Journal of Algebra 422 (2015) 450–486

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On selfinjective algebras of finite representation type with maximal almost split sequences ☆

Marta Błaszkiewicz, Andrzej Skowroński *

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

ARTICLE INFO

Article history:
Received 9 January 2014
Available online 11 October 2014
Communicated by Changchang Xi

Dedicated to Kunio Yamagata on the occasion of his 65th birthday

MSC:
16D50
16G10
16G60
16G70

Keywords:
Selfinjective algebra
Tilted algebra
Almost split sequence
Auslander–Reiten quiver
Repetitive category
Galois covering

ABSTRACT

We investigate the structure of finite dimensional selfinjective algebras of finite representation type over an arbitrary field having almost split sequences of modules whose middle term admits the maximal possible number of indecomposable direct summands.

☆ The authors gratefully acknowledge support from the research grant DEC-2011/02/A/ST1/00216 of the Polish National Science Center.

* Corresponding author.
E-mail addresses: marta22@mat.uni.torun.pl (M. Błaszkiewicz), skowron@mat.uni.torun.pl (A. Skowroński).

http://dx.doi.org/10.1016/j.jalgebra.2014.09.018
0021-8693/

Introduction and the main results

Throughout the paper, by an algebra we mean a basic indecomposable finite dimensional associative K-algebra with an identity over a (fixed) field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^{-1} the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We do not distinguish between a module in $\operatorname{ind} A$ and the vertex of Γ_A corresponding to it. An algebra A is of finite representation type if the category $\operatorname{ind} A$ admits only a finite number of pairwise nonisomorphic modules. It is well known that a hereditary algebra A is of finite representation type if and only if A is of Dynkin type, that is, the valued quiver Q_A of A is a Dynkin quiver of type $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{B}_n$ $(n \geq 2)$, $\mathbb{C}_n$ $(n \geq 3)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, or $\mathbb{G}_2$ (see [8–10]). A distinguished class of algebras of finite representation type is formed by the tilted algebras of Dynkin type, that is, the algebras of the form $\operatorname{End}_H(T)$ for a hereditary algebra H of Dynkin type and a (multiplicity-free) tilting module T in $\operatorname{mod} H$. For an algebra A, we denote by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, K)$. An algebra A is called *selfinjective* if $A \cong D(A)$ in $\operatorname{mod} A$, and *symmetric* if $A \cong D(A)$ as A–A-bimodules. We note that every algebra B is a quotient algebra of a symmetric algebra, namely the *trivial extension algebra* $T(B) = B \ltimes D(B)$ of B by its minimal injective cogenerator $D(B)$. Recall that $T(B) = B \oplus D(B)$ as K-vector space and the multiplication in $T(B)$ is defined by

$$(x, f)(y, g) = (xy, xg + fy)$$

for $x, y \in B$ and $f, g \in D(B)$.

A prominent role in the representation theory of algebras is played by almost split sequences introduced by M. Auslander and I. Reiten in [2] (see [3] for general theory and applications). For an algebra A and a nonprojective module X in $\operatorname{ind} A$, there is an almost split sequence

$$0 \longrightarrow \tau_A X \longrightarrow Y \longrightarrow X \longrightarrow 0,$$

with $\tau_A X$ a noninjective module in $\operatorname{ind} A$ called the *Auslander–Reiten translation* of X. Then we may associate to X the numerical invariant $\alpha(X)$ being the number of summands in a decomposition $Y = Y_1 \oplus \ldots \oplus Y_r$ of Y into a direct sum of modules $Y_1, \ldots, Y_r$ in $\operatorname{ind} A$. Then $\alpha(X)$ measures the complexity of homomorphisms in $\operatorname{mod} A$ with domain $\tau_A X$ and codomain X. It has been proved by R. Bautista and S. Brenner in [4] (see also [17] for an alternative proof) that, if A is of finite representation type and X is a nonprojective module in $\operatorname{ind} A$, then $\alpha(X) \leq 4$, and if $\alpha(X) = 4$, then the middle Y of an almost split sequence in $\operatorname{mod} A$ with the right term X admits an indecomposable projective–injective direct summand. It follows from general theory that, if P is an indecomposable projective–injective module in a module category $\operatorname{mod} A$, then there is in $\operatorname{mod} A$ an almost split sequence of the form

$$0 \longrightarrow \operatorname{rad} P \longrightarrow (\operatorname{rad} P/\operatorname{soc} P) \oplus P \longrightarrow P/\operatorname{soc} P \longrightarrow 0.$$

An almost split sequence in a module category $\operatorname{mod} A$ of an algebra A of finite representation type with $\alpha(X) = 4$ for its right term X is said to be a *maximal almost split sequence* in $\operatorname{mod} A$. Therefore, the module category $\operatorname{mod} A$ of an algebra A of finite representation type admits a maximal almost split sequence if and only if there is a projective–injective module P in $\operatorname{ind} A$ such that $\operatorname{rad} P/\operatorname{soc} P$ is a direct sum of three indecomposable modules.

We are concerned with the problem of describing the isomorphism classes of selfinjective algebras of finite representation type. For K algebraically closed, the problem was solved in the early 1980's by C. Riedtmann (see [7,20–22]) via the combinatorial classification of the Auslander–Reiten quivers of selfinjective algebras of finite representation type over K. Equivalently, Riedtmann's classification can be presented as follows (see [26, Section 3]): a nonsimple selfinjective algebra A over an algebraically closed field K is of finite representation type if and only if A is socle equivalent to an orbit algebra $\widehat{B}/G$, where $\widehat{B}$ is the repetitive category of a tilted algebra B of Dynkin type $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$. For an arbitrary field K, the problem seems to be difficult (see [5,33] for some results in this direction and [34, Section 12] for related open problems). An important known result towards solution of this general problem is the description of the stable Auslander–Reiten quiver Γ_A^s of a selfinjective algebra of finite representation type established by C. Riedtmann [20] and G. Todorov [36] (see also [35, Section IV.15]): Γ_A^s is isomorphic to the orbit quiver $\mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver of type $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{B}_n$ $(n \geq 2)$, $\mathbb{C}_n$ $(n \geq 3)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, or $\mathbb{G}_2$, and G is an admissible infinite cyclic group of automorphisms of the translation quiver $\mathbb{Z}\Delta$. Therefore, we may associate to any selfinjective algebra A of finite representation type a Dynkin graph $\Delta(A)$, called the *Dynkin type* of A, such that $\Gamma_A^s = \mathbb{Z}\Delta/G$ for a quiver Δ having $\Delta(A)$ as underlying graph. We also mention that, for a Dynkin quiver Δ and a tilted algebra B of type Δ, the orbit algebras $\widehat{B}/G$ are selfinjective algebras of finite representation type whose Dynkin type is the underlying graph of Δ.

The aim of this paper is to investigate the structure of selfinjective algebras of finite representation type whose module category admits a maximal almost split sequence.

Let A be a selfinjective algebra of finite representation type, and assume that $\operatorname{ind} A$ admits a projective–injective module P with $\operatorname{rad} P/\operatorname{soc} P$ being a direct sum of three indecomposable modules. We denote by Δ_P the full subquiver of Γ_A given by the module $\tau_A^{-1}(P/\operatorname{soc} P)$ and all modules X in $\operatorname{ind} A$ such that there is a nontrivial sectional path in Γ_A from $P/\operatorname{soc} P$ to X. We prove that Δ_P is a Dynkin quiver whose underlying graph is the Dynkin type $\Delta(A)$ of A. Moreover, let M_P be the direct sum of all modules lying on Δ_P.

In Section 3 we describe the family $\mathcal{B}_{max}$ of tilted algebras of Dynkin type having an indecomposable projective module with injective top and the radical being a direct sum of three projective modules. Recall also that two selfinjective algebras A and A' are called *socle equivalent* if the quotient algebras $A/\operatorname{soc} A$ and $A'/\operatorname{soc} A'$ are isomorphic (see [35, Section IV.6]).

The following theorem is the main result of the paper.

Theorem 1. *Let A be a selfinjective algebra over a field K. The following statements are equivalent.*

(i) *A is of finite representation type having an indecomposable projective module P with $\alpha(P/\operatorname{soc} P) = 4$ and $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$.*
(ii) *A is socle equivalent to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, where B is an algebra from the family $\mathcal{B}_{max}$, m a positive integer, $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$, and φ a rigid automorphism of $\widehat{B}$.*

Moreover, if K is an algebraically closed field, we may replace "socle equivalent" by "isomorphic".

We would like to point that in general we cannot replace in the above theorem "socle equivalent" by "isomorphic" (see Section 9).

The next two theorems suggest that possibly the statement (ii) of the above theorem provides description of all selfinjective algebras of finite representation type whose module category admits a maximal almost split sequence (equivalently, the assumption $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$ in the statement (i) is superfluous).

Theorem 2. *Let $A = \widehat{\Lambda}/G$ for a tilted algebra Λ of Dynkin type and G an admissible infinite cyclic group of automorphisms of $\widehat{\Lambda}$. The following statements are equivalent.*

(i) $\operatorname{mod} A$ *admits a maximal almost split sequence.*
(ii) *A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, where B is an algebra from the family $\mathcal{B}_{max}$, m a positive integer, $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$, and φ a rigid automorphism of $\widehat{B}$.*

Theorem 3. *Let A be a selfinjective algebra over an algebraically closed field K. The following statements are equivalent.*

(i) *A is of finite representation type and* $\operatorname{mod} A$ *admits a maximal almost split sequence.*
(ii) *A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, where B is an algebra from the family $\mathcal{B}_{max}$, m a positive integer, $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$, and φ a rigid automorphism of $\widehat{B}$.*

We also mention that for an algebra B from $\mathcal{B}_{max}$, a positive integer m, a rigid automorphism φ of $\widehat{B}$, and the associated selfinjective orbit algebra $A = \widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, the module category $\operatorname{mod} A$ has exactly m maximal almost split sequences.

The following corollary is a direct consequence of Theorem 1 (and its proof), Theorems 2.6, 4.3, and Corollary 4.4.

Corollary 4. *Let A be a selfinjective algebra of finite representation type and P be an indecomposable projective module in* $\operatorname{mod} A$ *with* $\alpha(P/\operatorname{soc} P) = 4$ *and* $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$. *The following statements are equivalent.*

(i) $\operatorname{mod} A$ *admits at least two maximal almost split sequences.*
(ii) *A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, where B is an algebra from the family $\mathcal{B}_{max}$, $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$, $m \geq 2$, and φ a rigid automorphism of $\widehat{B}$.*

In [19] Y. Ohnuki, K. Takeda and K. Yamagata proved that an orbit algebra $A = \widehat{B}/(\varphi\nu_{\widehat{B}})$, with φ a positive automorphism of $\widehat{B}$, is a symmetric algebra if and only if A is isomorphic to the trivial extension algebra $T(B)$ of B. Then we have the following consequence of the proof of Theorem 1 and Theorem 2.4(iv).

Corollary 5. *Let A be a symmetric algebra of finite representation type over a field K having an indecomposable projective module P with $\alpha(P/\operatorname{soc} P) = 4$ and* $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$. *Then A is socle equivalent to the trivial extension algebra $T(B)$ of an algebra B from the family $\mathcal{B}_{max}$. Moreover, if K is an algebraically closed field, we may replace "socle equivalent" by "isomorphic".*

The following corollary is a direct consequence of Theorem 2 and [19].

Corollary 6. *Let $A = \widehat{\Lambda}/G$ for a tilted algebra Λ of Dynkin type and G an admissible infinite cyclic group of automorphisms of $\widehat{\Lambda}$. The following statements are equivalent.*

(i) *A is a symmetric algebra and* $\operatorname{mod} A$ *admits a maximal almost split sequence.*
(ii) *A is isomorphic to the trivial extension algebra $T(B)$ of a tilted algebra B from the family $\mathcal{B}_{max}$.*

The paper is organized as follows. In Section 1 we recall the background on the orbit algebras of repetitive categories of algebras. Section 2 is devoted to presenting the theory of selfinjective algebras with deforming ideals developed by A. Skowroński and K. Yamagata, playing a prominent role in the proof of Theorem 1. In Section 3 we introduce the family $\mathcal{B}_{max}$ of tilted algebras of Dynkin type occurring in the main results of the paper as well as provide sufficient conditions for an algebra to be in the family $\mathcal{B}_{max}$. In Section 4 we recall properties of the repetitive categories of tilted algebras of Dynkin type and their orbit algebras. Moreover, we prove that the module categories of the repetitive categories of the tilted algebras from the family $\mathcal{B}_{max}$ have maximal almost split sequences. In Section 5 we describe properties of Dynkin quivers associated to maximal almost split sequences of selfinjective algebras of finite representation type. Sections 6, 7 and 8 are devoted to the proofs of Theorems 2, 3 and 1, respectively. In the final Section 9 we present examples illustrating the main results of the paper.

For basic background on the representation theory applied in the paper we refer to [1,3,23,35].

1. Orbit algebras of repetitive categories

Let B be an algebra and $1_B = e_1 + \cdots + e_n$ a decomposition of the identity of B into a sum of pairwise orthogonal primitive idempotents. We associate to B a selfinjective locally bounded K-category $\widehat{B}$, called the *repetitive category* of B (see [15]). The objects of $\widehat{B}$ are $e_{m,i}$, $m \in \mathbb{Z}$, $i \in \{1, \dots, n\}$, and the morphism spaces are defined as follows

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_j B e_i, & r = m, \\ D(e_i B e_j), & r = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $e_j B e_i = \mathrm{Hom}_B(e_i B, e_j B)$, $D(e_i B e_j) = e_j D(B) e_i$ and

$$\bigoplus_{(m,i)\in\mathbb{Z}\times\{1,\dots,n\}} \widehat{B}(e_{m,i}, e_{r,j}) = e_j B \oplus D(Be_j),$$

for any $r \in \mathbb{Z}$ and $j \in \{1, \dots, n\}$. We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ defined by

$$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i} \quad \text{for all } (m,i) \in \mathbb{Z} \times \{1, \dots, n\}.$$

An automorphism φ of the K-category $\widehat{B}$ is said to be:

- *positive* if, for each pair $(m,i) \in \mathbb{Z} \times \{1, \dots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and some $j \in \{1, \dots, n\}$;
- *rigid* if, for each pair $(m,i) \in \mathbb{Z} \times \{1, \dots, n\}$, there exists $j \in \{1, \dots, n\}$ such that $\varphi(e_{m,i}) = e_{m,j}$;
- *strictly positive* if it is positive but not rigid.

Then the automorphisms $\nu_{\widehat{B}}^r$, $r \geq 1$, are strictly positive automorphisms of $\widehat{B}$.

