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Computing the extensions of preinjective and preprojective Kronecker modules ☆

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ABSTRACT

Let I, I' be preinjective Kronecker modules (i.e. all their indecomposable components are preinjective). We describe the modules M for which there exists an exact sequence $0 \to I' \to M \to I \to 0$ by explicit, easy to check numerical conditions, resulting in an algorithm (linear in the number of indecomposable components) for the decision problem. We also propose a method to generate all extensions of I' by I and we give a different proof for a theorem in [13] providing numerical criteria in terms of Kronecker invariants for the existence of a monomorphism $f : I' \to I$. All these results apply dually to preprojective modules as well.

1. Introduction

Let $K : 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$ be the *Kronecker quiver* and κ an arbitrary field. The path algebra κK over the Kronecker quiver is the *Kronecker algebra*. We will consider the category of finite dimensional right modules over this algebra, the category of *Kronecker modules*, denoted by mod-κK. The category mod-κK can and will be identified with the category rep-κK of the finite dimensional κ-representations of the Kronecker quiver. Such a representation is defined as a quadruple $(M_1, M_2; f, g)$ where M_1, M_2 are finite dimensional κ-vector spaces (corresponding to the vertices) and $f, g : M_2 \to M_1$ are κ-linear maps (corresponding to the arrows). The dimension vector of a module (viewed as a representation) $M = (M_1, M_2; f, g) \in$ mod-κK is $\underline{\dim}(M) = (\dim_\kappa M_1, \dim_\kappa M_2)$.

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For a module $M \in \text{mod-}\kappa K$, tM will denote $M \oplus \cdots \oplus M$ (t times) and $[M]$ will mean the isomorphism class of M. For two modules $M, M' \in \text{mod-}\kappa K$ we will denote by $M' \hookrightarrow M$ the fact that M' can be embedded in M and by $M \twoheadrightarrow M'$ that M projects on M'. It is important to note that by writing "an extension of M' by M" we mean a Kronecker module X, which is a middle term of elements in $\mathrm{Ext}^1(M, M')$, where $\mathrm{Ext}^1(M, M')$ is the space of Yoneda extensions. Or simply put, an extension of M' by M is a module X for which there exists a short exact sequence $0 \to M' \to X \to M \to 0$.

The indecomposables in $\text{mod-}\kappa K$ are divided into three families: the preprojectives, the regulars and the preinjectives (see [1,2,9]). Because in the current paper we deal only with preprojective and preinjective Kronecker modules, we are going to introduce briefly only these (skipping regulars).

The *preinjective indecomposable Kronecker modules* are up to isomorphism uniquely determined by their dimension vectors. For $n \in \mathbb{N}$ we will denote by I_n the indecomposable preinjective Kronecker module of dimension $(n, n+1)$. So I_0, I_1 are the injective indecomposable modules (I_0 being simple). The module I_n can be identified with the linear representation

$$I_n : \kappa^n \underset{(0\ E_n)}{\overset{(E_n\ 0)}{\leftleftarrows}} \kappa^{n+1},$$

where choosing the canonical basis in κ^n and κ^{n+1}, the matrices of the two linear functions from κ^{n+1} to κ^n are $(0 \quad E_n)$ and $(E_n \quad 0)$ respectively. Here E_n denotes the n-dimensional identity matrix.

A *preinjective Kronecker module* is a module with all its indecomposable components preinjective. By Krull–Schmidt theorem a preinjective module up to isomorphism has the form $I_{b_1} \oplus \cdots \oplus I_{b_k}$, where $(b_1, \ldots, b_k)$ is a finite decreasing sequence of nonnegative integers.

The preprojective Kronecker modules are just the categorical dual of the preinjectives and are also determined up to isomorphism by their dimension vectors. For $n \in \mathbb{N}$, we will denote by P_n the *indecomposable preprojective Kronecker module* of dimension $(n+1, n)$, meaning that P_0 and P_1 will be the projective indecomposable modules with P_0 being simple. A module P_n can be identified with the linear representation

$$P_n : \kappa^{n+1} \underset{\binom{E_n}{0}}{\overset{\binom{0}{E_n}}{\leftleftarrows}} \kappa^n,$$

where the matrices of the two linear functions from κ^{n+1} to κ^n are $\begin{pmatrix} E_n \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ E_n \end{pmatrix}$ in the canonical basis.

A *preprojective Kronecker module* is a module with all its indecomposable components preprojective. A preprojective module has the form $P_{b_1} \oplus \cdots \oplus P_{b_k}$ up to isomorphism, where $(b_1, \ldots, b_k)$ is a finite increasing sequence of nonnegative integers.

In this paper we deal with a basic problem in the theory of Kronecker modules, the description of extensions of I' by I, where I, I' are preinjective modules (or dually, the description of extensions of P' by P, where P and P' are preprojective modules). It turns out that these extensions carry some interesting combinatorial properties. Based on results from [10] we have described, albeit implicitly, these extensions for the first time in [11] (see Theorems 2 and 17). In Section 2 we take the idea further, turning the implicit characterization into explicit, easy to check numerical conditions, in the form of the following theorem, the main result of our current work:

Theorem. *Let $a_1 \geqslant \cdots \geqslant a_p \geqslant 0$, $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$, $c_1 \geqslant \cdots \geqslant c_r \geqslant 0$ be decreasing sequences of nonnegative integers and let $B_j = \{l \in \{0, \ldots, n\} \mid \sum_{k=1}^{l} b_k + \sum_{k=1}^{j} a_k \geqslant \sum_{k=1}^{l+j} c_k\}$ for $1 \leqslant j \leqslant p$. Then there is a short exact sequence*

$$0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \to I \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0$$

if and only if $[I] = [I_{c_1} \oplus \cdots \oplus I_{c_r}]$, $r = p + n$, $\sum_{i=1}^{r} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$, $B_j \neq \emptyset$, $a_j \leqslant c_{\alpha_j}$ *and* $b_i \geqslant c_{\beta_i}$ *for* $1 \leqslant j \leqslant p$ *and* $1 \leqslant i \leqslant n$*, where*

$$\alpha_j = \begin{cases} \min B_1 + 1, & j = 1, \\ \max\{\alpha_{j-1} + 1, \min B_j + j\}, & 1 < j \leqslant p \end{cases}$$

and

$$\beta_i = \begin{cases} \min\{l \in \{1, \ldots, r\} \mid l \neq \alpha_j,\ 1 \leqslant j \leqslant p\}, & i = 1, \\ \min\{l \in \{\beta_{i-1} + 1, \ldots, r\} \mid l \neq \alpha_j,\ 1 \leqslant j \leqslant p\}, & 1 < i \leqslant n. \end{cases}$$

As shown in Section 3, the previous theorem yields a straightforward method for solving the decision problem, i.e. given $I, I', I'' \in \text{mod-}\kappa K$ preinjectives, decide whether I'' is as extension of I' by I. An implementation would result in a linear-time algorithm (linear in the number of indecomposable components of I''). A method for generating all the extensions of I' by I (efficiently enough to serve as a practical helping tool in further investigation of Kronecker modules) may also be given. In Section 4 we give a combinatorial proof of a theorem from [13] providing numerical criteria in terms of Kronecker invariants for the existence of a monomorphism $f : I' \to I$. We show how the same easy and explicit criteria can be derived only from Theorem 2 using nothing more than some inequalities involving integers. Finally, in Section 5 we enlist the relevant results dualized for the preprojective case.

In what follows we are going to briefly sketch the relation between Kronecker modules and matrix pencils. One of our motivations behind the current work is exactly this connection and the hope that these results represent some first steps towards solving an important open problem in matrix theory (see the statement of the challenge below).

Kronecker modules correspond to *matrix pencils* in linear algebra, so the Kronecker algebra relates representation theory with numerical linear algebra and matrix theory. Recall that a *matrix pencil* over a field κ is a matrix $A + \lambda B$ where A, B are matrices over κ of the same size and λ is an indeterminate.

Two pencils $A + \lambda B$, $A' + \lambda B'$ are *strictly equivalent*, denoted by $A + \lambda B \sim A' + \lambda B'$, if and only if there exist invertible, constant (λ independent) matrices P, Q such that $P(A' + \lambda B')Q = A + \lambda B$. Kronecker proved that pencils are uniquely determined up to strict equivalence by their *classical Kronecker invariants*, which are *the minimal indices for columns, the minimal indices for rows, the finite elementary divisors, the infinite elementary divisors* (see [4] for all the details).

A pencil $A' + \lambda B'$ is called *subpencil* of $A + \lambda B$ if and only if there are pencils $A_{12} + \lambda B_{12}$, $A_{21} + \lambda B_{21}$, $A_{22} + \lambda B_{22}$ such that

$$A + \lambda B \sim \begin{pmatrix} A' + \lambda B' & A_{12} + \lambda B_{12} \\ A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22} \end{pmatrix}.$$

In this case we also say that the subpencil can be completed to the bigger pencil. We speak about row completion when A_{12}, B_{12}, A_{22}, B_{22} are zero matrices and about column completion when A_{21}, B_{21}, A_{22}, B_{22} are zero.

