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Growth of subalgebras and subideals in free Lie algebras

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ABSTRACT

We investigate subalgebras in free Lie algebras, the main tool being relative growth and cogrowth functions. Our study reveals drastic differences in the behavior of proper finitely generated subalgebras and nonzero subideals. For instance, the *growth* of a proper finitely generated subalgebra H of a free Lie algebra L , with respect to any fixed free basis X , is exponentially small compared to the growth of the whole of L . Quite opposite, the *cogrowth* of any nonzero subideal S is exponentially small compared to the growth of L .

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1. Introduction

According to the classical Shirshov–Witt theorem, every subalgebra of a free Lie algebra is free. There are a number of ways to draw distinction between different subalgebras. In this paper we are interested in the (relative) growth and cogrowth of subalgebras.

Any free basis X of a free Lie algebra $L = L(X)$ defines an increasing filtration

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$$L^{(1)} \subset L^{(2)} \subset \dots \subset L^{(n)} \subset \dots,$$

where $\dim L^{(i)} < \infty$, if X is finite. The (relative) growth function of a subspace $H \subset L$ is an integer valued function of an integral argument whose n th value equals $\dim(H \cap L^{(n)})$. The n th value of the cogrowth function equals $\dim L^{(n)} / (H \cap L^{(n)})$.

Let us recall that if $\#X = m \geq 2$ then the growth of the whole of $L = L(X)$ is exponential. This easily follows from the classical Witt's formula [2, Theorem 3.1.3] for the dimension of the n th homogeneous component L_n of L

$$\dim L_n = \frac{1}{n} \sum_{d|n} \mu(d) m^{\frac{n}{d}} \sim \frac{1}{n} m^n. \quad (1)$$

Here μ is the Möbius function and the equivalence means that the ratio of the left hand side to the right hand side tends to 1 as $n \rightarrow \infty$. The exponentiality holds for the growths and the cogrowth of the most of subalgebras studied in this paper but a more precise comparison shows that the degree of exponentiality can be rather different.

To be precise, we call a function $f(n)$ of natural argument with nonnegative values *exponential*, if the *lower* limit of the function $\sqrt[n]{f(n)}$ is greater than 1. If $\lim_{n \rightarrow \infty} \sqrt[n]{f(n)}$ exists, we call it the (exponential) *base* of $f(n)$.

Let us say that a (growth) function $f_1(n)$ is *exponentially negligible* when compared to another (growth) function $f_2(n)$ if the ratio $\frac{f_2(n)}{f_1(n)}$ grows exponentially. We say that the growth or the cogrowth of a subalgebra H of a free Lie algebra $L = L(X)$, with a fixed free basis X , is exponentially negligible if the respective relative growth (cogrowth) function for H is exponentially negligible when compared with the growth function for L . The exponential base of the relative growth function for H in L with respect to a fixed free basis X is denoted by β_H .

In the study of the growth and cogrowth of subalgebras of free Lie algebras, the best explored is the case of ideals. The cogrowth of an ideal I of a free Lie algebra L is actually the growth of the factor-algebra L/I . This notion is one of the main in the theory of infinite-dimensional Lie algebras. Many classes of algebras are defined in terms of their growth. For a general monograph on the topic see [9]. In the theory of simple Lie algebras, confirming a conjecture by V. Kac [7] (see also his monograph [8]), O. Mathieu classified simple Lie algebras of polynomial growth [10]. The growth of solvable Lie algebras and algebras satisfying polynomial identities also was studied by a number of authors. For a survey of this area see [13].

Note that in the case of ideals, the cogrowth of a nonzero ideal I of a free Lie algebra $L = L(X)$ of rank ≥ 2 is exponentially negligible when compared with the growth of L . This easily follows from the results of [3, Subsection 2.2] (see also [11] for the case of graded ideals), where the respective claim has been established for free associative algebras. Indeed, consider the free associative algebra $A = A(X)$ as the universal enveloping algebra of L and the two-sided ideal J of A generated by I . Then $L/I \subset A/J$ by Poincaré–Birkhoff–Witt Theorem and the growth of A/J by [3, Proposition 8] is neg-

ligible. Since by Witt's formula (1) the exponent of the growth of L is the same as that of A , it follows that the growth of L/I is negligible, as well.

Much less explored is the case of subalgebras which are not ideals. In this paper we start with general finite-dimensional subalgebras. Note that in the case of graded subalgebras the proof of the result that follows can be found in [11].

Theorem 1. *The (relative) growth function $g_H(n)$ of any proper finitely generated subalgebra H of a free Lie algebra L of finite rank m is exponentially negligible.*

In the case of subalgebras which are *subideals* the situation is entirely opposite.

Let us recall that a subalgebra M of a Lie algebra L is named an ℓ -subideal, $\ell = 1, 2, \dots$, if there is a decreasing sequence

$$L \triangleright L_1 \triangleright \dots \triangleright L_{\ell-1} \triangleright M,$$

where each term is an ideal in the preceding term.

The general theory of infinite-dimensional Lie algebras with a sizeable portion of material devoted to subideals is exposed in a book [1]; however, our present topic, the *growth and cogrowth of subideals*, is not covered there.

Theorem 2. *The cogrowth of any nonzero subideal of a free Lie algebra of rank $m \geq 2$ is exponentially negligible.*

The result of Theorem 2 contrasts dramatically with the situation in the Group Theory where the cogrowth of a nontrivial subnormal subgroup of the free group F_r , $r \geq 2$, can be equivalent to the growth of F_r . For full details see [12].

When compared to the proof of the same result for the ideals, as given earlier, the proof of this result required much more sophistication and even the development of a new piece of technique. This technique is presented in Section 4. One of the immediate applications of the technique is the following result, which provides essential details to our claim of exponential negligibility of finitely generated subalgebras in Theorem 1.

Theorem 3. *Let L be a free Lie algebra of rank m , with a fixed basis X . Then the following are true.*

- (i) *The relative growth function $g_H(n)$ of a proper nonabelian subalgebra H of L , with respect to X , is exponential.*
- (ii) *If H is finitely generated nonabelian subalgebra of L then the exponential base β_H of g_H is an algebraic integer. If H is proper then $\beta_H < m$.*
- (iii) *For any real $m_0 \in [1, m]$ there is a subalgebra H of L such that $\beta_H = m_0$. One can choose H as a retract of an ideal of codimension 1 in L .*
- (iv) *For any real $m_0 \in [1, m]$ there is a 2-subideal S of L such that the exponential base $\beta_{L/S}$ of the relative cogrowth function $g_{L/S}$ equals m_0 . One can choose S as an ideal in an ideal M of codimension 1 in L , with free factor algebra M/S .*

We do not know if for any $m_0 \in [1, m]$ there is an ideal S with $\beta_{L/S} = m_0$.

To further emphasize the potential of the combinatorial approach to the questions of the growth/cogrowth, in Section 6, as an example, we calculate the cogrowth function for the minimal 2-subideal containing one of the free generators of a free Lie algebra of rank 2.

Theorem 1 gives a quantitative evaluation of the “size” of a finitely generated subalgebra H when compared to the “size” of the whole of the free Lie algebra L . An example showing how the “quantitative” **Theorems 1 and 2**, can be applied to obtain purely “qualitative” consequences, is as follows.

Corollary 1. *No proper finitely generated subalgebra K of a free Lie algebra L can contain a nonzero subideal H of L .*

Proof. We may assume that L has a finite rank $m \geq 2$. By **Theorem 1**, K is negligible. If $H \subset K$, then also H is negligible. However, this contradicts **Theorem 2**. $\square$

Apart from the sharp difference in the character of growth/cogrowth, there are other properties where finitely generated subalgebras and subideals differ drastically. Let us recall that given a subalgebra H of a Lie algebra G , the *idealizer* of H in G is the unique maximal subalgebra K containing H as an ideal. Clearly, any proper subideal is different from its idealizer in any Lie algebra. At the same time, an easy consequence of our **Theorem 1** is the following, probably known fact: *any nonzero finitely generated subalgebra of a nonabelian free Lie algebra is self-idealizing, that is, equal to its idealizer*. Using this property one could derive **Corollary 1** using a more traditional argument, like **Theorem 4** below. This result, for which we were unable to find a source in the literature, constitutes a natural counterpart to M. Hall’s theorem about finitely generated subgroups in the free groups [6].

Theorem 4. *Let H be a finitely generated subalgebra of a free Lie algebra L of finite rank. Then a free basis B_0 of H can be complemented to produce a free basis B of a subalgebra C of finite codimension in L . In other words, if K is the subalgebra generated by $B \setminus B_0$ then $C \cong H * K$, the free product of H and K .*

One can always choose $B \setminus B_0$, hence K , homogeneous. If H is itself homogeneous, one can choose B so that C contains some term of the lower central series of L .

From this theorem we easily obtain the following consequence.

Corollary 2. *Every proper finitely generated subalgebra H in a free Lie algebra L of finite rank is a free factor in strictly greater subalgebra of L ; in addition, if homogeneous, H is a free factor of a subideal of L .*

A comparison with Group Theory reveals the following differences. First, in Hall’s Theorem [6], saying that a finitely generated subgroup H of a free group F_r of rank

r is a free factor in a greater subgroup of F_r , one must stipulate that H is of *infinite index* in F_r . Second, in the case of Lie algebras, if H is proper nonzero, the free factor K appearing in Theorem 4 can never be chosen finitely generated; whereas in the case of Groups, Hall's Theorem asserts that the complementary free factor K is *always finitely generated*.

When dealing with the cogrowth of subideals, the following situation comes to mind quite naturally. If a subideal S contained a nonzero ideal I then the cogrowth of S would be exponentially negligible, without any further argument. The result that follows shows that this such situation is highly unlikely. In the statement of the result, we will use the following notation.

We denote the ideal of an algebra L generated by a set S , $\text{id}_L S$, is the minimal ideal of L that contains S . We also say that $\text{id}_L S$ is an *ideal closure* of S in L . If ℓ is a natural number, then an ℓ -subideal of L generated by S is the subalgebra $\text{id}_L^\ell S$ of L which is defined by induction as follows. We set $\text{id}_L^1 S = \text{id}_L S$ and if $\ell > 1$ then define $\text{id}_L^\ell S$ as the ideal of $\text{id}_L^{\ell-1} S$ generated by S . An easy inductive argument shows that $\text{id}_L^\ell S$ is contained in every ℓ -subideal of L containing S . We also say that $\text{id}_L^\ell S$ is an ℓ -subideal closure of S in L .

