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## Frobenius algebras of corepresentations and group-graded vector spaces



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### ABSTRACT

We consider Frobenius algebras in the monoidal category of right comodules over a Hopf algebra  $H$ . If  $H$  is a group Hopf algebra, we study a more general Frobenius type property, uncover the structure of graded Frobenius algebras, and investigate graded symmetric algebras. The graded Frobenius concept is related to Frobenius functors.

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## 1. Introduction and preliminaries

Frobenius algebras originate in the work of F.G. Frobenius on representation theory of finite groups. Their study was initiated by Brauer, Nesbitt and Nakayama, and then continued by Dieudonné, Eilenberg, Azumaya, Thrall, Kasch, etc., uncovering a rich representation theory. Also, Frobenius algebras have played a role in Hopf algebra theory (any finite dimensional Hopf algebra is Frobenius), cohomology rings of compact oriented manifolds, orbifold theories, solutions of the quantum Yang–Baxter equation, Jones polynomials, topological quantum field theory; see for instance [12,10]. A finite dimensional algebra  $A$  over a field  $k$  is Frobenius if  $A \simeq A^*$  as left  $A$ -modules. Abrams proved in [1,2] that  $A$  is Frobenius if and only if there is a coalgebra structure  $(A, \delta, \varepsilon_A)$  on the space  $A$  such that  $\delta$  is a morphism of  $A$ ,  $A$ -bimodules. Since Abrams' characterization makes sense in any monoidal category, this provided the right way to define the concept of Frobenius algebra in such a framework. The study of Frobenius algebras in monoidal categories was initiated and developed by Müger [14], Street [17], Fuchs and Stigner [8], Yamagami [18]. It is a challenging problem to understand the significance of Frobenius algebras in monoidal categories other than categories of vector spaces. Müger investigated in [14] Frobenius algebras arising in category theory and subfactor theory. Fuchs, Schweigert and Stigner [9] considered certain Frobenius algebras in the monoidal category of  $H$ -bimodules, where  $H$  is a finite dimensional factorizable ribbon Hopf algebra. In Section 2 we consider the monoidal category  $\mathcal{M}^H$  of right comodules (i.e. corepresentations) over a Hopf algebra  $H$ . We do not require that the antipode  $S$  is bijective, thus  $\mathcal{M}^H$  has left duals, but not necessarily right duals. If  $A$  is a finite dimensional algebra in this category, i.e. a right  $H$ -comodule algebra, then both  $A$  and its dual space  $A^*$  have natural structures of objects in the category  $\mathcal{M}_A^H$  of right Doi–Hopf modules. We say that  $A$  is right  $H$ -Frobenius if these two objects are isomorphic. On the other hand,  $A^*$  has a natural structure of a left Doi–Hopf module over the right  $H$ -comodule algebra  $A^{(S^2)}$ , which is just  $A$  as an algebra, and has the  $H$ -coaction shifted by  $S^2$ . We say that  $A$  is left  $H$ -Frobenius if  $A^*$  and  $A^{(S^2)}$  are isomorphic as such left Doi–Hopf modules. We give equivalent characterizations of these two Frobenius properties. In fact, the right  $H$ -Frobenius property is not new, it is equivalent to  $A$  being a Frobenius algebra in the category  $\mathcal{M}^H$ . The left  $H$ -Frobenius property seems to be specific to the category of  $H$ -comodules, there is no obvious such concept in an arbitrary monoidal category. We prove that  $A$  is left  $H$ -Frobenius if and only if  $A^{(S^2)}$  is a Frobenius algebra in the monoidal category  $\mathcal{M}^H$ . Also, we show that if  $A$  is right  $H$ -Frobenius, then  $A$  is left  $H$ -Frobenius; if  $S$  is injective, the converse also holds.

In the rest of the paper we specialize the study of Frobenius algebras in  $\mathcal{M}^H$  to the case where  $H$  is a group Hopf algebra  $kG$  of an arbitrary group  $G$ . In this case the corepresentations of  $H$  are the  $G$ -graded vector spaces, and an algebra in this category is just a  $G$ -graded algebra. A  $G$ -graded algebra  $A$  is called graded Frobenius if it is right (or equivalently left)  $kG$ -Frobenius. In fact we define a more general concept in Section 3. For  $\sigma \in G$ , we say that  $A$  is  $\sigma$ -graded Frobenius if  $A^*$  is isomorphic to the suspension

$A(\sigma)$  of  $A$  in the category  $A\text{-gr}$  of graded left  $A$ -modules. We study basic properties of  $\sigma$ -graded Frobenius algebras and give several characterizations of them. One of our main results says that a finite dimensional  $G$ -graded algebra  $A$  is  $\sigma$ -graded Frobenius if and only if  $A_\sigma \simeq A_e^*$  as left  $A_e$ -modules, and  $A$  is left  $\sigma$ -faithful. Here  $A_\sigma$  denotes the homogeneous component of degree  $\sigma$  of  $A$ , and  $e$  is the neutral element of  $G$ . In particular, this gives the structure of Frobenius algebras in the category of graded vector spaces. These are the finite dimensional  $G$ -graded algebras  $A$  which are left  $e$ -faithful and such that  $A_e$  is a Frobenius algebra. In particular, graded semisimple algebras are graded Frobenius. We give examples (in Section 6) of Frobenius algebras which are not  $\sigma$ -graded Frobenius for any  $\sigma$ . However, if  $A_e$  is a local ring, and  $A$  is Frobenius, we show that  $A$  is  $\sigma$ -graded Frobenius for some  $\sigma$ .

Among graded Frobenius algebras there are some objects with more symmetry: the graded symmetric algebras, which are just the symmetric algebras in the sovereign category of graded vector spaces, see [8]. In Section 5 we show that a graded division algebra  $\Delta$  is graded symmetric, provided that the center of  $\Delta_e$  consists of the central elements of  $\Delta$  of degree  $e$ . In Section 6 we give several examples to illustrate the concepts of  $\sigma$ -graded Frobenius, graded symmetric, Frobenius and symmetric, and connections between them. In particular we show that any matrix algebra (over a field), endowed with a grading such that the matrix units are homogeneous elements, is graded symmetric. Using the description of abelian gradings on matrix algebras over an algebraically closed field, given by Bahturin, Sehgal and Zaicev [3], we show that any such grading is graded symmetric.

Finally, in Section 7 we discuss the concept of graded Frobenius in relation to Frobenius functors. The property of being a Frobenius algebra is not Morita invariant. However a matrix algebra  $M_n(A)$  is Frobenius whenever  $A$  is so. We show that the converse also holds: if  $M_n(A)$  is Frobenius, then so is  $A$ . We characterize graded Frobenius algebras as those finite dimensional graded algebras  $A$  for which the functor  $U : A\text{-gr} \rightarrow k\text{-gr}$ , forgetting the  $A$ -action, is a Frobenius functor. We connect the graded Frobenius property for  $A$  and the Frobenius property for the functor  $(-)_e : A\text{-gr} \rightarrow A_e\text{-mod}$ . Finally, we give a new proof for a result of Bergen [4] stating that if  $H$  is a finite dimensional Hopf algebra acting on the finite dimensional algebra  $A$ , then the smash product  $A\#H$  is Frobenius if and only if so is  $A$ .

For definitions and notation we refer to [7,15] for coalgebras and Hopf algebras, and to [16] for graded algebras. All algebras, coalgebras and graded algebras are considered over a field  $k$ .

## 2. Frobenius algebras in categories of comodules

Let  $C$  be a coalgebra, and let  $M \in \mathcal{M}^C$  be a finite dimensional comodule with comodule structure map  $\rho : M \rightarrow M \otimes C$ . Then  $M$  is a left  $C^*$ -module in the standard way, and then  $M^*$  is a right  $C^*$ -module. By [7, Lemma 2.2.12] this is a rational right  $C^*$ -module, i.e. its module structure comes from a left  $C$ -comodule structure. Since we

need the details of the proof, we indicate how this comes out. Consider the natural isomorphism  $\gamma : M \otimes C \otimes M^* \rightarrow \text{Hom}(M, M \otimes C)$ ,  $\gamma(m \otimes c \otimes m^*)(x) = m^*(x)m \otimes c$  for any  $m, x \in M$ ,  $m^* \in M^*$ ,  $c \in C$ , and look at the inverse image of  $\rho$ , i.e.  $\rho = \gamma(\sum_i m_i \otimes c_i \otimes m_i^*)$ . Then  $\rho(m) = \sum_i m_i^*(m)m_i \otimes c_i$  for any  $m \in M$ , and a direct computation shows that  $m^*c^* = \sum_i c^*(c_i)m^*(m_i)m_i^*$  for any  $m^* \in M^*$  and  $c^* \in C^*$ , i.e. the right  $C^*$ -module structure of  $M^*$  comes from a left  $C$ -comodule structure given by  $\mu : M^* \rightarrow C \otimes M^*$ ,  $\mu(m^*) = \sum_i m^*(m_i)c_i \otimes m_i^*$ .

Now let  $H$  be a Hopf algebra and let  $A$  be a right  $H$ -comodule algebra with  $H$ -comodule structure denoted by  $a \mapsto \sum a_0 \otimes a_1$ . Let  $M \in {}_A\mathcal{M}^H$  be a finite dimensional left  $(A, H)$ -Doi–Hopf module, i.e. it is a left  $A$ -module and a right  $H$ -comodule with  $\rho : M \rightarrow M \otimes H$ ,  $\rho(m) = \sum m_0 \otimes m_1$ , such that  $\rho(am) = \sum a_0 m_0 \otimes a_1 m_1$  for any  $a \in A$ ,  $m \in M$ . As above, there exists  $\sum_i m_i \otimes h_i \otimes m_i^* \in M \otimes H \otimes M^*$  such that  $\rho(m) = \sum_i m_i^*(m)m_i \otimes h_i$  for any  $m \in M$ , and  $M^*$  is a left  $H$ -comodule by  $m^* \mapsto \sum_i m^*(m_i)h_i \otimes m_i^*$ . Using the antipode  $S$  of  $H$ ,  $M^*$  becomes a right  $H$ -comodule via  $\nu : M^* \rightarrow M^* \otimes H$ ,  $\nu(m^*) = \sum m_{(0)}^* \otimes m_{(1)}^* = \sum_i m^*(m_i)m_i^* \otimes S(h_i)$ . Note that since  $M$  is a left  $(A, H)$ -Doi–Hopf module, we have that

$$\sum_i m_i^*(am)m_i \otimes h_i = \sum_i m_i^*(m)a_0 m_i \otimes a_1 h_i \quad (1)$$

for any  $a \in A$  and  $m \in M$ .