There is an unsolved challenge in pencil theory with lots of applications in control theory (problems related to pole placement, non-regular feedback, dynamic feedback etc. may be formulated in terms of matrix pencils, for details see [8]):

Challenge: If $A + \lambda B$, $A' + \lambda B'$ are pencils over $\mathbb{C}$, find a necessary and sufficient condition in terms of their classical Kronecker invariants for $A' + \lambda B'$ to be a subpencil of $A + \lambda B$. Also construct the completion pencils $A_{12} + \lambda B_{12}$, $A_{21} + \lambda B_{21}$, $A_{22} + \lambda B_{22}$. A particular case of the challenge above is when we limit ourselves to column or row completions.

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Next we will translate all the terms above (taken from pencil theory) into the language of Kronecker modules (representations). A matrix pencil $A + \lambda B \in M_{m,n}(\kappa[\lambda])$ corresponds to the Kronecker module $M_{A,B} = (\kappa^m, \kappa^n; f_A, f_B)$, where choosing the canonical basis in κ^n and κ^m, the matrix of $f_A : \kappa^n \to \kappa^m$ (respectively of $f_B : \kappa^n \to \kappa^m$) is A (respectively B). The strict equivalence $A + \lambda B \sim A' + \lambda B'$ means the isomorphism of modules $M_{A,B} \cong M_{A',B'}$. It follows easily that a pencil $A' + \lambda B'$ is a subpencil of $A + \lambda B$ if and only if the module $M_{A',B'}$ is a *subfactor* of $M_{A,B}$ (i.e. there is a module N such that $M_{A',B'} \twoheadleftarrow N \hookrightarrow M_{A,B}$ or equivalently there is a module L such that $M_{A,B} \twoheadrightarrow L \hookleftarrow M_{A',B'}$, see [5]). In particular a pencil $A' + \lambda B'$ is a subpencil of $A + \lambda B$ by column completions if and only if $M_{A',B'} \hookrightarrow M_{A,B}$ with factor isomorphic to tI_0 where $t \in \mathbb{N}$ is arbitrary. Respectively, a pencil $A' + \lambda B'$ is a subpencil of $A + \lambda B$ by row completions if and only if $M_{A',B'} \twoheadleftarrow M_{A,B}$ with kernel isomorphic to tP_0 where $t \in \mathbb{N}$ is arbitrary.

A preinjective module $I_{b_1} \oplus \cdots \oplus I_{b_k}$ corresponds to the matrix pencil with the following classical Kronecker invariants:

- minimal indices for columns: $b_1, \ldots, b_k$;
- no minimal indices for rows, no finite elementary divisors, no infinite elementary divisors.

A preprojective module $P_{b_1} \oplus \cdots \oplus P_{b_k}$ corresponds to the matrix pencil with the following classical Kronecker invariants:

- minimal indices for rows: $b_1, \ldots, b_k$;
- no minimal indices for columns, no finite elementary divisors, no infinite elementary divisors.

This "subfactor–subpencil correspondence" has motivated us in the first place to study short exact sequences of Kronecker modules. Also, note that (explicit) knowledge of certain short exact sequences is sufficient for solving the matrix subpencil problem.

Certainly, the study of Kronecker modules is not interesting only because of the connection to matrix pencils. The Kronecker algebra is an important special case of tame hereditary algebra [2,9]. Moreover, the category mod-κK is derived equivalent with the category $\mathrm{Coh}(\mathbb{P}^1(\kappa))$ of coherent sheaves on the projective line (see [3]), as the Kronecker quiver K is just the Beilinson quiver for $\mathbb{P}^1$. The preinjective and the preprojective Kronecker modules correspond in $\mathrm{Coh}(\mathbb{P}^1(\kappa))$ to the indecomposable locally free coherent sheaves.

From now on, throughout the paper empty sums are considered to be zero. In case of integers a and b, by $\{a, \ldots, b\}$ we mean the set of all integers x, such that $a \leqslant x \leqslant b$, so if $a > b$, then $\{a, \ldots, b\} = \emptyset$.

2. Some results on preinjective Kronecker modules

For two preinjective modules I and I' we know that their extensions are also preinjective, in other words if we have a short exact sequence $0 \to I' \to X \to I \to 0$, then X is also preinjective (see [1,2,9]). In [11] we have given a somewhat implicit characterization based on results from [10], involving the Ringel–Hall algebra associated to the Kronecker algebra $\tilde{\kappa}K$, where $\tilde{\kappa}$ is a finite field. However, according to the main result from [13], the short exact sequences of preinjective (preprojective) Kronecker modules have the same form independently of the underlying field, so the finiteness of the base field is not a requirement anymore. Moreover, in [12] it has been shown that short exact sequences are field independent in general, not only in the preinjective (preprojective) case.

The extension of two indecomposable preinjectives has been described in [10] ($\lfloor x \rfloor$ is the largest integral value that is not greater than x):

Lemma 1. *Let I_{n_1} and I_{n_2} be two indecomposable preinjective Kronecker modules. We have a short exact sequence $0 \to I_{n_2} \to I \to I_{n_1} \to 0$ if and only if the conditions are met from one of the following two cases*:

(a) $n_1 \geqslant n_2$ *and* $[I] = [I_{n_1} \oplus I_{n_2}]$,
(b) $n_1 < n_2$ *and* $[I] \in \{[I_{n_2} \oplus I_{n_1}], [I_{n_2-1} \oplus I_{n_1+1}], \ldots, [I_{n_2-\lfloor \frac{n_2-n_1}{2} \rfloor} \oplus I_{n_1+\lfloor \frac{n_2-n_1}{2} \rfloor}]\}$.

One of the main results from [11] is a generalization of the previous lemma for the case of two arbitrary preinjective Kronecker modules:

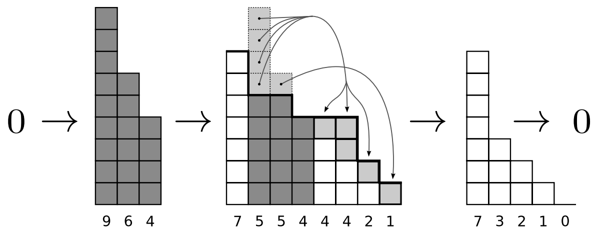
Theorem 2. *If $a_1 \geqslant \cdots \geqslant a_p \geqslant 0$, $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$ and $c_1 \geqslant \cdots \geqslant c_r \geqslant 0$ are nonnegative integers, then there exists a short exact sequence $0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \to I \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0$ if and only if $[I] = [I_{c_1} \oplus \cdots \oplus I_{c_r}]$, $r = n + p$, $\exists \beta : \{1, \ldots, n\} \to \{1, \ldots, n + p\}$, $\exists \alpha : \{1, \ldots, p\} \to \{1, \ldots, n + p\}$ both functions strictly increasing with $\operatorname{Im}\alpha \cap \operatorname{Im}\beta = \emptyset$ and $\exists m^i_j \geqslant 0$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, such that $\forall \ell \in \{1, \ldots, n + p\}$*

$$c_\ell = \begin{cases} b_i - \sum_{\beta(i)<\alpha(j),\, 1\leqslant j\leqslant p} m^i_j, \text{ where } i = \beta^{-1}(\ell), & \ell \in \operatorname{Im}\beta, \\ a_j + \sum_{\beta(i)<\alpha(j),\, 1\leqslant i\leqslant n} m^i_j, \text{ where } j = \alpha^{-1}(\ell), & \ell \in \operatorname{Im}\alpha. \end{cases} \tag{2.1}$$

Example 3. Using the previous theorem we can make sure that there exists a short exact sequence of the form:

$$0 \to I_9 \oplus I_6 \oplus I_4 \to I_7 \oplus I_5 \oplus I_5 \oplus I_4 \oplus I_4 \oplus I_4 \oplus I_2 \oplus I_1 \to I_7 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_0 \to 0.$$

Using the notation from the theorem, we have $p = 5$, $n = 3$, $r = 8$ and the strictly increasing functions are $\beta : \{1, 2, 3\} \to \{1, \ldots, 8\}$ with $\beta(1) = 2$, $\beta(2) = 3$, $\beta(3) = 4$ and $\alpha : \{1, \ldots, 5\} \to \{1, \ldots, 8\}$ with $\alpha(1) = 1$, $\alpha(2) = 5$, $\alpha(3) = 6$, $\alpha(4) = 7$, $\alpha(5) = 8$. For the values m^i_j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, we can choose $m^1_2 = m^1_4 = m^2_5 = 1$, $m^1_3 = 2$ and $m^i_j = 0$ in all other cases. With these values, Eq. (2.1) is satisfied (remind that empty sums are taken to be zero). We can illustrate this as follows:



So, less formally, Theorem 2 claims that a short exact sequence of the form $0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \to I_{c_1} \oplus \cdots \oplus I_{c_r} \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0$ exists if and only if the sequence $c_1 \geqslant \cdots \geqslant c_r \geqslant 0$ is obtained by merging the sequences $a_1 \geqslant \cdots \geqslant a_p \geqslant 0$ and $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$ and by applying the "box dropping rule" illustrated in the picture above. This rule says that in the middle term boxes can be dropped only to the right and only from columns corresponding to elements of the sequence $(b_1, \ldots, b_n)$ on top of columns corresponding to elements of the sequence $(a_1, \ldots, a_p)$. A value m^i_j denotes the number of boxes dropped from the column corresponding to the element b_i on top of the column corresponding to the element a_j and the functions α and β give the positions of the columns in the sequence $(c_1, \ldots, c_r)$ corresponding to the elements from the sequences $(a_1, \ldots, a_p)$ respectively $(b_1, \ldots, b_n)$.