Theorem 5. *Let L be a free Lie algebra, J a proper ideal of L , S a finite subset of J . Then for no $\ell \geq 1$ the ℓ -subideal closure $\text{id}_J^\ell S$ contains a nonzero ℓ -subideal of L .*

Corollary 3. *Let L be a free Lie algebra and S a finite subset of L whose ideal closure in L is different from L . Then for any $\ell \geq 1$, the $(\ell + 1)$ st subideal closure of S in L does not contain any nonzero ℓ -subideal of L .*

The restriction $\text{id}_L(S) \neq L$ in the statement of the above theorem is necessary because, otherwise, for every $\ell = 1, 2, \dots$, the ℓ -subideal closure of S equals L and the claim is not valid.

Corollary 4. *Let J be a proper ideal of a free Lie algebra L and S a finite subset of J . Then the ideal closure I of S in J does not contain nonzero ideals of L .*

We conclude this section with few remarks about the differences of the situation with the growth and cogrowth in the case of algebras as opposed to the case of groups. We will also give some hints as for the methods we have used or developed.

Two observations make the situation in the case of algebras so different from the case of groups (and semigroups!). In the case of algebras, we measure the growth using the *degree* of the elements while in the case of groups we deal with the *length*. So our first observation is that the degree of the sum of two elements never exceeds the degree of the summands, while the length of the product does not need to obey this rule. As for the second, the reader probably already noticed that in the proof of Corollary 1 we have used a simple general purely “algebraic” property: *The sum of the (relative) growth*

and cogrowth functions of a subalgebra always equals to the growth function of the whole algebra. Again, no analogue of this claim holds valid in the case of groups. As a result, dealing with the growth and cogrowth functions is by necessity more delicate in the case of groups.

Still, to prove our results about the growth of subalgebras in the free Lie algebras here, it was necessary to suggest at least two pieces of technique we think new.

One of them, suggested in Section 5, deals with the following issue. Although we define the growth functions for subalgebras in the free Lie algebra $L = L(X)$ using a finite fixed free basis X , in our proofs we need to switch from this fixed basis to other bases, which could be by necessity infinite. To preserve growth functions, we need to develop an approach to the growth with respect to infinite graded bases of free Lie algebras. This is done in Section 5 where certain analytic finiteness conditions are suggested that allow us to estimate the cogrowth when we use induction to pass to infinitely generated ideals and subideals and at the same time have the degrees of elements unchanged. The second piece of technique is produced in Lemma 16 where we examine what can be naturally called *shifting derivations* in associative and Lie algebras.

2. Preliminaries

In this section we give necessary definitions concerning the relative growth of subspaces in algebras. We also recall some results about the connection between associative and Lie algebras, bases of free Lie algebras and their subalgebras.

2.1. Relative growth of subspaces in finitely filtered spaces

Let V be a vector space over a field F with the filtration

$$\alpha : \quad \{0\} \subset V^{(0)} \subset V^{(1)} \subset \dots \subset V^{(n)} \subset \dots, \quad V = \bigcup_{n=0}^{\infty} V^{(n)}. \quad (2)$$

We say that α is a *finite ascending filtration* if all subspaces $V^{(n)}$ are finite-dimensional. The *growth function* of α , g_α , is then given by $g_\alpha(n) = \dim V^{(n)}$. The *graded growth function* of α , d_α , is given by $d_\alpha(n) = \dim V^{(n)}/V^{(n-1)}$ (assuming $V^{(-1)} = \{0\}$). The filtration α also defines a degree function $\deg_\alpha$ on the set of nonzero elements of L given by $\deg_\alpha(a) = n$ provided that $a \in V^{(n)} \setminus V^{(n-1)}$.

If W is subspace then the *subfiltration* $\alpha \cap W$ is a filtration formed by the subspaces $W^{(n)} = W \cap V^{(n)}$. The factor-space V/W acquires a *factor-filtration* α/W given by the subspaces $(V/W)^{(n)} = (V^{(n)} + W)/W \cong V^{(n)}/(W \cap V^{(n)})$.

In each of these cases the filtrations of the respective spaces are finite. Once α and g_α are fixed, we can define the *growth function relative to α* : $g_{\alpha \cap W}(n) = \dim W^{(n)}$, and *cogrowth function relative to α* : $g_{\alpha/W}(n) = \dim(V/W)^{(n)}$. If the filtrations are fixed than we denote the respective growth functions as g_V , g_W and $g_{V/W}$.

In each of these cases we have also *relative graded functions* d_V , d_W and $d_{V/W}$. Say, $d_{V/W}(n) = g_{V/W}(n) - g_{V/W}(n-1)$.

It often so happens that the space V is $\mathbb{Z}$ -graded: $V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_n \oplus \cdots$, and for each term of the filtration α one has $V^{(n)} = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_n$, for all $n = 1, 2, \dots$. The nonzero elements of V_n are called *homogeneous of degree n* and V_n the *homogeneous subspace of the grading of degree n* . If $v \in V$ then v can be uniquely written as the sum of some elements $v_n \in V_n$. We call v_n a homogeneous components of v of degree n . If n is the greatest such that $v_n \neq 0$ then we call v_n the *leading part* of v and write $v_n = \text{Lp}(v)$.

A subspace $U \subset V$ is called *graded* if U contains all homogeneous components of its elements. Equivalently, $U = (U \cap V_0) \oplus (U \cap V_1) \oplus (U \cap V_2) \oplus \cdots \oplus (U \cap V_n) \oplus \cdots$. If W is not graded then the linear span of the leading parts of nonzero elements of W is a graded subspace of V , which is denoted by $\text{gr } W$. For each natural n , the homogeneous component $(\text{gr } W)_n$ of this space is the linear span of the leading parts of all elements of degree n in W .

An easy but useful technical result is the following.

Lemma 1. *For the relative growth functions, one has*

- (a) $g_W = g_{\text{gr } W}$;
- (b) $g_{V/W} = g_{V/\text{gr } W}$.

Proof. (a) It follows from the definition of $(\text{gr } W)_n$ that the mapping $w + W^{(n-1)} \mapsto \text{Lp}(w)$ for $w \in W^{(n)} \setminus W^{(n-1)}$ and $w \mapsto 0$ for $w \in W^{(n-1)}$ is a well-defined linear isomorphism between $W^{(n)}/W^{(n-1)}$ and $(\text{gr } W)_n$. Hence

$$g_W(n) = \dim W^{(n)} = \sum_{i=0}^n \dim W^{(i)}/W^{(i-1)} = \sum_{i=0}^n \dim (\text{gr } W)_i = g_{\text{gr } W}(n).$$

- (b) Follows from (a) and the equality $\dim V^{(n)}/W^{(n)} = \dim V^{(n)} - g_W(n)$. $\square$

2.2. Linear and free bases of subalgebras in free Lie algebras

As usual, given an alphabet X , we denote by $W(X)$ the monoid of all words in X , including the empty word 1. In this section we will need to count the words in the subsets of $W(X)$, where X is not necessarily finite. If we do not impose any conditions, the number of words of the same length can be infinite. To cope with this, we consider the alphabets which are *finitely graded*, that is, $X = \bigsqcup_{i=1}^{\infty} X_n$ with each set X_n finite. Now in addition to the *lengths* of words, we can speak about the *degrees* of words. We define the degree of a word w in $W(X)$ by induction on its length if we set $\deg(1) = 0$, $\deg(x) = n$ for $x \in X_n$ and if $w = x_1 \cdots x_k$ then $\deg(w) = \deg(x_1) + \cdots + \deg(x_k)$. Let $W(X)_n$ be the set of all words of degree n . Now each number $d_X(n) = \#W(X)_n$ is

finite. We have $W(X) = \bigsqcup_{n=0}^{\infty} W(X)_n$ where $W(X)_n$ is the set of all words of degree n in X .

In its turn, the free associative algebra $A(X)$, which is a vector space with basis $W(X)$, acquires a $\mathbb{Z}$ -grading $A(X) = \bigoplus_{n=0}^{\infty} A(X)_n$, where $A(X)_n$ is the linear span of all words in $W(X)_n$. Now in addition to the grading, $A(X)$ have a natural *degree* filtration

$$A(X)^{(0)} \subset A(X)^{(1)} \subset \dots \subset A(X)^{(n)} \subset \dots$$

where $A(X)^{(n)} = A(X)_0 \oplus A(X)_1 \oplus \dots \oplus A(X)_n$, for each $n = 0, 1, 2, \dots$.

As mentioned earlier, every element $f \in A(X)$ acquires a degree if we set $\deg(f) = n$ in the case where $f \in A(X)^{(n)}$ but $f \notin A(X)^{(n-1)}$. In this case $f = \sum_{i=0}^n f_i$, where $f_i \in A(X)_i$, $f_n \neq 0$; $f_n = \text{Lp}(f)$ is the *leading part* of f .

In what follows we are going to study the growth and cogrowth of certain subalgebras H of a free Lie algebra $L = L(X)$. We view L as a graded subspace of $A = A(X)$ which is generated by X with respect to the bracket operation $[a, b] = ab - ba$. The cardinality $\#X$ of X is often called the *rank* of any of the system appearing in this text: $W(X)$, $A(X)$ or $L(X)$. Being a graded subspace of A , L becomes $\mathbb{Z}$ -graded: $L = \bigoplus_{n=1}^{\infty} L_n$ where $L_n = L \cap A_n$. Similarly, we have a degree filtration of L as follows:

$$L^{(1)} \subset L^{(2)} \subset \dots \subset L^{(n)} \subset \dots \quad (3)$$

where $L^{(n)} = L \cap A^{(n)}$ or $L^{(n)} = L_1 \oplus \dots \oplus L_n$, for each $n = 1, 2, \dots$. Every element $u \in L$ has degree which is the same whether we use the degree filtration of A or L . When we apply the notions of Subsection 2.1 to the free Lie algebra, we view (3) as the “base” filtration α of the formula (2) and then we know what is the degree, the leading part of an element of L and, given a subalgebra $H \subset L$, what is the associated graded subalgebra $\text{gr } H$. This is indeed a subalgebra because the commutator of two leading parts is either zero or the leading part of the commutator of two elements of H .

Using Lemma 1, one can see that the growth and the cogrowth of H and $\text{gr } H$ are the same.

In the next lemmas we consider the elements of the free Lie algebra $L(X)$ called the *commutators in X* of certain length. The commutators of length 1 are just the elements of X . If the commutators of all lengths less than $n > 1$ have been defined, then the commutators of length n are all the elements $[c, d]$ where c is a commutator of length k and d a commutator of length $n - k$.

Since we always view $L(X)$ as a subalgebra in the free associative algebra $A(X)$, recalling $[c, d] = cd - dc$, one can write each commutator as a linear combination of associative words, each of which has the same degree. Thus each commutator acquires a uniquely defined degree and $\deg([c, d]) = \deg(c) + \deg(d)$.

If we forget all brackets and commas on a commutator c , then resulting is an associative word w , we write $c = [w]$, called the *associative support* of c . We can also say that any commutator can be obtained by replacing brackets (and there inherent commas) on

an associative word. Replacing brackets in a certain way on certain associative words may produce a linear basis of the free Lie algebra $L(X)$. Let us describe a process due to A.I. Shirshov [15].