**Proposition 2.1.** *If  $M \in {}_A\mathcal{M}^H$  is finite dimensional, then  $M^* \in \mathcal{M}_{A^*}^H$ , i.e.  $M^*$  is a right  $(A, H)$ -Doi–Hopf module with the right  $H$ -comodule structure  $\nu$ .*

**Proof.** We have to show that  $\nu(m^*a) = \sum m_{(0)}^* a_0 \otimes m_{(1)}^* a_1$ . Taking into account the comodule structure formula, this is equivalent to

$$\sum_i m^*(am_i)m_i^* \otimes S(h_i) = \sum_i m^*(m_i)(m_i^* a_0) \otimes S(h_i)a_1 \quad (2)$$

Since  $M^* \otimes H \simeq \text{Hom}(M, H)$ , Eq. (2) is equivalent to the fact that for any  $m \in M$  we have that

$$\sum_i m^*(am_i)m_i^*(m)S(h_i) = \sum_i m^*(m_i)(m_i^* a_0)(m)S(h_i)a_1 \quad (3)$$

But we have that

$$\begin{aligned} \sum_i m^*(m_i)(m_i^* a_0)(m)S(h_i)a_1 &= \sum_i m^*(m_i)m_i^*(a_0 m)S(h_i)a_1 \\ &= \sum_i m^*(a_0 m_i)m_i^*(m)S(a_1 h_i)a_2 \quad (\text{by (1) for } a_0 \text{ and } m) \\ &= \sum_i m^*(am_i)m_i^*(m)S(h_i) \end{aligned}$$

Thus (3) is satisfied, and this ends the proof.  $\square$

If  $A$  is a right  $H$ -comodule algebra, we denote by  $\cdot$  the left  $H^*$ -action on  $A$  associated to the right  $H$ -coaction. We define a new object  $A^{(S^2)}$  as being just  $A$  as an algebra, and with a right  $H$ -coaction  $a \mapsto \sum a_0 \otimes S^2(a_1)$ . Then  $A^{(S^2)}$  is a right  $H$ -comodule algebra. We denote by  $*$  the associated left  $H^*$ -action on  $A$ . In a way similar to Proposition 2.1, one can see that if  $M \in \mathcal{M}_A^H$ , a right Doi–Hopf module, is finite dimensional, then  $M^* \in {}_{A^{(S^2)}}\mathcal{M}^H$ , where the  $H$ -coaction on  $M^*$  is given by the same formula as above for left Doi–Hopf modules.

Now assume that  $A$  is a finite dimensional right  $H$ -comodule algebra. As a particular case of the discussion above, we see that there exists  $\sum_i a_i \otimes h_i \otimes a_i^* \in A \otimes H \otimes A^*$  such that  $\sum a_0 \otimes a_1 = \sum_i a_i^*(a) a_i \otimes h_i$  for any  $a \in A$ . Then  $A^* \in \mathcal{M}^H$  with coaction  $a^* \mapsto \sum a_0^* \otimes a_1^* = \sum_i a_i^*(a_i) a_i^* \otimes S(h_i)$ . We keep this notation for the rest of the section.

We can regard  $A$  as a left  $(A, H)$ -Doi–Hopf module, and also as a right  $(A, H)$ -Doi–Hopf module. The first structure induces a structure of a right  $(A, H)$ -Doi–Hopf module on  $A^*$ , while the second one induces a structure of a left  $(A^{(S^2)}, H)$ -Doi–Hopf module on  $A^*$ .

**Definition 2.2.** The finite dimensional right  $H$ -comodule algebra  $A$  is called

- left  $H$ -Frobenius if  $A^{(S^2)} \simeq A^*$  in  ${}_{A^{(S^2)}}\mathcal{M}^H$ .
- right  $H$ -Frobenius if  $A \simeq A^*$  in  $\mathcal{M}_A^H$ .

The following characterizes the left  $H$ -Frobenius property. The first four equivalent conditions are in the spirit of the classical ones for Frobenius algebras, taking also care of the  $H$ -coaction. The last condition shows the connection to the concept of Frobenius algebra in the category of corepresentations of  $H$ .

**Theorem 2.3.** Let  $A$  be a finite dimensional right  $H$ -comodule algebra. The following assertions are equivalent.

- (1)  $A$  is left  $H$ -Frobenius.
- (2) There exists a non-degenerate associative bilinear form  $B : A \times A \rightarrow k$  with the property that  $B(b, (h^* S^2) \cdot a) = B((h^* S) \cdot b, a)$  for any  $a, b \in A$  and any  $h^* \in H^*$ .
- (3)  $A$  has a hyperplane  $\mathcal{H}$  which does not contain any non-zero left ideal of  $A$ , and  $(h^* S^2) \cdot A \subseteq \mathcal{H}$  for any  $h^* \in H^*$  with  $h^*(1) = 0$ .
- (4)  $A$  has a hyperplane  $\mathcal{H}$  which does not contain any non-zero subobject of  $A^{(S^2)}$  in  ${}_{A^{(S^2)}}\mathcal{M}^H$ , and  $(h^* S^2) \cdot A \subseteq \mathcal{H}$  for any  $h^* \in H^*$  with  $h^*(1) = 0$ .
- (5)  $A^{(S^2)}$  is a Frobenius algebra in the monoidal category  $\mathcal{M}^H$ .

**Proof.** If  $\theta : A \rightarrow A^*$  is a linear map, let  $B : A \times A \rightarrow k$  be the bilinear map defined by  $B(a, b) = \theta(b)(a)$  for any  $a, b \in A$ . We have that  $\theta(h^* * a)(b) = B(b, h^* * a) = B(b, (h^* S^2) \cdot a)$ , and

$$\begin{aligned}
(h^*\theta(a))(b) &= \sum_i h^*(S(h_i))\theta(a)(a_i)a_i^*(b) \\
&= \sum_i B(a_i, a)a_i^*(b)h^*(S(h_i)) \\
&= B\left(\sum_i a_i^*(b)h^*(S(h_i))a_i, a\right) \\
&= B((h^*S) \cdot b, a)
\end{aligned}$$

This shows that  $\theta$  is a morphism of left  $H^*$ -modules, or equivalently, of right  $H$ -comodules, if and only if  $B(b, (h^*S^2) \cdot a) = B((h^*S) \cdot b, a)$  for any  $a, b \in A$  and any  $h^* \in H^*$ . By the proof of [12, Theorem 3.15],  $\theta$  is an isomorphism of left  $A$ -modules if and only if  $B$  is associative and non-degenerate. Now it is clear that (1) and (2) are equivalent.

Now we show that (2)  $\Rightarrow$  (3). Let  $\mathcal{H} = \{a \in A \mid B(1, a) = 0\}$ . If  $I$  is a left ideal of  $A$  such that  $I \subseteq \mathcal{H}$ , then for any  $a \in I$  and  $x \in A$  we have that  $B(x, a) = B(1, xa) = 0$ , showing that  $a = 0$ . Thus  $I$  must be 0. On the other hand, if  $a \in A$  and  $h^* \in H^*$ , one has

$$\begin{aligned}
B(1, (h^*S^2) \cdot a - h^*(1)a) &= B(1, (h^*S^2) \cdot a) - h^*(1)B(1, a) \\
&= B((h^*S) \cdot 1, a) - h^*(1)B(1, a) \\
&= B((h^*S)(1)1, a) - h^*(1)B(1, a) \\
&= 0
\end{aligned}$$

so (3) holds.

(3)  $\Rightarrow$  (4) is obvious.

(4)  $\Rightarrow$  (1) Since for an arbitrary  $h^* \in H^*$  we have that  $(h^*S^2 - h^*(1)\varepsilon)(1) = 0$ , the last condition in (3) is equivalent to  $(h^*S^2) \cdot a - h^*(1)a \in \mathcal{H}$  for any  $a \in A$  and  $h^* \in H^*$ . Let  $\pi : A \rightarrow k$  be a linear map whose kernel is  $\mathcal{H}$ . Define  $\theta : A^{(S^2)} \rightarrow A^*$  by  $\theta(b)(a) = \pi(ab)$  for any  $a, b \in A$ , and let  $B(a, b) = \theta(b)(a)$  be the associated bilinear form. Then  $\theta(ay)(x) = \pi(xay) = \theta(y)(xa) = (a\theta(y))(x)$ , so  $\theta$  is a morphism of left  $A$ -modules.

On the other hand, if  $h^* \in H^*$  and  $a, b \in A$ , we have

$$\begin{aligned}
&((h^*S) \cdot b)a - b((h^*S^2) \cdot a) \\
&= \sum (h^*S)(b_1)b_0a - b((h^*S^2) \cdot a) \\
&= \sum (h^*S)(b_1a_1S(a_2))b_0a_0 - b((h^*S^2) \cdot a) \\
&= \sum (S(a_1) \rightharpoonup (h^*S)) \cdot (ba_0) - b((h^*S^2) \cdot a) \\
&= \sum ((S(a_1) \rightharpoonup (h^*S)) \cdot (ba_0) - (S(a_1) \rightharpoonup (h^*S))(1)ba_0)
\end{aligned}$$

$$\begin{aligned}
& + \sum (S(a_1) \rightharpoonup (h^*S))(1)ba_0 - b((h^*S^2) \cdot a) \\
& = \sum ((S(a_1) \rightharpoonup (h^*S)) \cdot (ba_0) - (S(a_1) \rightharpoonup (h^*S))(1)ba_0) \\
& \quad + \sum (h^*S^2)(a_1)ba_0 - \sum (h^*S^2)(a_1)ba_0 \\
& = \sum ((S(a_1) \rightharpoonup (h^*S)) \cdot (ba_0) - (S(a_1) \rightharpoonup (h^*S))(1)ba_0) \in \mathcal{H}
\end{aligned}$$

This shows that  $\pi(((h^*S) \cdot b)a) = \pi(b((h^*S^2) \cdot a))$ , which means that  $\theta$  is  $H^*$ -linear, or equivalently, right  $H$ -colinear. On the other hand  $I = \text{Ker}(\theta)$ , which is a subobject of  $A^{(S^2)}$  in  ${}_{A^{(S^2)}}\mathcal{M}^H$ , so it must be zero, showing that  $\theta$  is injective, hence it is an isomorphism.

(1)  $\Rightarrow$  (5) Let  $\theta : A^{(S^2)} \rightarrow A^*$  be an isomorphism in  ${}_{A^{(S^2)}}\mathcal{M}^H$ . In particular  $\theta$  is an isomorphism of left  $A$ -modules, so  $A$  is a Frobenius algebra. By [1,2],  $A$  is a Frobenius algebra in  $k\text{-mod}$ . Using the proof in the cited references,  $A$  has a coalgebra structure with comultiplication  $\delta = (\theta^{-1} \otimes \theta^{-1})\Delta^{cop}\theta$ , where  $\Delta^{cop}$  is the comultiplication co-opposite to the comultiplication  $\Delta$  of  $A^*$  induced by the multiplication of  $A$ , and counit  $\varepsilon_A = \theta(1_A)$ . Moreover,  $\delta$  is a morphism of  $A, A$ -bimodules.