In order to be able to handle this characterization computationally in the most efficient way possible, we must get rid of the condition requiring the existence of the nonnegative integers m^i_j above. In the following two lemmas, we replace the condition involving the existence by some inequalities depending only on the sequences $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_n)$ and $(c_1, \ldots, c_r)$ and on the functions α and β.

Lemma 4. *Let $\alpha : \{1, \ldots, p\} \to \{1, \ldots, n + p\}$ and $\beta : \{1, \ldots, n\} \to \{1, \ldots, n + p\}$ be two strictly increasing functions with $\operatorname{Im}\alpha \cap \operatorname{Im}\beta = \emptyset$, where $p, n > 0$. Let further $a_1 \geqslant \cdots \geqslant a_p \geqslant 0$, $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$ and $c_1 \geqslant \cdots \geqslant c_{n+p} \geqslant 0$ be decreasing sequences of nonnegative integers. Then $\exists m^i_j \geqslant 0$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, such that $\forall \ell \in \{1, \ldots, n + p\}$*

$$c_\ell = \begin{cases} b_i - \sum_{\beta(i)<\alpha(j),\, 1\leqslant j\leqslant p} m^i_j, \ \text{where } i=\beta^{-1}(\ell), & \ell\in \operatorname{Im}\beta, \\ a_j + \sum_{\beta(i)<\alpha(j),\, 1\leqslant i\leqslant n} m^i_j, \ \text{where } j=\alpha^{-1}(\ell), & \ell\in \operatorname{Im}\alpha \end{cases}$$

if and only if $b_i \geqslant c_{\beta(i)}$ *and* $a_j \leqslant c_{\alpha(j)}$ *for* $1\leqslant i\leqslant n$, $1\leqslant j\leqslant p$, $\sum_{i=1}^{n+p} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$ *and for any* $j\in\{1,\dots,p\}$ *the following inequality is satisfied*:

$$\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} (b_i - c_{\beta(i)}) \geqslant \sum_{k=1}^{j} (c_{\alpha(k)} - a_k). \tag{2.2}$$

Proof. "$\Longrightarrow$". The equality of sums $\sum_{i=1}^{n+p} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$ as well as the inequalities $b_i \geqslant c_{\beta(i)}$ and $a_j \leqslant c_{\alpha(j)}$ are obvious. Inequality (2.2) is again easy. Let $j\in\{1,\dots,p\}$ and we immediately get

$$\sum_{k=1}^{j}(c_{\alpha(k)}-a_k) = \sum_{k=1}^{j} \sum_{\substack{\beta(i)<\alpha(k)\\ 1\leqslant i\leqslant n}} m^i_k = \sum_{\substack{\beta(i)<\alpha(k)\\ 1\leqslant k\leqslant j\\ 1\leqslant i\leqslant n}} m^i_k \leqslant \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant k\leqslant p\\ 1\leqslant i\leqslant n}} m^i_k = \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} (b_i - c_{\beta(i)}).$$

"$\Longleftarrow$". We give a way to construct the values $m^i_j \geqslant 0$, for $1\leqslant i\leqslant n$, $1\leqslant j\leqslant p$ so that the sequence $(c_1,\dots,c_{n+p})$ can be written in the form shown in Eq. (2.1). Let us first introduce the following sets for $1\leqslant j\leqslant p$ and $1\leqslant i\leqslant n$:

$$M^i = \left\{ \sum_{k=1}^{i-1}(b_k - c_{\beta(k)}) + 1, \dots, \sum_{k=1}^{i}(b_k - c_{\beta(k)}) \right\},$$

$$M_j = \left\{ \sum_{k=1}^{j-1}(c_{\alpha(k)} - a_k) + 1, \dots, \sum_{k=1}^{j}(c_{\alpha(k)} - a_k) \right\}$$

and set $m^i_j = |M^i \cap M_j|$. In what follows, we are going to show that indeed, this is a valid choice.

First, note the following important properties of the sets M^i and M_j: $|M^i| = b_i - c_{\beta(i)}$, $|M_j| = c_{\alpha(j)} - a_j$, $M^{i_1}\cap M^{i_2} = M_{j_1}\cap M_{j_2} = \emptyset$ for any $i_1,i_2\in\{1,\dots,n\}$, $i_1\neq i_2$ and $j_1,j_2\in\{1,\dots,p\}$, $j_1\neq j_2$. Moreover, since $\sum_{j=1}^{p} a_j + \sum_{i=1}^{n} b_i = \sum_{k=1}^{p+n} c_k = \sum_{j=1}^{p} c_{\alpha(j)} + \sum_{i=1}^{n} c_{\beta(i)}$, $|\bigcup_{j=1}^{p} M_j| = \sum_{j=1}^{p}(c_{\alpha(j)} - a_j)$ and $|\bigcup_{i=1}^{n} M^i| = \sum_{i=1}^{n}(b_i - c_{\beta(i)})$, follows that $|\bigcup_{j=1}^{p} M_j| = |\bigcup_{i=1}^{n} M^i|$ and consequently $M = \bigcup_{j=1}^{p} M_j = \bigcup_{i=1}^{n} M^i$. So we may conclude that $\{M_j \mid 1\leqslant j\leqslant p\}$ and $\{M^i \mid 1\leqslant i\leqslant n\}$ are both partitions of the same set M.

Now, let us note some consequences of the inequality (2.2):

$$\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} (b_i - c_{\beta(i)}) = \left| \bigcup_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} M^i \right| \geqslant \sum_{k=1}^{j}(c_{\alpha(k)} - a_k) = \left| \bigcup_{k=1}^{j} M_k \right|,$$

from which (taking into account the definitions of M_j and M^i) follows, that

$$\bigcup_{k=1}^{j} M_k \subseteq \bigcup_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} M^i \tag{2.3}$$

and

$$M \setminus \left(\bigcup_{k=1}^{j} M_k\right) = \bigcup_{k=j+1}^{p} M_k \supseteq \bigcup_{\substack{\beta(i)>\alpha(j)\\ 1\leqslant i\leqslant n}} M^i = M \setminus \Bigg(\bigcup_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} M^i\Bigg). \tag{2.4}$$

Another immediate consequence is that $\beta(i) > \alpha(j) \Longrightarrow m_j^i = |M^i \cap M_j| = |\emptyset| = 0$, since by (2.4) we have $M^i \subseteq \bigcup_{k=j+1}^{p} M_k$ and $(\bigcup_{k=j+1}^{p} M_k) \cap M_j = \emptyset$.

Finally, we are ready to verify the correctness of our choice for the values m_j^i, $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$. First let $j \in \{1,\dots,p\}$. Using (2.3) we can write

$$\begin{aligned}\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} m_j^i &= \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} \left|M^i \cap M_j\right| = \Bigg|\bigcup_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} \left(M^i \cap M_j\right)\Bigg| = \Bigg|M_j \cap \Bigg(\bigcup_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant i\leqslant n}} M^i\Bigg)\Bigg| \\ &= |M_j| = c_{\alpha(j)} - a_j.\end{aligned}$$

Now let $i \in \{1,\dots,n\}$. Using $\beta(i) > \alpha(j) \Longrightarrow m_j^i = 0$ we have that

$$\begin{aligned}\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant j\leqslant p}} m_j^i &= \sum_{j=1}^{p} m_j^i = \sum_{j=1}^{p} \left|M^i \cap M_j\right| = \left|\bigcup_{j=1}^{p} \left(M^i \cap M_j\right)\right| = \left|M^i \cap \left(\bigcup_{j=1}^{p} M_j\right)\right| \\ &= \left|M^i \cap M\right| = \left|M^i\right| = b_i - c_{\beta(i)},\end{aligned}$$

so we can conclude that using this choice for the m_j^i we can obtain the sequence $(c_1,\dots,c_{n+p})$ in the requested form. □

Another, equivalent way of stating Lemma 4 is the following one, where the inequality (2.2) involving the sum of the first j elements from the sequence $(a_1,\dots,a_p)$ is replaced by the inequality (2.5), dealing instead with the sum of last i elements from the sequence $(b_1,\dots,b_n)$. We state Lemma 5 without proof, since everything goes analogously to the proof of Lemma 4.