Suppose we have a linear ordering $\leq$ on the set X of free generators of the free Lie algebra $L(X)$. We extend this ordering to the lexicographical ordering of the free monoid $W(X)$. Under this ordering, given two words u and v , then we say that $u \geq v$ if u is a prefix of v . If not, we say that $u < v$ if $u = wxu'$, $v = wyv'$ and $x < y$ where $x, y \in X$ and $w, u, v, u', v' \in W(X)$.

If c is a commutator then the *leading word* $\bar{c}$ of c is defined as the greatest word nontrivially entering the expansion of c as a linear combination of the elements of $W(X)$. Let us say that w is a *Lyndon–Shirshov* or simply *LS-word* if whenever $w = uv$, with u and v nonempty, then $w > vu$. (Notice that in this case we must also have $w > v$, that is, every LS-word is greater than its proper suffix.)

Let us call a commutator c an *LS-commutator* if $c = [w]$ where w is an LS-word and the following two conditions hold.

- (a) If $c = [c_1, c_2]$ then each c_i is an LS-commutator with LS-support w_i , $i = 1, 2$, and $w_1 > w_2$;
- (b) If c is as in (a) and $c_1 = [c'_1, c''_1]$ where w''_1 is the associative support of c''_1 then $w''_1 \leq w_2$.

In the remainder of this section, for the convenience of the reader, we state several results of A.I. Shirshov, which are now conveniently available in the collection [14]. They can also be found in the books [2] and [4].

Lemma 2. *On each LS-word w one can replace brackets in a unique way so that the resulting commutator $c = [w]$ is an LS-commutator (such that $\bar{c} = w$). The set of all LS-commutators is a linear basis of $L(X)$.*

Lemma 3. *Any associative word $v \neq 1$, which is not necessarily an LS-word, can be uniquely written as the product of LS-words $v = u_1 u_2 \cdots u_s$ so that $1 \neq u_1 \leq u_2 \leq \cdots \leq u_s$.*

To list a consequence of this fact, of importance to us, we introduce a convenient notation $[c_1, c_2, \dots, c_k]$, called the *left-normed commutator* of $c_1, c_2, \dots, c_k$. This is defined by induction, starting with just c_1 , if $k = 1$. If $k > 1$ then one sets $[c_1, c_2, \dots, c_{k-1}, c_k] = [[c_1, c_2, \dots, c_{k-1}], c_k]$.

Corollary 5. *If c is an LS-commutator and $w = \bar{c} = yv$ for some $y \in X$ and $w \in W(X)$ then y is the maximal letter among all letters involved in the expression of w and $c = [y, [u_1], [u_2], \dots, [u_s]]$, $u_1 \leq u_2 \leq \dots \leq u_s$, where each $[u_i]$ is the (unique) LS-commutator with the associative support u_i , $i = 1, 2, \dots, s$.*

A standard technique in the theory of free Lie algebras is the *elimination* of one or more elements of a free basis (sometimes called Lazard elimination [5]).

Lemma 4. *Let $L(X)$ be a free Lie algebra with free basis $X = Y \sqcup \{z\}$, where $Y \neq \emptyset$. Let $J = \text{id}_L Y$ be the ideal of L generated by Y . Let B be the set of all left normed commutators $[y, z, \dots, z]$, where $y \in Y$. Then J is a free Lie algebra with free basis B .*

3. Growth of finitely generated subalgebras of free Lie algebras

As before, $L = L(X)$ stands for a free Lie algebra over a field F with a free basis X . This section is devoted to the proof of Theorem 1.

Let us call a subset S of a free Lie algebra L *irreducible* if no leading part of any of its elements belongs to the subalgebra generated by the leading parts of the remaining elements of S . In his proof of the theorem on the freeness of subalgebras of free Lie algebras [15], Shirshov shows that any subalgebra M can be generated by an irreducible set S . Then he shows that any irreducible set is *independent*, that is, may serve as a free basis for a subalgebra it generates. We can summarize and slightly complement this as follows.

Lemma 5. *Let S be a subset of a free Lie algebra L , S' the set of leading parts of the elements in S . If S is irreducible then S' is independent. If S' is independent then S is also independent. Any subalgebra of a free Lie algebra can be generated by an irreducible set.*

Proof. We only need to explain the second claim. However, it is true in any algebra which is free in a variety of algebras and endowed with a grading with respect to a free basis, that any nontrivial relation between the elements of a subset S entails a nontrivial relation between the elements of S' . $\square$

Lemma 6. *Let M be a subalgebra of a free Lie algebra L , $M' = \text{gr } M$ the associated graded subalgebra. Let S be an irreducible free basis of M and $S' = \{\text{Lp}(s) \mid s \in S\}$ the set of leading parts of the elements in S . Then the map $\varphi : S \rightarrow S'$ given by $\varphi(s) = \text{Lp}(s)$ extends to a degree preserving isomorphism $\bar{\varphi} : M \rightarrow M'$.*

Proof. According to Lemma 5, S' is an independent set, hence a free basis of a subalgebra N . As a result, $\bar{\varphi}$ is an isomorphism from M to N . We only need to show that $N = M'$. Since S' and M' are homogeneous, it is sufficient to establish $\text{Lp}(\bar{\varphi}(u)) = \text{Lp}(u)$, for any $u \in M \setminus \{0\}$. To do this, let us express u as a Lie polynomial $u = f(s_1, \dots, s_m)$, in $s_1, \dots, s_m \in S$, and write $f = f_1 + f_2$, where f_1 is a linear combination of monomials of the highest degree if we count each variable s_i with the same degree as $\deg s_i$ in L . Then

$$\begin{aligned} \bar{\varphi}(u) &= \text{Lp}(\bar{\varphi}(f_1(s_1, \dots, s_m)) + \bar{\varphi}(f_2(s_1, \dots, s_m))) \\ &= \text{Lp}(f_1(s'_1, \dots, s'_m) + f_2(s'_1, \dots, s'_m)) = f_1(s'_1, \dots, s'_m) = \text{Lp}(u), \end{aligned}$$

as needed. $\square$

Now we proceed to the proof of [Theorem 1](#).

Proof. According to [Lemmas 1 and 6](#), given a finitely generated subalgebra M of L , the growth and cogrowth of a finitely generated homogeneous subalgebra $M' = \text{gr } M$ is the same as the growth and cogrowth of M . If M is proper then the same is true for M' . Therefore, in proving our theorem we may restrict ourselves to the case where M is a nonzero graded subalgebra. This entails, $m \geq 2$.

Let $\{z_1, \dots, z_k\}$ be a free homogeneous basis of M . Since M is proper in L , the elements of degree 1 among $\{z_1, \dots, z_k\}$ form a basis Z of $M \cap L_1$, which is proper in L_1 . Let us complement Z to a linear basis Y of L_1 . Then we will obtain a new free basis Y of L , which properly includes Z . As a result, the main degree filtration [\(3\)](#) does not change and so the relative growth functions of subalgebras do not change. The elements of degree 1 of this set are a proper part Z of a free basis Y . Hence there is an element of the new free basis, say y , which is not an element of Z .

As a result, without loss of generality, we may assume from the very beginning that the elements of degree 1 in the free homogeneous basis $\{z_1, \dots, z_k\}$ of M are a part of the fixed free basis $\{x_1, \dots, x_m\}$ of L and x_1 is not in M . Let us consider L as a subalgebra in the free associative algebra $A = A(x_1, \dots, x_m)$. Clearly, $z_1, \dots, z_k$ are the elements of the subalgebra B of A which is generated by finitely many monomials $u_1, \dots, u_t$ (all the monomials used to write the generators $z_1, \dots, z_k$). By our assumption, we know that none of these monomials equals x_1 . Moreover, we can actually assume that none of $u_1, \dots, u_t$ is a power of x_1 . Indeed, no such monomials can appear while writing the elements of the free Lie algebra $L(x_1, \dots, x_m)$ in the free associative algebra $A(x_1, \dots, x_m)$. It remains to show that the exponent of the growth of B is less than m .

Let us choose an integer d which is at least the double maximum of the degrees of all $u_1, \dots, u_t$ with respect to $x_1, \dots, x_m$. Since any of these monomials contains as a factor a letter different from x_1 , no product can contain a subword x_1^d . It is known (see, e.g., [\[3, Lemma 8\]](#)) that for every nonempty word w in $x_1, \dots, x_m$ there exist $C, \varepsilon > 0$ such that the number of words of length n which do not have w as a subword is bounded by $C(m - \varepsilon)^n$. $\square$

Remark 1. There are some consequences of [Theorem 1](#) that are well-known (see e.g. [\[2, Chapter 3\]](#)). For instance, because the growth of a subspace of finite codimension is equivalent to the growth of the whole space, we conclude that *in a free Lie algebra the proper nonzero subalgebras of finite codimension cannot be finitely generated*. Another consequence, already mentioned in Introduction, is the following: *in a free Lie algebra any nonzero finitely generated subalgebra is self-idealizing*.

Remark 2. [Theorem 1](#) fails in the case of associative algebras. A simple example of a proper finitely generated subalgebra whose growth is not exponentially negligible is the subalgebra B generated by all monomials of degree 2 and 3 in the free associative algebra

A of rank $m \geq 1$. In this case $\dim A/B = m + 1$, following because any number $n > 1$ can be written as $n = 2k + 3l$, where k and l are non-negative integers.

However, even if we assume A/B infinite-dimensional, we still can have examples of not exponentially negligible finitely generated subalgebras. For instance, one can proceed as follows. Let α be the standard filtration on A of rank $m \geq 1$ and B a subalgebra of A generated by all monomials of degree 2. We obviously have $\dim A/B = \infty$, but at the same time the value of the growth function for the filtration $\beta = \alpha \cap B$ is $g_\beta(n) = \frac{m^{n+2}-1}{m^2-1}$, if n is even. For α we have $g_\alpha(n) = \frac{m^{n+1}-1}{m-1}$. Clearly, B is *not* exponentially negligible.

4. Some properties of words in infinite alphabets

Infinite alphabets naturally appear when one applies a standard technique of elimination (see Lemma 4). Although it is not our goal to formally generalize certain results to the case of free Lie algebras of infinite rank, the logic of the proofs makes it necessary to consider such algebras, as an auxiliary tool.

In what follows, we will be considering only the alphabets with at *least two letters*. Let $k_n = \#X_n$ be the number of letters of degree n . In our analysis, we will be imposing on X some conditions, as follows.