We show that  $\delta$  and  $\varepsilon$  are morphisms in  $\mathcal{M}^H$ , and thus  $A^{(S^2)}$  is a Frobenius algebra in the category  $\mathcal{M}^H$ . Since  $\theta$  is an isomorphism in  $\mathcal{M}^H$ , in order to show that  $\delta$  is a morphism in  $\mathcal{M}^H$ , it is enough to show that  $\Delta^{cop}$  is so. As above, there exists  $\sum_i a_i \otimes h_i \otimes a_i^* \in A \otimes H \otimes A^*$  such that  $\sum a_0 \otimes a_1 = \sum_i a_i^*(a) a_i \otimes h_i$  for any  $a \in A$ . Then  $A^* \in \mathcal{M}^H$  with coaction  $a^* \mapsto \nu_1(a^*) = \sum a_0^* \otimes a_1^* = \sum_i a^*(a_i) a_i^* \otimes S(h_i)$ . Let  $\nu_2 : A^* \otimes A^* \rightarrow A^* \otimes A^* \otimes H$  be the induced coaction. Then

$$(\Delta^{cop} \otimes I)\nu_1(a^*) = \sum_i a^*(a_i)(a_i^*)_2 \otimes (a_i^*)_1 \otimes S(h_i)$$

and

$$\begin{aligned}
(\nu_2 \Delta^{cop})(a^*) &= \sum (a_2^*)_0 \otimes (a_1^*)_0 \otimes (a_2^*)_1 (a_1^*)_1 \\
&= \sum_{i,j} a_2^*(a_j) a_1^*(a_i) a_j^* \otimes a_i^* \otimes S(h_j) S(h_i) \\
&= \sum_{i,j} a^*(a_i a_j) a_j^* \otimes a_i^* \otimes S(h_i h_j)
\end{aligned}$$

If we use the natural isomorphism  $A^* \otimes A^* \otimes H \simeq \text{Hom}(A \otimes A, H)$ , we see that showing that  $(\Delta^{cop} \otimes I)\nu_1(a^*) = (\nu_2 \Delta^{cop})(a^*)$  is equivalent to showing that for any  $a, b \in A$  we have

$$\sum_i a^*(a_i)(a_i^*)_2(a)(a_i^*)_1(b)S(h_i) = \sum_{i,j} a^*(a_i a_j) a_j^*(a) a_i^*(b) S(h_i h_j) \quad (4)$$

Since  $\sum (ba)_0 \otimes (ba)_1 = \sum b_0 a_0 \otimes b_1 a_1$ , we have

$$\sum_i a_i^*(ba)a_i \otimes h_i = \sum_{i,j} a_i^*(b)a_j^*(a)a_i a_j \otimes h_i h_j \quad (5)$$

Now

$$\begin{aligned} \sum_{i,j} a^*(a_i a_j) a_j^*(a) a_i^*(b) S(h_i h_j) &= \sum_i a_i^*(ba) a^*(a_i) S(h_i) \quad (\text{by (5)}) \\ &= \sum_i a^*(a_i) (a_i^*)_1(b) (a_i^*)_2(a) S(h_i) \end{aligned}$$

which shows that (4) holds.

Now since  $\theta$  is  $H$ -colinear, we see that

$$\sum_i \theta(a)(a_i) a_i^* \otimes S(h_i) = \sum_i a_i^*(a) \theta(a_i) \otimes S^2(h_i)$$

for any  $a \in A$ . This implies that

$$\sum_i \theta(a)(a_i) a_i^*(1_A) S(h_i) = \sum_i a_i^*(a) \theta(a_i)(1_A) S^2(h_i) \quad (6)$$

Since  $\sum_i a_i^*(1) a_i \otimes h_i = 1_A \otimes 1_H$ , the left hand side in (6) equals  $\theta(a)(1_A) S(1_H) = \theta(a)(1_A) 1_H$ . Since  $\theta(x)(y) = (x\theta(1_A))(y) = \theta(1_A)(yx)$ , we obtain from (6) that

$$\theta(1_A)(a) 1_H = \sum_i a_i^*(a) \theta(1_A)(a_i) S^2(h_i)$$

which means that  $\varepsilon_A$  is  $H$ -colinear.

(5)  $\Rightarrow$  (3) Since  $A^{(S^2)}$  is Frobenius in  $\mathcal{M}^H$ , it has a coalgebra structure with comultiplication a morphism of  $A$ -bimodules. Let  $\varepsilon_{A^{(S^2)}} : A^{(S^2)} \rightarrow k$  be the counit of this coalgebra structure, which is a morphism in  $\mathcal{M}^H$ . Let  $\mathcal{H} = \text{Ker}(\varepsilon_{A^{(S^2)}})$ , a hyperplane of  $A$ . By [1,2],  $A$  is a Frobenius algebra with a non-degenerate associative bilinear form  $B : A \times A \rightarrow k$ ,  $B(b, a) = \varepsilon_{A^{(S^2)}}(ba)$ . Then  $\mathcal{H}$  does not contain non-zero left ideals of  $A$ , since  $Aa \subseteq \mathcal{H}$  implies that  $B(A, a) = 0$ , so then  $a = 0$ . On the other hand

$$\begin{aligned} \varepsilon_{A^{(S^2)}}((h^* S^2) \cdot a) &= \varepsilon_{A^{(S^2)}}\left(\sum (h^* S^2)(a_1) a_0\right) \\ &= (h^* S^2)\left(\sum \varepsilon_{A^{(S^2)}}(a_0) a_1\right) \\ &= (h^* S^2)(\varepsilon_{A^{(S^2)}}(a) 1) \\ &= (h^* S^2)(1) \varepsilon_{A^{(S^2)}}(a) \\ &= h^*(1) \varepsilon_{A^{(S^2)}}(a) \end{aligned}$$

This implies that  $(h^* S^2) \cdot A \subseteq \mathcal{H}$  for any  $h^* \in H^*$  with  $h^*(1) = 0$ .  $\square$



With arguments similar to the ones in the proof of [Theorem 2.3](#), one can prove the following. In fact, this essentially follows from the theory of Frobenius algebras in monoidal categories, see for instance [\[5, Theorem 5.1\]](#). Indeed, any object  $M$  of the category  $\mathcal{M}^H$  has a left dual  $M^*$ . Then a Frobenius algebra  $A$  in the category  $\mathcal{M}^H$ , which is known to have just  $A$  as a left dual, must be isomorphic to the left dual  $A^*$ , and this turns out to be even an isomorphism of right  $A$ -modules.

**Theorem 2.4.** *Let  $A$  be a finite dimensional right  $H$ -comodule algebra. The following assertions are equivalent.*

- (1)  $A$  is right  $H$ -Frobenius.
- (2) There exists a non-degenerate associative bilinear form  $B : A \times A \rightarrow k$  such that  $B(h^* \cdot a, b) = B(a, (h^* S) \cdot b)$  for any  $a, b \in A$  and any  $h^* \in H^*$ .
- (3)  $A$  has a hyperplane  $\mathcal{H}$  which does not contain any non-zero right ideal of  $A$ , and  $h^* \cdot A \subseteq \mathcal{H}$  for any  $h^* \in H^*$  with  $h^*(1) = 0$ .
- (4)  $A$  has a hyperplane  $\mathcal{H}$  which does not contain any non-zero subobject of  $A$  in  $\mathcal{M}_A^H$ , and  $h^* \cdot A \subseteq \mathcal{H}$  for any  $h^* \in H^*$  with  $h^*(1) = 0$ .
- (5)  $A$  is a Frobenius algebra in the monoidal category  $\mathcal{M}^H$ .

The connection between the left–right  $H$ -Frobenius properties is given by the following.

**Theorem 2.5.** *Let  $H$  be a Hopf algebra, and let  $A$  be a finite dimensional right  $H$ -comodule algebra. The following assertions hold.*

- (1) If  $A$  is right  $H$ -Frobenius, then  $A$  is left  $H$ -Frobenius.
- (2) If the antipode  $S$  is injective and  $A$  is left  $H$ -Frobenius, then  $A$  is also right  $H$ -Frobenius.

**Proof.** Let  $\rho : A \rightarrow A \otimes H$  be the  $H$ -comodule structure of  $A$ , and denote by  $\rho_{A \otimes A}$  the  $H$ -comodule structure on  $A \otimes A$ . Then the right  $H$ -comodule structures of  $A^{(S^2)}$ , respectively  $A^{(S^2)} \otimes A^{(S^2)}$ , are given by  $(I \otimes S^2)\rho$ , respectively  $(I \otimes I \otimes S^2)\rho_{A \otimes A}$ . Let  $\delta : A \rightarrow A \otimes A$  be a linear map. Consider the following diagram.

$$\begin{array}{ccccc}
 A & \xrightarrow{\rho} & A \otimes H & \xrightarrow{I \otimes S^2} & A \otimes H \\
 \delta \downarrow & & \downarrow \delta \otimes I & & \downarrow \delta \otimes I \\
 A \otimes A & \xrightarrow{\rho_{A \otimes A}} & A \otimes A \otimes H & \xrightarrow{I \otimes I \otimes S^2} & A \otimes A \otimes H
 \end{array}$$

If  $A$  is right Frobenius, one can choose such a  $\delta$  which is coassociative, a morphism of  $A$ -bimodules and a morphism of right  $H$ -comodules. Then the left square of the diagram is commutative, therefore, since the right square is always commutative, we get that the

big rectangle is commutative, showing that  $\delta$  is also a morphism of right  $H$ -comodules when regarded as  $\delta : A^{(S^2)} \rightarrow A^{(S^2)} \otimes A^{(S^2)}$ . A similar argument works for the counit, and we conclude that if  $A$  has a coalgebra structure in  $\mathcal{M}^H$  such that the comultiplication is a morphism of  $A$ -modules, then  $A^{(S^2)}$  has the same property. Thus (1) holds. For (2), we see that if the big rectangle commutes, then so does the left square, since  $S^2$  is injective, and the rest of the argument is as in (1).  $\square$

### 3. $\sigma$ -Graded Frobenius algebras

Let  $A = \bigoplus_{g \in G} A_g$  be a  $G$ -graded algebra, i.e. a  $k$ -algebra which is a direct sum of the subspaces  $A_g$ , such that  $A_g A_h \subseteq A_{gh}$  for any  $g, h \in G$ . This is equivalent to  $A$  being an algebra in  $\mathcal{M}^{kG}$ , the monoidal category of  $G$ -graded vector spaces. The category  ${}_A \mathcal{M}^{kG}$  of left Doi–Hopf modules is just the category of graded left  $A$ -modules, and we use the standard notation  $A - gr$  for it. The category of graded right  $A$ -modules is denoted by  $gr - A$ . If  $M = \bigoplus_{g \in G} M_g$  is a graded left  $A$ -module of finite support, then the dual space  $M^*$  is a graded right  $A$ -module, with the grading  $M^* = \bigoplus_{g \in G} (M^*)_g$ , where  $(M^*)_g = \{f \in M^* \mid f(M_h) = 0 \text{ for any } h \neq g^{-1}\}$ . In a similar way, the dual space of a graded right  $A$ -module of finite support is a graded left  $A$ -module. In particular, if  $A$  has finite support, then  $A^*$  is a graded left  $A$ -module and a graded right  $A$ -module, i.e. it is a left  $A$ , right  $A$ -graded bimodule.

If  $M = \bigoplus_{g \in G} M_g$  is a graded left  $A$ -module and  $\sigma \in G$ , then the  $\sigma$ -suspension of  $M$  is the graded left  $A$ -module  $M(\sigma)$ , which is just  $M$  as an  $A$ -module, and has the grading defined by  $M(\sigma)_g = M_{g\sigma}$  for any  $g \in G$ .

If  $M = \bigoplus_{g \in G} M_g$  is a graded right  $A$ -module and  $\sigma \in G$ , the  $\sigma$ -suspension of  $M$ , denoted by  $(\sigma)M$ , is the  $A$ -module  $M$  with the grading defined by  $(\sigma)M_g = M_{\sigma g}$ .