Lemma 5. *Let $\alpha : \{1,\dots,p\} \to \{1,\dots,n+p\}$ and $\beta : \{1,\dots,n\} \to \{1,\dots,n+p\}$ be two strictly increasing functions with $\operatorname{Im}\alpha \cap \operatorname{Im}\beta = \emptyset$, where $p, n > 0$. Let further $a_1 \geqslant \dots \geqslant a_p \geqslant 0$, $b_1 \geqslant \dots \geqslant b_n \geqslant 0$ and $c_1 \geqslant \dots \geqslant c_{n+p} \geqslant 0$ be decreasing sequences of nonnegative integers. Then $\exists m_j^i \geqslant 0$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, such that $\forall \ell \in \{1,\dots,n+p\}$*

$$c_\ell = \begin{cases} b_i - \sum_{\beta(i)<\alpha(j),\, 1\leqslant j\leqslant p} m_j^i, & \text{where } i = \beta^{-1}(\ell), \quad \ell \in \operatorname{Im}\beta, \\ a_j + \sum_{\beta(i)<\alpha(j),\, 1\leqslant i\leqslant n} m_j^i, & \text{where } j = \alpha^{-1}(\ell), \quad \ell \in \operatorname{Im}\alpha \end{cases}$$

if and only if $b_i \geqslant c_{\beta(i)}$ and $a_j \leqslant c_{\alpha(j)}$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, $\sum_{i=1}^{n+p} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$ and for any $i \in \{1,\dots,n\}$ the following inequality is satisfied:

$$\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leqslant j\leqslant p}} \left(c_{\alpha(j)} - a_j\right) \geqslant \sum_{k=i}^{n} \left(b_k - c_{\beta(k)}\right). \tag{2.5}$$

The only "uncertainty" left now from Theorem 2 is around the functions α and β. The following theorem clears everything up, characterizing the extension of preinjective Kronecker modules by explicit, easy to check numerical conditions, involving only the decreasing sequences of integers obtained from the dimension vectors of the respective modules.

Theorem 6. *Let $a_1 \geqslant \cdots \geqslant a_p \geqslant 0$, $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$, $c_1 \geqslant \cdots \geqslant c_r \geqslant 0$ be decreasing sequences of nonnegative integers and let $B_j = \{l \in \{0, \ldots, n\} \mid \sum_{k=1}^{l} b_k + \sum_{k=1}^{j} a_k \geqslant \sum_{k=1}^{l+j} c_k\}$ for $1 \leqslant j \leqslant p$. Then there is a short exact sequence*

$$0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \to I \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0$$

if and only if $[I] = [I_{c_1} \oplus \cdots \oplus I_{c_r}]$, $r = p + n$, $\sum_{i=1}^{r} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$, $B_j \neq \emptyset$, $a_j \leqslant c_{\alpha_j}$ and $b_i \geqslant c_{\beta_i}$ for $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$, where

$$\alpha_j = \begin{cases} \min B_1 + 1, & j = 1, \\ \max\{\alpha_{j-1} + 1, \min B_j + j\}, & 1 < j \leqslant p \end{cases}$$

and

$$\beta_i = \begin{cases} \min\{l \in \{1, \ldots, r\} \mid l \neq \alpha_j,\ 1 \leqslant j \leqslant p\}, & i = 1, \\ \min\{l \in \{\beta_{i-1} + 1, \ldots, r\} \mid l \neq \alpha_j,\ 1 \leqslant j \leqslant p\}, & 1 < i \leqslant n. \end{cases}$$

Proof. It is easy to see that α_j and β_i are both well defined: $a_j, \beta_i \in \{1, \ldots, p + n\}$, so they can be used to index the elements in the sequence $(c_1, \ldots, c_r)$. We are going to show that our statement is equivalent with Theorem 2.

"$\Longrightarrow$". First let us suppose that we have a short exact sequence $0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \to I_{c_1} \oplus \cdots \oplus I_{c_r} \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0$. Then $r = p + n$ and $\exists \beta : \{1, \ldots, n\} \to \{1, \ldots, p + n\}$, $\exists \alpha : \{1, \ldots, p\} \to \{1, \ldots, p+n\}$ both functions strictly increasing with $\operatorname{Im} \alpha \cap \operatorname{Im} \beta = \emptyset$ and $\exists m_j^i \geqslant 0$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, such that $\forall \ell \in \{1, \ldots, p + n\}$

$$c_\ell = \begin{cases} b_i - \sum_{\beta(i)<\alpha(j),\, 1 \leqslant j \leqslant p} m_j^i, \text{ where } i = \beta^{-1}(\ell), & \ell \in \operatorname{Im} \beta, \\ a_j + \sum_{\beta(i)<\alpha(j),\, 1 \leqslant i \leqslant n} m_j^i, \text{ where } j = \alpha^{-1}(\ell), & \ell \in \operatorname{Im} \alpha. \end{cases}$$

Using Lemma 4, the equality of sums follows immediately: $\sum_{i=1}^{r} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$.

Now, let us show that $B_j \neq \emptyset$. So let $j \in \{1, \ldots, p\}$ and consider the following inequality:

$$\sum_{\substack{\beta(i)<\alpha(j) \\ 1 \leqslant i \leqslant n}} (b_i - c_{\beta(i)}) \geqslant \sum_{k=1}^{j} (c_{\alpha(k)} - a_k),$$

which also holds by Lemma 4.

We reorder it a bit to get

$$\sum_{\substack{\beta(i)<\alpha(j) \\ 1 \leqslant i \leqslant n}} b_i + \sum_{k=1}^{j} a_k \geqslant \sum_{k=1}^{j} c_{\alpha(k)} + \sum_{\substack{\beta(i)<\alpha(j) \\ 1 \leqslant i \leqslant n}} c_{\beta(i)}$$

and by letting $l_j = |\{i \in \{1, \ldots, n\} \mid \beta(i) < \alpha(j)\}|$, we get

$$\sum_{k=1}^{l_j} b_k + \sum_{k=1}^{j} a_k \geqslant \sum_{k=1}^{j} c_{\alpha(k)} + \sum_{k=1}^{l_j} c_{\beta(k)} = \sum_{k=1}^{j+l_j} c_k,$$

exactly what we wanted, the last equality being true, because $\operatorname{Im}\alpha \cap \operatorname{Im}\beta = \emptyset$ and both functions α and β are strictly increasing. Now, obviously $0 \leqslant l_j \leqslant n \Longrightarrow l_j \in B_j$, so the set $B_j = \{l \in \{0,\dots,n\} \mid \sum_{k=1}^{l} b_k + \sum_{k=1}^{j} a_k \geqslant \sum_{k=1}^{l+j} c_k\}$ is nonempty.

It only remained to show that $a_j \leqslant c_{\alpha_j}$ and $b_i \geqslant c_{\beta_i}$ for $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$. Observe that the function α can be written in a similar way to the values α_j, i.e.

$$\alpha(j) = \begin{cases} l_1 + 1, & j = i, \\ \max\{\alpha(j-1)+1, l_j + j\}, & 1 < j \leqslant p \end{cases}$$

and β can be given in terms of the function α in a similar way to β_i:

$$\beta(i) = \begin{cases} \min\{l \in \{1,\dots,r\} \mid l \notin \operatorname{Im}\alpha\}, & i = 1, \\ \min\{l \in \{\beta(i-1)+1,\dots,r\} \mid l \notin \operatorname{Im}\alpha\}, & 1 < i \leqslant n. \end{cases}$$

Now, since $\min B_j \leqslant l_j$ for any $j \in \{1,\dots,p\}$, it follows easily that $\alpha(j) \geqslant \alpha_j$ and consequently $\beta(i) \leqslant \beta_i$ for $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$. But from Lemma 4 we know that $a_j \leqslant c_{\alpha(j)}$ and $b_i \geqslant c_{\beta(i)}$, and since $c_1 \geqslant \cdots \geqslant c_r$, we can conclude that $a_j \leqslant c_{\alpha(j)} \leqslant c_{\alpha_j}$ and $b_i \geqslant c_{\beta(i)} \geqslant c_{\beta_i}$.