Condition G. *Either $k_2 = k_3 = \dots = 0$ or, for every i , if $k_i > 0$ then $k_{i+1} > 0$.*

To formulate another condition we first introduce a function of one real variable ζ (z is a positive parameter):

$$F(\zeta) = F_z^X(\zeta) = \sum_{i=1}^{\infty} \frac{k_i}{(z - \zeta)^i}. \quad (4)$$

Its domain is the set of real values of ζ for which the series on the right hand side converges.

Recall (Subsection 2.2) that $d_X(n)$ stands for the number of words of degree n in the free monoid $W(X)$. A simple fact is the following.

Lemma 7. *If $F(0) \leq 1$ then $d_X(n) \leq z^n$, for all natural values of n .*

Proof. Induction on $n = 0, 1, \dots$. We have $d_X(0) = 1$ and for $n \geq 1$, the number of words of degree n with the last letter from X_i equals $d_X(n - i)k_i$. Applying induction and our hypothesis, we obtain:

$$d_X(n) = \sum_{i=1}^n k_i d_X(n - i) \leq \sum_{i=1}^n k_i z^{n-i} \leq z^n \sum_{i=1}^{\infty} \frac{k_i}{z^i} = z^n F(0) \leq z^n. \quad \square$$

Lemma 8. *Let for some real $z > 1$ the series $\sum_{i=1}^{\infty} \frac{k_i}{z^i}$ converge to a number $\alpha > 1$. Denote by δ the greatest common divisor of all integers in the set $I = \{i \mid k_i \neq 0\}$. Then*

there is a positive constant c such that for every sufficiently large n divisible by δ , we have $d_X(n) > cz^n$.

Proof. Since the series converges to $\alpha > 1$, there is N such that $\sum_{i=1}^N \frac{k_i}{z^i} > 1$. It is well known that there is an integer M such that every $n > M$ divisible by δ is a linear combination of the numbers from I with non-negative integral coefficients. Without loss of generality, let us assume that both M and N are divisible by δ . It follows that there exists a word of degree n over the graded alphabet X , and so there is a small real number $c > 0$ such that $d_X(n) > cz^n$ for every $n = \delta t$ belonging to the segment $[M, M + N]$, where t is a positive integer.

The assertion of the lemma will be proved by induction on t , where $n = \delta t$, with base $t = M/\delta$ guaranteed above by the choice of c . Moreover, for the inductive step, we may assume that $n > M + N$ and so $t \geq (M + N)/\delta$. Now we have

$$d_X(n) = \sum_{i=1}^n k_i d_X(n-i) \geq \sum_{i=1}^N k_i d_X(n-i) \quad (5)$$

The right-hand side of (5) can be rewritten as $\sum_{i=1}^{N/\delta} k_{\delta i} d_X(\delta t - \delta i)$ because obviously $d_X(j) = 0$ if j is not a multiple of δ . Note that $t > t - i \geq t - N/\delta > M/\delta$ for $1 \leq i \leq N/\delta$, and so by the inductive hypothesis, $d_X(\delta t - \delta i) > cz^{\delta t - \delta i} = c \frac{z^n}{z^{\delta i}}$. Taking (5) into account, we obtain $d_X(n) > cz^n \sum_{i=1}^{N/\delta} \frac{k_{\delta i}}{z^{\delta i}}$. Again, the sum on the right-hand side is equal to $\sum_{i=1}^N \frac{k_i}{z^i}$, that is, greater than 1 by the choice of N . Hence $d_X(n) > cz^n$ as required. $\square$

In the next result, we denote by $g_A(n)$ the growth function of $A(X)$ with respect to the degree filtration defined by the graded set X .

Lemma 9. *The function $g_A(n)$ is superexponential if the series $F(0)$ diverges for every positive z . Otherwise, provided that $\#X > 1$, the growth of $g_A(n)$ is exponential. The base of the exponent of the growth can be determined as z_0 in the following.*

$$\lim_{n \rightarrow \infty} \sqrt[n]{g_X(n)} = z_0 = \inf \left\{ z > 1 \mid \sum_{i=1}^{\infty} \frac{k_i}{z^i} \leq 1 \right\}. \quad (6)$$

Moreover, $d_X(n) \leq z^n$ for any $n \geq 0$.

Proof. By Lemma 7, $d_X(n) \leq z^n$ for any $z > z_0$, whence $g_X(n) \leq c_1 z^n$ for a positive constant c_1 . By Lemma 8, $d_X(n) > c(z')^n$ for any sufficiently large n divisible by δ if $z' < z_0$. Therefore $g_X(n) > c_2(z')^n$ for a positive c_2 and every large enough n . To finish the proof of formula (6), we should extract the n th roots from the inequalities obtained for $g_X(n)$ and pass to the limit, using that z and z' can be chosen arbitrary close to z_0 . The second statement is proved in Lemma 7. $\square$

To further proceed we need to impose one more condition on the set X .

Condition W_z (z is a real number > 1). $F(\zeta)$ is defined in a neighborhood of 0 and $F(0) \leq 1$.

Our main Lemma in this section deals with the following sets. Given a nonempty word $u \in W(X)$, let $N(u)$ be the set of all words in $W(X)$ that do not include u as a subword. We also set $N(u)_n = N(u) \cap W(X)_n$ and define $f_u(n) = \#N(u)_n$, that is, $f_u(n)$ is the number of all words of degree n without subword u .

Lemma 10. *Let X satisfy Conditions G and W_z . Then for every nonempty word $u \in W(X)$ there exist positive constants C and ε , such that*

$$f_u(n) < C(z - \varepsilon)^n.$$

Proof. With u fixed, let us write $f(n) = f_u(n)$. If u is a subword of u' then, clearly, $N_n(u) \subset N_n(u')$. Since X has at least two letters, it is easy to include u as a subword in a word u' such that no proper prefix of u' is a suffix of u' . Therefore we may assume from the very beginning that the word u itself enjoys this property.

It follows from Condition W_z , that $F(\zeta)$ is continuous in a neighborhood of 0. Let us choose a small positive $\varepsilon \in (0; 1)$ such that $F(\varepsilon) < 1 + \frac{1}{z^\ell}$, where $\ell = \deg(u)$. Clearly one can choose C so that $f(n) < C(z - \varepsilon)^n$ for every $n = 1, \dots, \ell - 1$. To show that the same inequality also holds for all $n \geq \ell$, we first apply the argument of Lemma 7. Namely, we use

$$N(u)_n \subset \bigsqcup_{i=1}^n N(u)_{n-i} X_i, \quad (7)$$

to produce an upper bound

$$f(n) \leq \sum_{i=1}^n f(n-i) k_i.$$

Let us note that the union on the right hand side of (7) contains some words with subword u . For example, such are the words in $N(u)_{n-\ell}u$. Indeed, if $u = u'x$ with $x \in X_s$, and $v \in N_{n-\ell}(u)$ then $vu = (vu')x \in N_{n-s}(u)X_s$. This follows because, considering v has no subword u , if vu' has such a subword, then $v = v'p$, $u = pq$ and $u' = qr$, for some words v', p, q, r , q nonempty. Then a nonempty word q would be a suffix and a prefix of u at the same time, which contradicts our assumption about u . Since the number of the words in $N(u)_{n-\ell}u$ is $f(n - \ell)$, the stronger inequality holds:

$$f(n) \leq -f(n - \ell) + \sum_{i=1}^n k_i f(n - i) \quad (8)$$

Now according to our [Condition G](#) we either have G1: $k_2 = k_3 = \dots = 0$ or G2: for every i , if $k_i > 0$ then $k_{i+1} > 0$. Let us handle these cases separately.

In the case G1, all letters are contained in X_1 and $f(n-1) \leq f(n-\ell)z^{\ell-1}$ by [Lemma 7](#) since in this case, every word of degree $n-1$ is a product of a word of degree $n-\ell$ and a word of degree $\ell-1$. Therefore we obtain from [\(8\)](#):

$$f(n) \leq -f(n-1)z^{1-\ell} + \sum_{i=1}^n k_i f(n-i) \leq \sum_{i=1}^n k'_i f(n-i), \quad (9)$$

where $k'_1 = k_1 - z^{1-\ell} \geq 0$ and $k'_j = k_j$ for $j \neq 1$. Note that in this case,

$$\frac{k'_1}{z-\varepsilon} = \frac{k_1}{z-\varepsilon} - \frac{1}{z^{\ell-1}(z-\varepsilon)} \leq \frac{k_1}{z-\varepsilon} - \frac{1}{z^\ell}. \quad (10)$$

In the case G2, since the last letter x of u is in X_s , we have $k_s > 0$. Also $\ell = \deg(u) = \deg(u'x) \geq \deg(x) = s$. As a result, we have $k_\ell > 0$ and so we can write

$$f(n) \leq -f(n-\ell) + \sum_{i=1}^n k_i f(n-i) \leq \sum_{i=1}^n k'_i f(n-i), \quad (11)$$

where $k'_\ell = k_\ell - 1 \geq 0$ and $k'_j = k_j$ for $j \neq \ell$. In this case we have

$$\frac{k'_\ell}{(z-\varepsilon)^\ell} \leq \frac{k_\ell}{(z-\varepsilon)^\ell} - \frac{1}{z^\ell}. \quad (12)$$

Now we can make conclusions which are the same both for G1 and G2. Since the coefficients k'_i are non-negative, we conclude from the inductive hypothesis and Eqs. [\(9\)](#) and [\(11\)](#) that

$$f(n) \leq \sum_{i=1}^n k'_i f(n-i) \leq \sum_{i=1}^n C k'_i (z-\varepsilon)^{n-i} = C(z-\varepsilon)^n \sum_{i=1}^n \frac{k'_i}{(z-\varepsilon)^i} \quad (13)$$

Hence, both in G1 and G2, it follows from Eq. [\(13\)](#), the definition of the coefficients k'_i and Eqs. [\(10\)](#) and [\(12\)](#) that

$$\begin{aligned} f(n) &\leq C(z-\varepsilon)^n \left(-\frac{1}{z^\ell} + \sum_{i=1}^n \frac{k_i}{(z-\varepsilon)^i} \right) \\ &\leq C(z-\varepsilon)^n \left(-\frac{1}{z^\ell} + F(\varepsilon) \right) \\ &< C(z-\varepsilon)^n, \end{aligned}$$

because $F(\varepsilon) < 1 + \frac{1}{z^\ell}$. The proof is now complete. $\square$

5. Cogrowth of subideals

This section contains the proof of [Theorem 2](#). We will be using the notions and results about free associative and Lie algebras given in [Subsection 2.2](#).