#### Lemma 3.1.

- 1) If  $M \in A - gr$  has finite support and  $\sigma \in G$ , then  $M(\sigma)^* = (\sigma^{-1})M^*$  in  $gr - A$ .
- 2) If  $M \in gr - A$  has finite support and  $\sigma \in G$ , then  $(\sigma)M^* = M^*(\sigma^{-1})$  in  $A - gr$ .

**Proof.** Let  $g \in G$ . We have that

$$\begin{aligned} (M(\sigma)^*)_g &= \{f \in M^* \mid f(M(\sigma)_\lambda) = 0 \text{ for any } \lambda \neq g^{-1}\} \\ &= \{f \in M^* \mid f(M_{\lambda\sigma}) = 0 \text{ for any } \lambda \neq g^{-1}\} \\ &= \{f \in M^* \mid f(M_\tau) = 0 \text{ for any } \tau \neq g^{-1}\sigma\} \\ &= (M^*)_{\sigma^{-1}g} \\ &= ((\sigma^{-1})M^*)_g \end{aligned}$$

and (1) follows. (2) is similar.  $\square$

**Lemma 3.2.** *If  $M$  is a finite dimensional left (respectively right) graded  $A$ -module, then  $(M^*)^* \simeq M$  in  $A - gr$  (respectively  $gr - A$ ).*

**Proof.** It is easy to check that the natural isomorphism  $M \simeq M^{**}$  of  $A$ -modules is in fact a graded isomorphism.  $\square$

**Proposition 3.3.** *Let  $A$  be a finite dimensional  $G$ -graded algebra, and let  $\sigma \in G$ . The following conditions are equivalent.*

- (i)  $A(\sigma) \simeq A^*$  in  $A - gr$ .
- (ii)  $(\sigma)A \simeq A^*$  in  $gr - A$ .

**Proof.** (i)  $\Rightarrow$  (ii) Since  $A(\sigma) \simeq A^*$  in  $A - gr$ , by using [Lemmas 3.1 and 3.2](#), we have that  $A \simeq A^{**} \simeq (A(\sigma))^* \simeq (\sigma^{-1})A^*$  in  $gr - A$ . Therefore  $A^* \simeq (\sigma)A$  in  $gr - A$ .

(ii)  $\Rightarrow$  (i) is similar.  $\square$

**Definition 3.4.** A finite dimensional  $G$ -graded algebra  $A$  is called  $\sigma$ -graded Frobenius if it satisfies the equivalent conditions in [Proposition 3.3](#). Clearly,  $e$ -graded Frobenius means just (left, or equivalently right)  $kG$ -Frobenius, and we simply say in this case that  $A$  is graded Frobenius.

**Remark 3.5.** (1) If  $A$  is  $\sigma$ -graded Frobenius for some  $\sigma$ , then obviously  $A$  is Frobenius as an algebra.

(2) If  $A$  is  $\sigma$ -graded Frobenius, and  $H$  is a subgroup of  $G$  such that  $\sigma \in H$ , then  $A_H = \bigoplus_{g \in H} A_g$  is  $\sigma$ -graded Frobenius as an  $H$ -graded algebra. Indeed, this follows from the fact that an isomorphism  $A(\sigma) \simeq A^*$  of  $G$ -graded left  $A$ -modules, induces, by restriction to the components of degrees in  $H$ , an isomorphism of  $H$ -graded left  $A_H$ -modules  $A_H(\sigma) \simeq A_H^*$ . In particular, if  $A$  is graded Frobenius, then  $A_e$  is a Frobenius algebra.

(3) If  $A$  is  $\sigma$ -graded Frobenius, then for  $\tau \in G$  we have that  $A$  is  $\tau$ -graded Frobenius if and only if  $\tau \in \sigma G\{A\}$ , where  $G\{A\} = \{g \in G \mid A(g) \simeq A \text{ in } A - gr\}$  is the inertia group of  $A$ .

(4) If  $A = \bigoplus_{g \in G} A_g$  is a  $G$ -graded algebra and  $H$  is a normal subgroup of  $G$ , then  $A$  also has a  $G/H$ -grading given by  $A_{\hat{\sigma}} = \bigoplus_{g \in \hat{\sigma}} A_g$  for any  $\hat{\sigma} \in G/H$ . If  $A$  is graded Frobenius with the  $G$ -grading, then it is also graded Frobenius with the  $G/H$ -grading.

#### 4. Characterizing $\sigma$ -graded Frobenius

The characterization of the Frobenius property in the ungraded case (see for example [\[12, Theorem 3.15\]](#)) carries on to the graded case. The proof is on the same line. For  $\sigma = e$ , the result is just a particular case of [Theorem 2.3](#).

**Theorem 4.1.** *Let  $A = \bigoplus_{g \in G} A_g$  be a finite dimensional  $G$ -graded algebra, and let  $\sigma \in G$ . The following assertions are equivalent.*

- (i)  $A$  is  $\sigma$ -graded Frobenius.
- (ii) There exists a non-degenerate associative bilinear form  $B : A \times A \rightarrow k$  such that  $B(r_\tau, r_\mu) = 0$  for any  $r_\tau \in A_\tau$  and  $r_\mu \in A_\mu$  with  $\tau\mu \neq \sigma$ .
- (iii) There exists a hyperplane  $\mathcal{H}$  in  $A$  such that  $A_\tau \subseteq \mathcal{H}$  for any  $\tau \neq \sigma$ , and  $\mathcal{H}$  does not contain nonzero graded left ideals.

**Proof.** (i)  $\Rightarrow$  (ii) Since  $A(\sigma) \simeq A^*$ , we have that  $A \simeq A^*(\sigma^{-1})$  in  $A - gr$ . Let  $\phi : A \rightarrow A^*(\sigma)$  be an isomorphism in  $A - gr$ . Then  $B : A \times A \rightarrow k$ ,  $B(x, y) = \phi(y)(x)$  is associative and non-degenerate. Let  $r_\tau \in A_\tau$  and  $r_\mu \in A_\mu$  with  $\tau\mu \neq \sigma$ . Then  $\phi(r_\mu) \in A^*(\sigma^{-1})_\mu = A^*_{\mu\sigma^{-1}}$ , so  $\phi(r_\mu)(A_\lambda) = 0$  for any  $\lambda \neq (\mu\sigma^{-1})^{-1} = \sigma\mu^{-1}$ . Since  $\tau \neq \sigma\mu^{-1}$ , we have that  $B(r_\tau, r_\mu) = \phi(r_\mu)(r_\tau) = 0$ .

(ii)  $\Rightarrow$  (iii) As in the ungraded case, the set  $\mathcal{H} = \{r \in A \mid B(1, r) = 0\}$  is a hyperplane which does not contain nonzero left ideals of  $A$ , so then neither nonzero graded left ideals. Since  $B(1, r_\tau) = 0$  for any  $\tau \neq \sigma$ , we see that  $A_\tau \subseteq \mathcal{H}$  for any  $\tau \neq \sigma$ .

(iii)  $\Rightarrow$  (i) Let  $\pi : A \rightarrow A$  be a linear map with  $\text{Ker}(\pi) = \mathcal{H}$ . Define  $\phi : A(\sigma) \rightarrow A^*$  by  $\phi(y)(x) = \pi(xy)$  for any  $x, y \in A$ , which is a morphism of left  $A$ -modules. We show that  $\phi$  is a morphism in  $A - gr$ . Indeed, let  $r_{\tau\sigma} \in A_{\tau\sigma} = A(\sigma)_\tau$ . Then for  $\lambda \neq \tau^{-1}$  we have that  $\lambda\tau\sigma \neq \sigma$ , so  $A_{\lambda\tau\sigma} \subseteq \mathcal{H}$ . Then  $\phi(r_{\tau\sigma})(A_\lambda) = \pi(A_\lambda r_{\tau\sigma}) \subseteq \pi(A_{\lambda\tau\sigma}) = 0$ , thus  $\phi(r_{\tau\sigma}) \in A^*_\tau$ .

Finally,  $\phi$  is injective, and hence an isomorphism. Indeed, if the graded left ideal  $\text{Ker}(\phi)$  is nonzero, let  $y \in \text{Ker}(\phi)$  be a nonzero homogeneous element. Since  $\pi(xy) = \phi(y)(x) = 0$  for any  $x \in A$ , we have that  $Ay \subseteq \text{Ker}(\pi) = \mathcal{H}$ , a contradiction.  $\square$

**Corollary 4.2.** *Let  $A$  be a strongly graded finite dimensional  $G$ -graded algebra. Then  $A$  is graded Frobenius if and only if  $A_e$  is Frobenius.*

**Proof.** If  $A$  is graded Frobenius, then  $A_e$  is Frobenius by Remark 3.5(2). Conversely, assume that  $A_e$  is a Frobenius algebra, and let  $\mathcal{T}$  be a hyperplane in  $A_e$  which does not contain nonzero left ideals. Then  $\mathcal{H} = \mathcal{T} \oplus (\bigoplus_{g \in G \setminus \{e\}} A_g)$  is a hyperplane in  $A$ , and clearly  $A_g \subseteq \mathcal{H}$  for any  $g \neq e$ . If  $I$  is a graded left ideal of  $A$  contained in  $\mathcal{H}$ , then  $I_e \subseteq \mathcal{T}$ . Since  $I_e$  is a left ideal in  $A_e$ , it must be zero. Then  $I$  is also zero, since  $A$  is strongly graded.  $\square$

Let  $A = \bigoplus_{g \in G} A_g$  be a finite dimensional  $G$ -graded algebra. Let  $\sigma \in G$ , and let  $\mathcal{C}_\sigma$  be the localizing subcategory of  $A - gr$  consisting of all objects  $M$  such that  $M_\sigma = 0$ . Let  $t_{\mathcal{C}_\sigma}$  be the associated radical, i.e. for a graded  $A$ -module  $M$ ,  $t_{\mathcal{C}_\sigma}(M)$  is the largest graded submodule of  $M$  whose homogeneous component of degree  $\sigma$  is zero. A graded left  $A$ -module  $M$  is called  $\sigma$ -faithful if  $t_{\mathcal{C}_\sigma}(M) = 0$ . The graded ring  $A$  is called left  $\sigma$ -faithful if it is  $\sigma$ -faithful as a left graded  $A$ -module, i.e. for any non-zero  $a_g \in A_g$ ,  $g \in G$ , we

have that  $A_{\sigma g^{-1}} a_g \neq 0$ . Similarly,  $A$  is called right  $\sigma$ -faithful if for any non-zero  $a_g \in A_g$ ,  $g \in G$ , we have that  $a_g A_{g^{-1}\sigma} \neq 0$ .