"$\Longleftarrow$". Obviously $\alpha_j \neq \beta_i$ for $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$, moreover $\alpha_1 < \cdots < \alpha_p$, $\beta_1 < \cdots < \beta_n$ and $a_j, \beta_i \in \{1,\dots,p+n\}$, making them a natural choice for the functions $\alpha : \{1,\dots,p\} \to \{1,\dots,p+n\}$ and $\beta : \{1,\dots,n\} \to \{1,\dots,p+n\}$. So let $\alpha(j) = \alpha_j$ and $\beta(i) = \beta_i$. With this choice for the functions α and β we already know that $a_j \leqslant c_{\alpha(j)}$ and $b_i \geqslant c_{\beta(i)}$ and we have that for any $j \in \{1,\dots,p\}$, $\min B_j = |\{i \in \{1,\dots,n+b\} \mid \beta(i) < \alpha(j)\}|$, meaning that the steps already shown in the first part of the proof involving the inequalities may also be done in reverse order:

$$\sum_{\substack{\beta(i)<\alpha(j) \\ 1\leqslant i\leqslant n}} b_i + \sum_{k=1}^{j} a_k = \sum_{k=1}^{\min B_j} b_k + \sum_{k=1}^{j} a_k \geqslant \sum_{k=1}^{\min B_j + j} c_k = \sum_{k=1}^{j} c_{\alpha(k)} + \sum_{\substack{\beta(i)<\alpha(j) \\ 1\leqslant i\leqslant n}} c_{\beta(i)},$$

leading again to

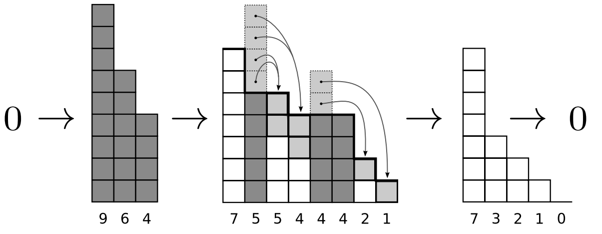
$$\sum_{\substack{\beta(i)<\alpha(j) \\ 1\leqslant i\leqslant n}} (b_i - c_{\beta(i)}) \geqslant \sum_{k=1}^{j} (c_{\alpha(k)} - a_k),$$

meaning that all the conditions in Lemma 4 are satisfied, so $\exists m_j^i \geqslant 0$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$ such that $\forall \ell \in \{1,\dots,p+n\}$, c_ℓ can be written according to (2.1) and we are done. $\square$

Example 7. Let us show how this theorem works using the same short exact sequence as in Example 3:

$$0 \to I_9 \oplus I_6 \oplus I_4 \to I_7 \oplus I_5 \oplus I_5 \oplus I_4 \oplus I_4 \oplus I_4 \oplus I_2 \oplus I_1 \to I_7 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_0 \to 0.$$

Using the notation from the theorem, we have $p = 5$, $n = 3$, $r = 8$, $B_1 = \{0,1,2,3\}$, $B_2 = B_3 = \{1,2,3\}$, $B_4 = B_5 = \{3\}$ and consequently $\alpha_1 = 1$, $\alpha_2 = 3$, $\alpha_3 = 4$, $\alpha_4 = 7$, $\alpha_5 = 8$, $\beta_1 = 2$, $\beta_2 = 5$, $\beta_3 = 6$ and the inequalities $a_j \leqslant c_{\alpha_j}$ and $b_i \geqslant c_{\beta_i}$ for $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$ hold. We can illustrate this case as follows:



Comparing this image with the one from Example 3, one can get an intuition about what Theorem 6 does: it gives an explicit way of choosing the positions of the columns corresponding to the sequences $(a_1, \dots, a_p)$ and $(b_1, \dots, b_n)$ in the sequence $(c_1, \dots, c_r)$ such that the "box dropping" can be carried out, obeying the rule explained after Example 3.

Example 8. $I_8 \oplus I_5 \oplus I_5 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_1$ is not an extension of $I_9 \oplus I_6 \oplus I_4$ by $I_7 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_0$ because $B_1 = B_2 = \{1, 2, 3\}$, $B_3 = \{2, 3\}$, $B_4 = B_5 = \{3\}$ and consequently $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 5$, $\alpha_4 = 7$, $\alpha_5 = 8$, $\beta_1 = 1$, $\beta_2 = 4$, $\beta_3 = 6$, but $7 = a_1 > c_{\alpha_1} = 5$.

Similarly, $I_9 \oplus I_8 \oplus I_6 \oplus I_5 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_0$ is not an extension of $I_9 \oplus I_6 \oplus I_4$ by $I_7 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_0$ because $B_1 = \emptyset$.

As we have already done with Lemmas 4 and 5, we give the analogue version of this theorem without proof (the proof involves the same steps, but using Lemma 5 instead of Lemma 4).

Theorem 9. *Let $a_1 \geqslant \dots \geqslant a_p \geqslant 0$, $b_1 \geqslant \dots \geqslant b_n \geqslant 0$, $c_1 \geqslant \dots \geqslant c_r \geqslant 0$ be decreasing sequences of non-negative integers and let $B^i = \{l \in \{0, \dots, p\} \mid \sum_{k=i}^n b_k + \sum_{k=1}^l a_{p+1-k} \leqslant \sum_{k=r-(n-i+l)}^r c_k\}$ for $1 \leqslant i \leqslant n$. Then there exists a short exact sequence $0 \to I_{b_1} \oplus \dots \oplus I_{b_n} \to I \to I_{a_1} \oplus \dots \oplus I_{a_p} \to 0$ if and only if $[I] = [I_{c_1} \oplus \dots \oplus I_{c_r}]$, $r = p + n$, $\sum_{i=1}^r c_i = \sum_{i=1}^p a_i + \sum_{i=1}^n b_i$, $B^i \neq \emptyset$, $a_j \leqslant c_{\alpha_j}$ and $b_i \geqslant c_{\beta_i}$ for $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$, where*

$$\beta_i = \begin{cases} r - \min B^n, & i = n, \\ \min\{\beta_{i+1} - 1, r - (n - i + \min B^i)\}, & 1 \leqslant i < n \end{cases}$$

and

$$\alpha_j = \begin{cases} \max\{l \in \{1, \dots, r\} \mid l \neq \beta_i,\ 1 \leqslant i \leqslant n\}, & j = p, \\ \max\{l \in \{1, \dots, \alpha_{j+1} - 1\} \mid l \neq \beta_i,\ 1 \leqslant i \leqslant n\}, & 1 \leqslant j < p. \end{cases}$$

3. Computing the extensions

Theorem 6 may seem thorny at first sight, so we are going to show how to use it in order to obtain an algorithm which decides in linear time whether a certain preinjective Kronecker module is an extension of two other preinjective Kronecker modules.

Suppose we are given three preinjective modules $I_{a_1} \oplus \dots \oplus I_{a_p}$, $I_{b_1} \oplus \dots \oplus I_{b_n}$ and $I_{c_1} \oplus \dots \oplus I_{c_r}$ and we want to decide if a short exact sequence of the form $0 \to I_{b_1} \oplus \dots \oplus I_{b_n} \to I_{c_1} \oplus \dots \oplus I_{c_r} \to I_{a_1} \oplus \dots \oplus I_{a_p} \to 0$ can be written. Obviously, if $r \neq p + n$ or $\sum_{k=1}^r c_r \neq \sum_{j=1}^p a_j + \sum_{i=1}^n b_i$, the answer is a quick and unhesitating no, so in what follows, we suppose that $r = p + n$ and $\sum_{k=1}^r c_r = \sum_{j=1}^p a_j + \sum_{i=1}^n b_i$ both hold, and we work only with the decreasing sequences $(a_1, \dots, a_p)$, $(b_1, \dots, b_n)$ and $(c_1, \dots, c_r)$.

The practical use of Theorem 6 involves for every element of the sequence $(c_1, \dots, c_r)$ a choice of one element either from the sequence $(a_1, \dots, a_p)$ or from $(b_1, \dots, b_n)$ – in other words, for every indecomposable summand of the module $I_{c_1} \oplus \dots \oplus I_{c_r}$ there is a corresponding indecomposable summand either from $I_{a_1} \oplus \dots \oplus I_{a_p}$ or from $I_{b_1} \oplus \dots \oplus I_{b_n}$. The values α_j and β_i (for $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$) in Theorem 6 describe this choice and indicate the positions in the sequence $(c_1, \dots, c_r)$

against which elements from the sequence $(a_1, \dots, a_p)$ respectively $(b_1, \dots, b_n)$ must be checked. From the definition of the set B_j, one can observe that the condition $B_j \neq \emptyset$ practically says that for each a_j there should be a certain number of elements from the sequence $(b_1, \dots, b_n)$ such that positioned in front of a_j, inequality (2.2) from Lemma 4 is satisfied with α_j being actually the smallest such position (i.e. a minimal number of elements b_i put in front of a_j to satisfy the inequalities). This justifies the strategy of choosing between the elements of the sequences $(a_1, \dots, a_p)$ and $(b_1, \dots, b_n)$: first try with a value a_j and only if that fails (one of the inequalities cannot be satisfied) try with a b_i. If none of the choices are possible, then the whole construction fails, meaning there is no exact sequence involving the three modules.

So, let us set the initial values $j = i = k = 1$ for the integers used to index elements from the sequences $(a_1, \dots, a_p)$, $(b_1, \dots, b_n)$ respectively $(c_1, \dots, c_r)$. In a practical implementation one can repeat the following steps for all successive values of $1 \leqslant k \leqslant r$:

(1) If $j \leqslant p$ and $a_j \leqslant c_k$ and $(a_1 + \dots + a_{j-1}) + (b_1 + \dots + b_{i-1}) + a_j \geqslant c_1 + \dots + c_k$, then increase j by one.
(2) Else, if $i \leqslant n$ and $b_i \geqslant c_k$ and $(a_1 + \dots + a_{j-1}) + (b_1 + \dots + b_{i-1}) + b_i \geqslant c_1 + \dots + c_k$, then increase i by one.
(3) If none of the steps above can be carried out then stop with a negative answer.