A group G of automorphisms of $\widehat{B}$ is said to be *admissible* if G acts freely on the set of objects of $\widehat{B}$ and has finitely many orbits. Then, following P. Gabriel [12], we may consider the orbit category $\widehat{B}/G$ of $\widehat{B}$ with respect to G whose objects are the G-orbits of objects in $\widehat{B}$, and the morphism spaces are given by

$$(\widehat{B}/G)(a,b) = \left\{ (f_{y,x}) \in \prod_{(x,y)\in a\times b} \widehat{B}(x,y) \;\middle|\; g f_{y,x} = f_{gy,gx},\ \forall_{g\in G,\,(x,y)\in a\times b} \right\}$$

for all objects a, b of $\widehat{B}/G$. Since $\widehat{B}/G$ has finitely many objects and the morphism spaces in $\widehat{B}/G$ are finite dimensional, we have the associated finite dimensional selfinjective K-algebra $\bigoplus(\widehat{B}/G)$ which is the direct sum of all morphism spaces in $\widehat{B}/G$, called the *orbit algebra* of $\widehat{B}$ with respect to G. We will identify $\widehat{B}/G$ with $\bigoplus(\widehat{B}/G)$. For example, for each positive integer r, the infinite cyclic group $(\nu_{\widehat{B}}^r)$ generated by the r-th power $\nu_{\widehat{B}}^r$ of $\nu_{\widehat{B}}$ is an admissible group of automorphisms of $\widehat{B}$, and we have the associated selfinjective orbit algebra

$$T(B)^{(r)} = \widehat{B}/\big(\nu_{\widehat{B}}^r\big) = \left\{ \begin{array}{c} \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ f_2 & b_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & f_3 & b_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f_{r-1} & b_{r-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \dots, b_{r-1} \in B,\ f_1, \dots, f_{r-1} \in D(B) \end{array} \right\},$$

called the *r-fold trivial extension algebra of* B. In particular, $T(B)^{(1)} \cong T(B) = B \ltimes D(B)$ is the trivial extension of B by the injective cogenerator $D(B)$.

2. Selfinjective algebras with deforming ideals

In this section we present criteria for selfinjective algebras to be socle equivalent to orbit algebras of the repetitive categories of algebras with respect to infinite cyclic automorphism groups, playing fundamental role in the proof of Theorem 1.

Let A be a selfinjective algebra. For a subset X of A, we may consider the left annihilator $l_A(X) = \{a \in A \mid aX = 0\}$ of X in A and the right annihilator $r_A(X) = \{a \in A \mid Xa = 0\}$ of X in A. Then by a theorem due to T. Nakayama (see [35, Theorem IV.6.10]) the annihilator operation l_A induces a Galois correspondence from the lattice of right ideals of A to the lattice of left ideals of A, and r_A is the inverse Galois correspondence to l_A. Let I be an ideal of A, $B = A/I$, and e an idempotent of A such that $e + I$ is the identity of B. We may assume that $1_A = e_1 + \dots + e_r$ with $e_1, \dots, e_r$ pairwise orthogonal primitive idempotents of A, $e = e_1 + \dots + e_n$ for some $n \le r$, and $\{e_i \mid 1 \le i \le n\}$ is the set of all idempotents in $\{e_i \mid 1 \le i \le r\}$ which are not in I. Then such an idempotent e is uniquely determined by I up to an inner automorphism of A, and is called a *residual identity* of $B = A/I$. Observe also that $B \cong eAe/eIe$.

We have the following lemma from [32, Lemma 5.1].

Lemma 2.1. *Let A be a selfinjective algebra, I an ideal of A, and e an idempotent of A such that $l_A(I) = Ie$ or $r_A(I) = eI$. Then e is a residual identity of A/I.*

We recall also the following proposition proved in [27, Proposition 2.3].

Proposition 2.2. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$, e a residual identity of B, and assume that $IeI = 0$. The following conditions are equivalent.*

(i) *Ie is an injective cogenerator in* $\operatorname{mod} B$.
(ii) *eI is an injective cogenerator in* $\operatorname{mod} B^{\mathrm{op}}$.
(iii) $l_A(I) = Ie$.
(iv) $r_A(I) = eI$.

Moreover, under these equivalent conditions, we have $\operatorname{soc} A \subseteq I$ *and* $l_{eAe}(I) = eIe = r_{eAe}(I)$.

The following theorem proved in [29, Theorem 3.8] (sufficiency part) and [32, Theorem 5.3] (necessity part) will be fundamental for our considerations.

Theorem 2.3. *Let A be a selfinjective algebra. The following conditions are equivalent.*

(i) *A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is an algebra and φ is a positive automorphism of $\widehat{B}$.*
(ii) *There is an ideal I of A and an idempotent e of A such that*
  (1) $r_A(I) = eI$;
  (2) *the canonical algebra epimorphism $eAe \to eAe/eIe$ is a retraction.*

Moreover, in this case, B is isomorphic to A/I.

Let A be a selfinjective algebra, I an ideal of A, and e a residual identity of A/I. Following [27], I is said to be a *deforming ideal* of A if the following conditions are satisfied:

(D1) $l_{eAe}(I) = eIe = r_{eAe}(I)$;
(D2) the valued quiver $Q_{A/I}$ of A/I is acyclic.

Assume I is a deforming ideal of A. Then we have a canonical isomorphism of algebras $eAe/eIe \to A/I$ and I can be considered as an (eAe/eIe)–(eAe/eIe)-bimodule. Denote by $A[I]$ the direct sum of K-vector spaces $(eAe/eIe) \oplus I$ with the multiplication

$$(b, x) \cdot (c, y) = (bc, by + xc + xy)$$

for $b, c \in eAe/eIe$ and $x, y \in I$. Then $A[I]$ is a K-algebra with the identity $(e+eIe, 1_A-e)$, and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider I as an ideal of $A[I]$. Observe that $e = (e + eIe, 0)$ is a residual identity of $A[I]/I = eAe/eIe \cong A/I$, $eA[I]e = (eAe/eIe) \oplus eIe$ and the canonical algebra epimorphism $eA[I]e \to eA[I]e/eIe$ is a retraction.

The following properties of the algebra $A[I]$ were established in [27, Theorem 4.1], [28, Theorem 3] and [33, Lemma 3.1].

Theorem 2.4. *Let A be a selfinjective algebra and I a deforming ideal of A. The following statements hold.*

(i) *$A[I]$ is a selfinjective algebra with the same Nakayama permutation as A and I is a deforming ideal of $A[I]$.*
(ii) *A and $A[I]$ are socle equivalent.*
(iii) *A and $A[I]$ are stably equivalent.*
(iv) *$A[I]$ is a symmetric algebra if A is a symmetric algebra.*

We note that if A is a selfinjective algebra, I an ideal of A, $B = A/I$, e an idempotent of A such that $r_A(I) = eI$, and the valued quiver Q_B of B is acyclic, then by Lemma 2.1 and Proposition 2.2, I is a deforming ideal of A and e is a residual identity of B.

The following theorem proved in [29, Theorem 4.1] shows the importance of the algebras $A[I]$.

Theorem 2.5. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$ and e an idempotent of A. Assume that $r_A(I) = eI$ and Q_B is acyclic. Then $A[I]$ is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.*

We point out that there are selfinjective algebras A with deforming ideals I such that the algebras A and $A[I]$ are not isomorphic (see [29, Example 4.2]), and A is not isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is an algebra and φ is a positive automorphism of $\widehat{B}$ (see [30, Proposition 4]).

The following result proved in [31, Proposition 3.2] describes a situation when the algebras A and $A[I]$ are isomorphic.

Theorem 2.6. *Let A be a selfinjective algebra with a deforming ideal I, $B = A/I$, e be a residual identity of B, and ν the Nakayama permutation of A. Assume that $IeI = 0$ and $e_i \neq e_{\nu(i)}$, for any primitive summand e_i of e. Then the algebras A and $A[I]$ are isomorphic. In particular, A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.*

Moreover, we have the following consequence of [27, Theorem 3.2].

Theorem 2.7. *Let A be a selfinjective algebra over an algebraically closed field K and I a deforming ideal of A. Then A and $A[I]$ are isomorphic.*

3. Tilted algebras of Dynkin type

Let A be an algebra. Following [6,13] a module T in $\operatorname{mod} A$ is said to be a *tilting module* if $\operatorname{pd}_A T \leq 1$, $\operatorname{Ext}^1_A(T,T) = 0$ and T is a direct sum of n pairwise nonisomorphic indecomposable modules, where n is the rank of the Grothendieck group $K_0(A)$ of A (equivalently, the number of vertices of the quiver Q_A of A). In case H is a hereditary algebra and T is a tilting module in $\operatorname{mod} H$, the endomorphism algebra $B = \operatorname{End}_H(T)$ is called a *tilted algebra (of type Q_H)*. Then the images $\operatorname{Hom}_H(T, I_i)$ of indecomposable injective modules I_i, $1 \le i \le n$ (with n the rank of $K_0(H)$), via the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$ form a faithful section Δ_T of a connected component $\mathcal{C}_T$ of Γ_B, called the *connecting component* of Γ_B determined by T, which connects the torsion-free part $\mathcal{Y}(T) = \{Y \in \operatorname{mod} B \mid \operatorname{Tor}_1^B(Y, T) = 0\}$ and the torsion part $\mathcal{X}(T) = \{X \in \operatorname{mod} B \mid X \otimes_B T = 0\}$ of $\operatorname{mod} B$ (see [1,13]). Recall that a full valued subquiver Δ of a connected component $\mathcal{C}$ of an Auslander–Reiten quiver Γ_A is called a *section* if Δ is an acyclic convex full subquiver of $\mathcal{C}$ intersecting every τ_A-orbit in $\mathcal{C}$ exactly once. Moreover, a subquiver Δ of Γ_A is called *faithful* if the direct sum of all indecomposable modules lying on Δ is a faithful A-module.

The following criterion established independently by S. Liu [16] and A. Skowroński [25] (see also [1, Theorem VIII.5.6] and its proof) provides a criterion for an algebra to be a tilted algebra.

Theorem 3.1. *An algebra B is a tilted algebra if and only if Γ_B contains a connected component $\mathcal{C}$ with a faithful section Δ such that* $\operatorname{Hom}_B(X, \tau_B Y) = 0$ *for all modules X and Y lying on Δ. Moreover, in this case, the following statements hold.*

(i) *The direct sum T_Δ^* of all modules lying on Δ is a tilting module in* $\operatorname{mod} B$.
(ii) $H_\Delta = \operatorname{End}_B(T_\Delta^*)$ *is a hereditary algebra.*
(iii) $T_\Delta = D(T_\Delta^*)$ *is a tilting module in* $\operatorname{mod} H_\Delta$.
(iv) *There is a canonical isomorphism of algebras $\sigma : B \to \operatorname{End}_{H_\Delta}(T_\Delta)$ such that $\sigma(b)(f)(t^*) = f(t^* b)$ for $b \in B$, $f \in T_\Delta$ and $t^* \in T_\Delta^*$.*
(v) *The component $\mathcal{C}$ is the connecting component $\mathcal{C}_{T_\Delta}$ of Γ_B and Δ the section Δ_{T_Δ} determined by T_Δ.*

We introduce now a family $\mathcal{B}_{max}$ of tilted algebras of Dynkin type, occurring in Theorems 1, 2, 3 and Corollaries 4, 5, 6.

A *K-species* is a system $\mathbb{M} = (F_i, {}_iM_j)_{1 \le i,j \le n}$, where $F_1, \ldots, F_n$ are division K-algebras and, for each pair $i, j \in \{1, \ldots, n\}$, ${}_iM_j$ is an F_i–F_j-bimodule on which K acts centrally and $\dim_K {}_iM_j$ is finite (see [8,11]). We may associate to such a K-species $\mathbb{M}$ the valued quiver $Q_{\mathbb{M}}$ defined as follows:

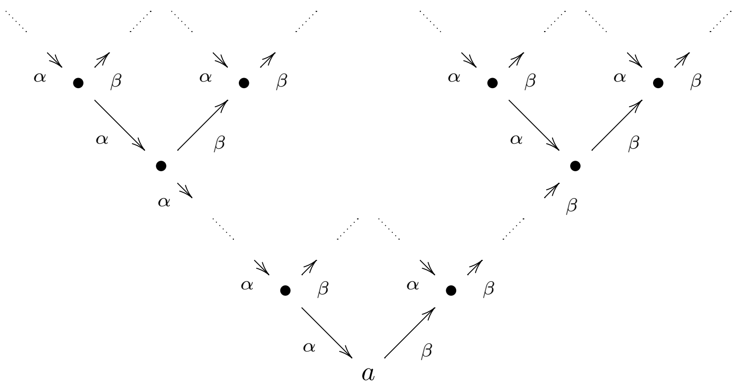
(1) The vertices of $Q_{\mathbb{M}}$ are $1, 2, \ldots, n$.
(2) For two vertices i and j in $Q_{\mathbb{M}}$, there exists an arrow $i \longrightarrow j$ if and only if ${}_iM_j \ne 0$. Moreover, we associate to an arrow of $Q_{\mathbb{M}}$ the valuation (d_{ij}, d'_{ij}), with $d_{ij} = \dim_{F_j} {}_iM_j$ and $d'_{ij} = \dim_{F_i} {}_iM_j$, so we have in $Q_{\mathbb{M}}$ the valued arrow

$$i \xrightarrow{(d_{ij}, d'_{ij})} j.$$

We will write $i \longrightarrow j$ instead of $i \xrightarrow{(1,1)} j$. A K-species $\mathbb{M}$ is said to be *acyclic* if the valued quiver $Q_{\mathbb{M}}$ is acyclic (has no oriented cycles).

Let $\mathbb{M} = (F_i, {}_iM_j)_{1\le i,j\le n}$ be a K-species. Consider the K-algebra $R = R_{\mathbb{M}} = \prod_{i=1}^n F_i$, the R–R-bimodule $M = M_{\mathbb{M}} = \bigoplus_{i,j=1}^n {}_iM_j$, and the associated tensor algebra $T(\mathbb{M}) = T_R(M)$ of M over R. Then $T(\mathbb{M})$ is a finite dimensional hereditary K-algebra if and only if the quiver $Q_{\mathbb{M}}$ is acyclic. Moreover, if it is the case, then $Q_{\mathbb{M}}$ is the valued quiver $Q_{T(\mathbb{M})}$ of the algebra $T(\mathbb{M})$.

By a *branch* we mean a finite connected full subquiver $\mathcal{L}$, containing the lowest vertex a, of the following infinite tree

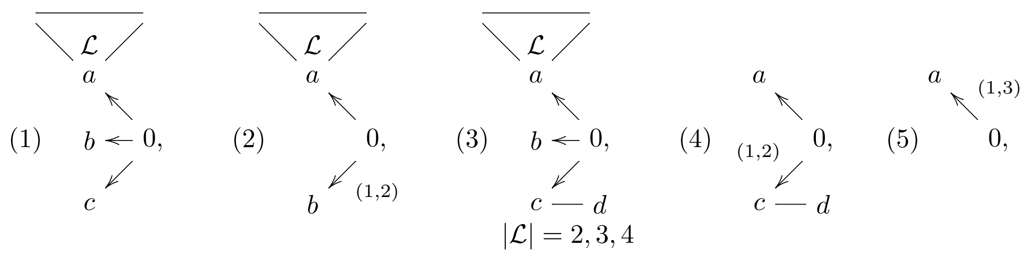


bound by all possible zero relations of the form $\alpha\beta = 0$. Then $|\mathcal{L}|$ is the number of vertices of $\mathcal{L}$.

We denote by $\mathcal{B}_{max}$ the family of all algebras

$$B(\mathbb{M}, \mathcal{L}) = T(\mathbb{M})/I(\mathbb{M}, \mathcal{L}),$$

where $\mathbb{M}$ is a K-species with $Q_{\mathbb{M}}$ of one of the forms



where c —— d denotes $c \longrightarrow d$ or $c \longleftarrow d$, $\mathcal{L}$ is a branch with the lowest vertex a, and $I(\mathbb{M}, \mathcal{L})$ is the ideal in the tensor algebra $T(\mathbb{M})$ of $\mathbb{M}$ generated by ${}_iM_j \otimes_{R_{\mathbb{M}}} {}_jM_k$ for all paths $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ in $\mathcal{L}$. The following proposition shows that $\mathcal{B}_{max}$ consists of tilted algebras of Dynkin type.