We recall from [16, Section 2.5] that the coinduced functor  $Coind : A_e\text{-mod} \rightarrow A\text{-gr}$  associates to a left  $A_e$ -module  $N$  the left  $A$ -module  $Coind(N) = Hom_{A_e}(A, N)$  (with  $A$ -module structure induced by the right  $A$ -module structure of  $A$ ), with the grading such that the homogeneous component of degree  $g$  consists of all the maps  $f \in Hom_{A_e}(A, N)$  vanishing on  $A_h$  for any  $h \neq g^{-1}$ . A right adjoint of the functor  $(-)_\sigma : A\text{-gr} \rightarrow A_e\text{-mod}$ , which associates to a graded module its homogeneous component of degree  $\sigma$ , is  $T_{\sigma^{-1}} \circ Coind$ , where  $T_{\sigma^{-1}} : A\text{-gr} \rightarrow A\text{-gr}$  is the isomorphism of categories taking a graded module  $M$  to its  $\sigma^{-1}$ -suspension  $M(\sigma^{-1})$ . The unit of this adjunction is defined as follows. For  $M \in A\text{-gr}$ , let  $\nu_M : M \rightarrow Coind(M_\sigma)(\sigma^{-1})$ ,  $\nu_M(x_\lambda)(a) = a_{\sigma\lambda^{-1}}x_\lambda$  for any  $\lambda \in G$ ,  $x_\lambda \in M_\lambda$ ,  $a \in A$ . Then  $\nu_M$  is a morphism in  $A\text{-gr}$ , moreover  $Ker(\nu_M) = t_{C_\sigma}(M)$ . Now we have the following characterization of the  $\sigma$ -graded Frobenius property.

**Theorem 4.3.** *Let  $A = \bigoplus_{g \in G} A_g$  be a finite dimensional  $G$ -graded algebra, and let  $\sigma \in G$ . The following assertions are equivalent.*

- (i)  $A$  is  $\sigma$ -graded Frobenius.
- (ii)  $A_\sigma \simeq A_e^*$  as left  $A_e$ -modules, and  $A$  is left  $\sigma$ -faithful.
- (iii)  $A_\sigma \simeq A_e^*$  as right  $A_e$ -modules, and  $A$  is right  $\sigma$ -faithful.

**Proof.** (i)  $\Rightarrow$  (ii) An isomorphism  $A(\sigma) \simeq A^*$  of graded left  $A$ -modules induces an isomorphism of left  $A_e$ -modules between the homogeneous components of degree  $e$ , i.e.  $A_\sigma \simeq A_e^*$ .

On the other hand,  $A^*$  is  $e$ -faithful. Indeed, if  $f \in A_g^*$ ,  $g \in G$ , and  $A_{g^{-1}}f = 0$ , then for any  $r_{g^{-1}} \in A_{g^{-1}}$  we have  $f(r_{g^{-1}}) = f(1r_{g^{-1}}) = (r_{g^{-1}}f)(1) = 0$ . But  $f(A_h) = 0$  for any  $h \neq g^{-1}$ , so  $f$  must be zero. Now  $A(\sigma)$  is  $e$ -faithful, which implies that  $A$  is  $\sigma$ -faithful.

(ii)  $\Rightarrow$  (i) Let  $\theta : A_\sigma \rightarrow A_e^*$  be an isomorphism of left  $A_e$ -modules, and let

$$\bar{\theta} : Hom_{A_e}(A, A_\sigma) \rightarrow Hom_{A_e}(A, A_e^*), \quad \bar{\theta}(f) = \theta f$$

be the induced isomorphism of left  $A$ -modules. Now let

$$\phi : Hom_{A_e}(A, A_e^*) \rightarrow Hom_k(A_e \otimes_{A_e} A, k)$$

defined by

$$\phi(F)(a \otimes r) = F(r)(a) \quad \text{for } F \in Hom_{A_e}(A, A_e^*), \quad r \in A, \quad a \in A_e,$$

be the natural isomorphism of left  $A$ -modules. Finally let  $\gamma : A \rightarrow A_e \otimes_{A_e} A$  be the natural isomorphism, and

$$\bar{\gamma} : Hom_k(A_e \otimes_{A_e} A, k) \rightarrow Hom_k(A, k), \quad \bar{\gamma}(f) = f\gamma$$

be the induced isomorphism of left  $A$ -modules.

Therefore  $\bar{\gamma}\phi\bar{\theta} : \text{Coind}(A_\sigma) \rightarrow A^*$  is an isomorphism of left  $A$ -modules. We show that it is also a morphism in  $A - gr$ . Indeed, let  $u \in \text{Coind}(A_\sigma)_g$ , so  $u : A \rightarrow A_\sigma$  and  $u(A_h) = 0$  for any  $h \neq g^{-1}$ . Then for  $a_h \in A_h$ , with  $h \neq g^{-1}$ , we have

$$\begin{aligned} ((\bar{\gamma}\phi\bar{\theta})(u))(a_h) &= (\bar{\gamma}\phi(\theta u))(a_h) \\ &= ((\phi(\theta u))\gamma)(a_h) \\ &= \phi(\theta u)(1 \otimes a_h) \\ &= (\theta u)(a_h)(1) \\ &= \theta(u(a_h))(1) \\ &= 0 \end{aligned}$$

By applying the isomorphism of categories  $T_{\sigma^{-1}}$ , we can regard as  $\bar{\gamma}\phi\bar{\theta} : \text{Coind}(A_\sigma)(\sigma^{-1}) \rightarrow A^*(\sigma^{-1})$ , a graded isomorphism. On the other hand  $\nu_A : A \rightarrow \text{Coind}(A_\sigma)(\sigma^{-1})$  is an injective morphism of graded  $A$ -modules, since its kernel is  $t_{C_\sigma}(A)$  and  $A$  is left  $\sigma$ -faithful. Hence  $\bar{\gamma}\phi\bar{\theta}\nu_A : A \rightarrow A^*(\sigma^{-1})$  is an injective morphism in  $A - gr$ . Since  $A$  and  $A^*$  have the same dimension, this injective morphism must be bijective, so then  $A \simeq A^*(\sigma^{-1})$ , and  $A(\sigma) \simeq A^*$ .

(i)  $\Leftrightarrow$  (iii) is similar to (i)  $\Leftrightarrow$  (ii).  $\square$

In particular, we obtain the description of Frobenius algebras in the category of  $G$ -graded vector spaces.

**Corollary 4.4.** *Let  $A$  be a graded algebra. Then  $A$  is graded Frobenius if and only if  $A_e$  is Frobenius and  $A$  is left (or right)  $e$ -faithful.*

We recall that a graded algebra  $A$  is called graded semisimple if  $A$  is a direct sum of minimal graded left ideals. This is equivalent to  $A - gr$  being a semisimple category.

**Corollary 4.5.** *A graded semisimple algebra is graded Frobenius. In particular, a graded division algebra is graded Frobenius.*

**Proof.** If  $A$  is graded semisimple, then it is easy to see that  $A_e$  is a semisimple algebra, in particular it is Frobenius. On the other hand, by [16, Proposition 2.9.6], if  $A$  is graded semisimple, the  $A$  is  $e$ -faithful. Now  $A$  is graded Frobenius by Theorem 4.3.  $\square$

In Section 6 we will give examples graded algebras which are Frobenius, but not graded Frobenius. However, we have the following.

**Theorem 4.6.** *Let  $A = \bigoplus_{g \in G} A_g$  be a finite dimensional graded algebra such that  $A$  is a Frobenius algebra and  $A_e$  is a local ring. Then there exists  $\sigma \in G$  such that  $A$  is  $\sigma$ -graded Frobenius.*

**Proof.** The graded left  $A$ -module  $A$  is indecomposable. Indeed, if  $A = X \oplus Y$  in  $A - gr$ , then  $X = Au$  and  $Y = Av$  for some orthogonal idempotents  $u, v$  in  $A_e$  such that  $u + v = 1$ . Since  $A_e$  is local, we have  $u = 0$  or  $v = 0$ , so  $X = 0$  or  $Y = 0$ . We also have that  $A^*$  is an indecomposable graded left module. Indeed,  $A^* = X \oplus Y$  in  $A - gr$  implies that  $A^{**} = X^* \oplus Y^*$  in  $gr - A$ . Since  $A^{**} \simeq A$  in  $gr - A$ , and  $A$  is indecomposable in  $gr - A$  (the argument above works also on the right), we see that  $X^* = 0$  or  $Y^* = 0$ , therefore  $X = 0$  or  $Y = 0$ .

Since  $A$  is Frobenius, the left  $A$ -modules  $A$  and  $A^*$  are injective, hence they are also injective objects in  $A - gr$ , by [16, Corollary 2.3.2]. Now we consider the forgetful functor  $U : A - gr \rightarrow A - mod$ , which has a right adjoint  $F$  (see [16, Theorem 2.5.1]). It is known (see [16, Proposition 2.5.4]) that for any  $M \in A - gr$ , we have that  $F(U(M)) \simeq \bigoplus_{g \in G} M(g)$ . Then if we apply  $F$  to the isomorphism of left  $A$ -modules  $A \simeq A^*$ , we obtain an isomorphism  $\bigoplus_{g \in G} A(g) \simeq \bigoplus_{g \in G} A^*(g)$  in  $A - gr$ . In both sides we have direct sums of injective indecomposable objects in  $A - gr$ , so by Azumaya Theorem, we have that the decompositions are equivalent, in particular there is  $\sigma \in G$  such that  $A^* \simeq A(\sigma)$ .  $\square$

If we drop the condition that  $A_e$  is local, the above result does not hold anymore, see Example 6.3. As a biproduct of the method in the proof of Theorem 4.6 we obtain the following.

**Proposition 4.7.** *Let  $A$  be an algebra graded by a finite group  $G$ , and let  $M$  and  $N$  be finite dimensional graded left  $A$ -modules such that  $M$  is graded indecomposable. If  $M \simeq N$  as  $A$ -modules, then  $M \simeq N(\sigma)$  as graded  $A$ -modules for some  $\sigma \in G$ .*

**Proof.** If  $F$  is the right adjoint of the forgetful functor  $U : A - gr \rightarrow A - mod$ , then we have  $F(M) \simeq F(N)$ , so then  $\bigoplus_{g \in G} M(g) \simeq \bigoplus_{g \in G} N(g)$  in  $A - gr$ . Since each  $M(g)$  is indecomposable in  $A - gr$  and the two sides have the same finite number of summands, the Krull–Schmidt Theorem shows that  $N$  must be graded indecomposable and  $M \simeq N(\sigma)$  for some  $\sigma$ .  $\square$

## 5. Graded symmetric algebras

We consider a special class of graded Frobenius algebras. These are just the symmetric algebras in the sovereign monoidal category of  $G$ -graded vector spaces, see [8].

**Definition 5.1.** A finite dimensional graded algebra  $A$  is called graded symmetric if  $A$  and  $A^*$  are isomorphic as graded left- $A$ , right- $A$  bimodules.

Obviously, if  $A$  is graded symmetric, then  $A$  is symmetric as an algebra, and also  $A$  is graded Frobenius. If  $A$  is graded symmetric and  $H$  is a subgroup of the grading group  $G$ , then  $A_H$  is a graded symmetric algebra of type  $H$ . In particular  $A_e$  is a symmetric algebra

whenever  $A$  is graded symmetric. The following characterizes the graded symmetric property. The proof is adapted from the ungraded case (see [12, Theorem 16.54]).