Finally, if one of the first two steps can be made for $k = r$ too, then return a positive answer, i.e. we have a $0 \to I_{b_1} \oplus \dots \oplus I_{b_n} \to I_{c_1} \oplus \dots \oplus I_{c_r} \to I_{a_1} \oplus \dots \oplus I_{a_p} \to 0$ exact.

It is trivial to see that the algorithm is linear in the number of indecomposables (i.e. in $r = n + p$), since the only cycle in the algorithm runs at most r times and the partial sums $a_1 + \dots + a_j$, $b_1 + \dots + b_i$ and $c_1 + \dots + c_k$ can be computed one term at a time at every iteration.

Remark 10. One immediate application of the algorithm we have just presented is that it can be readily used to decide in linear time if a matrix pencil $A' + \lambda B'$ having only minimal indices for columns in the set of its classical Kronecker invariants, is a subpencil or not by column completions of $A + \lambda B$ of the same type (see introduction). Such a matrix pencil $A' + \lambda B'$ is subpencil by column completion of $A + \lambda B$ if and only if a short exact sequence of the form $0 \to M_{A',B'} \to M_{A,B} \to nI_0 \to 0$ exists, where both $M_{A,B}$ and $M_{A',B'}$ are preinjectives and n is the difference between the number of columns of the matrices $A + \lambda B$ and $A' + \lambda B'$.

To develop an algorithm for generating all the extensions X in the short exact sequence $0 \to I' \to X \to I \to 0$, we could use of course "brute force" and generate all possible modules while checking every one in part with the previous method. But we can do a little better than that for example by using the method of non-recursive backtracking (also known as "iterative backtracking") to generate all the possible modules X. In general, using the backtracking method, one can find all solutions to some computational problem, by incrementally building solution candidates, and abandoning each partial candidate as soon as it is determined that the candidate cannot possibly be completed to a valid solution (see [7]).

In our case the space of possible solutions (or candidates) is a subset of the set all decreasing sequences of nonnegative integers $(c_1, \dots, c_r)$ with a fixed length and a fixed sum, i.e. $r = p + n$ and $\sum_{k=1}^{r} c_k = s = \sum_{j=1}^{p} a_j + \sum_{i=1}^{n} b_i$. First, observe that we have the following recursive relation between the elements of any such sequence ($\lceil x \rceil$ meaning the smallest integral value that is not less than x):

$$\begin{cases} \left\lceil \dfrac{s}{r} \right\rceil \leqslant c_k \leqslant s & \text{if } k = 1, \\ \left\lceil \dfrac{s - \sum_{l=1}^{k-1} c_l}{r - k + 1} \right\rceil \leqslant c_k \leqslant \min\left\{ c_{k-1}, s - \displaystyle\sum_{l=1}^{k-1} c_l \right\} & \text{if } 1 < k \leqslant r. \end{cases} \tag{3.1}$$

So a sequence of nonnegative integers $(c_1, \dots, c_r)$ having length r is decreasing and has the sum equal to $s \geqslant 0$, if and only if relation (3.1) is satisfied by its elements.

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Thus, we say that a decreasing sequence of nonnegative integers $(c_1, \dots, c_k)$ with $1 \leqslant k \leqslant r$ and $c_1 + \dots + c_k \leqslant s$ is a valid partial candidate if $\sum_{l=1}^{j'} a_l + \sum_{l=1}^{i'} b_l \geqslant c_1 + \dots + c_k$ for some $j' \in \{0, \dots, p\}$ and $i' \in \{0, \dots, n\}$ such that $i' + j' = k$, $a_j \leqslant c_{\alpha_j}$ and $b_i \geqslant c_{\beta_i}$ for all $j \in \{1, \dots, j'\}$ and $i \in \{1, \dots, i'\}$, where α_j and β_i are the indices defined in Theorem 6, moreover relation (3.1) is satisfied by all the elements of the sequence $(c_1, \dots, c_k)$. If $k = r$ and $(c_1, \dots, c_k)$ is a valid partial candidate, then it is also a solution. If $k < r$, then we try to extend it to a valid partial candidate of length $k + 1$ by computing the set $S_{k+1}(c_1, \dots, c_k)$ of all possible values for c_{k+1} turning $(c_1, \dots, c_{k+1})$ into a valid partial candidate.

Starting at level 1 (with $k = 1$) and working on the sequence $(c_1, \dots, c_r)$, i.e. assigning different values to an array of integers denoted by $c_1, \dots, c_r$, the general steps of the backtracking algorithm at a certain level k are:

(1) If $k > r$, add the current sequence $(c_1, \dots, c_r)$ to the result set and backtrack (return to the previous level by decreasing k by one).
(2) If $k \leqslant r$ and level k has been reached from the previous level then:
  (a) Compute the set $S_k(c_1, \dots, c_{k-1})$ of nonnegative integers, such that $\forall c_k \in S_k(c_1, \dots, c_{k-1})$, the sequence $(c_1, \dots, c_k)$ is a valid partial candidate.
  (b) If $S_k(c_1, \dots, c_{k-1}) = \emptyset$, then backtrack (since $(c_1, \dots, c_{k-1})$ cannot possibly be completed to a valid solution).
  (c) Else, assign a value from $S_k(c_1, \dots, c_{k-1})$ to c_k (for example the greatest) and pass to the next level (i.e. increase k by one).
(3) If $k \leqslant r$ and level k has been reached from the next, higher level (as a result of a backtracking) then:
  (a) If there are any unused values in the set $S_k(c_1, \dots, c_{k-1})$, assign a new value (for example the greatest unused value) to c_k and pass to the next level.
  (b) If all the elements of the set $S_k(c_1, \dots, c_{k-1})$ have been tried, backtrack.

The algorithm terminates when k gets assigned the value 0.

Remark 11. The purpose of this algorithm is to serve as a helping tool in investigating the extensions of preinjective Kronecker modules by generating examples for modules with smaller dimensions. While the decision problem was solved in linear time, the number of extensions can be huge. In worst case, when we want to compute all the possible modules I in the short exact sequence $0 \to mI_0 \to I \to I_n \to 0$ where $m \geqslant n$, the number of isoclasses $[I]$ is $\mathcal{P}(n)$, with $\mathcal{P}(n)$ being the number of partitions of the integer n. It is well known that $\mathcal{P}(n)$ can be estimated by the Hardy–Ramanujan asymptotic formula [6]:

$$\mathcal{P}(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

While in most of the cases the number of extensions is much smaller, $\mathcal{P}(n)$ – with n being the sum of the dimensions of the two preinjectives – can be used as very rough and overly pessimistic estimate for the upper bound, that is for the worse one can expect in terms of output size and running time. In practice however, we have found that using the method described, one can generate almost instantly extensions when they are up to around 100 000 in number, so the algorithm fits its purpose quiet well.

Example 12. Using a non-recursive (iterative) backtracking implementation in GAP [14] of our method it turns out that:

(1) There are 18 possible isoclasses of modules $[I]$ such that there is a short exact sequence

$$0 \to I_5 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \to I \to I_4 \oplus I_3 \oplus I_1 \oplus I_0 \to 0,$$

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namely the following: $[I] \in \{[I_5 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_0], [I_5 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_1 \oplus I_1 \oplus I_1 \oplus I_1], [I_5 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_0], [I_5 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_1], [I_5 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1], [I_4 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_0], [I_4 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_1 \oplus I_1 \oplus I_1 \oplus I_1], [I_4 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_0], [I_4 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_1], [I_4 \oplus I_4 \oplus I_4 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1], [I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_0], [I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_1], [I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1], [I_4 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1], [I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_0], [I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_1 \oplus I_1 \oplus I_1], [I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1], [I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1]\}$.

(2) There are 102 501 possible isoclasses $[I]$ such that there is a short exact sequence

$$0 \to I_{26} \oplus I_{12} \oplus I_{10} \oplus I_8 \oplus I_4 \to I \to I_{19} \oplus I_{15} \oplus I_8 \oplus I_4 \oplus I_1 \oplus I_1 \to 0,$$

all of which were generated in 2 seconds on a laptop computer.

(3) There are 3 322 698 possible isoclasses $[I]$ such that a short exact sequence of the form

$$0 \to I_{20} \oplus I_{19} \oplus I_{18} \oplus I_{10} \oplus I_8 \oplus I_3 \oplus I_2 \oplus I_2 \to I$$
$$\to I_{16} \oplus I_{11} \oplus I_7 \oplus I_6 \oplus I_3 \oplus I_1 \oplus I_1 \oplus I_0 \to 0$$

exists, and all the 3 322 698 extensions were generated in just under 2 minutes on a laptop computer.

4. Embedding preinjective Kronecker modules

In [13] we have given simple numerical criteria in terms of Kronecker invariants for the existence of a monomorphism $I' \hookrightarrow I$ between two preinjective modules using a homological proof. We will reprove this theorem using only Theorem 2 and Lemma 5.