Lemma 11. *Let $L = L(X)$ be a free Lie algebra with free linearly ordered basis X . Let w be an LS-word, $x \in X$ the maximal letter involved in w and $c = [w]$ the (unique) LS-commutator with associative support w . If x^ℓ is a subword of w for some $\ell > 0$, then c is an element of the ℓ -subideal closure $\text{id}_L^\ell x$ of L generated by x .*

Proof. By definition of the lexicographical order, it follows that $w = x^\ell w'$. According to [Corollary 5](#), $c = [x, [u_1], [u_2], \dots, [u_s]]$, where $u_1 \leq u_2 \leq \dots \leq u_s$. If $\ell = 1$ then c is an element of the ideal generated by x . So we can proceed by induction on ℓ with basis $\ell = 1$. Assume $\ell > 1$. Since $[u_1]$ and c are LS-commutators, u_1 cannot be a power of x , hence u_1 has a proper prefix $x^{\ell-1}$. By the property of the lexicographical order, $x^{\ell-1}$ is also a prefix in each $u_2, \dots, u_s$. Then by induction $[u_1], \dots, [u_s]$ are all in the $(\ell - 1)$ -subideal generated by x , hence, c is an element of the ℓ -subideal $\text{id}_L^\ell x$ generated by x . $\square$

Lemma 12. *Let $L = L(X)$ be a free Lie algebra with the graded free basis X satisfying [Conditions G and W_z](#). Let c be a nonzero linear combination of several free generators of the same degree in L . Then for any $\ell \geq 1$, there are $C, \varepsilon > 0$ such that the cogrowth function of the ℓ -subideal H generated by c in L does not exceed $C(m - \varepsilon)^n$.*

Proof. We will use the fact that given a linear order on X , a linear basis of L_n is given by LS-commutators of degree n . Hence the linear basis of $L \bmod H$ can be chosen as a subset of the set of LS-commutators.

Let us first consider the case where $c = y$, an element of X . We can order X so that y is the greatest element with respect to this order. According to [Lemma 11](#), then any LS-commutator whose associative support contains a subword y^ℓ is an element of H . According to [Lemma 10](#), there are positive C, ε such that the number of associative words without a subword y^ℓ is bounded from above by $C(m - \varepsilon)^n$, for any n . Thus $d_{L/H}(n) < C(m - \varepsilon)^n$. Since $g_{L/H}(n) = d_{L/H}(1) + \dots + d_{L/H}(n)$, we immediately observe that there is $C' > 0$ such that $g_{L/H}(n) \leq C'(m - \varepsilon)^n$, as needed.

Now assume c is not necessarily one letter and c involves a generator y with a nonzero coefficient. We have $\deg(c) = \deg(y)$. According to [\[2, Lemma 2.4.1\]](#), the set $X' = (X \setminus \{y\}) \cup \{c\}$ is another set of free generators of L . Notice that the degrees of elements of L with respect to X' remain to be the same as with respect to X . By the previous paragraph, $d_{L/H}(n) < C(m - \varepsilon)^n$, and the proof is complete. $\square$

Lemma 13. *Let $L = L(X)$ be a free Lie algebra with the graded free basis X satisfying [Conditions G and W_z](#). Let $w \in L$ be any non-zero homogeneous element. Then for any $\ell \geq 1$, there are $C, \varepsilon > 0$ such that the cogrowth function of the ℓ -subideal H generated by w in L does not exceed $C(m - \varepsilon)^n$.*

Proof. Let us consider a linear order on X such that $x < y$ if $\deg(x) < \deg(y)$. We choose the minimal generator $x \in X$. The minimal ideal L' of L containing all other generators has codimension 1 in L . By Lemma 4, the Lie algebra L' has a free basis X' consisting of all commutators z of the form $[y, x, \dots, x]$, where $y \in X \setminus \{x\}$. The free basis X' of L' is a graded subset of L and therefore the degree of every element of L' with respect to X' is equal to its degree with respect to X .

Note that also X' satisfies Condition W_z . Indeed, assume $\deg(x) = s$. Removing x from X we decrease the value $F(0)$ by $\frac{1}{m^s}$. Then adding all the commutators $[y, x, \dots, x]$ to $X \setminus \{x\}$ we multiply the contribution of every such $y \in X \setminus \{x\}$ by $(1 + \frac{1}{m^s} + \frac{1}{m^{2s}} + \dots)$. Hence the $F_m^{X'}(0)$ does not exceed

$$\left(1 - \frac{1}{m^s}\right) \left(1 + \frac{1}{m^s} + \frac{1}{m^{2s}} + \dots\right) = 1.$$

Also the function $F_m^{X'}(\zeta)$ for X' remains defined in a neighborhood of 0. Indeed, we obtain this new function $F_m^{X'}(\zeta)$ by first subtracting $\frac{1}{(m-\zeta)^s}$ from the old function $F_m^X(\zeta)$. Then we multiply the difference obtained by the function

$$\left(1 + \frac{1}{(m-\zeta)^s} + \frac{1}{(m-\zeta)^{2s}} + \dots\right) = \left(1 - \frac{1}{(m-\zeta)^s}\right)^{-1},$$

which is defined in the neighborhood of zero. Thus all components of Condition W_z remain in place.

Because we remove x of the smallest degree in X , also Condition G still holds for X' .

We will repeat the above construction applying it to L' and X' , and so on. After t steps we obtain the Lie algebra $L^{(t)}$ and its free basis $X^{(t)}$.

Now let us show that $\bigcap_{t=1}^{\infty} L^{(t)} = \{0\}$. Indeed, by our construction, the number of generators of degree 1 in X' is strictly less (if any) than in X (at the same time it is possible that we get more generators of degree 2). After several steps of application of our construction, there are no more generators of degree 1. Then, in the same manner, we get rid of all generators of degree 2, and so on. As a result, if u is an element of degree k and on the t th step we got rid of all generators of degree less than or equal to k , we have $u \notin L^{(t)}$.

So let us assume that $w \in L^{(t)} \setminus L^{(t+1)}$. We want to show that w is a linear combination of some elements of $X^{(t)}$. Suppose to the contrary, that the expression for w in $X^{(t)}$ nontrivially includes some commutators of length n where $n > 1$. Because w is homogeneous, it cannot include commutators of length 1 equal to the generators of the minimal degree, and so $w \in L^{(t+1)}$ by definition of the transition $L^{(t)} \rightarrow L^{(t+1)}$. This contradiction proves that a finite sequence of transitions described above will bring us to the case where w is a linear combination of the free generators of some $L^{(t)}$. This case has been dealt with in Lemma 12.

We have that the ℓ -subideal H' generated by w in $L^{(t)}$ has codimension growth in $L^{(t)}$ with respect to $X^{(t)}$ at most $C(m - \varepsilon)^n$. The same is true with respect to X ,

since the degrees of the elements in $L^{(t)}$ with respect to these bases are equal. Since the codimension of $L^{(t)}$ in L is t , the codimension growth of H' with respect to X in the entire L is bounded by $t + C(m - \varepsilon)^n$, where t does not depend on n . Since $H' \subset H$, we have the same upper bound for the cogrowth of H , and altering C we complete the induction. $\square$

We are now in position to complete the proof of one of the main results of this paper, [Theorem 2](#).

Proof. Every nonzero subideal S contains, for some $\ell \geq 1$, an ℓ -subideal generated by one nonzero element w . If we prove our claim for such subideals S , we will be finished. Since we already have this proven in [Lemma 13](#) for the case where w is homogeneous (in that case, with respect to the standard grading associated with the finite free basis of L), using Part (b) of [Lemma 1](#) it is sufficient to show the following.

Let w be a nonzero element of subalgebra H and u its leading part. Then for $\ell \geq 1$ the ℓ -subideal K_ℓ generated by u in $K = \text{gr } H$ is contained in $\text{gr } H_\ell$, where H_ℓ is an ℓ -subideal generated by w in H .

Indeed, by definition of K_1 , any nonzero homogeneous element $v \in K_1$ is a linear combination of nonzero commutators $[u, u_1, \dots, u_s]$, where all u_i are homogeneous elements in K . In this case each u_i is a leading part of an element $w_i \in H$. Therefore, v itself is a leading part of a respective linear combination of commutators $[w, w_1, \dots, w_s] \in H_1$, so that $K_1 \subset \text{gr } H_1$.

If $\ell > 1$ then we have $K_{\ell-1} \leq \text{gr } H_{\ell-1}$, by induction. Since K_ℓ is an ideal in $K_{\ell-1}$ generated by the element u , it is contained in the ideal M generated by u in $\text{gr } H_{\ell-1}$. But H_ℓ is an ideal generated by w in $H_{\ell-1}$. Hence, if we apply what we have proved for the ideals (that is, when $\ell = 1$), we obtain $M \subset \text{gr } H_\ell$. As a result, $K_\ell \subset \text{gr } H_\ell$, as needed. $\square$

6. An example

In this section, based on [Lemmas 11 and 14](#), we find the cogrowth function for the 2-subideal $\text{id}_L^2 x$ generated by x in the free Lie algebra $L = L(x, y)$. We will need the following converse to [Lemma 11](#) in our particular case.

Lemma 14. *Let $L = L(x, y)$. Then the set T of LS-commutators, different from x , whose associative supports do not have x^2 as a subword, are linearly independent modulo $H = \text{id}_L^2 x$.*

Proof. Any LS-commutator without x^2 in the support (excluding x and y) is an element of the subalgebra M of L with free basis $\{[x, y], [x, y, y], \dots\}$. At the same time, M is a free factor in the ideal $N = \langle M, x \rangle = \text{id}_L^1 x$. Both claims follow by [Lemma 4](#). Therefore, M trivially intersects the ideal generated by x in N which is exactly $H = \text{id}_L^2 x$. It

then follows that the set T of all commutators in question, excluding y , being linearly independent in L and M , remains linearly independent modulo $\text{id}_L^2 x$. Restoring y to the set does not change the situation because $y \notin \langle M, x \rangle$, while all the rest elements of T are in $\langle M, x \rangle$ which also contains $\text{id}_L^2 x$. $\square$

We will use a linear basis of L composed of LS-commutators. It follows from Lemmas 11 and 14, that we need to find the growth of the set of associative LS-words different from x , that do not have x^2 as a subword.

We first count the number of words of length n each being the product of the subwords y and yx . Let a_n denote the number of such words with suffix x and b_n with suffix y . Then $a_n = a_{n-2} + b_{n-2}$ (these are obtained by attaching yx to the words of length $n-2$) and $b_n = a_{n-1} + b_{n-1}$. Then also $b_{n-1} = a_{n-2} + b_{n-2} = a_n$. It follows that b_n is the n th term of the Fibonacci sequence $1, 1, 2, 3, 5, \dots$, while a_n is the $(n-1)$ st term of the same sequence (we assume the zeroth term being 0).

The total number of these words of length n is now $a_n + b_n = b_{n-1} + b_n$, hence the $(n+1)$ st term of the Fibonacci sequence. Actually, we have counted the number of words w of degree n , starting with y , such that no cyclic shift of w contains x^2 , as a subword. We will say that w does not contain a *cyclic subword* x^2 .