Proposition 3.2. *Let B be an algebra from the family $\mathcal{B}_{max}$. Then B is a tilted algebra of one of the Dynkin types:*

(i) $\mathbb{D}_n$ $(n \geq 4)$, *if* $Q_{\mathbb{M}}$ *is of type* (1) *and* $|\mathcal{L}| = n - 3$;
(ii) $\mathbb{B}_n$ $(n \geq 3)$, *if* $Q_{\mathbb{M}}$ *is of type* (2) *and* $|\mathcal{L}| = n - 2$;
(iii) $\mathbb{E}_6$, *if* $Q_{\mathbb{M}}$ *is of type* (3) *and* $|\mathcal{L}| = 2$;
(iv) $\mathbb{E}_7$, *if* $Q_{\mathbb{M}}$ *is of type* (3) *and* $|\mathcal{L}| = 3$;
(v) $\mathbb{E}_8$, *if* $Q_{\mathbb{M}}$ *is of type* (3) *and* $|\mathcal{L}| = 4$;
(vi) $\mathbb{F}_4$, *if* $Q_{\mathbb{M}}$ *is of type* (4);
(vii) $\mathbb{G}_2$, *if* $Q_{\mathbb{M}}$ *is of type* (5).

Proof. Let $B = B(\mathbb{M}, \mathcal{L})$. Let $\mathbb{M}(\mathcal{L})$ be the restriction of the K-species $\mathbb{M}$ to the vertices of the branch $\mathcal{L}$. Consider the algebra $C = T(\mathbb{M}(\mathcal{L}))/I(\mathcal{L})$, where $T(\mathbb{M}(\mathcal{L}))$ is the tensor algebra of $\mathbb{M}(\mathcal{L})$ and $I(\mathcal{L})$ is the ideal in $T(\mathbb{M}(\mathcal{L}))$ generated by ${}_iM_j \otimes_{R_{\mathbb{M}(\mathcal{L})}} {}_jM_k$ for all paths $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ in $\mathcal{L}$. Then, applying [24, Theorem XVI.2.3], we conclude that C is a tilted algebra of the equioriented type $\mathbb{A}_m$ with $m = |\mathcal{L}|$, and Γ_C admits a unique section Σ (of equioriented type $\mathbb{A}_m$) whose source is the indecomposable projective C-module $P_C(a)$ at the vertex a while the sink is the indecomposable injective C-module $I_C(a)$ at the vertex a. Then Γ_B admits a section Δ consisting of the indecomposable C-modules lying on Σ and the indecomposable projective B-modules $P(0)$, $P(b)$, $P(c)$, $P(d)$ at the vertices 0, b, c, d, respectively. We note that $P_C(a)$ is the indecomposable projective B-module at the vertex a. Since Σ is a faithful section in Γ_C, Δ is a faithful translation subquiver of Γ_B. In fact, Γ_B is a finite acyclic quiver and Δ intersects each τ_B-orbit of Γ_B exactly once. Hence Δ is a faithful section of Γ_B, and $\mathrm{Hom}_B(X, \tau_B Y) = 0$ for all indecomposable modules X and Y lying on Δ. Therefore, applying Theorem 3.1, we conclude that B is a tilted algebra of Dynkin type Δ^{op}. It follows from definition of Δ that Δ^{op} is one of the Dynkin types required in (i)–(vii). We also note that Γ_B admits the section $\tau_B^{-1}\Delta$ formed by the indecomposable modules $\tau_B^{-1}X$ with X indecomposable module lying on Δ, because no module lying on Δ is an injective B-module. □

The following theorem will be crucial for the proofs of Theorems 1 and 2.

Theorem 3.3. *Let B be a basic indecomposable finite dimensional algebra over a field K, P an indecomposable projective noninjective module in* $\mathrm{mod}\,B$ *such that $\alpha(\tau_B^{-1}P) = 3$, and Δ_P the full valued subquiver of Γ_B given by the module $\tau_B^{-1}P$ and the indecomposable modules in* $\mathrm{mod}\,B$ *lying on a nontrivial sectional path in Γ_B starting at P. Assume that the following conditions are satisfied.*

(1) *Δ_P is a Dynkin quiver.*
(2) *Δ_P does not contain a projective module.*
(3) *$\Delta_P \backslash \{\tau_B^{-1}P\}$ does not contain an injective module.*
(4) *For any arrow $V \xrightarrow{(d,d')} U$ in Γ_B with U in Δ_P, V belongs to Δ_P or to $\tau_B\Delta_P$.*

(5) *For any arrow* $U \xrightarrow{(e,e')} W$ *in* Γ_B *with* U *in* Δ_P, W *belongs to* Δ_P *or to* $\tau_B^{-1}\Delta_P$.
(6) *The direct sum* M_P *of indecomposable modules lying on* Δ_P *is a faithful module in* $\operatorname{mod} B$ *with* $\operatorname{Hom}_B(M_P, \tau_B M_P) = 0$.

Then the following statements hold.

(i) M_P *is a tilting module in* $\operatorname{mod} B$.
(ii) B *is a tilted algebra from* $\mathcal{B}_{max}$.
(iii) $\tau_B^{-1}P$ *is not an injective module.*

Proof. We abbreviate $\Delta = \Delta_P$ and $M = M_P$. Since $\alpha(\tau_B^{-1}P) = 3$ and Δ is a Dynkin quiver, we conclude that Δ is one of the quivers:

$$\begin{tikzcd}
 & & & & X_m \\
 & & & X_{m-1} \arrow[ur] & \\
 & & \cdots \arrow[ur] & & \\
 & X_2 \arrow[ur] & & & \\
X_1 \arrow[ur] \arrow[dr] & & & & \\
Y_1 \arrow[r] & \tau_B^{-1}P, & & & \\
Z_1 \arrow[u, phantom] \arrow[ur] & & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & & X_m \\
 & & & X_{m-1} \arrow[ur] & \\
 & & \cdots \arrow[ur] & & \\
 & X_2 \arrow[ur] & & & \\
X_1 \arrow[ur] \arrow[dr] & & & & \\
 & \tau_B^{-1}P, & & & \\
Y_1 \arrow[ur, "{(1,2)}"'] & & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_2 \\
X_1 \arrow[ur] \arrow[dr] & \\
Y_1 \arrow[r] & \tau_B^{-1}P, \\
Z_1 \arrow[ur] \arrow[dr] & \\
 & Z_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_3 \\
 & X_2 \arrow[ur] & \\
X_1 \arrow[ur] \arrow[dr] & & \\
Y_1 \arrow[r] & \tau_B^{-1}P, & \\
Z_1 \arrow[ur] \arrow[dr] & & \\
 & Z_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & X_4 \\
 & & X_3 \arrow[ur] & \\
 & X_2 \arrow[ur] & & \\
X_1 \arrow[ur] \arrow[dr] & & & \\
Y_1 \arrow[r] & \tau_B^{-1}P, & & \\
Z_1 \arrow[ur] \arrow[dr] & & & \\
 & Z_2 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_2 \\
X_1 \arrow[ur] \arrow[dr, "{(1,2)}"] & \\
 & \tau_B^{-1}P, \\
Y_1 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[dr, "{(1,3)}"] & \\
 & \tau_B^{-1}P.
\end{tikzcd}$$

It follows from (6) and [1, Lemma VIII.5.1, Corollary IV.2.14] that $\mathrm{pd}_B M \leq 1$ and $\mathrm{Ext}^1_B(M,M) \cong D\,\mathrm{Hom}_B(M,\tau_B M) = 0$. We claim that $\mathrm{id}_B M \leq 1$. Since M is a faithful B-module, it is enough to show that $\mathrm{Hom}_B(\tau_B^{-1}M,M) = 0$, again by [1, Lemma VIII.5.1]. Observe that, by (3), (5) and the shape of Δ, we have an epimorphism $M^t \to \tau_B^{-1}M$ in $\mathrm{mod}\, B$ for some positive integer t. Suppose now that there exist modules L and U on Δ such that $\mathrm{Hom}_B(\tau_B^{-1}L,U) \neq 0$. Then using (4) and [1, Lemma VIII.5.4] we conclude that $\mathrm{Hom}_B(\tau_B^{-1}L,\tau_B M) \neq 0$. But then $\mathrm{Hom}_B(M,\tau_B M) \neq 0$, because there is an epimorphism $M^t \to \tau_B^{-1}M$, a contradiction. Hence, indeed $\mathrm{id}_B M \leq 1$.

We will show now that M is a tilting module in $\mathrm{mod}\, B$.

Let $f_1, \dots f_d$ be a basis of $\mathrm{Hom}_B(B,M)$. Then we have a monomorphism $f : B \to M^d$ in $\mathrm{mod}\, B$, induced by $f_1, \dots f_d$, and a short exact sequence

$$0 \longrightarrow B \xrightarrow{f} M^d \xrightarrow{g} N \longrightarrow 0$$

in $\mathrm{mod}\, B$, where $N = \mathrm{Coker}\, f$ and g a canonical epimorphism. We give now the standard arguments showing that $M \oplus N$ is a tilting module in $\mathrm{mod}\, B$. Since B is a projective module in $\mathrm{mod}\, B$, we have $\mathrm{Ext}^2_B(N,-) \cong \mathrm{Ext}^2_B(M^d,-)$ and hence $\mathrm{pd}_B N \leq 1$, because $\mathrm{pd}_B M \leq 1$. Hence, $\mathrm{pd}_B(M \oplus N) \leq 1$. Applying $\mathrm{Hom}_B(-,M)$ to the above short exact sequence, we obtain a short exact sequence in $\mathrm{mod}\, K$ of the form

$$\mathrm{Hom}_B\big(M^d,M\big) \xrightarrow{\mathrm{Hom}_B(f,M)} \mathrm{Hom}_B(B,M) \longrightarrow \mathrm{Ext}^1_B(N,M) \longrightarrow \mathrm{Ext}^1_B\big(M^d,M\big),$$

where $\mathrm{Ext}^1_B(M^d,M) = 0$ and $\mathrm{Hom}_B(f,M)$ is an epimorphism by the choice of f, and so $\mathrm{Ext}^1_B(N,M) = 0$. Applying now $\mathrm{Hom}_B(N,-)$, we obtain an epimorphism $\mathrm{Ext}^1_B(N,g) : \mathrm{Ext}^1_B(N,M^d) \to \mathrm{Ext}^1_B(N,N)$, because $\mathrm{pd}_B N \leq 1$ implies $\mathrm{Ext}^2_B(N,B) = 0$, and consequently $\mathrm{Ext}^1_B(N,N) = 0$. Finally, applying $\mathrm{Hom}_B(M,-)$, we obtain an epimorphism $\mathrm{Ext}^1_B(M,g) : \mathrm{Ext}^1_B(M,M^d) \to \mathrm{Ext}^1_B(M,N)$, because $\mathrm{pd}_B M \leq 1$ implies $\mathrm{Ext}^2_B(M,B) = 0$, and hence $\mathrm{Ext}^1_B(M,N) = 0$. Summing up, we have $\mathrm{pd}_B(M \oplus N) \leq 1$ and $\mathrm{Ext}^1_B(M \oplus N, M \oplus N) = 0$, and so $M \oplus N$ is a tilting module in $\mathrm{mod}\, B$.

We will show now that N belongs to the additive category $\mathrm{add}\, M$ of M. Assume to the contrary that there exists an indecomposable summand W of N which is not in $\mathrm{add}\, M$, or equivalently W does not lie on Δ. Clearly, we have $\mathrm{Hom}_B(M,W) \neq 0$, because N is a quotient module of M^d. Hence $\mathrm{Hom}_B(V,W) \neq 0$ for an indecomposable module V from Δ. Applying now (5) and [1, Lemma VIII.5.4] we conclude that $\mathrm{Hom}_B(\tau_B^{-1}M,W) \neq 0$. Since $\mathrm{id}_B M \leq 1$, applying [1, Corollary IV.2.14], we obtain that $\mathrm{Ext}^1_B(W,M) \cong D\,\mathrm{Hom}_B(\tau_B^{-1}M,W) \neq 0$, which contradicts $\mathrm{Ext}^1_B(N,M) = 0$. Therefore, M is a tilting module in $\mathrm{mod}\, B$. Moreover, the rank of $K_0(B)$ coincides with the number of indecomposable modules lying on Δ.

Let $H = \mathrm{End}_B(M)$. We will prove that H is a hereditary algebra. Let Q be an indecomposable projective module in $\mathrm{mod}\, H$, R an indecomposable right H-submodule of Q, and $f : R \to Q$ the inclusion homomorphism. We claim that R is a projective module. The tilting module M induces the torsion pair $(\mathcal{T}(M), \mathcal{F}(M))$ in $\mathrm{mod}\, B$, with $\mathcal{T}(M) = \{U \in \mathrm{mod}\, B \mid \mathrm{Ext}^1_B(M,U) = 0\}$ and $\mathcal{F}(M) = \{W \in \mathrm{mod}\, B \mid \mathrm{Hom}_B(M,W) = 0\}$, and the torsion pair $(\mathcal{X}(M), \mathcal{Y}(M))$ in $\mathrm{mod}\, H$, with $\mathcal{X}(M) = \{X \in \mathrm{mod}\, H \mid X \otimes_H M = 0\}$ and $\mathcal{Y}(M) = \{Y \in \mathrm{mod}\, H \mid \mathrm{Tor}^H_1(Y,M) = 0\}$. Since Q belongs to $\mathcal{Y}(M)$ and the torsion-free class $\mathcal{Y}(M)$ is closed under submodules, we conclude that R belongs to $\mathcal{Y}(M)$. Moreover, the functor $\mathrm{Hom}_B(M,-) : \mathrm{mod}\, B \to \mathrm{mod}\, H$ induces an equivalence of categories $\mathcal{T}(M) \xrightarrow{\sim} \mathcal{Y}(M)$. Hence there exists a homomorphism $g : V \to U$ in $\mathrm{mod}\, B$ with V, U indecomposable modules from $\mathcal{T}(M)$, U from Δ, such that $\mathrm{Hom}_B(M,V) = R$, $\mathrm{Hom}_B(M,U) = Q$, and $\mathrm{Hom}_B(M,g) = f$. Take now a nonzero homomorphism $h : Q' \to R$ in $\mathrm{mod}\, H$ with Q' an indecomposable projective module. Then there exists a nonzero homomorphism $u : V' \to V$ in $\mathrm{mod}\, B$ such that V' is in Δ, $\mathrm{Hom}_B(M,V') = Q'$, and $\mathrm{Hom}_B(M,u) = h$. Since f is a monomorphism, we conclude that $fh \neq 0$, and hence $gu \neq 0$. We claim that V lies on Δ. Suppose V is not on Δ. Then, applying (4) and [1, Lemma VIII.5.4], we conclude that there exist homomorphisms $p : V \to W$ and $q : W \to U$ in $\mathrm{mod}\, B$, with W being a direct sum of modules from $\tau_B\Delta$, such that $g = qp$. But then $qpu = gu \neq 0$ implies $pu \neq 0$, and hence $\mathrm{Hom}_B(M, \tau_B M) \neq 0$, a contradiction. Hence V belongs to Δ, and consequently $R = \mathrm{Hom}_B(M,V)$ is a projective module in $\mathrm{mod}\, H$. This shows that every right H-submodule of Q is projective. Therefore, H is a hereditary algebra whose quiver Q_H is the opposite quiver Δ^{op} of Δ. It follows also from the Brenner–Butler tilting theorem [1, Theorem VI.3.8] that $T = D(M)$ is a tilting module in $\mathrm{mod}\, H$ and there is a canonical K-algebra isomorphism $B \xrightarrow{\sim} \mathrm{End}_H(T)$. In particular, we conclude that B is a tilted algebra of Dynkin type, and hence Γ_B is a finite acyclic quiver. In fact, Δ is the section Δ_T of Γ_B given by the images $\mathrm{Hom}_H(T,I)$ of the indecomposable injective modules I in $\mathrm{mod}\, H$. Indeed, we have isomorphisms of right B-modules

$$\begin{aligned}\mathrm{Hom}_H\bigl(T, D(B)\bigr) &= \mathrm{Hom}_H\bigl(D(M), D(B)\bigr)\\ &\cong \mathrm{Hom}_{H^{\mathrm{op}}}(B, M)\\ &\cong M.\end{aligned}$$

