delta \ker \lambda_x = 1$. If not, since $0 \neq x^{m-1}M \subseteq \ker \lambda_x$, we can find two linearly independent (over Δ)

elements $x^{m-1}u, v \in \ker \lambda_x$. Notice that the elements $u, xu, x^2u, \dots, x^{m-1}u, v \in M$ are also Δ-independent. To this end, suppose that $\varphi_0, \varphi_1, \dots, \varphi_{m-1}, \varphi \in \Delta$ satisfy

$$\sum_{i=0}^{m-1} (x^i u)\varphi_i + v\varphi = 0 \quad \text{that is} \quad \sum_{i=0}^{m-1} x^i(u\varphi_i) + v\varphi = 0.$$

If we assume that $\varphi_i = 0$ for all $i < k$ (where $0 \leqslant k < m-1$), then multiplying the above equality on the left by x^{m-1-k} yields $x^{m-1}(u\varphi_k) = (x^{m-1}u)\varphi_k = 0$. Since $x^{m-1}u \neq 0$, $\varphi_k = 0$. At the end we get $(x^{m-1}u)\varphi_{m-1} + v\varphi = 0$; thus $\varphi_{m-1} = \varphi = 0$, according to our assumption that $x^{m-1}u$ and v are linearly independent over Δ. Therefore the Jacobson Density Theorem implies that there exists $r \in R$ such that

$$rv = ru = rxu = \cdots = rx^{m-2}u = 0, \qquad rx^{m-1}u = u.$$

Observe that the element $T(r) = \sum_{i=0}^{m-1} x^i r x^{m-1-i} \in R^d$ satisfies

$$T(r)u = rx^{m-1}u = u \quad \text{and} \quad T(r)v = 0.$$

Thus $(1 - T(r))u = 0 = T(r)v$, so $T(r)$ and $1 - T(r)$ are not invertible in a local ring R^d. The contradiction, just obtained, shows that the kernel of λ_x is one dimensional, as claimed. Notice that there is a sequence

$$M \xrightarrow{\lambda_x} xM \xrightarrow{\lambda_x} x^2M \xrightarrow{\lambda_x} \cdots \xrightarrow{\lambda_x} x^{m-1}M = \ker \lambda_x,$$

of Δ-linear surjections, so $\dim M_\Delta \leqslant m < \infty$. On the other hand if $0 \neq x^{m-1}u \in \ker \lambda_x$, then the elements $u, xu, \dots, x^{m-1}u$ are linearly independent over Δ, so $\dim M_\Delta = m$. Therefore R is isomorphic to the ring $M_m(\Delta)$ of all $m \times m$ matrices over the division ring Δ.

Returning to the general case, we can conclude that R contains orthogonal idempotents $e_1, \dots, e_l \in R^d$ such that $e_1 + \cdots + e_l = 1$ and each $e_i R e_i$ is simple Artinian. In light of Lemma 2.2, the ring R is semisimple Artinian. □

We can now prove our second main result.

Proof of Theorem B. Let R be a semiprimitive algebra over a field K and d be an algebraic derivation of R. Suppose that the subalgebra of constants R^d is semilocal. By Theorem 2.4 the 0-eigenspace R_0 of d is left Artinian and [12, Proposition 2.4] shows that R is left Artinian. This proves the implication (1) ⇒ (2). The equivalency (2) ⇔ (3) follows directly from [11, Theorem 4.7]. The part (3) ⇒ (1) is obvious. □

Theorem B can be strengthened if we assume some natural restrictions on characteristic.

Corollary 2.5. *Suppose that either R is a semiprime $n!$-torsion free ring with a nilpotent derivation ($d^n = 0$) or R is a semiprime algebra over a field K with a derivation d satisfying the identity*

$$\alpha_0 d^n + \alpha_1 d^{n+1} + \cdots + \alpha_k d^{n+k} = 0,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k \in K$, $\alpha_0 \neq 0$ and the characteristic $\operatorname{char} K$ of K does not divide $n!$. The following conditions are equivalent:

(1) *$R^d/P(R^d)$ is left Artinian.*
(2) *R^d is semilocal and R is semiprimitive.*
(3) *R is semisimple Artinian.*
(4) *R^d is left and right Artinian.*

Proof. In light of Theorem B it suffices to show that (1) implies (2). Our assumptions guarantee that the Jacobson radical $J(R)$ is d-stable (cf. [2, Theorem 7]). By Theorem 2.1(a) if $J(R) \neq 0$, then $J(R)^d = J(R) \cap R^d$ is a nonnilpotent ideal of R^d. Notice that the ideal $J(R)^d$ is quasi-invertible, so $J(R)^d \subseteq J(R^d)$. Thus $J(R^d)$ is also nonnilpotent. By assumption $R^d/P(R^d)$ is left Artinian, so $R^d/P(R^d)$ is semisimple Artinian, and consequently $P(R^d) = J(R^d)$. However, by Theorem 2.1(b) the prime radical of R^d is nilpotent. Therefore $J(R) = 0$, $J(R^d) = P(R^d)$ and R^d is semilocal. □

We conclude the paper with some remark concerning Goldie and dual Goldie dimensions.

Remark 2.6. Recall that the dual Goldie dimension of an R-module M is defined as the supremum of positive integers k such that M can be mapped homomorphically onto the product of k nonzero R-modules. Sarath and Varadarajan proved in [15] that a ring with identity R has finite dual Goldie dimension as a left module over itself if and only if R is semilocal. From this point of view Theorem B and Corollary 2.5 say that if R is either semiprimitive or semiprime with some restriction on characteristic, then R^d has finite dual Goldie dimension if and only if R has finite dual Goldie dimension. We should also point out that there is a strong relationship between the Goldie dimensions of R^d and R when R is semiprime and d is algebraic. In particular, it is proved in [12] that if R is semiprime and $d^n = 0$, then $\dim R^d \leqslant \dim R \leqslant n \dim R^d$.

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