The set of words we have just counted is the set of all words of length n without subword x^2 , with the exclusion of those words that have prefix x . It is obvious that the number of words of length n without cyclic subword x^2 with prefix x is equal to the number of words without cyclic subword x^2 with suffix x . Each of these latter can already be written via y and yx and has suffix x . Earlier, we denoted the number of such words as a_n . As a result, we have the total number c_n of the words of length n without cyclic subwords x^2 equal to $(a_n + b_n) + a_n$ (with the exception of x). Thus, c_n is the sum of the $(n+1)$ st and $(n-1)$ st terms of the Fibonacci sequence.

Now we want to delete from the above number those subwords that are powers of shorter words. Notice that every proper power is a proper prime power. Note that if v is a word without cyclic subwords x^2 then each power of v enjoys the same property. Now the computation of the number of words which are prime powers is routine. If $p|n$, where p is prime, then $c_{n/p}$ of our words will be p th proper powers. These must be subtracted, for each prime divisor p of n .

If $d = p_1 p_2$ is the product of two different primes, then the d th powers have appeared in the above process twice, so while considering such powers we have to correct this error, etc. All this leads to a formula for the number d_n of words of length n without cyclic subwords x^2 , where the Möbius function plays the role of the “corrector”: $d_n = \sum_{d|n} \mu(d) c_{n/d}$. Although we considered only those divisors d which split as the product of pairwise different prime divisors, we have included in the preceding formula all divisors $d|n$ simply because $\mu(d) = 0$ as soon as d is divisible by a square of a prime number.

The set of all words we just counted splits into n -element subsets consisting of cyclic shifts of one of its elements. The greatest word in this subset, in the sense of the LEX-grading such that $x > y$, is an LS-word (see Subsection 2.2). As a result, the number of

Table 1Cogrowth of the 2-subideal generated by x in $L(x, y)$.

n	1	2	3	4	5	6	7	8	9	10
$d_{L/S}(n)$	1	1	1	1	2	2	4	5	8	11
$g_{L/S}(n)$	1	2	3	4	6	8	12	17	25	36
n	11	12	13	14	15	16	17	18	19	20
$d_{L/S}(n)$	18	25	40	58	90	135	210	316	492	750
$g_{L/S}(n)$	54	79	119	177	267	402	612	928	1420	2170

all LS-words of length n in the alphabet x, y , without cyclic subwords x^2 is now given by the formula $(1/n) \sum_{d|n} \mu(d) c_{n/d}$ (as earlier, x is not counted here). Since no LS-word of length ≥ 2 can have suffix x , this formula also gives the total number of LS-words of length n , without subword x^2 .

Proposition 1. *Let $L = L(X)$ be a free Lie algebra of rank 2, with free basis X , x a letter in X , S a 2-subideal of L generated by x . Let $\text{Fib}(n)$ denote the n th Fibonacci number. Then the graded function $d_{L/S}$ of the relative cogrowth of S in L is given by*

$$d_{L/S}(n) = \frac{1}{n} \sum_{d|n} \mu(d) \left(\text{Fib}\left(\frac{n}{d} - 1\right) + \text{Fib}\left(\frac{n}{d} + 1\right) \right). \quad (14)$$

We can see from (14) that the growth of the functions $d_{L/S}$ and $g_{L/S}$ is exponential, with the base of exponent $\lim_{n \rightarrow \infty} \sqrt[n]{g_{L/S}(n)}$ being the same number as for the Fibonacci sequence, that is, $\frac{1+\sqrt{5}}{2}$. Recall (see (1)) that for the whole of L the base of exponent is 2.

The first values of the functions $d_{L/S}$ and $g_{L/S}$ are given in Table 1.

7. Computing the exponential base of the relative growth

The approach suggested in Section 4 allows us to prove our claims about the exponential base of the growth of (not necessarily finitely generated) subalgebras in free Lie algebras, as stated in Theorem 3. To be able to apply the results of Section 4, we need an auxiliary lemma.

Lemma 15. *Let A and L be a free associative algebra and a free Lie algebra, with the same graded free basis Y . Suppose $\lim_{n \rightarrow \infty} \sqrt[n]{g_A(n)}$ exists and > 1 . Then $\beta_L = \lim_{n \rightarrow \infty} \sqrt[n]{g_L(n)}$ exists and is equal to $\lim_{n \rightarrow \infty} \sqrt[n]{g_A(n)}$.*

Proof. If $d|n$ then the product of any d words of degree n/d is a word of degree n , which implies $d_A(n/d)^d \leq d_A(n)$. If additionally, $d \geq 2$ then $d_A(n/d) \leq \sqrt[d]{d_A(n)} \leq \sqrt{g_A(n)}$. But $d_A(n/d)$ is the number of words of degree n which are d -powers. After summation over all divisors $d \geq 2$, we will find that the number of words of degree n does not exceed $n\sqrt{g_A(n)}$.

Now the existence of $\lim_{n \rightarrow \infty} \sqrt[n]{g_A(n)} > 1$ means that $\sqrt{g_A(m)} < g_A(n)^{2/3}$, for all sufficiently great n and all $m \leq n$. Therefore, after summation over $m \leq n$, we will have for large enough n that the number of proper powers among the words of degree at most n is bounded from above by $n^2 g_A(n)^{2/3}$, which is $o(g_A(n))$, because $g_A(n)$ is exponential. It follows that the number of words of degree $\leq n$, which are not proper powers is $g_A(n)(1 - o(1))$.

Since among the cyclic shifts of any such word of degree $m \leq n$ precisely one is an LS-word, the number of these latter takes the form of $g_A(n)(1 - o(1))\gamma(n)$, where $\gamma(n) \in [1/n, 1]$. By Lemma 2, the number of LS-words of length at most n is $g_L(n)$. Taking the n th roots of these numbers, we arrive at the following:

$$\lim_{n \rightarrow \infty} \sqrt[n]{g_L(n)} = \lim_{n \rightarrow \infty} \sqrt[n]{g_A(n)},$$

as claimed. $\square$

Now we can proceed to the proof of Theorem 3.

Proof. Recall that we are dealing with the relative growth a subalgebra H of a free Lie algebra L of rank m with a nongraded free basis X . Using Lemmas 1 and 6 allows us to always assume that H is a homogeneous subalgebra generated by an irreducible free basis Y , in which the number of elements of degree i equals k_i , for $i = 1, 2, \dots$. The relative growth of H in L is the same as the absolute growth of a free Lie algebra $L(Y)$, where Y is viewed as an abstract graded set. The degree of each $y \in Y$ equals the degree of y with respect to X in L . Let us embed $L(Y)$ in a free associative algebra $A(Y)$. Lemma 15 allows us to translate our claims about H to the claims about $A(Y)$.

We start with Claim (i), where H is not necessarily finitely generated but is non-abelian. Let us use Lemma 9 (notice that in that Lemma the graded set of generators is denoted by X !) Since $\dim L_i \leq m^i$, it follows that $k_i \leq m^i$, for each $i = 1, 2, \dots$. It follows that for any $z > m$ the series $F(0)$, as a function of z , converges, hence z_0 in formula (6) exists and provides us with the exponential base for the growth of $A(Y)$, hence for the growth of $L(Y)$ (Lemma 15), hence for the relative growth of H in L . As noted in Lemma 9, if $\#Y > 1$, the growth is exponential. Thus we have proved Claim (i).

In the proof of Claim (ii), the function $F(\zeta) = F_z^Y(\zeta)$ takes the form of

$$F(\zeta) = \frac{k_1}{z - \zeta} + \frac{k_2}{(z - \zeta)^2} + \dots + \frac{k_d}{(z - \zeta)^d}, \quad (15)$$

where d is the maximal degree of the elements in Y , and each k_i is the number of elements of degree i , $i = 1, 2, \dots, d$. As in the proof of Claim (i), we use Lemma 9. To be able to apply formula (6), we need to determine zeros of $F(0) - 1$, as a function of z . One can write $F(0)$, as a function of the real argument $z > 0$ in the form $\frac{f(z)}{z^d}$, where $f(z)$ is a polynomial of degree less than d , with nonnegative coefficients and positive free term k_d . Using the Descartes Rule of Signs, the equation $z^d - f(z) = 0$ has a unique positive

root z_0 . Since z_0 is a root of a monic polynomial with integral coefficients, our claim about the algebraic integrality of the exponential base of H follows. The exponential base is less than m , which follows from the exponential negligibility of [Theorem 1](#).

We prove (iii) for the case $m = 2$, the general case being quite analogous. Choose $m_0 \in (1, 2)$. Let M be the subalgebra of codimension 1, with free basis $u_1 = x$, $u_2 = [x, y], \dots$, given in [Lemma 4](#). For this subalgebra, $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$. Suppose we could discard few summands in the series so that

$$\sum_{i=1}^{\infty} \frac{k_i}{m_0^i} = 1, \quad (16)$$

where $k_i = 0$ if u_i is discarded and $k_i = 1$ otherwise. Let B be the subalgebra of M generated by those u_i for which $k_i = 1$. Then, according to our previous argument, m_0 will be the exponential base for the B .

To find the sequence $k_1, k_2, \dots$ so that (16) holds, let us assume by induction, that we have chosen $k_1, \dots, k_j$ so that $1 - \sum_{i=1}^j \frac{k_i}{m_0^i} \in [0, \frac{1}{m_0^j})$. Let us set $a_j = 1 - \sum_{i=1}^j \frac{k_i}{m_0^i}$. If $a_j < \frac{1}{m_0^{j+1}}$, then we set $k_{j+1} = 0$, otherwise, $k_{j+1} = 1$. Since $m_0 < 2$, in either case, a_{j+1} is in the desired interval. Since $m_0 > 1$, it follows that $a_j \rightarrow 0$. Hence $\sum_{i=1}^{\infty} \frac{k_i}{m_0^i} = 1$, proving Claim (iii).

Claim (iv) is an easy consequence of the construction in the proof of Claim (iii). Indeed, the subalgebra H in that proof is generated by a subset of the set of the canonical free basis of M . Let S be the ideal of M generated by the complement of that set. We have a vector space decomposition $L = H \oplus S \oplus \langle y \rangle$, where all subspaces are homogeneous. In this case, for any $n \geq 1$, $L^{(n)} = (L^{(n)} \cap H) \oplus (L^{(n)} \cap S) \oplus \langle y \rangle$. Hence,

$$g_{L/S}(n) = \dim L^{(n)} / (L^{(n)} \cap S) = \dim(L^{(n)} \cap H) + 1 = g_H(n) + 1.$$

It follows that the exponential base of $g_{L/S}$ is the same as that for g_H . Finally, $M/S \cong H$, a free Lie algebra. $\square$

We conclude this section by exhibiting a positive algebraic integer which is not the exponential base of any finitely generated subalgebra. For instance, if H is generated in $L(x, y)$ by $x, [x, y]$ then by Claim (ii) in [Theorem 3](#), we have to solve $1 + z = z^2$. As a result, we obtain $z_0 = \frac{1+\sqrt{5}}{2}$. As a side remark, not every positive algebraic integer $< m$ is the exponential base for the growth of a finitely generated subalgebra in a free Lie algebra of rank m . For instance, $\lambda = (5 - \sqrt{5})/2$, which is a positive root of $z^2 - 5z + 5$, cannot serve as such a base. Indeed, if λ is the solution to the equation with integral coefficients $f(z) = z^d$ then another root $\mu = (5 + \sqrt{5})/2 > 0$ of $z^2 - 5z + 5$ is also a solution. At the same time, by the argument in the proof of Claim (ii) the positive solution z_0 of $f(z) = z^d$ must be unique!