Our next aim is to prove that B belongs to the family $\mathcal{B}_{max}$. We know that H is the tensor algebra $T(\mathbb{M})$ of a K-species $\mathbb{M}$ whose quiver $Q_{\mathbb{M}} = Q_H$ is one of the forms:

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & & & a_m \arrow[dl] \\
 & & & a_{m-1} \arrow[dl] & \\
 & & \cdots \arrow[dl] & & \\
 & a_2 \arrow[dl] & & & \\
a_1 & & & & \\
b_1 & 0, \arrow[ul] \arrow[l] \arrow[dl] & & & \\
c_1 & & & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & & & a_m \arrow[dl] \\
 & & & a_{m-1} \arrow[dl] & \\
 & & \cdots \arrow[dl] & & \\
 & a_2 \arrow[dl] & & & \\
a_1 & & & & \\
 & 0, \arrow[ul] \arrow[dl] & & & \\
b_1 & {}^{(1,2)} & & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & a_2 \arrow[dl] \\
a_1 & \\
b_1 & 0, \arrow[ul] \arrow[l] \arrow[dl] \\
c_1 & \\
 & c_2 \arrow[ul]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & a_3 \arrow[dl] \\
 & a_2 \arrow[dl] & \\
a_1 & & \\
b_1 & 0, \arrow[ul] \arrow[l] \arrow[dl] & \\
c_1 & & \\
 & c_2 \arrow[ul] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & & a_4 \arrow[dl] \\
 & & a_3 \arrow[dl] & \\
 & a_2 \arrow[dl] & & \\
a_1 & & & \\
b_1 & 0, \arrow[ul] \arrow[l] \arrow[dl] & & \\
c_1 & & & \\
 & c_2 \arrow[ul] & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & a_2 \arrow[dl] \\
a_1 & {}^{(1,2)} \\
 & 0, \arrow[ul] \arrow[dl] \\
b_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
a_1 & {}^{(1,3)} \\
 & 0. \arrow[ul]
\end{tikzcd}
$$

For each vertex x of $Q_H = Q_{\mathbb{M}}$, we denote by $I(x)$ the indecomposable injective module in $\operatorname{mod} H$ at the vertex x. Since $\Delta = \Delta_T$, we have the equalities $\tau_B^{-1} P = \operatorname{Hom}_H(T, I(0))$, $X_i = \operatorname{Hom}_H(T, I(a_i))$, $Y_1 = \operatorname{Hom}_H(T, I(b_1))$, and $Z_j = \operatorname{Hom}_H(T, I(c_j))$. It follows from (2) that Δ has no projective module, and hence T has no injective direct summand. In particular, we conclude that $\operatorname{Ext}_H^1(T, \tau_H I(0)) \cong D \operatorname{Hom}_H(I(0), T) = 0$, and so $\tau_H I(0)$ belongs to the torsion part $\mathcal{T}(T)$ of $\operatorname{mod} H$ induced by T. Then we have isomorphisms

$$
\operatorname{Hom}_H\big(T, \tau_H I(0)\big) \cong \tau_B \operatorname{Hom}_H\big(T, I(0)\big) = \tau_B\big(\tau_B^{-1} P\big) = P,
$$

and consequently $\tau_H I(0)$ is the direct summand of T. Let T' be an indecomposable direct summand of T nonisomorphic to $\tau_H I(0)$. Then we have isomorphisms

$$
0 = \operatorname{Ext}_H^1\big(\tau_H I(0), T'\big) \cong D \operatorname{Hom}_H\big(T', \tau_H^2 I(0)\big) \cong D \operatorname{Hom}_H\big(\tau_H^{-2} T', I(0)\big).
$$

This implies that T' belongs to one of the cones $\mathcal{C}(\tau_H I(a_1))$, $\mathcal{C}(\tau_H I(b_1))$, $\mathcal{C}(\tau_H I(c_1))$ in Γ_H induced by the modules $\tau_H I(a_1)$, $\tau_H I(b_1)$, $\tau_H I(c_1)$, respectively:

$$\mathcal{C}\big(\tau_H I(a_1)\big): \quad \begin{tikzcd}[column sep=tiny]
\tau_H^m I(a_m) \arrow[dr] & & \tau_H^{m-1} I(a_m) \arrow[dr] & & & & & & \tau_H^2 I(a_m) \arrow[dr] & & \tau_H I(a_m) \\
& \tau_H^{m-1} I(a_{m-1}) \arrow[ur] \arrow[dr] & & {} \arrow[dr, dotted, no head] & & & & {} \arrow[ur] & & \tau_H I(a_{m-1}) \arrow[ur] \\
& & {} \arrow[dr, dotted, no head] & & {} \arrow[dr] & & {} \arrow[ur, dotted, no head] & & {} \arrow[ur] \\
& & & {} \arrow[dr] & & \tau_H^2 I(a_3) \arrow[ur] \arrow[dr] & & {} \arrow[ur, dotted, no head] \\
& & & & \tau_H^2 I(a_2) \arrow[ur] \arrow[dr] & & \tau_H I(a_2) \arrow[ur] \\
& & & & & \tau_H I(a_1) \arrow[ur]
\end{tikzcd}$$

$$\mathcal{C}\big(\tau_H I(b_1)\big): \quad \tau_H I(b_1)$$

$$\mathcal{C}\big(\tau_H I(c_1)\big): \quad \tau_H I(c_1) \qquad \text{or} \qquad \begin{tikzcd}[column sep=small]
\tau_H^2 I(c_2) \arrow[dr] & & \tau_H I(c_2) \\
& \tau_H I(c_1) \arrow[ur]
\end{tikzcd}$$

We note that the full subcategory $\operatorname{ind}\mathcal{C}(\tau_H I(a_1))$ of $\operatorname{mod} H$ given by the indecomposable modules from the cone $\mathcal{C}(\tau_H I(a_1))$ is equivalent to the category of indecomposable modules over the tensor algebra $T(\mathbb{M}(a_1))$ of the K-species $\mathbb{M}(a_1)$ with the equioriented quiver

$$Q_{\mathbb{M}(a_1)} : 1 \longleftarrow 2 \longleftarrow \ldots \longleftarrow m-1 \longleftarrow m,$$

and all division K-algebras and nonzero bimodules in $\mathbb{M}(a_1)$ being isomorphic to the division endomorphism algebra $F_{a_1} = \operatorname{End}_H(\tau_H I(a_1)) \cong \operatorname{End}_H(I(a_1))$. Similarly, if the cone $\mathcal{C}(\tau_H I(c_1))$ has the three indecomposable modules, then the full subcategory of $\operatorname{mod} H$ given by these modules is equivalent to the category of indecomposable modules over the tensor algebra $T(\mathbb{M}(c_1))$ of the K-species $\mathbb{M}(c_1)$ with the quiver

$$Q_{\mathbb{M}(c_1)} : 1 \longleftarrow 2,$$

and the division K-algebras and the nonzero bimodule in $\mathbb{M}(c_1)$ being isomorphic to the division algebra $F_{c_1} = \operatorname{End}_H(\tau_H I(c_1)) \cong \operatorname{End}_H(I(c_1))$. Therefore, the tilting module T has a decomposition

$$T = \tau_H I(0) \oplus T_a \oplus T_b \oplus T_c,$$

where T_a, T_b, T_c are direct sums of indecomposable modules lying in the cones $\mathcal{C}(\tau_H I(a_1))$, $\mathcal{C}(\tau_H I(b_1))$, $\mathcal{C}(\tau_H I(c_1))$, respectively. Observe also that for any indecomposable modules U in $\mathcal{C}(\tau_H I(a_1))$, V in $\mathcal{C}(\tau_H I(b_1))$, W in $\mathcal{C}(\tau_H I(c_1))$, we have

$$\begin{aligned}
\operatorname{Ext}_H^1(U \oplus V \oplus W, U \oplus V \oplus W) &\cong D\operatorname{Hom}_H(U \oplus V \oplus W, \tau_H U \oplus \tau_H V \oplus \tau_H W) = 0,\\
\operatorname{Ext}_H^1\big(U \oplus V \oplus W, \tau_H I(0)\big) &\cong D\operatorname{Hom}_H\big(I(0), U \oplus V \oplus W\big) = 0,\\
\operatorname{Ext}_H^1\big(\tau_H I(0), U \oplus V \oplus W\big) &\cong D\operatorname{Hom}_H\big(U \oplus V \oplus W, \tau_H^2 I(0)\big) = 0.
\end{aligned}$$

Further, since $B \cong \operatorname{End}_H(T)$ is a basic algebra, T is a direct sum of pairwise nonisomorphic indecomposable modules, and the number of them is equal to the rank of $K_0(H)$, or equivalently, the number of τ_H-orbits in Γ_H. Moreover, T_a, T_b, T_c are partial tilting modules from the additive categories $\operatorname{add}\mathcal{C}(\tau_H I(a_1))$, $\operatorname{add}\mathcal{C}(\tau_H I(b_1))$, $\operatorname{add}\mathcal{C}(\tau_H I(c_1))$ of the cones $\mathcal{C}(\tau_H I(a_1))$, $\mathcal{C}(\tau_H I(b_1))$, $\mathcal{C}(\tau_H I(c_1))$, respectively. Hence, the numbers of indecomposable direct summands of T_a, T_b, T_c are less than or equal to the depths of the cones $\mathcal{C}(\tau_H I(a_1))$, $\mathcal{C}(\tau_H I(b_1))$, $\mathcal{C}(\tau_H I(c_1))$, respectively. Then it follows that the numbers of indecomposable direct summands of T_a, T_b, T_c are exactly the depths of $\mathcal{C}(\tau_H I(a_1))$, $\mathcal{C}(\tau_H I(b_1))$, $\mathcal{C}(\tau_H I(c_1))$, and $\tau_H I(a_1)$, $\tau_H I(b_1)$, $\tau_H I(c_1)$ are direct summands of T_a, T_b, T_c, respectively. Applying now the classification of tilted algebras of an equioriented type $\mathbb{A}_m$ (see [24, Section XVI.2]), we conclude that $B = \operatorname{End}_H(T) = \operatorname{End}_H(\tau_H I(0) \oplus T_a \oplus T_b \oplus T_c)$ is a tilted algebra from the family $\mathcal{B}_{max}$.

Finally, we prove that $\tau_B^{-1} P$ is not an injective module in $\operatorname{mod} B$. Suppose $\tau_B^{-1} P = \operatorname{Hom}_H(T, I(0))$ is an injective module. Then it follows from [1, Lemma VI.4.9] that the indecomposable projective module $P(0)$ at the vertex 0 of Q_H is a direct summand of T, and hence is isomorphic to $\tau_H I(0)$, or belongs to one of the cones $\mathcal{C}(\tau_H I(a_1))$, $\mathcal{C}(\tau_H I(b_1))$, $\mathcal{C}(\tau_H I(c_1))$. But it is not possible, because $P(0)$ has simple top and $\operatorname{rad} P(0)$ is a direct sum of three indecomposable projective modules. Therefore, $\tau_B^{-1} P$ is not an injective module in $\operatorname{mod} B$. □

4. Selfinjective algebras of Dynkin type

Let B be a triangular algebra (the quiver Q_B is acyclic) and $e_1, \dots, e_n$ be pairwise orthogonal primitive idempotents of B with $1_B = e_1 + \cdots + e_n$. We identify B with the full subcategory B_0 of the repetitive category $\widehat{B}$ given by the objects $e_{0,i}$, $1 \le j \le n$. For a sink i of Q_B, the *reflection* $S_i^+ B$ of B at i is the full subcategory of $\widehat{B}$ given by the objects

$$e_{0,j}, \quad 1 \le j \le n, \quad j \ne i, \quad \text{and} \quad e_{1,i} = \nu_{\widehat{B}}(e_{0,i}).$$

Then the quiver $Q_{S_i^+ B}$ of $S_i^+ B$ is the reflection $\sigma_i^+ Q_B$ of Q_B at i (see [15]). Observe that $\widehat{B} = \widehat{S_i^+ B}$. By a *reflection sequences of sinks* of Q_B we mean a sequence $i_1, \dots, i_t$ of vertices of Q_B such that i_s is a sink of $\sigma^+_{i_{s-1}} \dots \sigma^+_{i_1} Q_B$ for all s in $\{1, \dots, t\}$. Moreover, for a sink i of Q_B, we denote by $T^+_i B$ the full subcategory of $\widehat{B}$ given by the objects

$$e_{0,j}, \quad 1 \leq j \leq n, \quad \text{and} \quad e_{1,i} = \nu_{\widehat{B}}(e_{0,i}).$$

Observe that $T^+_i B$ is the one-point extension $B[I_B(i)]$ of B by the indecomposable injective B-module $I_B(i)$ at the vertex i. By a finite dimensional $\widehat{B}$-module we mean a contravariant K-linear functor M from $\widehat{B}$ to the category of K-vector spaces such that $\sum_{x \in ob\widehat{B}} \dim_K M(x)$ is finite. We denote by $\operatorname{mod} \widehat{B}$ the category of all finite dimensional $\widehat{B}$-modules. Finally, for a module M in $\operatorname{mod} \widehat{B}$, we denote by $\operatorname{supp}(M)$ the full subcategory of $\widehat{B}$ formed by all objects x with $M(x) \neq 0$, and call the *support of* M.

The following consequences of results proved in [14,15] describe the supports of finite dimensional indecomposable modules over the repetitive categories $\widehat{B}$ of tilted algebras B of Dynkin type.

Theorem 4.1. *Let B be a tilted algebra of Dynkin type and n the rank of $K_0(B)$. Then there exists a reflection sequence $i_1, \dots, i_n$ of sinks of Q_B such that the following statements hold.*

(i) $S^+_{i_n} \dots S^+_{i_1} B = \nu_{\widehat{B}}(B)$.
(ii) *For every indecomposable nonprojective module M in* $\operatorname{mod} \widehat{B}$, $\operatorname{supp}(M)$ *is contained in one of the full subcategories of $\widehat{B}$*

$$\nu^m_{\widehat{B}}\big(S^+_{i_r} \dots S^+_{i_1} B\big), \quad r \in \{1, \dots, n\},\ m \in \mathbb{Z}.$$

(iii) *For every indecomposable projective module P in* $\operatorname{mod} \widehat{B}$, $\operatorname{supp}(P)$ *is contained in one of the full subcategories of $\widehat{B}$*

$$\nu^m_{\widehat{B}}\big(T^+_{i_r} S^+_{i_{r-1}} \dots S^+_{i_1} B\big), \quad r \in \{1, \dots, n\},\ m \in \mathbb{Z}.$$

It follows from Theorem 4.1 that the repetitive category $\widehat{B}$ of a tilted algebra B of Dynkin type is a locally representation-finite category [12], that is, for any object x in $\widehat{B}$ the number of isomorphism classes of indecomposable modules N in $\operatorname{mod} \widehat{B}$ with $N(x) \neq 0$ is finite. Then we obtain the following consequence of [12, Theorem 3.6].