**Theorem 5.2.** *Let  $A = \bigoplus_{g \in G} A_g$  be a finite dimensional  $G$ -graded algebra. The following assertions are equivalent.*

- (i)  $A$  is graded symmetric.
- (ii) There exists a non-degenerate associative symmetric bilinear form  $B : A \times A \rightarrow k$  such that  $B(r_\tau, r_\mu) = 0$  for any  $r_\tau \in A_\tau$  and  $r_\mu \in A_\mu$  with  $\tau\mu \neq e$ .
- (iii) There exists a linear map  $\lambda : A \rightarrow k$  such that  $\lambda(xy) = \lambda(yx)$  for any  $x, y \in A$ ,  $\text{Ker}(\lambda)$  does not contain non-zero graded left ideals, and  $\lambda(x_\sigma y_\tau) = 0$  for any  $x_\sigma \in A_\sigma, y_\tau \in A_\tau$  with  $\sigma\tau \neq e$ .

**Proof.** (i)  $\Rightarrow$  (ii) Let  $\phi : A \rightarrow A^*$  be an isomorphism of graded bimodules. As in the ungraded case,  $B : A \times A \rightarrow k, B(x, y) = \phi(y)(x)$  is a non-degenerate associative symmetric bilinear form. As in the proof of Theorem 4.1, (i)  $\Rightarrow$  (ii), we see that  $B(r_\tau, r_\mu) = 0$  for any  $r_\tau \in A_\tau$  and  $r_\mu \in A_\mu$  with  $\tau\mu \neq e$ .

(ii)  $\Rightarrow$  (iii) As in the ungraded case, the linear map  $\lambda : A \rightarrow k, \lambda(x) = B(x, 1)$  satisfies  $\lambda(xy) = \lambda(yx)$  for any  $x, y \in A$ , and moreover,  $\text{Ker}(\lambda)$  does not even contain non-zero left ideals. If  $x_\sigma \in A_\sigma, y_\tau \in A_\tau$  with  $\sigma\tau \neq e$ , then  $\lambda(x_\sigma y_\tau) = B(x_\sigma y_\tau, 1) = B(x_\sigma, y_\tau) = 0$ .

(iii)  $\Rightarrow$  (i) Let  $\phi : A \rightarrow A^*, \phi(y)(x) = \lambda(xy)$ . Then  $\phi$  is an isomorphism of left  $A$ , right  $A$ -bimodules, as in the ungraded case. If  $y_\tau \in A_\tau$ , then  $\phi(y_\tau)(x_\sigma) = 0$  for any  $x_\sigma \in A_\sigma$  with  $\sigma \neq \tau^{-1}$ . This shows that  $\phi(y_\tau) \in A_\tau^*$ , so  $\phi$  is graded.  $\square$

**Remark 5.3.** Let  $A$  be a strongly graded algebra. We showed in Corollary 4.2 that  $A$  is graded Frobenius if and only if  $A_e$  is Frobenius. For the symmetric property, this equivalence does not hold anymore. If  $A$  is graded symmetric, we have seen that  $A_e$  must be symmetric. We give an example showing that the converse is not true. Assume that the base field  $k$  has characteristic  $\neq 2$ , and let  $R$  be a symmetric algebra and an algebra automorphism  $\alpha$  of order 2 of  $R$  such that the invariant subalgebra  $R^\alpha$  is not symmetric (see [12, Exercise 32, p. 457]). Then the cyclic group  $C_2 = \{e, \alpha\}$  acts as automorphisms on  $R$ , and let  $A = R * C_2$  be the associated skew group algebra. We have that  $u = \frac{1}{2}(e + \alpha)$  is an idempotent of  $A$  and  $uAu \simeq R^\alpha$ , which is not symmetric. By [12, Exercise 25, p. 456],  $A$  cannot be symmetric, so then it is not graded symmetric when regarded as a  $C_2$ -graded algebra. However  $A_e \simeq R$  is symmetric.

This example also shows that there is not a version of Theorem 4.3 for the graded symmetric property. It is clear that if  $A$  is a finite dimensional  $G$ -graded algebra which is graded symmetric, then  $A_e$  is symmetric and  $A$  is  $e$ -faithful (to the left and to the right). The  $e$ -faithfulness follows from the fact that  $A$  is graded Frobenius, using Theorem 4.3. However, the converse is not true. It is possible that  $A_e$  is symmetric and  $A$  is (left and right)  $e$ -faithful, but  $A$  is not graded symmetric. Indeed, let  $A$  be the  $C_2$ -graded algebra constructed in the first part of this example. Then  $A_e = R$  is symmetric, and  $A$  is left



and right  $e$ -faithful, since it is strongly graded. However,  $A$  is not graded symmetric (in fact not even symmetric).

Now we discuss tensor products of graded Frobenius (symmetric) algebras. Assume that  $A = \bigoplus_{g \in G} A_g$  and  $B = \bigoplus_{g \in G} B_g$  are finite dimensional  $G$ -graded algebras such that  $\sigma\tau = \tau\sigma$  for any  $\sigma \in \text{supp}(A)$  and  $\tau \in \text{supp}(B)$ . Then the tensor product of algebras  $A \otimes B$  is  $G$ -graded, with the grading given by  $(A \otimes B)_\sigma = \bigoplus_{gh=\sigma} A_g \otimes B_h$ . Under these conditions we have the following.

**Proposition 5.4.** *If  $A$  and  $B$  are graded Frobenius (respectively graded symmetric), then so is  $A \otimes B$ .*

**Proof.** Let  $\phi : A \rightarrow A^*$  be an isomorphism of graded left  $A$ -modules (respectively bimodules), and let  $\psi : B \rightarrow B^*$  be an isomorphism of graded left  $B$ -modules (respectively bimodules). Let  $F = \gamma(\phi \otimes \psi)$ , where  $\gamma : A^* \otimes B^* \rightarrow (A \otimes B)^*$  is the natural linear isomorphism. Then it is easy to check that  $F$  is an isomorphism of left  $A \otimes B$ -modules (respectively bimodules), and that  $F$  is moreover graded.  $\square$

It is known that a division algebra is symmetric, see [11, Example 16.59]. At this point we do not know whether any graded division algebra is graded symmetric. The following gives a partial answer.

**Theorem 5.5.** *Let  $\Delta = \bigoplus_{g \in G} \Delta_g$  be a graded division  $k$ -algebra such that  $Z(\Delta_e) = Z(\Delta) \cap \Delta_e$ . Then  $\Delta$  is graded symmetric.*

**Proof.** We claim that the  $k$ -subspace of  $\Delta_e$  spanned by all the commutators of the form  $xy - yx$  with  $x \in \Delta_g$ ,  $y \in \Delta_{g^{-1}}$ ,  $g \in G$ , is not the whole of  $\Delta_e$ . Then we can choose a hyperplane  $H$  of  $\Delta$  containing all these commutators and all homogeneous components  $\Delta_g$  with  $g \neq e$ . Noting that the only graded left ideals of  $\Delta$  are 0 and  $\Delta$ , we have that a linear map  $\lambda : \Delta \rightarrow k$  with  $\text{Ker}(\lambda) = H$  satisfies the conditions in Theorem 5.2(3), thus making  $\Delta$  a graded symmetric algebra. Let  $F = Z(\Delta_e) \subseteq Z(\Delta)$ . Obviously  $k \subseteq F$ . The claim above obviously follows if we show that the  $F$ -subspace  $S$  of  $\Delta_e$  spanned by all commutators as above is not the whole of  $\Delta_e$ .

Assume otherwise that  $S = \Delta_e$ . If  $x \in \Delta_g$  and  $y \in \Delta_{g^{-1}}$  with  $g \neq e$ , and  $x, y \neq 0$ , then  $xy - yx = xyxx^{-1} - yx = xux^{-1} - u$ , where  $u = yx \in \Delta_e$ . Thus each such commutator is of the form  $\phi(u) - u$ , where  $\phi$  is an  $F$ -automorphism of  $\Delta_e$  (note that the inner automorphism of  $\Delta_e$  associated to some  $x \in \Delta_g \setminus \{0\}$  is an  $F$ -automorphism since  $F \subseteq Z(\Delta)$ ). Therefore  $\Delta_e$  is the  $F$ -span of some commutators  $xy - yx$  with  $x, y \in \Delta_e$ , and some elements of the form  $\phi(u) - u$ , where  $\phi$  is an  $F$ -automorphism of  $\Delta_e$ , and  $u \in \Delta_e$ . Let  $\bar{F}$  be the algebraic closure of  $F$ . Lifting to  $\bar{F}$ , we obtain that  $\bar{F} \otimes_F \Delta_e$  is  $\bar{F}$ -spanned by some elements of the form  $ab - ba$ , and some elements of the form  $\psi(a) - a$ , where

$a, b \in \bar{F} \otimes_F \Delta_e$  and  $\psi$  is an automorphism of the  $\bar{F}$ -algebra  $\bar{F} \otimes_F \Delta_e$ . As a consequence of [11, Theorem 15.1] we have that  $\bar{F} \otimes_F \Delta_e$  is isomorphic to a matrix algebra  $M_r(\bar{F})$  as an  $\bar{F}$ -algebra. But if  $a, b \in M_r(\bar{F})$ , then  $ab - ba$  has trace 0, and since an automorphism of  $M_r(\bar{F})$  is inner, an element of the form  $\psi(a) - a$ , where  $a \in M_r(\bar{F})$  and  $\psi$  is an automorphism of the  $\bar{F}$ -algebra  $M_r(\bar{F})$ , has also trace 0. This is a contradiction, since a family of elements of trace 0 of  $M_r(\bar{F})$  cannot span  $M_r(\bar{F})$ , and this ends the proof.  $\square$

The following shows that any graded division algebra has a distinguished graded sub-division algebra which is graded symmetric.

**Corollary 5.6.** *Let  $A$  be a  $G$ -graded division algebra. Then  $\Delta = C_A(A_e)$ , the centralizer of  $A_e$  in  $A$ , is a graded division algebra which is graded symmetric.*

**Proof.** It is clear that  $\Delta$  is a graded division algebra. If we take into account Theorem 5.5, it is enough to show that  $Z(\Delta_e) = Z(\Delta) \cap \Delta_e$ . To see this, if we take  $a \in Z(\Delta_e) = \Delta_e$ , then  $ay = ya$  for any  $y \in \Delta$ , so  $a \in Z(\Delta)$ . Thus  $Z(\Delta_e) \subseteq Z(\Delta) \cap \Delta_e$ . The converse inclusion always holds.  $\square$

## 6. Examples

In this section we give examples to illustrate the concepts of  $\sigma$ -graded Frobenius algebra and graded symmetric algebra, and their connection to the Frobenius and the symmetric properties.

**Example 6.1.** Let  $R$  be a finitely generated  $k$ -algebra, and  $R^*$  the dual space of  $R$ , with the usual bimodule structure. The trivial extension  $A = R \oplus R^*$  is the algebra with multiplication defined by  $(r, f)(r', f') = (rr', rf' + fr')$ . It is known (see [12, Example 16.60]) that  $A$  is a symmetric algebra. On the other hand,  $A$  is a  $\mathbf{Z}_2$ -graded algebra, with homogeneous components  $A_0 = R \oplus 0 \simeq R$ ,  $A_1 = 0 \oplus R^*$ .