8. Free complements

The main contents of this section is the proof of [Theorem 4](#).

Proof. We first consider the case where H is a homogeneous subalgebra in L with a homogeneous set B_0 of free generators. Since B_0 is finite, there is t equal to the maximum of degrees of all elements in B_0 . For the homogeneous component L_t of L we have $L_t = (H \cap L_t) \oplus M_t$, for some (homogeneous) complementary subspace M_t . Let us denote by B_1 a linear basis of M_t and consider the subalgebra H_1 generated by H and B_1 . We want to prove that $B(1) = B_0 \sqcup B_1$ is a free basis for H_1 .

For this, it is necessary and sufficient (see [\[2, Theorem 4.2.11\]](#)) to show that $B(1)$ generates H_1 (this is clear) and that the elements of $B(1)$ are linearly independent modulo the commutator subalgebra $[H_1, H_1]$. Let us denote by L^m the m th term of the lower central series for L . We have $L^m = L_m \oplus L_{m+1} \oplus \dots$. Since $[M_t, H] \subset L^{t+1}$, we have that $[H_1, H_1] \subset [H, H] \oplus L^{t+1}$. So a nontrivial linear dependence of the elements of $B(1)$ modulo $[H_1, H_1]$ would imply a nontrivial linear dependence of the elements of $B(1)$ modulo $[H, H] + L^{t+1}$. The subspaces $[H, H]$ and L^{t+1} are homogeneous. It follows then that we can consider the linear dependence only for homogeneous elements of $B(1)$ of the same degree d .

If $d < t$ then neither L^{t+1} nor B_1 have homogeneous elements of degree d . Thus we have a nontrivial linear dependence of some elements of B_0 modulo $[H, H]$. This is not possible because B_0 is a free basis of H . The case $d > t$ is vacuous because neither B_0 nor B_1 have elements of such degree. Now if $d = t$ then neither B_0 nor L^{t+1} have elements of degree t . As a result, we have a nontrivial linear dependence between the elements of B_1 modulo $[H, H] \cap L_t \subset H \cap L_t$. This is not possible by the choice of B_1 as a basis of a direct complement to $H \cap L_t$ in L_t . Thus we have proved that $B(1)$ is indeed a free basis for H_1 .

Since H_1 is homogeneous and the degrees of elements in $B(1)$ are less than $t + 1$, we can repeat the previous construction by replacing t with $t + 1$. So we write $L_{t+1} = (H \cap L_{t+1}) \oplus M_{t+1}$, choose a linear basis B_2 in M_{t+1} , set $B(2) = B_0 \sqcup B_1 \sqcup B_2$, introduce the subalgebra H_2 generated by $B(2)$, prove that $B(2)$ is a free basis of H_2 , etc. The union $C = \bigcup_{i=1}^{\infty} H_i$ is a free Lie subalgebra with basis $B = \bigsqcup_{i=0}^{\infty} B_i$. Clearly, H is a free factor in C .

By construction, any homogeneous polynomial of degree $\geq t$ is an element of C . It follows that the codimension of C in L is at most the dimension of the t th term $L^{(t)}$ of the degree filtration of L . Since L is finitely generated, we have $\dim L^{(t)} \leq \infty$, proving that C is of finite codimension in L .

Now suppose that H is not necessarily homogeneous. Recalling [Section 3](#), we choose a finite irreducible set S of free generators for H and consider the set S' of leading parts of the elements of this set. Let H' be the subalgebra generated by S' . By [Lemma 5](#), S' is independent. Let us set $B_0 = S'$ and proceed as just above to produce a homogeneous subalgebra C' of finite codimension in L with the set B' of free generators such that

$B' = B_0 \sqcup \overline{B_0}$, where $\overline{B_0} = \bigsqcup_{i=1}^{\infty} B_i$. Now let us consider $B = S \sqcup \overline{B_0}$. Then the set of the leading parts of elements in B is B' . Since B' is independent, by Lemma 5, the same is true for B .

Hence B is the free basis of a subalgebra C . Invoking Lemma 6, we have that $C' = \text{gr } C$. If K is a subalgebra generated by $\overline{B_0}$, we have $C = H * K$. Notice that by construction the subalgebra K is homogeneous.

It remains to note that $\dim L/C < \infty$, which follows by Part (b) of Lemma 1. Now the proof is complete. $\square$

It is worth noting that in a particular case where the subalgebra H of a free Lie algebra L is generated by its homogeneous component of degree t , it is true that H is a free factor in the t th term L^t of the lower central series of L . Probably, this result was already known to A.I. Shirshov.

9. Triviality of ideals in subideals

We start this section with a fairly general result about the derivations of free algebras, which we think could be of interest in its own.

Lemma 16. *Let A (respectively, L) be the free associative (respectively, Lie) algebra with free generators $x_1, x_2, \dots$ over an arbitrary field and D its derivation given on the generators by the rule $Dx_i = x_{i+1}$ ($i = 1, 2, \dots$). Suppose a is a nonzero element of A (of L). Then for every $k \geq 1$, there exists $n \geq 0$ such that the element $D^n(a)$ does not belong to the ideal I_k (to the Lie ideal J_k) of A (resp., of L) generated by the set $\{x_1, \dots, x_k\}$.*

Proof. Let us begin with the associative case. We may assume that a is a homogeneous element of degree $c \geq 1$ and a is an element of a subalgebra A_l generated by $x_1, \dots, x_l$ for some $l \geq k$. We can write

$$a = \sum_{1 \leq i_1 \leq l, \dots, 1 \leq i_c \leq l} \alpha_{i_1, \dots, i_c} x_{i_1} \cdots x_{i_c}$$

for some scalars $\alpha_{i_1, \dots, i_c}$. We say that a c -tuple $(i_1, \dots, i_c)$ is an element of the support $\text{Supp } a$ if $\alpha_{i_1, \dots, i_c} \neq 0$.

Let us define a sequence of numbers $K_1, K_2, \dots, K_c$ by setting $K_d = (1/2)(2k+2)^{2^{c-d}}$, for any of $d = 1, 2, \dots, c$. Then $K_c = k+1$ and $K_d = 2K_{d+1}^2$, for any $1 \leq d < c$. Our lemma is an easy consequence of the following.

Claim. *For any d , $1 \leq d \leq c$, there is $n \geq 1$ such that the support of $D^n(a)$ has a c -tuple $(i_1, \dots, i_c)$ with $\min\{i_1, \dots, i_d\} \geq K_d$.*

The estimate for the value of n can be recovered from the proof that follows. Once proven, our Lemma follows because $K_c > k$.

For the proof, let us use induction by d .

In the case $d = 1$, our task is easy. Indeed, let $(i_1, \dots, i_c) \in \text{Supp } a$ with the greatest possible i_1 . Then the only c -tuple $(i_1 + 1, i_2, \dots, i_c)$ comes to $\text{Supp } D(a)$ from

$$D(\alpha_{i_1, i_2, \dots, i_c} x_{i_1} x_{i_2} \cdots x_{i_c}) = \alpha_{i_1, i_2, \dots, i_c} (x_{i_1+1} \cdots x_{i_c} + x_{i_1} x_{i_2+1} \cdots x_{i_c} + \dots),$$

and $i_1 + 1$ is the greatest first index in the members of $\text{Supp } D(a)$. Then we repeat the argument and have $(i_1 + K_1, i_2, \dots, i_c) \in \text{Supp } D^n(a)$ if $n = K_1$.

Now assume that the statement is true for some d ($1 \leq d < c$). In this case, there is n such that the support of $b = D^n(a)$ includes a c -tuple $(i_1, \dots, i_c)$ with $\min\{i_1, \dots, i_d\} \geq K_d$. Let us split $\text{Supp } b$ as the disjoint union

$$\text{Supp } b = \left(\bigsqcup_{j \geq K_{d+1}} S_j \right) \sqcup S,$$

where $(i_1, \dots, i_c) \in S_j$ if $\min\{i_1, \dots, i_d\} = j$, $j \geq K_{d+1}$. The c -tuples in S satisfy $\min\{i_1, \dots, i_d\} < K_{d+1}$. The maximal j with non-empty S_j satisfies the inequality $j \geq K_d$ because of the way we have obtained b .

To proceed, we introduce a function $f(j)$ ($j \geq K_{d+1}$) whose j th value equals to the maximal value of the $(d+1)$ st coordinate of the tuples in the subset S_j ; we set, by definition, $f(j) = 1$ if the subset S_j is empty.

If $f(j) \geq K_{d+1}$ for some $j \geq K_{d+1}$, then there is a c -tuple $(i_1, \dots, i_c) \in S_j$ such that $\min\{i_1, \dots, i_{d+1}\} \geq K_{d+1}$, which is sufficient to complete the induction step. Therefore, we further assume that $f(j) < K_{d+1}$ for every $j \geq K_{d+1}$, and so $f(j)$ takes less than K_{d+1} different values.

It follows that the length of any decreasing sequence $j(1) > j(2) > \dots$ such that the values $f(j(1)) < f(j(2)) < \dots$ increase, is less than K_{d+1} . Recall that the maximal value of j with non-empty S_j is at least K_d . Let us choose a particular decreasing sequence as follows. There exists $j(1) \geq K_d$ such that $f(j(1)) \geq f(q)$ for every $q \geq j(1)$. If for all $j < j(1)$ we have $f(j) \leq f(j(1))$, then we are done. Otherwise we keep working and define $j(2)$ as the maximal index such that $j(2) < j(1)$ while $f(j(2)) > f(j(1))$. Proceeding in the same way, we eventually arrive at a finite sequence $j(1) > j(2) > \dots > j(s)$ with $f(j(1)) < f(j(2)) < \dots < f(j(s))$, which cannot be extended any further. Here we additionally have $s < K_{d+1}$ and $f(j(t)) \geq f(q)$, for arbitrary $t \leq s$ and $q \geq j(t)$.