Theorem 4.2. *Let B be a tilted algebra of Dynkin type, G an admissible infinite cyclic automorphism group of $\widehat{B}$, and $A = \widehat{B}/G$ the associated selfinjective orbit algebra. Then the following statements hold.*

(i) *The push-down functor F_λ :* $\operatorname{mod} \widehat{B} \to \operatorname{mod} A$, *associated to the Galois covering $F : \widehat{B} \to \widehat{B}/G = A$, is dense and preserves the almost split sequences.*

(ii) *The Auslander–Reiten quiver* Γ_A *of* A *is the orbit quiver* $\Gamma_{\widehat{B}}/G$ *of* $\Gamma_{\widehat{B}}$ *with respect to the induced action of* G *on* $\Gamma_{\widehat{B}}$.

(iii) A *is of finite representation type.*

Theorem 4.3. *Let* B *be an algebra from the family* $\mathcal{B}_{max}$. *The following statements hold.*

(i) *For any integer* m, *the indecomposable projective module* $P(m,0)$ *in* $\operatorname{mod}\widehat{B}$ *given by the idempotent* $e_{m,0} = \nu_{\widehat{B}}^m(e_{0,0})$ *has the property* $\alpha(P(m,0)/\operatorname{soc}P(m,0)) = 4$.

(ii) *For any indecomposable projective module* P *in* $\operatorname{mod}\widehat{B}$ *nonisomorphic to a module* $P(m,0)$, $m \in \mathbb{Z}$, *we have* $\alpha(P/\operatorname{soc}P) \le 3$.

Proof. For a vertex i of Q_B and $m \in \mathbb{Z}$, we denote by $P(m,i)$ the indecomposable projective module in $\operatorname{mod}\widehat{B}$ given by the idempotent $e_{m,i} = \nu_{\widehat{B}}^m(e_{0,i})$ and by $S(m,i)$ the top of $P(m,i)$. Let $a = a_0 \to a_1 \to \ldots \to a_r$, with $r \ge 0$, be the maximal path in Q_B starting from a, and $c = c_0 \longrightarrow c_s$, with $s \in \{0,1\}$, the maximal path in Q_B starting from c (if B is of type (1), (3) or (4)). For each $m \in \mathbb{Z}$, we denote by $U(m,a)$ the uniserial module in $\operatorname{mod}\widehat{B}$ with the radical layers $\operatorname{rad}^i U(m,a)/\operatorname{rad}^{i+1} U(m,a) = S(m,a_i)$ for $i \in \{0,\ldots,r\}$. Similarly, for each $m \in \mathbb{Z}$, we denote by $U(m,c)$ the uniserial module in $\operatorname{mod}\widehat{B}$ of length two with $U(m,c)/\operatorname{rad}U(m,c) = S(m,c)$ and $\operatorname{rad}U(m,c) = S(m,d)$, if $s = 1$, and the simple module $S(m,c)$, if $s = 0$.

Let n be the rank of $K_0(B)$. Then there is a reflection sequence $i_1,\ldots,i_n$ of sinks of Q_B with $i_n = 0$ such that

(α) $S^+_{i_n}\ldots S^+_{i_1}B = \nu_{\widehat{B}}(B)$.

(β) For $D = S^+_{i_{n-1}}\ldots S^+_{i_1}B$, the indecomposable injective module $I_D(0)$ at the sink $0 = i_n$ of $Q_D = \sigma^+_{i_{n-1}}\ldots\sigma^+_{i_1}Q_B$ has the property $I_D(0)/\operatorname{soc}I_D(0) = U_1 \oplus U_2 \oplus U_3$, where U_1, U_2, U_3 are uniserial modules in $\operatorname{mod}\widehat{B}$ of the forms:
- $U_1 = U(1,a)$;
- $U_2 = U_3 = S(1,b)$ if B is of type (2);
- $U_2 = U_3 = U(1,c)$ if B is of type (4);
- $U_2 = S(1,b)$ and $U_3 = U(1,c)$ if B is of type (1) or (3);
- $U_1 = U_2 = U_3 = U(1,a) = S(1,a)$ if B is of type (5).

Then the indecomposable projective module $P(1,0)$ in $\operatorname{mod}\widehat{B}$ has the property $\alpha(P(1,0)/\operatorname{soc}P(1,0)) = 4$, because $\operatorname{rad}P(1,0) = I_D(0)$. On the other hand, for $r \in \{1,\ldots,n-1\}$, and the indecomposable injective module $I_{S^+_{i_{r-1}}\ldots S^+_{i_1}B}(i_r)$ in $\operatorname{mod}(S^+_{i_{r-1}}\ldots S^+_{i_1}B)$ at the sink i_r of $\sigma^+_{i_{r-1}}\ldots\sigma^+_{i_1}Q_B$, we have $I_{S^+_{i_{r-1}}\ldots S^+_{i_1}B}(i_r)/\operatorname{soc}I_{S^+_{i_{r-1}}\ldots S^+_{i_1}B}(i_r)$ being a direct sum of at most two uniserial modules, and consequently $\alpha(P(1,i_r)/\operatorname{soc}P(1,i_r)) \le 3$. Since the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ induces an automorphism of the module category $\operatorname{mod} \widehat{B}$, we have $P(m,i) = \nu_{\widehat{B}}^{m-1} P(1,i)$ for any vertex i of Q_B and $m \in \mathbb{Z}$. Then the statements (i) and (ii) follow. $\square$

Corollary 4.4. *Let B be an algebra from the family $\mathcal{B}_{max}$ and G an admissible group of automorphisms of $\widehat{B}$. Then G is an infinite cyclic group generated by an automorphism $\varphi\nu_{\widehat{B}}^{m}$, for a positive integer m and a rigid automorphism φ of $\widehat{B}$.*

Proof. Let g be an element of G. Then, for any indecomposable projective module P in $\operatorname{mod} \widehat{B}$, we have $\alpha(g(P)/\operatorname{soc} g(P)) = \alpha(P/\operatorname{soc} P)$, because g is an automorphism of $\operatorname{mod} \widehat{B}$. In particular, it follows from Theorem 4.3 that there exists an integer m_g such that $g(P(m,0)) \cong P(m+m_g,0)$ for any $m \in \mathbb{Z}$. Observe that then $g^{-1}(P(m,0)) \cong P(m-m_g,0)$. Assume $m_g = 0$. Then $g(P(m,0)) \cong P(m,0)$, and hence $g(e_{m,0}) = e_{m,0}$ for any $m \in \mathbb{Z}$. Since G acts freely on the objects of $\widehat{B}$, we obtain $g = 1$. We also note that G is infinite because $\widehat{B}$ has infinitely many objects and there are only finitely many G-orbits of objects in $\widehat{B}$. Take now $g \in G$ with m_g positive and minimal. Then, for any $h \in G \backslash \{1\}$, we have $m_h = sm_g + r$ with $s \in \mathbb{Z}$ and $r \in \{0,\dots,m_g-1\}$. Consider $f = hg^{-s} \in G$. Then $m_f = 0$, and hence $h = g^s$. This shows that G is an infinite cyclic group generated by g. On the other hand, the automorphism $\varphi = g\nu_{\widehat{B}}^{-m_g}$ of $\widehat{B}$ has the property $\varphi(P(m,0)) = P(m,0)$ for any $m \in \mathbb{Z}$. This forces the equalities $\varphi(\nu_{\widehat{B}}^{m}(B)) = \nu_{\widehat{B}}^{m}(B)$ for $m \in \mathbb{Z}$, and so φ is a rigid automorphism of $\widehat{B}$. Therefore, $g = \varphi\nu_{\widehat{B}}^{m_g}$, as required. $\square$

5. Dynkin quivers of maximal almost split sequences

In this section we describe some properties of the quiver Δ_P associated to an indecomposable projective module P occurring in a maximal almost split sequence of a selfinjective algebra of finite representation type.

Proposition 5.1. *Let A be a selfinjective algebra of finite representation type and P be an indecomposable projective module in* $\operatorname{mod} A$ *with $\alpha(P/\operatorname{soc} P) = 4$. Moreover, let*

$$X = X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_r$$

be a sectional path of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ *with X a direct summand of* $\operatorname{rad} P/\operatorname{soc} P$. *Then $X_0, X_1, \dots, X_r$ are nonprojective modules.*

Proof. Observe first that X is not projective, being a proper submodule of the indecomposable module $P/\operatorname{soc} P$. Assume to the contrary that one of the modules $X_1,\dots,X_r$ is projective. We have two cases to consider.

(1) Assume $X_1 \neq P/\operatorname{soc} P$. Then Γ_A contains a full valued subquiver of the form

$$\begin{tikzcd}
 & Y_0 = P/\operatorname{soc} P \arrow[dr] & & & & \\
X = X_0 \arrow[ur, "{(1,d)}"] \arrow[dr] & & Y_1 \arrow[dr] & & & \\
 & X_1 \arrow[ur] \arrow[dr] & & \ddots \arrow[dr] & & \\
 & & \ddots \arrow[dr] & & Y_{m-1} \arrow[dr] & \\
 & & & X_{m-1} \arrow[ur] \arrow[dr] & & Y_m \\
 & & & & X_m \arrow[ur] &
\end{tikzcd}$$

with $d \in \{1, 2\}$, $1 \leq m \leq r$, $X_0, X_1, \ldots, X_m$ nonprojective, and $X_m = \operatorname{rad} Q$ for an indecomposable projective module Q. Then Y_m is an indecomposable direct summand of $\operatorname{rad} Q/\operatorname{soc} Q$, and hence $l(X_m) > l(Y_m)$. Moreover, we have the equalities

$$l(X_{i-1}) + l(Y_i) = l(X_i) + l(Y_{i-1})$$

for all $i \in \{1, \ldots, m\}$. Then $l(X_m) > l(Y_m)$ implies that $l(X) = l(X_0) > l(Y_0) = l(P/\operatorname{soc} P)$, a contradiction because X is a proper submodule of $P/\operatorname{soc} P$.

(2) Assume $X_1 = P/\operatorname{soc} P$. Clearly, X_1 is not projective. Then Γ_A contains a full valued subquiver of the form

$$\begin{tikzcd}
 & V_2 \arrow[dr] & & & & \\
X_2 \arrow[ur, "{(1,d)}"] \arrow[dr] & & V_3 \arrow[dr] & & & \\
 & X_3 \arrow[ur] \arrow[dr] & & \ddots \arrow[dr] & & \\
 & & \ddots \arrow[dr] & & V_{m-1} \arrow[dr] & \\
 & & & X_{m-1} \arrow[ur] \arrow[dr] & & V_m \\
 & & & & X_m \arrow[ur] &
\end{tikzcd}$$

with $d \in \{1, 2\}$, $2 \leq m \leq r$, $X_2, X_3, \dots, X_m$ nonprojective, and $X_m = \operatorname{rad} Q$ for an indecomposable projective module Q. Then V_m is an indecomposable direct summand of $\operatorname{rad} Q / \operatorname{soc} Q$, and hence $l(X_m) > l(V_m)$. Moreover, if $m \geq 3$, we have the equalities

$$l(X_{j-1}) + l(V_j) = l(X_j) + l(V_{j-1})$$

for all $j \in \{3, \dots, m\}$. Then $l(X_m) > l(V_m)$ implies that $l(X_2) > l(V_2)$. Since $\alpha(P/\operatorname{soc} P) = 4$ we have in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow \operatorname{rad} P \longrightarrow U_1 \oplus U_2 \oplus U_3 \oplus P \longrightarrow P/\operatorname{soc} P \longrightarrow 0$$

with U_1, U_2, U_3 indecomposable modules (possibly not pairwise nonisomorphic). Observe also that, for any $i \in \{1, 2, 3\}$, we have in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow U_i \longrightarrow P/\operatorname{soc} P \oplus W_i \longrightarrow \tau_A^{-1} U_i \longrightarrow 0,$$

and hence $l(U_i) + l(\tau_A^{-1} U_i) \geq l(P/\operatorname{soc} P)$. Moreover, $\operatorname{rad}(P/\operatorname{soc} P) = \operatorname{rad} P/\operatorname{soc} P = U_1 \oplus U_2 \oplus U_3$ has nonsimple socle, and hence $P/\operatorname{soc} P$ is not the radical of an indecomposable projective module in $\operatorname{mod} A$. Hence we have in $\operatorname{mod} A$ an almost split sequence of the form

$$0 \longrightarrow P/\operatorname{soc} P \longrightarrow \tau_A^{-1} U_1 \oplus \tau_A^{-1} U_2 \oplus \tau_A^{-1} U_3 \longrightarrow \tau_A^{-1}(P/\operatorname{soc} P) \longrightarrow 0,$$

and then

$$l(P/\operatorname{soc} P) + l\big(\tau_A^{-1}(P/\operatorname{soc} P)\big) = l\big(\tau_A^{-1} U_1\big) + l\big(\tau_A^{-1} U_2\big) + l\big(\tau_A^{-1} U_3\big).$$

Note that $X_2 = \tau_A^{-1} U_i$ for some $i \in \{1, 2, 3\}$. We may assume that $X_2 = \tau_A^{-1} U_1$. Then $l(\tau_A^{-1} U_1) = l(X_2) > l(V_2) = l(\tau_A^{-1}(P/\operatorname{soc} P))$ implies that $l(P/\operatorname{soc} P) > l(\tau_A^{-1} U_2) + l(\tau_A^{-1} U_3)$. Moreover, we have the inequality

$$l(U_2) + l\big(\tau_A^{-1} U_2\big) + l(U_3) + l\big(\tau_A^{-1} U_3\big) \geq 2 l(P/\operatorname{soc} P).$$

This leads to the inequality $l(U_2) + l(U_3) > l(P/\operatorname{soc} P)$, a contradiction because $U_2 \oplus U_3$ is a direct summand of $\operatorname{rad} P/\operatorname{soc} P$. $\square$

The following theorem is a direct consequence of Proposition 5.1 and description of the stable Auslander–Reiten quiver of a selfinjective algebra of finite representation type, established by C. Riedtmann [20] and G. Todorov [36] (see also [35, Theorem IV.15.6]).

Theorem 5.2. *Let A be a selfinjective algebra of finite representation type, P an indecomposable projective module in* $\operatorname{mod} A$ *with $\alpha(P/\operatorname{soc} P) = 4$, and Δ_P the full valued subquiver of Γ_A given by the module $\tau_A^{-1}(P/\operatorname{soc} P)$ and all modules X in* $\operatorname{ind} A$ *lying* *on a nontrivial sectional path in Γ_A from $P/\operatorname{soc} P$ to X. Then the following statements hold.*

(i) *Δ_P consists of nonprojective modules.*
(ii) *Δ_P is a Dynkin quiver whose underlying graph $\overline{\Delta_P}$ is the type $\Delta(A)$ of A.*
(iii) *There is no arrow $Q \to X$ in Γ_A with Q a projective module and X a module on Δ_P.*
(iv) *There is no arrow $X \to Q$ in Γ_A with Q a projective module and X a module on Δ_P different from $\tau_A^{-1}(P/\operatorname{soc} P)$.*

6. Proof of Theorem 2

Let Λ be a tilted algebra of Dynkin type, G an admissible infinite cyclic group of automorphisms of $\widehat{\Lambda}$, and $A = \widehat{\Lambda}/G$ the associated orbit algebra. Then it follows from Theorem 4.2 that A is a selfinjective algebra of finite representation type, the push-down functor $F_\lambda : \operatorname{mod} \widehat{\Lambda} \to \operatorname{mod} A$ associated to the Galois covering $F : \widehat{\Lambda} \to \widehat{\Lambda}/G = A$ is dense and preserves almost split sequences, and the Auslander–Reiten quiver Γ_A of A is the orbit quiver $\Gamma_{\widehat{\Lambda}}/G$ of $\Gamma_{\widehat{\Lambda}}$ with respect to the induced action of G on $\Gamma_{\widehat{\Lambda}}$.