Theorem 13. *Let $b_1 \geqslant \cdots \geqslant b_n > 0$ and $c_1 \geqslant \cdots \geqslant c_r > 0$ be decreasing sequences of integers. We have a monomorphism*

$$f : I_{b_1} \oplus \cdots \oplus I_{b_n} \oplus bI_0 \to I_{c_1} \oplus \cdots \oplus I_{c_r} \oplus cI_0$$

if and only if $b \leqslant c$ and $b_i + \cdots + b_n \leqslant \sum_{c_k \leqslant b_i} c_k$ for $1 \leqslant i \leqslant n$.

Proof. We know that the existence of a monomorphism f is equivalent with the existence of a short exact sequence

$$0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \oplus bI_0 \to I_{c_1} \oplus \cdots \oplus I_{c_r} \oplus cI_0 \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0.$$

"$\Longrightarrow$". If $I_{c_1} \oplus \cdots \oplus I_{c_r} \oplus cI_0$ is an extension of $I_{b_1} \oplus \cdots \oplus I_{b_n} \oplus bI_0$ by $I_{a_1} \oplus \cdots \oplus I_{a_p}$, where $c_1 \geqslant \cdots \geqslant c_r > c_{r+1} = \cdots = c_{r+c} = 0$ and $b_1 \geqslant \cdots \geqslant b_n > b_{n+1} = \cdots = b_{n+b} = 0$, then by Theorem 2 and Lemma 5 we know that $r + c = p + n + b$, $\exists \alpha : \{1, \ldots, p\} \to \{1, \ldots, p + n + b\}$ and $\exists \beta : \{1, \ldots, n\} \to \{1, \ldots, p + n + b\}$ two strictly increasing functions with $\operatorname{Im} \alpha \cap \operatorname{Im} \beta = \emptyset$ such that $b_i \geqslant c_{\beta(i)}$ and the inequality $\sum_{\beta(i) < \alpha(j),\, 1 \leqslant j \leqslant p} (c_{\alpha(j)} - a_j) \geqslant \sum_{k=i}^{n+b} (b_k - c_{\beta(k)})$ is satisfied for $i \in \{1, \ldots, n + b\}$.

It is clear that $0 = b_{n+1} \geqslant c_{\beta(n+1)} \Longrightarrow c_{\beta(n+1)} = 0 \Longrightarrow c_{\beta(n+1)} = \cdots = c_{r+c} = 0$, so $r + 1 \leqslant \beta(n + 1)$. But since we must have $\beta(n + 1) \leqslant p + n + 1 = r + c - b + 1$ (otherwise we would get $\beta(n + b) \notin \{1, \ldots, p + n + b\}$), follows that $b \leqslant c$. Let now $i \in \{1, \ldots, n + b\}$ and consider the inequality $\sum_{\beta(i) < \alpha(j),\, 1 \leqslant j \leqslant p} (c_{\alpha(j)} - a_j) \geqslant \sum_{k=i}^{n+b} (b_k - c_{\beta(k)})$. By reordering and leaving out a sum we get

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$$\sum_{k=i}^{n} b_k = \sum_{k=i}^{n+b} b_k \leqslant \sum_{\substack{\beta(i)<\alpha(j)\\1\leqslant j\leqslant p}} c_{\alpha(j)} + \sum_{k=i}^{n+b} c_{\beta(k)} - \sum_{\substack{\beta(i)<\alpha(j)\\1\leqslant j\leqslant p}} a_j \leqslant \sum_{\beta(i)\leqslant k\leqslant r+c} c_k$$

and because $b_i \geqslant c_{\beta(i)} \geqslant \cdots \geqslant c_{r+c}$ follows that $b_i + \cdots + b_n \leqslant \sum_{\beta(i)\leqslant k\leqslant r+c} c_k \leqslant \sum_{c_k \leqslant b_i} c_k$.

"$\Longleftarrow$". Easy recursive argument shows that in the inequality $b_i + \cdots + b_n \leqslant \sum_{c_k\leqslant b_i} c_k$ the sum on the right hand side contains at least as many summands as the sum on the left hand side, i.e. for any $i \in \{1,\dots,n\}$, $|\{c_k \mid c_k \leqslant b_i,\ 1 \leqslant k \leqslant r\}| \geqslant n - i + 1$. This also means, that $r \geqslant n$ and (since $c \geqslant b$) $r + c \geqslant n + b$. So we can choose a strictly increasing sequence of integers $(l_1,\dots,l_n)$ such that $1 \leqslant l_1 < \cdots < l_n \leqslant r$ and for any $i \in \{1,\dots,n\}$, $b_i + \cdots + b_n \leqslant \sum_{k=l_i}^{r} c_k$. If $r + c = n + b$, then clearly $r = n$, $c = b$, $b_i = c_i$ for $1 \leqslant i \leqslant n$ and f is just the identity, i.e. the trivial embedding of $I_{b_1} \oplus \cdots \oplus I_{b_n} \oplus bI_0$ in itself. From now on we suppose that $r + c > n + b$.

Let us now define a strictly increasing function $\beta' : \{1,\dots,n\} \to \{1,\dots,r+c-b\}$, $\beta'(i) = l_i$, so that our inequality now reads $\sum_{k=\beta'(i)}^{r} c_k \geqslant \sum_{k=i}^{n} b_k$. With our choice for β' we know that $b_i \geqslant c_{\beta'(i)}$, $1 \leqslant i \leqslant n$, so extracting these values from the previous inequality we get

$$\sum_{\substack{\beta'(i)\leqslant k\leqslant r\\ k\notin \operatorname{Im}\beta'}} c_k \geqslant \sum_{k=i}^{n} (b_k - c_{\beta'(k)}),$$

which looks promising because it begins to resemble the inequality (2.5) from Lemma 5. To make it look even better, let us introduce the strictly increasing function $\alpha' : \{1,\dots,p\} \to \{1,\dots,r+c-b\}$,

$$\alpha'(j) = \begin{cases} \min\{l \in \{1,\dots,r\} \mid l \neq \beta'(i),\ 1 \leqslant i \leqslant n\}, & j = 1,\\ \min\{l \in \{\alpha'(j-1)+1,\dots,r\} \mid l \neq \beta'(i),\ 1 \leqslant i \leqslant n\}, & 1 < j \leqslant p, \end{cases}$$

where $0 < p = r + c - (n+b)$. Using it we have so far

$$\sum_{\substack{\beta'(i)<\alpha'(j)\\1\leqslant j\leqslant p}} c_{\alpha'(j)} \geqslant \sum_{k=i}^{n} (b_k - c_{\beta'(k)}),$$

with only the sequence $a_1 \geqslant \cdots \geqslant a_p \geqslant 0$ missing. We claim that there exist decreasing sequences such that inequality (2.5) holds and in what follows, we are going to show how to construct one. We borrow the idea from the proof of Lemma 4, so let in this case

$$M^b = \left\{ \sum_{k=1}^{p} c_{\alpha'(k)} - \sum_{k=1}^{n} (b_k - c_{\beta'(k)}) + 1, \dots, \sum_{k=1}^{p} c_{\alpha'(k)} \right\},$$

$$M_j = \left\{ \sum_{k=1}^{j-1} c_{\alpha'(k)} + 1, \dots, \sum_{k=1}^{j} c_{\alpha'(k)} \right\}$$

and set $a_j = |M_j \setminus M^b|$. It is clear that for $j \in \{1,\dots,p\}$, $|M_j| = c_{\alpha'(j)}$, $|M^b| = \sum_{k=1}^{n} (b_k - c_{\beta'(k)})$, $M^b \subseteq \bigcup_{j=1}^{p} M_j$ and $M_{j_1} \cap M_{j_2} = \emptyset$ for $j_1, j_2 \in \{1,\dots,p\}$, $j_1 \neq j_2$, so we have

$$\sum_{j=1}^{p} a_j = \sum_{j=1}^{p} |M_j \setminus M^b| = \left| \bigcup_{j=1}^{p} (M_j \setminus M^b) \right| = \left| \left(\bigcup_{j=1}^{p} M_j \right) \setminus M^b \right|$$
$$= \left| \left(\bigcup_{j=1}^{p} M_j \right) \right| - |M^b| = \sum_{j=1}^{p} c_{\alpha'(j)} - \sum_{k=1}^{n} (b_k - c_{\beta'(k)}) = \sum_{k=1}^{r} c_k - \sum_{k=1}^{n} b_k.$$