We have that  $A$  is not  $\hat{0}$ -faithful, since  $A_1(0, f) = 0$  for any  $f \in R^*$ . On the other hand,  $A$  is  $\hat{1}$ -faithful. Indeed, if  $A_0(0, f) = 0$ , then  $Rf = 0$ , so  $f = 0$ ; also, if  $A_1(r, 0) = 0$ , then  $R^*r = 0$ , so  $f(r) = (fr)(1) = 0$  for any  $f \in R^*$ , showing that  $r = 0$ .

We conclude that  $A$  is Frobenius (since it is symmetric), but it is not graded Frobenius (since it is not  $\hat{0}$ -faithful). However,  $A$  is  $\hat{1}$ -graded Frobenius. This also gives an example of a graded algebra which is symmetric, but not graded symmetric.

**Example 6.2.** Let  $u$  and  $v$  be non-zero elements of  $k$ , and let  $A$  be the set of all matrices of the form  $\begin{pmatrix} a & b & c & d \\ 0 & a & 0 & uc \\ 0 & 0 & a & vb \\ 0 & 0 & 0 & a \end{pmatrix}$ , with  $a, b, c, d \in k$ . Then  $A$  is a Frobenius algebra, and  $A$  is symmetric if and only if  $u = v$  (see [12, Example 16.66 and Exercise 16.29]). This example is due to Nakayama and Nesbitt. Let

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have that  $X^2 = Y^2 = Z^2 = XZ = ZX = YZ = ZY = 0$ ,  $XY = uZ$  and  $YX = vZ$ , therefore  $A$  is a  $\mathbf{Z}_4$ -graded algebra with grading given by  $A_{\hat{0}} = kI_4$ ,  $A_{\hat{1}} = kX$ ,  $A_{\hat{2}} = kY$ ,  $A_{\hat{3}} = kZ$ . Since  $A_{\hat{3}}X = 0$ ,  $A_{\hat{3}}Y = 0$  and  $A_{\hat{1}}X = 0$ , we see that  $A$  is not  $g$ -faithful if  $g \in \{\hat{0}, \hat{1}, \hat{2}\}$ . On the other hand,  $A$  is  $\hat{3}$ -faithful. Then  $A$  is  $\hat{3}$ -graded Frobenius, but it is not  $g$ -graded Frobenius if  $g \in \{\hat{0}, \hat{1}, \hat{2}\}$ . If  $u = v$ , then  $A$  is symmetric, but not graded symmetric.

The following shows that the condition that  $A_e$  is local cannot be dropped in [Theorem 4.6](#).

**Example 6.3.** Let  $R$  be a finite dimensional algebra which is decomposable as an algebra, i.e.  $R = R_1 \times R_2$  for some algebras  $R_1$  and  $R_2$ . Then  $R^* = R_1^* \oplus R_2^*$ , and we have that  $R_2R_1^* = R_1^*R_2 = 0$  and  $R_1R_2^* = R_2^*R_1 = 0$  in the  $R$ ,  $R$ -bimodule structure of  $R^*$ . Let  $A = R \oplus R^*$  be the trivial extension, which is an algebra as in [Example 6.1](#). We consider another grading on  $A$ . More precisely,  $A$  is  $\mathbf{Z}_3$ -graded with  $A_{\hat{0}} = R$ ,  $A_{\hat{1}} = R_1^*$  and  $A_{\hat{2}} = R_2^*$ . Then  $A$  is not  $\hat{0}$ -faithful since  $A_{\hat{2}}f = 0$  for any  $f \in R_1^* = A_{\hat{1}}$ . Also,  $A$  is neither  $\hat{1}$ -faithful since  $A_{\hat{2}}f = 0$  for any  $f \in R_2^* = A_{\hat{2}}$ , nor  $\hat{2}$ -faithful since  $A_{\hat{1}}f = 0$  for any  $f \in R_1^* = A_{\hat{1}}$ . Therefore  $A$  is not  $\sigma$ -faithful, and by [Theorem 4.1](#)  $A$  cannot be  $\sigma$ -graded Frobenius for any  $\sigma$ . On the other hand  $A$  is a symmetric algebra, so it is also Frobenius.

Let us note that in fact  $A$  may be regarded as a  $G$ -graded algebra for any group  $G$  of order at least 3. Indeed, if  $e$  is the neutral element of  $G$ , and  $u, v$  are two other different elements of  $G$ , then  $A_e = A_{\hat{0}}$ ,  $A_u = A_{\hat{1}}$  and  $A_v = A_{\hat{2}}$  defines such a grading.

**Example 6.4.** (1) Let  $A = M_n(k)$  be a matrix algebra. There is a special class of gradings on  $A$ , called good gradings, for which any matrix unit  $e_{ij}$  is a homogeneous element. Under certain conditions on  $n, k$  and  $G$ , any  $G$ -grading on  $A$  is isomorphic to a good grading. We have that any good grading on  $A$  makes  $A$  a graded symmetric algebra. Indeed, the trace map  $tr : A \rightarrow k$  satisfies  $tr(xy) = tr(yx)$  for any  $x, y \in A$ , and it is easy to see that  $tr(yx) = 0$  for any  $y \in A$  implies that  $x = 0$  (just take  $y = e_{ij}$  for all  $i, j$ ), so the kernel of  $tr$  does not contain non-zero left ideals. Since all  $e_{ii}$ 's are homogeneous elements of degree  $e$  (the neutral element of  $G$ ), we see that  $tr(xy) = 0$  if  $x \in A_{\sigma}, y \in A_{\tau}$  with  $\sigma\tau \neq e$ .

(2) There is another special type of gradings on  $A = M_n(k)$ , called fine gradings. These are gradings for which any homogeneous component of  $A$  has dimension at most 1. Such a grading makes  $A$  a graded division algebra. By [Theorem 5.5](#) we see that  $A$  is graded symmetric.

(3) If  $G$  is abelian and  $k$  is algebraically closed, using (1) and (2) above and the result of [3], which describes any  $G$ -grading on  $A = M_n(k)$  as a tensor product of a matrix algebra with a good grading and a matrix algebra with a fine grading, we obtain that any  $G$ -grading on  $A$  makes it a graded symmetric algebra.

## 7. Frobenius functors and applications to (graded) Frobenius algebras

A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called Frobenius if it is a left and a right adjoint of the same functor  $G : \mathcal{B} \rightarrow \mathcal{A}$ . Frobenius functors originate in the work of Morita. A functor defining an equivalence of categories is Frobenius, and the composition of two Frobenius functors is a Frobenius functor. It is known that a finite dimensional algebra  $A$  is Frobenius if and only if the restriction of scalars functor  $A - \text{mod} \rightarrow k - \text{mod}$  is a Frobenius functor, see for example [6, Theorem 28].

It is well known that if  $A$  is a Frobenius algebra, then the matrix algebra  $M_n(A)$  is Frobenius, too. This follows from the fact that  $M_n(A) \simeq A \otimes M_n(k)$ , a tensor product of Frobenius algebras. We note that this also follows as an application of Frobenius functors. Indeed, consider the following commutative (up to a natural isomorphism) diagram of categories and functors.

$$\begin{array}{ccc} A - \text{mod} & \xrightarrow{F} & M_n(A) - \text{mod} \\ U_1 \downarrow & & \downarrow U_2 \\ k - \text{mod} & \xrightarrow{(-)^n} & k - \text{mod} \end{array}$$

where  $U_1$  and  $U_2$  are the restriction of scalars functors,  $F = \text{Col}_n(-)$  is the standard equivalence of categories, and  $(-)^n$  is the functor taking a vector space  $V$  to the direct sum of  $n$  copies of  $V$ , and acting accordingly on morphisms. It is clear that  $(-)^n$  is a Frobenius functor, since it is natural isomorphic to a functor of the form  $V \otimes (-)$ , where  $V$  is a vector space of dimension  $n$ , and this has the left and right adjoint  $V^* \otimes (-)$ . Now if  $A$  is Frobenius, then  $U_1$  is a Frobenius functor, so then is  $(-)^n U_1$ . Then  $U_2 F$  is a Frobenius functor, and since  $F$  is an equivalence, we obtain that  $U_2$  is also Frobenius. Thus  $M_n(A)$  is a Frobenius algebra.

We show that the converse is also true, i.e. if  $M_n(A)$  is a Frobenius algebra, then so is  $A$ . We first need the following.

**Lemma 7.1.** *Let  $A$  be an algebra and let  $M$  be a left  $A$ -module. If  $n$  is a positive integer, then there is an isomorphism  $\text{Hom}_{M_n(A)}(\text{Col}_n(M), M_n(A)) \simeq \text{Row}_n(\text{Hom}_A(M, A))$  of right  $M_n(A)$ -modules.*

**Proof.** Let  $f : \text{Col}_n(M) \rightarrow M_n(A)$  be a morphism of left  $M_n(A)$ -modules. Let  $m \in M$  and  $X(m)$  the column having  $m$  on the first spot, and 0 elsewhere. Since  $e_{11}X(m) =$

$X(m)$ , we have that  $f(X(m)) = f(e_{11}X(m)) = e_{11}f(X(m))$ , showing that  $f(X(m))$  has zeroes on any row except the first one. Thus there are  $f_1(m), \dots, f_n(m) \in A$  such that

$$f(X(m)) = \begin{pmatrix} f_1(m) & f_2(m) & \dots & f_n(m) \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Since  $f$  is a morphism of left  $M_n(A)$ -modules, it is easy to see that  $f_1, \dots, f_n \in \text{Hom}_A(M, A)$ . Now  $\begin{pmatrix} m_1 \\ m_2 \\ \dots \\ m_n \end{pmatrix} = e_{11}X(m_1) + e_{12}X(m_2) + \dots + e_{1n}X(m_n)$ , so then

$$\begin{aligned} f\left(\begin{pmatrix} m_1 \\ m_2 \\ \dots \\ m_n \end{pmatrix}\right) &= e_{11}f(X(m_1)) + e_{12}f(X(m_2)) + \dots + e_{1n}f(X(m_n)) \\ &= \begin{pmatrix} f_1(m_1) & f_2(m_1) & \dots & f_n(m_1) \\ f_1(m_2) & f_2(m_2) & \dots & f_n(m_2) \\ \dots & \dots & \dots & \dots \\ f_1(m_n) & f_2(m_n) & \dots & f_n(m_n) \end{pmatrix} \end{aligned}$$

Now define

$$\phi : \text{Hom}_{M_n(A)}(\text{Col}_n(M), M_n(A)) \rightarrow \text{Row}_n(\text{Hom}_A(M, A)), \quad \phi(f) = (f_1, \dots, f_n)$$

It is straightforward to check that  $\phi$  is a morphism of right  $M_n(A)$ -modules. Obviously the kernel of  $\phi$  is zero, so  $\phi$  is injective. To see that  $\phi$  is surjective, let  $u_1, \dots, u_n \in \text{Hom}_A(M, A)$ , and define  $g : \text{Col}_n(M) \rightarrow M_n(A)$  by

$$g\left(\begin{pmatrix} m_1 \\ m_2 \\ \dots \\ m_n \end{pmatrix}\right) = \begin{pmatrix} u_1(m_1) & u_2(m_1) & \dots & u_n(m_1) \\ u_1(m_2) & u_2(m_2) & \dots & u_n(m_2) \\ \dots & \dots & \dots & \dots \\ u_1(m_n) & u_2(m_n) & \dots & u_n(m_n) \end{pmatrix}$$

Then it is easy to check that  $g$  is a morphism of left  $M_n(A)$ -modules and  $\phi(g) = (u_1, \dots, u_n)$ , so  $\phi$  is also surjective.  $\square$

Now we can prove the following.