Assume $\operatorname{mod} A$ admits a maximal almost split sequence. Then there exists an indecomposable projective module P in $\operatorname{mod} A$ such that $\alpha(P/\operatorname{soc} P) = 4$. Let Δ_P be the full valued subquiver of Γ_A given by the module $\tau_A^{-1}(P/\operatorname{soc} P)$ and all modules X in $\operatorname{ind} A$ lying on a nontrivial sectional path in Γ_A from $P/\operatorname{soc} P$ to X. Since the push-down functor F_λ is dense and preserves almost split sequences, there exists an indecomposable projective module P^* in $\operatorname{mod} \widehat{\Lambda}$ such that $F_\lambda(P^*) = P$ and $\alpha(P^*/\operatorname{soc} P^*) = \alpha(P/\operatorname{soc} P) = 4$. Denote by Δ_{P^*} the full valued subquiver of $\Gamma_{\widehat{\Lambda}}$ given by the module $\tau_{\widehat{\Lambda}}^{-1}(P^*/\operatorname{soc} P^*)$ and all modules X^* in $\operatorname{ind} \widehat{\Lambda}$ lying on a nontrivial sectional path in $\Gamma_{\widehat{\Lambda}}$ from $P^*/\operatorname{soc} P^*$ to X^*. Observe that $F_\lambda(\Delta_{P^*}) = \Delta_P$. It follows from Theorem 5.2 that Δ_P is a Dynkin quiver consisting of nonprojective modules, and the underlying graph $\overline{\Delta_P}$ of Δ_P is the Dynkin type $\Delta(A)$ of A. Then Δ_{P^*} is a Dynkin quiver isomorphic to Δ_P, consisting of nonprojective modules, and the stable Auslander–Reiten quiver $\Gamma_{\widehat{\Lambda}}^s$ of $\Gamma_{\widehat{\Lambda}}$ is isomorphic to the translation quiver $\mathbb{Z}\Delta_{P^*}$. Let M_{P^*} be the direct sum of all indecomposable modules lying on Δ_{P^*}. Observe that, by Theorem 4.1, M_{P^*} is a module over a bounded full subcategory $\Lambda[-n, n]$ of $\widehat{\Lambda}$ given by all objects of $\nu_{\widehat{\Lambda}}^r(\Lambda)$ for integers r with $-n \le r \le n$, for some positive integer n. Let $I = I_{P^*}$ be the right annihilator of M_{P^*} in $\Lambda[-n, n]$, and $B = \Lambda[-n, n]/I$. Then M_{P^*} is a faithful B-module. Moreover, since Δ_{P^*} consists of nonprojective modules, we infer that $\overline{P} = P^*/\operatorname{soc} P^*$ is an indecomposable projective module in $\operatorname{mod} B$ and Δ_{P^*} coincides with the full translation subquiver $\Delta_{\overline{P}}$ of Γ_B given by the module $\tau_B^{-1}\overline{P} = \tau_{\widehat{\Lambda}}^{-1}(P^*/\operatorname{soc} P^*)$ and the indecomposable modules $\overline{X}$ in $\operatorname{ind} B$ lying on a nontrivial sectional path in Γ_B starting from $\overline{P} = P^*/\operatorname{soc} P^*$. Applying now Theorem 5.2 to $\Delta_{\overline{P}}$ we conclude that $\alpha(\tau_B^{-1}\overline{P}) = 3$ and $\Delta_{\overline{P}} = \Delta_{P^*}$ satisfies the assumptions of Theorem 3.3. Therefore, B is a tilted algebra from the family $\mathcal{B}_{max}$, and the full subquiver of Γ_B given by the modules from

$\tau_B\Delta_{P^*} \cup \Delta_{P^*} \cup \tau_B^{-1}\Delta_{P^*}$ coincides with the full subquiver of $\Gamma_{\widehat{\Lambda}}$ given by the modules from $\tau_{\widehat{\Lambda}}\Delta_{P^*} \cup \Delta_{P^*} \cup \tau_{\widehat{\Lambda}}^{-1}\Delta_{P^*}$. Then it follows from [14] and [15] that $\operatorname{mod}\widehat{\Lambda} = \operatorname{mod}\widehat{B}$, and consequently $\widehat{\Lambda} \cong \widehat{B}$. Applying now Corollary 4.4 we conclude that $A = \widehat{\Lambda}/G$ is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, for a positive integer m and a rigid automorphism φ of $\widehat{B}$.

Therefore the implication (i) $\Rightarrow$ (ii) in Theorem 2 holds. The converse implication (ii) $\Rightarrow$ (i) in Theorem 2 is a direct consequence of Theorems 4.2 and 4.3.

7. Proof of Theorem 3

Let K be an algebraically closed field. By general theory established by C. Riedtmann in [7,21,22] the class of all (basic, indecomposable) selfinjective algebras of finite representation type over K splits into two disjoint classes: the standard algebras having simply connected Galois covering and the remaining nonstandard algebras. The standard algebras A, nonisomorphic to K, are exactly the orbit algebras $\widehat{B}/G$, where B is a tilted algebra of type $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$ (see [26, Theorem 3.5]). The nonstandard selfinjective algebras of finite representation type occur only for K of characteristic 2, and can be described as modified Brauer tree algebras (see [22,37] and [26, Theorem 3.8]). By Theorem 2, in order to prove Theorem 3, it is enough to show that the module categories of nonstandard algebras of finite representation type over K do not admit maximal almost split sequences.

Assume K is of characteristic 2 and A is a nonstandard selfinjective algebra of finite representation type over K. Then there exists a tilted algebra B of Dynkin type $\mathbb{D}_{3n}$ $(n \geq 2)$, and a positive automorphism ψ of $\widehat{B}$ such that $\psi^3 = \nu_{\widehat{B}}$ and A is socle equivalent to the orbit algebra $\overline{A} = \widehat{B}/(\psi)$ (the standard form of A). Observe now that $\operatorname{mod} A$ admits an indecomposable projective module P with $\alpha(P/\operatorname{soc} P) = 4$ if and only if $\operatorname{mod}\overline{A}$ admits an indecomposable projective module $\overline{P}$ with $\alpha(\overline{P}/\operatorname{soc}\overline{P}) = 4$. Suppose that $\operatorname{mod}\overline{A}$ admits an indecomposable projective module $\overline{P}$ with $\alpha(\overline{P}/\operatorname{soc}\overline{P}) = 4$. Then it follows from Theorem 2 that $\overline{A}$ is isomorphic to an orbit algebra $\widehat{B^*}/(\varphi\nu_{\widehat{B^*}}^m)$, where B^* is a tilted algebra from the family $\mathcal{B}_{max}$, m is a positive integer, and φ is a rigid automorphism of $\widehat{B^*}$. But then $\widehat{B} \cong \widehat{B^*}$, and consequently there is no positive automorphism ψ of $\widehat{B}$ with $\psi^3 = \nu_{\widehat{B}}$ (see Corollary 4.4). Therefore, $\operatorname{mod} A$ does not admit a maximal almost split sequence. The fact that $\alpha(P/\operatorname{soc} P) \leq 3$ for any indecomposable projective module P over the nonstandard algebra A follows also from the description of A as a modified Brauer tree algebra (see [26, Section 3.6]).

8. Proof of Theorem 1

Let A be a finite dimensional basic indecomposable selfinjective K-algebra over a field K. We show first that (ii) implies (i).

Assume A is socle equivalent to an orbit algebra $A' = \widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, where B is an algebra from the family $\mathcal{B}_{max}$, m a positive integer, and φ a rigid automorphism of $\widehat{B}$. Then there is an isomorphism of algebras $\phi : A/\operatorname{soc} A \to A'/\operatorname{soc} A'$ and the induced isomorphism of module categories $\Phi : \operatorname{mod}(A/\operatorname{soc} A) \to \operatorname{mod}(A'/\operatorname{soc} A')$. Obviously, A is of finite representation type, because A' is of finite representation type, by Theorem 4.2. It follows from Theorem 2 that $\operatorname{mod} A'$ admits a maximal almost split sequence, and consequently there is an indecomposable projective module P' in $\operatorname{mod} A'$ such that $\alpha(P'/\operatorname{soc} P') = 4$. Let P be the indecomposable projective module in $\operatorname{mod} A$ such that $\Phi(P/\operatorname{soc} P) = P'/\operatorname{soc} P'$. Observe that $\alpha(P/\operatorname{soc} P) = \alpha(P'/\operatorname{soc} P') = 4$, because the Auslander–Reiten quivers Γ_A and $\Gamma_{A'}$ are isomorphic. Let $\Delta_{P'}$ be the full subquiver of $\Gamma_{A'}$ given by the module $\tau_{A'}^{-1}(P'/\operatorname{soc} P')$ and all modules X' in $\operatorname{ind} A'$ lying on a nontrivial sectional path in $\Gamma_{A'}$ from $P'/\operatorname{soc} P'$ to X'. It follows from Theorem 5.2 that $\Delta_{P'}$ consists of nonprojective modules in $\operatorname{mod} A'$, and consequently $\Delta_{P'}$ is a full translation subquiver of $\Gamma_{A'/\operatorname{soc} A'}$. Then $\Delta_{P'} = \Phi(\Delta_P)$ for the full subquiver Δ_P of $\Gamma_{A/\operatorname{soc} A}$ given by the module $\tau_A^{-1}(P/\operatorname{soc} P)$ and all modules X in $\operatorname{ind} A$ lying on a nontrivial sectional path in Γ_A from $P/\operatorname{soc} P$ to X. Observe that, by Theorem 5.2, Δ_P consists of nonprojective modules in $\operatorname{mod} A$, and hence is a full subquiver of $\Gamma_{A/\operatorname{soc} A}$. Let M_P be the direct sum of all modules in $\operatorname{ind} A$ lying on Δ_P and $M_{P'}$ the direct sum of all modules in $\operatorname{ind} A'$ lying on $\Delta_{P'}$. Then M_P is a module in $\operatorname{mod} A/\operatorname{soc} A$, $M_{P'}$ is a module in $\operatorname{mod} A'/\operatorname{soc} A'$, and $\Phi(M_P) = M_{P'}$. Moreover, we have $\tau_A M_P = \tau_{A/\operatorname{soc} A} M_P$, $\tau_{A'} M_{P'} = \tau_{A'/\operatorname{soc} A'} M_{P'}$, $\Phi(\tau_{A/\operatorname{soc} A} M_P) = \tau_{A'/\operatorname{soc} A'} M_{P'}$, and an isomorphism of K-vector spaces

$$\operatorname{Hom}_{A/\operatorname{soc} A}(M_P, \tau_{A/\operatorname{soc} A} M_P) \xrightarrow{\sim} \operatorname{Hom}_{A'/\operatorname{soc} A'}(M_{P'}, \tau_{A'/\operatorname{soc} A'} M_{P'})$$

induced by Φ. Clearly, we have $\operatorname{Hom}_A(M_P, \tau_A M_P) = \operatorname{Hom}_{A/\operatorname{soc} A}(M_P, \tau_{A/\operatorname{soc} A} M_P)$ and $\operatorname{Hom}_{A'}(M_{P'}, \tau_{A'} M_{P'}) = \operatorname{Hom}_{A'/\operatorname{soc} A'}(M_{P'}, \tau_{A'/\operatorname{soc} A'} M_{P'})$. Therefore, in order to prove that $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$, it is enough to show that $\operatorname{Hom}_{A'}(M_{P'}, \tau_{A'} M_{P'}) = 0$. Consider the Galois covering $F : \widehat{B} \to \widehat{B}/G = A'$, where G is the infinite cyclic group generated by $g = \varphi\nu_{\widehat{B}}^m$, and the associated push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A'$. Since F_λ is dense and preserves almost split sequences, there exists an indecomposable projective module P^* in $\operatorname{mod} \widehat{B}$ such that $\alpha(P^*/\operatorname{soc} P^*) = \alpha(P'/\operatorname{soc} P') = 4$ and $\Delta_{P'} = F_\lambda(\Delta_{P^*})$ for the full subquiver Δ_{P^*} of $\Gamma_{\widehat{B}}$ given by the module $\tau_{\widehat{B}}^{-1}(P^*/\operatorname{soc} P^*)$ and all modules X^* in $\operatorname{ind} \widehat{B}$ lying on a nontrivial sectional path in $\Gamma_{\widehat{B}}$ from $P^*/\operatorname{soc} P^*$ to X^*. Clearly, then $M_{P'} = F_\lambda(M_{P^*})$, where M_{P^*} is the direct sum of all modules in $\operatorname{ind} \widehat{B}$ lying on Δ_{P^*}. We also note that $\tau_{A'} M_{P'} = F_\lambda(\tau_{\widehat{B}} M_{P^*})$. Further, the push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A'$ is a Galois covering of module categories (see [12, Theorem 3.6]), and hence it induces an isomorphism of K-vector spaces

$$\bigoplus_{r\in\mathbb{Z}} \operatorname{Hom}_{\widehat{B}}\big(M_{P^*}, g^r(\tau_{\widehat{B}} M_{P^*})\big) \xrightarrow{\sim} \operatorname{Hom}_{A'}(M_{P'}, \tau_{A'} M_{P'}).$$

Moreover, we have $g^r(\tau_{\widehat{B}} M_{P^*}) = \tau_{\widehat{B}} g^r(M_{P^*})$ for any $r \in \mathbb{Z}$. Since $\alpha(P^*/\operatorname{soc} P^*) = 4$, applying Theorem 4.3, we conclude that M_{P^*} is a module in a module category $\operatorname{mod} \nu^s_{\widehat{B}}(B)$, for some integer s. We may assume without loss of generality that $s = 0$. We also note that $\operatorname{supp}(\tau_{\widehat{B}} M_{P^*}) = \operatorname{supp}(M_{P^*})$, because $\tau_{\widehat{B}} M_{P^*} = \tau_B M_{P^*}$ and every homomorphism from an indecomposable projective $\widehat{B}$-module to M_{P^*} factors through a module of the form $(\tau_{\widehat{B}} M_{P^*})^t$ for some integer t. On the other hand, $g = \varphi \nu^m_{\widehat{B}}$ with $m \geq 1$ and φ a rigid automorphism of $\widehat{B}$. Hence, applying Theorem 4.1, we conclude that the modules M_{P^*} and $\tau_{\widehat{B}} g^r(M_{P^*})$, with $r \in \mathbb{Z} \backslash \{0\}$, have disjoint supports, and consequently $\operatorname{Hom}_{\widehat{B}}(M_{P^*}, g^r(\tau_{\widehat{B}} M_{P^*})) = 0$. We have also $\operatorname{Hom}_{\widehat{B}}(M_{P^*}, \tau_{\widehat{B}} M_{P^*}) = 0$, because the quiver $\Gamma_{\widehat{B}}$ is acyclic and every nonzero nonisomorphism between modules in $\operatorname{ind} \widehat{B}$ is a finite sum of compositions of irreducible homomorphisms between indecomposable modules. Summing up, we conclude that $\operatorname{Hom}_{\widehat{B}}(M_{P^*}, g^r(\tau_{\widehat{B}} M_{P^*})) = 0$ for any $r \in \mathbb{Z}$, and consequently $\operatorname{Hom}_{A'}(M_{P'}, \tau_{A'} M_{P'}) = 0$. Therefore, we obtain $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$. This finishes the proof of implication (ii) $\Rightarrow$ (i).