Notice that it is not possible that every number in the sequence $j(1) - j(2), \dots, j(s-1) - j(s), j(s)$ is strictly less than $2K_{d+1}$. For, if this were the case, we would be able to write

$$K_d \leq j(1) = (j(1) - j(2)) + \dots + (j(s-1) - j(s)) + j(s) < 2K_{d+1}K_{d+1}.$$

Therefore, either $j(t) - j(t+1) > 2K_{d+1}$ for some $t < s$ or $j(s) > 2K_{d+1}$. In both cases, we can conclude that there is a number m such that the following are true:

$$m \geq 2K_{d+1}, \quad \text{and} \quad f(m) \geq f(q) \quad \text{for any } q \geq m - K_{d+1}.$$

Now we take $(i_1, \dots, i_d, f(m), \dots, i_c) \in S_m$, consider the respective monomial $u = x_{i_1} \cdots x_{i_d} x_{f(m)} \cdots x_{i_c}$, entering b with non-zero coefficient, and compute the elements $D^{K_{d+1}}(u)$ and $D^{K_{d+1}}(b)$. The support of $D^{K_{d+1}}(u)$ includes the c -tuple

$$\mathbf{i} = (i_1, \dots, i_d, f(m) + K_{d+1}, i_{d+2}, \dots, i_c).$$

This follows exactly as in the case $d = 1$ above. The first $d + 1$ indexes of this tuple are not less than K_{d+1} , by the definition of S_m . Therefore to complete the induction, it suffices to prove that for any monomial v occurring in b with nonzero coefficient, the tuple $\mathbf{i}$ does not appear in the support of $D^{K_{d+1}}(v)$. Let $\text{Supp } v = \{(i'_1, \dots, i'_d, \dots, i'_c)\}$.

Case 1: $\text{Supp } v \in S_q$, where $q \geq m - K_{d+1}$. By the choice of m , $f(m) \geq f(q)$. Our argument, as just above, shows that $D^{K_{d+1}}(v)$ has at most one monomial with multi-index of the form

$$\mathbf{i}' = (i'_1, \dots, i'_d, f(m) + K_{d+1}, \dots, i'_c).$$

Since $\mathbf{i}' \neq \mathbf{i}$ we have $\mathbf{i} \notin \text{Supp } D^{K_{d+1}}(v)$.

Case 2: $\text{Supp } v \in S_q$, where $q < m - K_{d+1}$. According to the definitions of S_q and S_m , we have $\min(i'_1, \dots, i'_d) = q < m - K_{d+1}$. Each of the K_{d+1} derivations can increase the minimum of the first d indices in the members of the support at most by 1. Hence $\mathbf{i} \notin \text{Supp } D^{K_{d+1}}(v)$.

Case 3: $\text{Supp } v \in S$. Then by the definitions of D and S , one of the first d indices of any term of $D^{K_{d+1}}(v)$ is less than $2K_{d+1}$. Hence, by the definition of m and S_m , $\mathbf{i} \notin \text{Supp } D^{K_{d+1}}(v)$.

Thus the proof is complete in the case of associative algebras. The case of Lie algebras reduces to the associative case because the free generators of A also freely generate L with respect to the bracket operation, $J_k \subset I_k$, and the derivation D of L extends to a derivation of A (see [5, Chapter II]). $\square$

Theorem 6. *Let J be a proper ideal of a free Lie algebra L and S a finite subset of J . Let $I = \text{id}_J S$ be the ideal closure of S in J . Then for every $z \in L \setminus J$ and for every nonzero $a \in I$ there is n such that $[a, z, \dots, z]$ (n times) is not an element of I .*

Proof. One may assume that $L = \langle J, z \rangle$ where J is an ideal of codimension 1 in L . Then an element $z' = z - v$, where $v \in J$ can be included in a free basis of L . If for any number n of occurrences of z' we had $b_n = [a, z', \dots, z'] \in I$ then we would also have that all $a_n = [a, z, \dots, z] \in I$. To prove this, we first note that, if we replace z by $z' + v$ then we will have the sum of left-normed commutators $w = [a, u_1, u_2, \dots, u_n]$, where each u_i is either z' or v . Using the Jacobi identity $[v, s, t] = [v, t, s] + [v, [s, t]]$, we can change places

z' and v in w , if $z = t$ follows $v = s$ in w . The resulting commutators would still be of the form of w but now u_i are either z' or some elements of J . If there is still some z' to the right of some u_i , we repeat the process. When all z' are to the left of all u_i we have the elements of the form $[a, z', \dots, z', u_1, \dots, u_k]$. The initial portion $[a, z', \dots, z']$ is in I , by our assumption. Since I an ideal of J , the whole commutator is in I , as well.

Hence, we may assume that z is an element of the free basis of L . Then the inner derivation $\text{ad } z$ shifts the free generators of J (see Lemma 4), so we are tempted to apply Lemma 16. However, the direct application is not possible, because there can be many different “orbits” for the action of $\text{ad } z$ on the free basis of J . Hence, we need an easy modification of Lemma 16, as follows.

Lemma 17. *Let A (respectively, L) be the free associative (respectively, Lie) algebra with the set X of free generators split as a disjoint union of subsets $X^{(j)} = \{x_1^{(j)}, x_2^{(j)}, \dots\}$ over arbitrary field and δ its derivation given on the generators by the rule $\delta x_i^{(j)} = x_{i+1}^{(j)}$ ($i = 1, 2, \dots$). Suppose a is a nonzero element of A (of L). Then for every finite subset S of X , there exists $n \geq 0$ such that the element $\delta^n(a)$ does not belong to the ideal I_S (to the Lie ideal J_S) of A (resp., of L) generated by S .*

Proof. This is a direct consequence of Lemma 16. Indeed, considering, say, the associative case, let us assume that, for any n , we have $\delta^n(a) \in I_S$. Let us consider a homomorphism of associative algebras φ from $A(X)$ to the free associative algebra $A(x_1, x_2, \dots)$ of Lemma 16, defined in the following way. Suppose an expression for a in $A(X)$ includes only at most m first letters from each of subalphabets $X^{(j)}$, $j = 1, 2, \dots$. Then we set $\varphi(x_i^{(1)}) = x_i$, $\varphi(x_i^{(2)}) = x_{i+m}$, $\varphi(x_i^{(3)}) = x_{i+2m}$, etc., where $i = 1, 2, \dots$. Clearly, under this homomorphism, $\varphi(a) \neq 0$ because we simply applied a bijective change of the sets of free variable involved in a . Now both algebras $A(X)$ and $A(x_1, x_2, \dots)$ are equipped with derivations, δ in the former case and D , as in Lemma 16, in the latter, that is, $D(x_i) = x_{i+1}$, $i = 1, 2, \dots$. Clearly, φ is a differential homomorphism, that is $\varphi \circ \delta = D \circ \varphi$.

Since we assumed that $\delta^n(a) \in I_S$, for all n , applying φ would mean that $D^n(\varphi(a)) \in I_{\varphi(S)}$. This comes as contradiction to Lemma 16. $\square$

Clearly, Theorem 6 follows from Lemma 17. $\square$

Now we came closer to the proof of Theorem 5 from the Introduction.

Proof. Let $J(i)$ be the i -closure of a fixed nonzero element or a finite set S of elements of $J = J(0)$, as in Theorem 5, and $L \supset M_1 \supset \dots$ an arbitrary subideal series in L with nonzero terms. We need to show that M_ℓ is not contained in $J(\ell)$.

Let us first prove that setting $N_i = J(i-1) \cap M_i$, $i = 1, 2, \dots$, the series $L \supset N_1 \supset N_2 \dots$ is also a subideal series with nonzero terms. First of all, if $P \cap Q = \{0\}$ for $\{0\} \neq P, Q \triangleleft L$, $a \in P$, $b \in Q$ then $[a, b] = 0$. Now the subalgebra M generated by a, b

is free; since a, b commute, M must be abelian, hence $\dim M = 1$. So a, b are scalar multiples of each other, a contradiction with $P \cap Q = \{0\}$. Therefore, the intersection N_1 of two nonzero ideals M_1 and $J = J(0)$ of a free Lie algebra L is again nonzero. Now since N_1 and $J(1)$ are two nonzero ideals of $J(0)$, their intersection $N_1 \cap J(1)$ is nonzero. Similarly, $M_2 \cap N_1$ is nonzero. Hence $(M_2 \cap N_1) \cap (N_1 \cap J(1))$ is nonzero, being the intersection of two nonzero ideals of N_1 . As a result, $N_2 = M_2 \cap J(1) \neq \{0\}$. Continuing in the same way, we find that all $N_i = M_i \cap J(i-1)$ are nonzero.

As a result, when we resume proving [Theorem 5](#), we may assume that $M_i \subset J(i-1)$, for all i . So we need to prove that $M_\ell \not\subset J(\ell)$, for any $\ell = 1, 2, \dots$. Assume the contrary, that is, $M_\ell \subset J(\ell)$, for some $\ell = 1, 2, \dots$. If $\ell = 1$, we can apply [Theorem 6](#), because J is a proper ideal in L while $J(1)$ is an ideal generated in $J = J(0)$ by a finite subset of elements. It follows that $J(1)$ does not contain an ideal M_1 of the whole algebra L . Now let us assume that $\ell > 1$. Then by induction we know that $M_{\ell-1}$ is not contained in $J(\ell-1)$. At the same time, by our assumption, $M_{\ell-1}$ is contained in $J(\ell-2)$. Therefore, there is $z \in M_{\ell-1} \setminus J(\ell-1)$ such that $z \in J(\ell-2)$. Since $J(\ell-1)$ is an ideal in $J(\ell-2)$, it follows that $\text{ad } z$ leaves $J(\ell-1)$ invariant. So we can conclude that $J(\ell-1)$ is a proper ideal in the free subalgebra $P = \langle z, J(\ell-1) \rangle$. Further, $J(\ell)$ is a finitely generated ideal in $J(\ell-1)$. Since $z \in M_{\ell-1}$, we have $[M_\ell, z] \subset M_\ell \subset J(\ell-1)$. As a result, M_ℓ is an ideal of P . Thus, in the same way as in case $\ell = 1$, we arrive at a contradiction with [Theorem 6](#), now applied to a free Lie algebra P , a proper ideal $J(\ell-1)$ of P , a finitely generated ideal closure $J(\ell)$ in $J(\ell-1)$ and an ideal M_ℓ of P contained in $J(\ell)$. Hence, the proof of [Theorem 5](#) is complete. $\square$

Remark 3. In the case of associative algebras we have a very simple example when there is an ideal I generated by finitely many elements in a proper ideal J of a free associative algebra A , and still containing a nonzero ideal of A . For this one can take A with free basis $\{x\}$, J the principal ideal of A generated by x^2 , $S = \{x^2, x^3\}$, I the ideal of J generated by S . The reader would easily check that the nonzero ideal of A contained in I is $\dots I$ itself!

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