**Proposition 7.2.** *Let  $A$  be a finite dimensional algebra such that  $M_n(A)$  is a Frobenius algebra for some positive integer  $n$ . Then  $A$  is a Frobenius algebra.*

**Proof.** Since  $M_n(A)$  is Frobenius, it is left self-injective, i.e. it is quasi-Frobenius. This property is invariant under Morita equivalence, so we have that  $A$  is also quasi-Frobenius.

On the other hand, let  $M$  be a finite dimensional left  $A$ -module. Then  $Col_n(M)$  is a finite dimensional left  $M_n(A)$ -module. Since  $M_n(A)$  is Frobenius, [12, Theorem 16.34] shows that

$$\dim_k Hom_{M_n(A)}(Col_n(M), M_n(A)) = \dim_k Col_n(M)$$

By Lemma 7.1 we have that  $\dim_k Hom_{M_n(A)}(Col_n(M), M_n(A)) = \dim_k Row_n(Hom_A(M, A)) = n \dim_k Hom_A(M, A)$ . Since  $\dim_k Col_n(M) = n \dim_k M$ , we obtain that  $\dim_k Hom_A(M, A) = \dim_k M$ . Now  $A$  is Frobenius by [12, Theorem 16.33].  $\square$

We will present a graded version of the result characterizing the Frobenius property by Frobenius functors. Let  $A = \bigoplus_{g \in G} A_g$  be a  $G$ -graded finite dimensional algebra. We can regard the basic field  $k$  as a  $G$ -graded algebra with trivial grading, i.e. the homogeneous component of degree  $e$  is the whole of  $k$  and all the other homogeneous components are zero. Let  $k - gr$  be the associated category of graded modules, which is in fact the category of  $G$ -graded vector spaces. A  $G$ -graded vector space is just a vector space which is a direct sum of subspaces indexed by  $G$ . Let  $U : A - gr \rightarrow k - gr$  be the forgetful functor, which forgets the  $A$ -action, but preserves the  $G$ -grading.

We define the functor  $F : k - gr \rightarrow A - gr$  by  $F(V) = A \otimes V$ , with  $A$ -action on the first tensor position, and grading given by  $F(V)_g = \bigoplus_{\sigma\tau=g} A_\sigma \otimes V_\tau$  for any  $g \in G$ . On morphisms  $F$  acts as the tensor product with  $Id_A$ .

We also define the functor  $T : k - gr \rightarrow A - gr$  by  $T(V) = Hom(A, V)$ , with the  $A$ -action induced by the right  $A$ -module structure of  $A$ , and the grading given by  $T(V)_\sigma = \{f \in Hom(A, V) \mid f(A_g) \subseteq V_{g\sigma} \text{ for any } g \in G\}$ . On morphisms  $T$  acts as  $Hom(Id_A, -)$ . It is straightforward to check that  $F$  is a left adjoint of  $U$ , and  $T$  is a right adjoint of  $U$ . Now we have the following characterization of the graded Frobenius property.

**Proposition 7.3.**  *$A$  is graded Frobenius if and only if the forgetful functor  $U : A - gr \rightarrow k - gr$  is a Frobenius functor.*

**Proof.**  $U$  is a Frobenius functor if and only if its left and right adjoints,  $F$  and  $T$ , are naturally isomorphic. Assume that  $U$  is Frobenius, thus there is an isomorphism  $\eta(V) : A \otimes V \rightarrow Hom(A, V)$  of graded  $A$ -modules, natural in  $V$ , for each graded vector space  $V$ . In particular, for  $V = k$ , with grading concentrated in the degree  $e$ -component, we obtain an isomorphism  $A \otimes k \simeq Hom(A, k)$  of graded  $A$ -modules. Since  $A \otimes k \simeq A$ , this yields an isomorphism of graded left  $A$ -modules  $A \simeq A^*$ , i.e.  $A$  is graded Frobenius.

Conversely, assume that  $A$  is graded Frobenius, and let  $\theta : A \rightarrow A^*$  be an isomorphism in  $A - gr$ . Then for any  $V \in k - gr$  define  $\eta(V)$  as the composition

$$\eta(V) : A \otimes V \xrightarrow{\theta \otimes I} A^* \otimes V \xrightarrow{\gamma(V)} Hom(A, V)$$

where  $\gamma(V)$  is the natural linear isomorphism defined by  $\gamma(V)(f \otimes v)(r) = f(r)v$  for any  $f \in A^*$ ,  $v \in V$  and  $r \in A$ . It is straightforward to check that  $\gamma(V)$  is in fact an isomorphism of graded  $A$ -modules, and since  $\theta \otimes I$  is an isomorphism in  $A - gr$ , we obtain that  $\eta(V)$  is also an isomorphism in  $A - gr$ . Obviously  $\eta(V)$  is natural in  $V$ , and this shows that  $\eta$  is a natural isomorphism between  $F$  and  $T$ .  $\square$

If  $A$  is a graded algebra, another functor providing good information about  $A$  is  $(-)_e : A - gr \rightarrow A_e - mod$ . For instance,  $A$  is strongly graded if and only if  $(-)_e$  is an equivalence of categories. We investigate whether  $(-)_e$  provides information about Frobenius properties on  $A$ . In this direction we have the following.

**Theorem 7.4.** *Let  $A$  be a finite dimensional  $G$ -graded algebra. The following assertions are true.*

- (1) *If  $A$  is graded Frobenius and  $A$  is a projective left  $A_e$ -module, then  $(-)_e$  is a Frobenius functor.*
- (2) *If  $(-)_e$  is a Frobenius functor and  $A_e$  is a Frobenius algebra, then  $A$  is graded Frobenius.*

**Proof.** (1) Assume that  $A$  is graded Frobenius. By the proof of Theorem 4.3 (for  $\sigma = e$ ), we see that  $\mu : A \rightarrow Hom_{A_e}(A, A_e)$ ,  $\mu(r_\sigma)(a) = a_{\sigma^{-1}}r_\sigma$  for any  $\sigma \in G, r_\sigma \in A_\sigma$  and  $a \in A$ , is an isomorphism in  $A - gr$ . Moreover

$$\begin{aligned} \mu(r_\sigma b_e)(a) &= a_{\sigma^{-1}}r_\sigma b_e \\ &= \mu(r_\sigma)(a)b_e \\ &= (\mu(r_\sigma)b_e)(a) \end{aligned}$$

for any  $b_e \in A_e$ , showing that  $\mu$  is also a morphism of right  $A_e$ -modules. Now we use [13, Theorem 3.4] and find that  $(-)_e$  is a Frobenius functor.

(2) We have seen that a right adjoint of  $(-)_e$  is  $Coind$ . A left adjoint of  $(-)_e$  is the induced functor  $Ind : A_e - mod \rightarrow A - gr$ ,  $Ind(N) = A \otimes_{R_e} N$ , with grading  $Ind(N)_g = A_g \otimes_{A_e} N$ , and acting on morphisms as  $Id_A \otimes_{A_e} -$ . There is a natural transformation  $\eta : Ind \rightarrow Coind$  defined by

$$\eta(N) : Ind(N) \rightarrow Coind(N), \quad \eta(N)(r \otimes x)(b) = \sum_g (b_{g^{-1}}r_g)x \quad \text{for } r, b \in A, x \in N$$

By [13, Theorem 3.4] the functor  $(-)_e$  is Frobenius if and only if  $\eta(A_e)$  is an isomorphism. Since  $Ker(\eta(A_e)) = t_e(A)$ , we see that  $A$  is  $e$ -faithful. Now  $A$  is graded Frobenius by Theorem 4.3.  $\square$

For many ring properties “ $P$ ” it is true that for an algebra  $A$  graded by a finite group  $G$  we have that  $A$  satisfies the graded version of the property “ $P$ ” if and only if

the smash product  $A\#(kG)^*$  satisfies “ $P$ ”. If “ $P$ ” is the property of being Frobenius, then this connection does not hold. The reason is that the Frobenius property is not a categorical one, more precisely it is not invariant under Morita equivalence, despite that it is preserved between  $A$  and  $M_n(A)$ . In fact it was shown in [4] that for a finite dimensional Hopf algebra  $H$  acting on a finite dimensional algebra  $A$ , we have that the smash product  $A\#H$  is Frobenius if and only if  $A$  is Frobenius. Now we give a new short proof of this result, by using Frobenius functors.

**Theorem 7.5.** *Let  $H$  be a finite dimensional Hopf algebra and let  $A$  be a finite dimensional left  $H$ -module algebra. Then  $A\#H$  is Frobenius if and only if  $A$  is Frobenius.*

**Proof.** Since  $H$  is finite dimensional,  $A$  is a right  $H^*$ -comodule algebra, and we can consider the category of generalized Hopf modules  ${}_A\mathcal{M}^{H^*}$ . There is an isomorphism of categories  $P : A\#H - \text{mod} \rightarrow {}_A\mathcal{M}^{H^*}$ , which associates to a left  $A\#H$ -module  $M$  the space  $M$  with the left  $A$ -module structure obtained by restriction of scalars via  $A \subseteq A\#H$ , and the right  $H^*$ -comodule structure coming from the left  $H$ -module structure. If  $U : {}_A\mathcal{M}^{H^*} \rightarrow A - \text{mod}$  is the functor forgetting the  $H^*$ -comodule structure, then  $UP$  is just the restriction of scalars, and it is known that  $U$  is a Frobenius functor.

If  $A$  is Frobenius, then the restriction of scalars  $U_1 : A - \text{mod} \rightarrow k - \text{mod}$  is a Frobenius functor, so then  $U_1UP$  is a Frobenius functor, as a composition of Frobenius functors. But  $U_1UP : A\#H - \text{mod} \rightarrow k - \text{mod}$  is just the restriction of scalars, and we obtain that  $A\#H$  is Frobenius.

Conversely, assume that  $A\#H$  is Frobenius. Then  $A\#H$  is a left  $H^*$ -module algebra, and by the above considerations  $(A\#H)\#H^*$  is Frobenius. Now  $(A\#H)\#H^* \simeq M_n(R)$ , where  $n = \dim(H)$ , by the duality theorem for finite dimensional Hopf algebras (see [15, Corollary 9.4.17]), and we obtain that  $A$  is Frobenius by Proposition 7.2.  $\square$

We note that if  $A$  is a  $G$ -graded algebra and  $X$  is a finite  $G$ -set, then there is an associated smash product  $A\#X$ , see [16, p. 216]. We have that if  $A$  is Frobenius, then so is  $A\#X$ . Indeed, the category of left modules over  $A\#X$  is isomorphic to the category of generalized Doi–Hopf modules  ${}_A\mathcal{M}^{kX}$ , which is nothing but the category of  $A$ -modules graded by the  $G$ -set  $X$ . The forgetful functor  ${}_A\mathcal{M}^{kX} \rightarrow {}_A\mathcal{M}$  is Frobenius (see for instance [6, Theorem 54]), and arguments similar to the ones above can be now applied.

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