Assume now that A is of finite representation type and $\operatorname{mod} A$ admits an indecomposable projective module P with $\alpha(P/\operatorname{soc} P) = 4$ and $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$. Let $I_P = r_A(M_P)$ and $B_P = A/I_P$. Then it follows from Theorems 3.3 and 5.2 that:

(a) M_P is a tilting module in $\operatorname{mod} B_P$;
(b) B_P is a tilted algebra of Dynkin type from the family $\mathcal{B}_{max}$;
(c) Δ_P is a section of Γ_{B_P} without projective and injective modules;
(d) the full translation subquiver of Γ_{B_P} given by all modules from $\tau_{B_P}\Delta_P \cup \Delta_P \cup \tau^{-1}_{B_P}\Delta_P$ is the full translation subquiver of Γ_A given by all modules from $\tau_A \Delta_P \cup \Delta_P \cup \tau^{-1}_A \Delta_P$.

A crucial step for proving the implication (i) $\Rightarrow$ (ii) of Theorem 1 is the following theorem.

Theorem 8.1. *The ideal I_P is a deforming ideal of A with $r_A(I_P) = e_P I_P$ for an idempotent e_P of A.*

We will prove the above theorem in several steps. Let $e_1, \ldots, e_r$ be a set of pairwise orthogonal primitive idempotents of A such that $1_A = e_1 + \cdots + e_r$ and $e_P = e_1 + \cdots + e_n$, for some $n \leq r$, is a residual identity of $B_P = A/I_P$. We abbreviate $\Delta = \Delta_P$, $M = M_P$, $I = I_P$, $B = B_P$, and $e = e_P$. We denote by J the trace ideal of M in A, that is, the ideal of A generated by the images of all homomorphisms from M to A, and by J' the trace ideal of the left A-module $D(M)$ in A. Observe that $I = l_A(D(M))$. Then we have the following lemma.

Lemma 8.2. *We have $J \subseteq I$ and $J' \subseteq I$.*

Proof. First we show that $J \subseteq I$. By definition, there exists an epimorphism $\varphi : M^s \to J$ for some integer $s \geq 1$. Suppose that J is not contained in I. Then there exists a homomorphism $f : A \to M$ in $\operatorname{mod} A$ with $f(J) \neq 0$. Observe that, by (c) and (d), we have $\tau_A M = \tau_B M$ and $\tau_A M$, M do not contain projective direct summands. Then, applying [1, Lemma VIII.5.4], we conclude that there are a positive integer t and homomorphisms $g : A \to (\tau_A M)^t$, $h : (\tau_A M)^t \to M$ such that $f = hg$. But then $hg\varphi = f\varphi \neq 0$ because $J = \operatorname{Im}\varphi$, and hence $g\varphi \neq 0$. This implies $\operatorname{Hom}_A(M, \tau_A M) \neq 0$, contradicting the assumption imposed on M. Therefore $J \subseteq I$.

Suppose now that J' is not contained in I. Then there is a homomorphism $f' : A \to D(M)$ in $\operatorname{mod} A^{\mathrm{op}}$ such that $f'(J') \neq 0$. Moreover, we have in $\operatorname{mod} A^{\mathrm{op}}$ an epimorphism $\varphi' : D(M)^m \to J'$ for some integer $m \geq 1$. Then $f'w'\varphi' \neq 0$ for $w' : J' \to A$ the inclusion homomorphism in $\operatorname{mod} A^{\mathrm{op}}$. Applying the duality functor $D : \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod} A$ we obtain homomorphisms

$$D\bigl(D(M)\bigr) \xrightarrow{D(f')} D(A) \xrightarrow{D(w')} D(J') \xrightarrow{D(\varphi')} D\bigl(D(M)^m\bigr)$$

in $\operatorname{mod} A$, where $D(D(M)) \cong M$, $D(D(M)^m) \cong M^m$, $D(A) \cong A$, and

$$D(\varphi')D(w')D(f') = D(f'w'\varphi') \neq 0.$$

Then, as in the first part of the proof, we conclude that $\operatorname{Hom}_A(M, \tau_A M) \neq 0$, a contradiction. Hence $J' \subseteq I$. □

Lemma 8.3. *We have $l_A(I) = J$, $r_A(I) = J'$ and $I = r_A(J) = l_A(J')$.*

Proof. We prove that $l_A(I) = J$ and $I = r_A(J)$. Since J is a right B-module, we have $JI = 0$, and hence $I \subseteq r_A(J)$. In order to show the converse inclusion, take a monomorphism $u : M \to A_A^t$ for some integer $t \geq 1$, and let $u_i : M \to A$ be the composite of u with the projection of A_A^t on the i-th component. Then there is a monomorphism $v : M \to \bigoplus_{i=1}^t \operatorname{Im} u_i$ induced by u. Further, by definition of J, $\bigoplus_{i=1}^t \operatorname{Im} u_i$ is contained in $\bigoplus_{i=1}^t J$. This leads to the inclusions

$$r_A(J) = r_A\left(\bigoplus_{i=1}^t J\right) \subseteq r_A(M) = I.$$

Therefore, $I = r_A(J)$. Moreover, applying a theorem by T. Nakayama (see [35, Corollary IV.6.11]), we obtain that $l_A(I) = l_A(r_A(J)) = J$.

Applying similar arguments, one shows the equalities $I = l_A(J')$ and $r_A(I) = r_A(l_A(J')) = J'$. □

Lemma 8.4. *We have $eIe = eJe$. In particular, $(eIe)^2 = 0$.*

Proof. Since e is a residual identity of $B = A/I$, we have $B \cong eAe/eIe$. In particular, we conclude that M is a module in $\operatorname{mod} eAe$ with $r_{eAe}(M) = eIe$. Observe also that eJe is the trace ideal of M in eAe, generated by the images of all homomorphisms from M to eAe in $\operatorname{mod} eAe$. It follows from Lemma 8.2 that $eJe = eJ$ is an ideal of eAe with $eJe \subseteq eIe \subseteq \operatorname{rad} eAe$. Let $\Lambda = eAe/eJe$. Then M is a sincere module in $\operatorname{mod} \Lambda$. We will prove that M is a faithful module in $\operatorname{mod} \Lambda$. Observe that then $eIe/eJe = r_\Lambda(M) = 0$, and consequently $eIe = eJe$. Clearly, then $(eIe)^2 = (eJe)(eIe) = 0$, because $JI = 0$.

We shall first show that $\operatorname{id}_\Lambda M \leq 1$. Consider the exact sequence

$$0 \longrightarrow eJe \xrightarrow{u} eAe \xrightarrow{v} \Lambda \longrightarrow 0$$

in $\operatorname{mod} \Lambda$, where u is the inclusion homomorphism and v is the canonical epimorphism. Applying the functor $\operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, -) : \operatorname{mod} eAe \to \operatorname{mod} K$ to this sequence, we get the exact sequence in $\operatorname{mod} K$ of the form

$$\begin{aligned} \operatorname{Hom}_{eAe}\big(\tau_{eAe}^{-1} M, eJe\big) &\xrightarrow{\alpha} \operatorname{Hom}_{eAe}\big(\tau_{eAe}^{-1} M, eAe\big) \\ &\xrightarrow{\beta} \operatorname{Hom}_{eAe}\big(\tau_{eAe}^{-1} M, \Lambda\big) \\ &\xrightarrow{\gamma} \operatorname{Ext}^1_{eAe}\big(\tau_{eAe}^{-1} M, eJe\big) \end{aligned}$$

where $\alpha = \operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, u)$, $\beta = \operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, v)$, and γ is the connecting homomorphism. Observe that there is an epimorphism $M^t \to \tau_{eAe}^{-1} M$ in $\operatorname{mod} eAe$ for some positive integer t. Indeed, it follows from the properties (c) and (d) of M that $\tau_{eAe}^{-1} M = \tau_B^{-1} M$ and the full translation subquiver of Γ_B given by the indecomposable direct summands of M and $\tau_B^{-1} M$ is the full translation subquiver of Γ_{eAe} given by the indecomposable direct summands of M and $\tau_{eAe}^{-1} M$. Moreover, no indecomposable projective module in $\operatorname{mod} eAe$ is a direct summand of $\tau_{eAe}^{-1} M$. Then a projective cover $Q \to \tau_{eAe}^{-1} M$ of $\tau_{eAe}^{-1} M$ in $\operatorname{mod} eAe$ factors through a module of the form M^t, and the claim follows. Observe that then the image of every homomorphism $g : \tau_{eAe}^{-1} M \to eAe$ in $\operatorname{mod} eAe$ is contained in eJe, and hence α is an isomorphism. This implies that γ is a monomorphism. Further, applying [1, Lemma VIII.5.4(b)], we conclude that every homomorphism $f : M \to eAe$ in $\operatorname{mod} eAe$ factors through a module of the form $(\tau_{eAe}^{-1} M)^s$ for some positive integer s. Hence there is an epimorphism $(\tau_{eAe}^{-1} M)^m \to eJe$ in $\operatorname{mod} eAe$. We claim that then $\operatorname{Hom}_{eAe}(eJe, M) = 0$. Assume $\operatorname{Hom}_{eAe}(eJe, M) \neq 0$. Since there is an epimorphism $M^t \to \tau_{eAe}^{-1} M$, we conclude that there exist homomorphisms $\varphi : M \to \tau_{eAe}^{-1} M$ and $\psi : \tau_{eAe}^{-1} M \to M$ such that $\psi\varphi \neq 0$. Applying [1, Lemma VIII.5.4(a)] again, we conclude that φ factors through a module of the form $(\tau_{eAe} M)^p$, for some positive integer p, and hence $\operatorname{Hom}_A(M, \tau_A M) = \operatorname{Hom}_{eAe}(M, \tau_{eAe} M) \neq 0$, a contradiction. Thus $\operatorname{Hom}_{eAe}(eJe, M) = 0$. Then we have $\operatorname{Ext}^1_{eAe}(\tau_{eAe}^{-1} M, eJe) \cong D\overline{\operatorname{Hom}}_{eAe}(eJe, M) = 0$. Summing up, we conclude that $\operatorname{Hom}_\Lambda(\tau_\Lambda^{-1} M, \Lambda) = \operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, \Lambda) = 0$, or equivalently, $\operatorname{id}_\Lambda M \leq 1$. Clearly, we have $\operatorname{Ext}^1_\Lambda(M, M) = D\overline{\operatorname{Hom}}_\Lambda(M, \tau_\Lambda M) = D\overline{\operatorname{Hom}}_{eAe}(M, \tau_{eAe} M) = D\overline{\operatorname{Hom}}_A(M, \tau_A M) = 0$. Since the rank of $K_0(\Lambda)$ is the rank of $K_0(B)$, we conclude that M is a cotilting module in $\operatorname{mod} \Lambda$, and hence $D(M)$ is a tilting module in $\operatorname{mod} \Lambda^{\mathrm{op}}$. In particular, $D(M)$ is a faithful module in $\operatorname{mod} \Lambda^{\mathrm{op}}$. Then we obtain the required fact $r_\Lambda(M) = l_{\Lambda^{\mathrm{op}}}(D(M)) = 0$. □

Lemma 8.5. *Let f be a primitive idempotent in I such that $fJ \neq fAe$. Then $L = fAeAf + fJ + fAeAfAe + eAf + eIe$ is an ideal of $F = (e+f)A(e+f)$, and $N = fAe/fLe$ is a module in* $\operatorname{mod} B$ *such that* $\operatorname{Hom}_B(N, M) = 0$ *and* $\operatorname{Hom}_B(M, N) \neq 0$.

Proof. It follows from Lemma 8.4 that $fAeIe \subseteq fJ$. Then the fact that L is an ideal of F is a direct consequence of $fJ \subseteq fAe$. Observe also that $fLe = fJ + fAeAfAe$, $fLf \subseteq \operatorname{rad}(fAf)$, $eLe = eIe$, and $eLf = eAf$. We have $N \neq 0$. Indeed, if $fAe = fLe$ then, since $eAfAe \subseteq \operatorname{rad}(eAe)$, we obtain $fAe = fJ + fAe(\operatorname{rad}(eAe))$, and so $fAe = fJ$, by the Nakayama lemma [35, Lemma I.3.3], which contradicts our assumption. Further, $B = eAe/eIe$ and $(fAe)(eIe) = fAeJ \subseteq fJ \subseteq fLe$, and hence N is a right B-module. Moreover, N is also a left module over $S = fAf/fLf$ and F/L is isomorphic to the triangular matrix algebra

$$\Lambda = \begin{pmatrix} S & N \\ 0 & B \end{pmatrix}.$$

Involving the properties (c) and (d) of M, we conclude that for every indecomposable direct summand X of M, we have in $\operatorname{mod} B$ an almost split sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

which is also an almost split sequence in $\operatorname{mod} A$, and hence an almost split sequence in $\operatorname{mod} \Lambda$. Applying now [27, Lemma 5.6] (or [24, Theorem XV.1.6]), we conclude that $\operatorname{Hom}_B(N, M) = 0$. Then we obtain $\operatorname{Hom}_B(M, N) \neq 0$, because every indecomposable module in $\operatorname{mod} B$ is either generated or cogenerated by M. □

Proposition 8.6. *We have $Ie = J$ and $eI = J'$.*

Proof. This follows exactly as [27, Proposition 5.9] by applying Lemmas 8.2, 8.3, 8.4, 8.5. □

Proof of Theorem 8.1. It follows from Lemma 8.3 and Proposition 8.6 that $r_A(I) = J' = eI$ and $l_A(I) = J = Ie$. In particular, we have $IeI = 0$, because $JI = 0$. Then, applying Proposition 2.2, we conclude that $\operatorname{soc} A \subseteq I$ and $l_{eAe}(I) = eIe = r_{eAe}(I)$. Moreover, the valued quiver $Q_{A/I}$ of $A/I = B$ is acyclic, because B is a tilted algebra. Therefore, I is a deforming ideal of A with $r_A(I) = eI$.

We complete now the proof of implication (i) ⇒ (ii) of Theorem 1. It follows from Theorem 2.5 that the algebra $A[I]$ associated to I is isomorphic to an orbit algebra $\widehat{B}/(\psi\nu_{\widehat{B}})$ for some positive automorphism ψ of $\widehat{B}$. Moreover, since B is a tilted algebra from the family $\mathcal{B}_{max}$, applying Corollary 4.4, we conclude that $(\psi\nu_{\widehat{B}}) = (\varphi\nu_{\widehat{B}}^m)$ for a positive integer m and a rigid automorphism φ of $\widehat{B}$. Finally, by Theorem 2.4, we obtain that A is socle equivalent to $A[I]$, and consequently A is socle equivalent to $\widehat{B}/(\varphi\nu_{\widehat{B}}^m)$. Moreover, if K is an algebraically closed field, then A and $A[I]$ are isomorphic, by Theorem 2.7, and hence A is isomorphic to $\widehat{B}/(\varphi\nu_{\widehat{B}}^m)$. $\square$

9. Examples

In this section we present examples illustrating the main results of the paper.