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Observe that (because $M^b \subseteq \bigcup_{j=1}^{p} M_j$) the set $M_j \setminus M^b$ can be either M_j (always the case when $\alpha'(j) < \beta'(1)$) or some M such that $\emptyset \subsetneq M \subsetneq M_j$ (for at most one j, where $\alpha'(j) > \beta'(1)$) or the empty set. Formally,

$$M_j \setminus M^b = \begin{cases} M_j, & 1 \leqslant j < j', \\ M \subseteq M_j, & j = j', \\ \emptyset, & j' < j \leqslant p \end{cases} \quad\Longrightarrow\quad a_j = \begin{cases} c_{\alpha'(j)}, & 1 \leqslant j < j', \\ a_j \leqslant c_{\alpha'(j)}, & j = j', \\ 0, & j' < j \leqslant p, \end{cases}$$

for some $j' \in \{1, \dots, p\}$, such that $\alpha'(j') > \beta'(1)$. Now it is clear that $a_1 \geqslant \cdots \geqslant a_p$ and $a_j \leqslant c_{\alpha'(j)}$, $1 \leqslant j \leqslant p$. We have already seen that $\sum_{k=1}^{n}(b_k - c_{\beta'(k)}) = \sum_{j=1}^{p} c_{\alpha'(j)} - \sum_{j=1}^{p} a_j$. Knowing the structure of the sequence $(a_1, \dots, a_p)$ we can write

$$\begin{aligned} \sum_{k=1}^{n}(b_k - c_{\beta'(k)}) &= \sum_{j=1}^{p} c_{\alpha'(j)} - \left(\sum_{j=1}^{j'-1} a_j + a_{j'} + \sum_{j=j'+1}^{p} a_j\right) \\ &= \underbrace{\sum_{j=1}^{j'-1}(c_{\alpha'(j)} - a_j)}_{=0} + (c_{\alpha'(j')} - a_{j'}) + \sum_{j=j'+1}^{p} (\underbrace{c_{\alpha'(j)} - a_j}_{=c_{\alpha'(j)}}) \end{aligned}$$

and because $\sum_{k=i}^{n}(b_k - c_{\beta'(k)}) \leqslant \sum_{k=1}^{n}(b_k - c_{\beta'(k)})$ for all $i \in \{1, \dots, n\}$,

$$\sum_{\substack{\beta'(i)<\alpha'(j)\\ 1\leqslant j\leqslant p}} (c_{\alpha'(j)} - a_j) \geqslant \sum_{k=i}^{n}(b_k - c_{\beta'(k)})$$

follows.

Consider now the functions $\alpha : \{1, \dots, p\} \to \{1, \dots, r + c\}$, $\alpha(j) = \alpha'(j)$ and $\beta : \{1, \dots, n + b\} \to \{1, \dots, r + c\}$,

$$\beta(i) = \begin{cases} \beta'(i), & 1 \leqslant i \leqslant n, \\ r + c - b + (i - n), & n < i \leqslant n + b. \end{cases}$$

Functions α and β are strictly increasing, clearly satisfy $\operatorname{Im}\alpha \cap \operatorname{Im}\beta = \emptyset$, and since $c_{r+1} = \cdots = c_{r+c} = b_{n+1} = \cdots = b_{n+b} = 0$, we have that $b_i \geqslant c_{\beta(i)}$ and $a_j \leqslant c_{\alpha(j)}$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, $\sum_{i=1}^{r+c} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n+b} b_i$, moreover, inequality $\sum_{\beta'(i)<\alpha'(j),\, 1\leqslant j\leqslant p}(c_{\alpha'(j)} - a_j) \geqslant \sum_{k=i}^{n}(b_k - c_{\beta'(k)})$ can be extended to $\sum_{\beta(i)<\alpha(j),\, 1\leqslant j\leqslant p}(c_{\alpha(j)} - a_j) \geqslant \sum_{k=i}^{n+b}(b_k - c_{\beta(k)})$ and the existence of the exact sequence $0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \oplus bI_0 \to I_{c_1} \oplus \cdots \oplus I_{c_r} \oplus cI_0 \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0$ follows by Lemma 5 and Theorem 2. $\square$

Remark 14. If we know that $I' \hookrightarrow I$, then the possible factors I/I' are explicitly described by the result in Theorem 6. So, if we are given three preinjective Kronecker modules I, I' and I'' such that $I' \hookrightarrow I$, then we can decide in linear time if $[I''] \in \{[I/\operatorname{Im} f] \mid f : I' \to I \text{ is a monomorphism}\}$.

Remark 15. Based on Theorem 6 a similar method to that described in Section 3 can be developed to generate all factors I/I', when $I' \hookrightarrow I$ is given.

5. The preprojective case

For two preprojective modules P and P' we know that their extensions are also preprojective, in other words if we have a short exact sequence $0 \to P' \to Y \to P \to 0$, then Y is also preprojective.

Since the preprojective modules are the duals of preinjectives, all the results of the previous section apply. So we will just restate the main results, applied for the case of preprojectives. The dual of Lemma 1 is the following:

Lemma 16. *Let P_{n_1} and P_{n_2} be two indecomposable preprojective Kronecker modules. We have a short exact sequence $0 \to P_{n_1} \to P \to P_{n_2} \to 0$ if and only if the conditions are met from one of the following two cases*:

(a) $n_1 \geqslant n_2$ *and* $[P] = [P_{n_2} \oplus P_{n_1}]$,
(b) $n_1 < n_2$ *and* $[P] \in \{[P_{n_1} \oplus P_{n_2}], [P_{n_1+1} \oplus P_{n_2-1}], \dots, [P_{n_1+\lfloor \frac{n_2-n_1}{2} \rfloor} \oplus P_{n_2-\lfloor \frac{n_2-n_1}{2} \rfloor}]\}$.

The dual of Theorem 2 is the following:

Theorem 17. *If $a_1 \geqslant \dots \geqslant a_p \geqslant 0$, $b_1 \geqslant \dots \geqslant b_n \geqslant 0$ and $c_1 \geqslant \dots \geqslant c_r \geqslant 0$ are nonnegative integers, then there exists a short exact sequence $0 \to P_{a_p} \oplus \dots \oplus P_{a_1} \to P \to P_{b_n} \oplus \dots \oplus P_{b_1} \to 0$ if and only if $[P] = [P_{c_r} \oplus \dots \oplus P_{c_1}]$, $r = n + p$, $\exists \beta : \{1, \dots, n\} \to \{1, \dots, n + p\}$, $\exists \alpha : \{1, \dots, p\} \to \{1, \dots, n + p\}$ both functions strictly increasing with $\operatorname{Im} \alpha \cap \operatorname{Im} \beta = \emptyset$ and $\exists m^i_j \geqslant 0$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant p$, such that $\forall \ell \in \{1, \dots, n + p\}$*

$$c_\ell = \begin{cases} b_i - \sum_{\beta(i)<\alpha(j),\, 1\leqslant j\leqslant p} m^i_j, \text{ where } i = \beta^{-1}(\ell), & \ell \in \operatorname{Im} \beta, \\ a_j + \sum_{\beta(i)<\alpha(j),\, 1\leqslant i\leqslant n} m^i_j, \text{ where } j = \alpha^{-1}(\ell), & \ell \in \operatorname{Im} \alpha. \end{cases}$$

The dual of Theorem 6:

Theorem 18. *Let $a_1 \geqslant \dots \geqslant a_p \geqslant 0$, $b_1 \geqslant \dots \geqslant b_n \geqslant 0$, $c_1 \geqslant \dots \geqslant c_r \geqslant 0$ be decreasing sequences of non-negative integers and let $B_j = \{l \in \{0, \dots, n\} \mid \sum_{k=1}^{l} b_k + \sum_{k=1}^{j} a_k \geqslant \sum_{k=1}^{l+j} c_k\}$ for $1 \leqslant j \leqslant p$. Then there is a short exact sequence*

$$0 \to P_{a_p} \oplus \dots \oplus P_{a_1} \to P \to P_{b_n} \oplus \dots \oplus P_{b_1} \to 0$$

if and only if $[P] = [P_{c_r} \oplus \dots \oplus P_{c_1}]$, $r = p + n$, $\sum_{i=1}^{r} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$, $B_j \neq \emptyset$, $a_j \leqslant c_{\alpha_j}$ and $b_i \geqslant c_{\beta_i}$ for $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$, where

$$\alpha_j = \begin{cases} \min B_1 + 1, & j = 1, \\ \max\{\alpha_{j-1} + 1, \min B_j + j\}, & 1 < j \leqslant p \end{cases}$$

and

$$\beta_i = \begin{cases} \min\{l \in \{1, \dots, r\} \mid l \neq \alpha_j,\ 1 \leqslant j \leqslant p\}, & i = 1, \\ \min\{l \in \{\beta_{i-1} + 1, \dots, r\} \mid l \neq \alpha_j,\ 1 \leqslant j \leqslant p\}, & 1 < i \leqslant n. \end{cases}$$

Dually to Theorem 13, in the case or preprojective modules we get easy and explicit criteria for the existence of a projection (instead of embedding):

Theorem 19. *Let $b_1 \geqslant \dots \geqslant b_n > 0$ and $c_1 \geqslant \dots \geqslant c_r > 0$ be decreasing sequences of integers. We have an epimorphism*

$$g : cP_0 \oplus P_{c_r} \oplus \dots \oplus P_{c_1} \to bP_0 \oplus P_{b_n} \oplus \dots \oplus P_{b_1}$$

if and only if $b \leqslant c$ and $b_i + \dots + b_n \leqslant \sum_{c_k \leqslant b_i} c_k$ for $1 \leqslant i \leqslant n$.

Remark 20. As it can be seen, the algorithms described in Section 3 will work in the case of preprojective modules as well, after switching over the order of arguments and reversing the indices, conforming to Theorems 17 and 18.

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