Example 9.1. Let Q be the quiver

$$\begin{tikzcd}
 & 0 \arrow[dl, "\alpha_1"'] \arrow[d, "\alpha_2"] \arrow[dr, "\alpha_3"] & \\
1 & 2 & 3
\end{tikzcd}$$

and $B = KQ$ the path algebra of Q over a field K. We note that B belongs to the family $\mathcal{B}_{max}$. The repetitive category $\widehat{B}$ of B is the bound quiver category $K\widehat{Q}/\widehat{I}$, where $\widehat{Q}$ is the quiver

$$\begin{tikzcd}
\vdots \arrow[dr, "\beta_{m+2,1}"'] & \vdots \arrow[d, "\beta_{m+2,2}"] & \vdots \arrow[dl, "\beta_{m+2,3}"] \\
 & (m+1,0) \arrow[dl, "\alpha_{m+1,1}"'] \arrow[d, "\alpha_{m+1,2}"] \arrow[dr, "\alpha_{m+1,3}"] & \\
(m+1,1) \arrow[dr, "\beta_{m+1,1}"'] & (m+1,2) \arrow[d, "\beta_{m+1,2}"] & (m+1,3) \arrow[dl, "\beta_{m+1,3}"] \\
 & (m,0) \arrow[dl, "\alpha_{m,1}"'] \arrow[d, "\alpha_{m,2}"] \arrow[dr, "\alpha_{m,3}"] & \\
(m,1) \arrow[dr, "\beta_{m,1}"'] & (m,2) \arrow[d, "\beta_{m,2}"] & (m,3) \arrow[dl, "\beta_{m,3}"] \\
 & (m-1,0) \arrow[dl, "\alpha_{m-1,1}"'] \arrow[d, "\alpha_{m-1,2}"] \arrow[dr, "\alpha_{m-1,3}"] & \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

and $\widehat{I}$ is the ideal in the path category $K\widehat{Q}$ generated by the elements $\alpha_{m,i}\beta_{m,i}-\alpha_{m,j}\beta_{m,j}$, $\beta_{m,i}\alpha_{m-1,j}$, for $m \in \mathbb{Z}$ and $i \neq j$ in $\{1,2,3\}$. For each $(m,i) \in \mathbb{Z} \times \{1,2,3\}$, denote by $e_{m,i}$ the object of $\widehat{B}$ corresponding to the vertex (m,i) of $\widehat{Q}$. Then the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ is given by $\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$ for any $(m,i) \in \mathbb{Z} \times \{1,2,3\}$. Consider the automorphisms φ and ψ of $\widehat{B}$ given by the cyclic permutations of objects of $\widehat{B}$

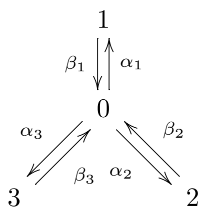
$$\varphi = (e_{m,1}, e_{m,3}), \quad \text{for all } m \in \mathbb{Z},$$

$$\psi = (e_{m,1}, e_{m,2}, e_{m,3}), \quad \text{for all } m \in \mathbb{Z}.$$

Then

$$T(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^r), \quad \widehat{B}/(\varphi\nu_{\widehat{B}}^r), \quad \widehat{B}/(\psi\nu_{\widehat{B}}^r),$$

for $r \in \mathbb{N}^+ = \{1,2,\dots\}$, form a complete family of pairwise nonisomorphic selfinjective orbit algebras of finite representation type, created by the algebra B, whose module categories admit maximal almost split sequences. In particular, $A = \widehat{B}/(\nu_{\widehat{B}}) = T(B)$, $A' = \widehat{B}/(\varphi\nu_{\widehat{B}})$, $A'' = \widehat{B}/(\psi\nu_{\widehat{B}})$ are isomorphic to the bound quiver algebras $K\Omega/J$, $K\Omega/J'$, $K\Omega/J''$, respectively, where Ω is the quiver



and J, J', J'' are the ideals

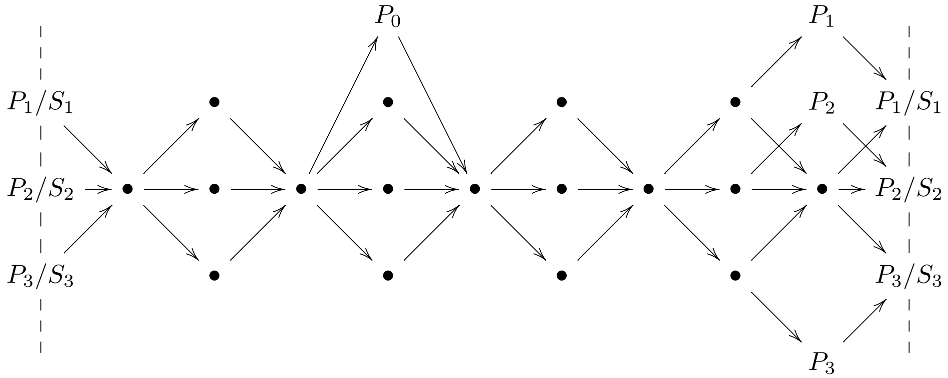
$$J = \langle \alpha_1\beta_1 - \alpha_2\beta_2, \alpha_2\beta_2 - \alpha_3\beta_3, \beta_1\alpha_2, \beta_1\alpha_3, \beta_2\alpha_3, \beta_2\alpha_1, \beta_3\alpha_1, \beta_3\alpha_2 \rangle,$$

$$J' = \langle \alpha_1\beta_1 - \alpha_2\beta_2, \alpha_2\beta_2 - \alpha_3\beta_3, \beta_1\alpha_1, \beta_1\alpha_2, \beta_2\alpha_3, \beta_2\alpha_1, \beta_3\alpha_2, \beta_3\alpha_3 \rangle,$$

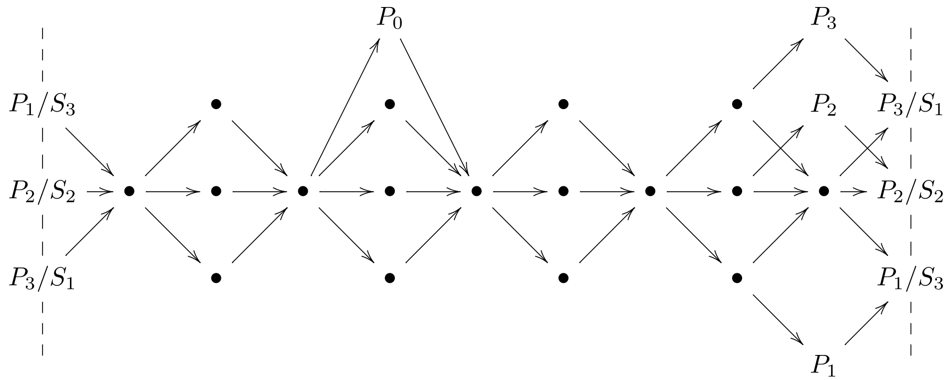
$$J'' = \langle \alpha_1\beta_1 - \alpha_2\beta_2, \alpha_2\beta_2 - \alpha_3\beta_3, \beta_1\alpha_1, \beta_1\alpha_3, \beta_2\alpha_1, \beta_2\alpha_2, \beta_3\alpha_2, \beta_3\alpha_3 \rangle.$$

Moreover, the Auslander–Reiten quivers Γ_A, $\Gamma_{A'}$, $\Gamma_{A''}$ of A, A', A'', respectively, are of the forms

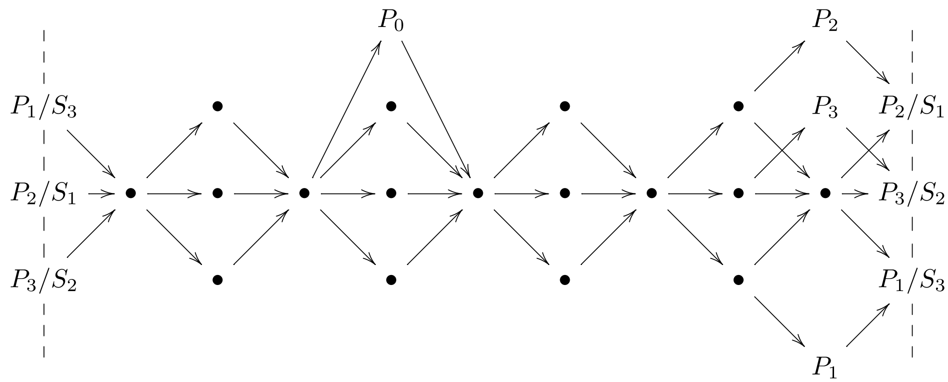
Γ_A



$\Gamma_{A'}$



$\Gamma_{A''}$



(see [26, 3.3]).

We exhibit now the classes of selfinjective algebras of finite representation type with maximal almost split sequences which are not orbit algebras of repetitive categories of algebras, showing that we cannot replace in Theorem 1 "socle equivalent" by "isomorphic".

Example 9.2. Let L be a finite field extension of K, Q a finite acyclic quiver, and Q_0 the set of vertices of Q. Then the path algebra $H = LQ$ of Q over L is a finite dimensional hereditary K-algebra whose quiver Q_H coincides with Q. Moreover, the H–H-bimodules $D(H) = \mathrm{Hom}_K(H, K)$ and $\mathrm{Hom}_L(H, L)$ are canonically isomorphic, so we may identify them. We denote by e_i, $i \in Q_0$, the canonical set of pairwise orthogonal primitive idempotents in H, and by e_i^*, $i \in Q_0$, the associated dual elements in $D(H)$.

Let $\alpha : L \times L \to L$ be a 2-cocycle of the K-algebra L, that is, a K-bilinear map satisfying the condition

$$\lambda_1\alpha(\lambda_2, \lambda_3) - \alpha(\lambda_1\lambda_2, \lambda_3) + \alpha(\lambda_1, \lambda_2\lambda_3) - \alpha(\lambda_1, \lambda_2)\lambda_3 = 0$$

for all $\lambda_1, \lambda_2, \lambda_3 \in L$. Then we have the associated Hochschild extension

$$0 \longrightarrow L \longrightarrow T(L, \alpha) \longrightarrow L \longrightarrow 0$$

such that $T(L, \alpha) = L \oplus L$ as K vector space and the multiplication is defined by

$$(\lambda, \xi)(\mu, \eta) = \big(\lambda\mu, \lambda\eta + \xi\mu + \alpha(\lambda, \mu)\big)$$

for all $\lambda, \mu, \xi, \eta \in L$. We may associate to α the 2-cocycle $\hat{\alpha} : H \times H \to D(H)$ given by

$$\hat{\alpha}(a, b) = \sum_{i \in Q_0} \alpha(e_i a e_i, e_i b e_i) e_i^*$$

for $a, b \in H$. Consider the associated Hochschild extension

$$0 \longrightarrow D(H) \longrightarrow T(H, \hat{\alpha}) \longrightarrow H \longrightarrow 0.$$

Recall that $T(H, \hat{\alpha}) = H \oplus D(H)$ as K-vector space and the multiplication is defined by

$$(a, f)(b, g) = \big(ab, ag + fb + \hat{\alpha}(a, b)\big)$$

for $a, b \in H$ and $f, g \in D(H)$. It follows from [38] that $T(H, \hat{\alpha})$ is a finite dimensional selfinjective K-algebra, the elements $(e_i, -\alpha(1, 1)e_i^*)$, $i \in Q_0$, form a set of pairwise orthogonal primitive idempotents of $T(H, \hat{\alpha})$, whose sum $(1, -\alpha(1, 1)\sum_{i \in Q_0} e_i^*)$ is the identity of $T(H, \hat{\alpha})$. Then we have the following facts:

(1) The extension

$$0 \longrightarrow D(H) \longrightarrow T(H, \hat{\alpha}) \longrightarrow H \longrightarrow 0$$

is splittable if and only if the extension

$$0 \longrightarrow L \longrightarrow T(L, \alpha) \longrightarrow L \longrightarrow 0$$

is splittable (see [18, Proposition 3.8], [27, Proposition 6.1], [34, Theorem 5.8]).

(2) $T(H, \hat{\alpha})$ is a symmetric algebra if and only if $T(L, \alpha)$ is a symmetric algebra (see [18, Theorem 3.9]).
(3) $T(H, \hat{\alpha})$ is socle equivalent to the trivial extension algebra $T(H) = H \ltimes D(H)$, because the elements e_i^*, $i \in Q_0$, belong to the socle of $T(H, \hat{\alpha})$.

Assume now that $H = LQ$ is a hereditary algebra (over K) from the family $\mathcal{B}_{max}$. Then $T(H, \hat{\alpha})$ is a selfinjective algebra of finite representation type and $\operatorname{mod} T(H, \hat{\alpha})$ admits exactly one maximal almost split sequence, because $T(H, \hat{\alpha})$ is socle equivalent to $T(H) = \widehat{H}/(\nu_{\widehat{H}})$ (see Theorem 2 and Corollary 6). We consider now two choices of the finite extension fields.

(i) Let $K = \mathbb{Z}_p(u, v)$ be the rational function field in two indeterminates u, v over the prime field $\mathbb{Z}_p$ of characteristic $p > 0$, and L be the finite extension field $K[X, Y]/(X^p - u, Y^p - v)$ of K. Moreover, let x and y be the residue classes of X and Y in L, respectively. Then L has a K-basis given by the elements of the form $x^l y^m$, $l, m \in \{0, 1, \ldots, p-1\}$. Consider the 2-cocycle (see [18, Example 3.10]) $\alpha : L \times L \to L$ given by

$$\alpha\left(x^l y^m, x^r y^s\right) = ls x^{l+r} y^{m+s}$$

for $l, m, r, s \in \{0, 1, \ldots, p-1\}$. Observe that $\alpha(x, y) = xy \neq 0 = \alpha(y, x)$. Then it follows from [18, Lemma 1.3] that the extension

$$0 \longrightarrow L \longrightarrow T(L, \alpha) \longrightarrow L \longrightarrow 0$$

is not splittable. Moreover, $T(L, \alpha)$ is a symmetric algebra, because L is an extension of K by two elements (see [18, Theorem 4.2]). Therefore, by (1)–(3), $T(H, \hat{\alpha})$ is a symmetric algebra which is socle equivalent but not isomorphic to the trivial extension algebra $T(H)$.

(ii) Let $K = \mathbb{Z}_p(u, v, w)$ be the rational function field in three indeterminates u, v, w over the prime field $\mathbb{Z}_p$ of characteristic $p > 0$, and L be the finite extension field $K[X, Y, Z]/(X^p - u, Y^p - v, Z^p - w)$ of K. Moreover, let x, y, z be the residue classes of X, Y, Z in L, respectively. Then L has a K-basis given by the elements of the form $x^l y^m z^n$, $l, m, n \in \{0, 1, \ldots, p-1\}$. Consider the 2-cocycle (see [18, Lemma 4.5 and comments below it]) $\alpha : L \times L \to L$ given by

$$\alpha\left(x^l y^m z^n, x^r y^s z^t\right) = x^{l+r-1} y^{m+s-1} z^{n+t-1}(lsz + mtxy)$$

for $l, m, n, r, s, t \in \{0, 1, \ldots, p-1\}$. Observe that $\alpha(xy, yz) = y(z + xy) \neq 0 = \alpha(yz, xy)$. Hence, applying [18, Lemma 1.3] again, we conclude that the extension

$$0 \longrightarrow L \longrightarrow T(L, \alpha) \longrightarrow L \longrightarrow 0$$

is not splittable. Moreover, $T(L, \alpha)$ is not a symmetric algebra, by [18, Proposition 4.6 and comments below it]. Therefore, by (1)–(3), $T(H, \hat{\alpha})$ is not a symmetric algebra and is socle equivalent but not isomorphic to the trivial extension algebra $T(H)$.

We also note that, in the both cases (i) and (ii), $T(H,\hat{\alpha})$ is a selfinjective algebra of finite representation type, $\operatorname{mod} T(H,\hat{\alpha})$ admits a maximal almost split sequence, and $T(H,\hat{\alpha})$ is not isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a K-algebra and φ is a positive automorphism of $\widehat{B}$ (see [30, Proposition 4]).

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