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Graded algebras with polynomial growth of their codimensions

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ABSTRACT

Let A be an algebra over a field of characteristic 0 and assume A is graded by a finite group G. We study combinatorial and asymptotic properties of the G-graded polynomial identities of A provided A is of polynomial growth of the sequence of its graded codimensions. Roughly speaking this means that the ideal of graded identities is "very large". We relate the polynomial growth of the codimensions to the module structure of the multilinear elements in the relatively free G-graded algebra in the variety generated by A. We describe the irreducible modules that can appear in the decomposition, we show that their multiplicities are eventually constant depending on the shape obtained by the corresponding multipartition after removing its first row. We relate, moreover, the polynomial growth to the colengths. Finally we describe in detail the algebras whose graded codimensions are of linear growth.

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Introduction

All algebras we consider are associative, not necessarily unitary, and over a fixed field F of characteristic 0. Let A be an associative algebra satisfying a polynomial identity (also called a PI algebra), and let G be a finite group. Assume that A is G-graded, then A satisfies non-trivial G-graded polynomial identities. Denote by P_n the vector space of all multilinear polynomials of degree n in the variables $x_1, \ldots, x_n$ in the free associative algebra $F\langle X\rangle$ freely generated over F by $X = \{x_1, x_2, \ldots\}$. It is well known that in order to study the polynomial identities of A one may consider only the multilinear ones (as long as the characteristic of the base field equals 0). If A is an algebra and $\mathrm{Id}(A)$ is its T-ideal, that is the ideal of its polynomial identities in $F\langle X\rangle$ then $\mathrm{Id}(A)$ is generated as a T-ideal by the elements in $\mathrm{Id}(A) \cap P_n$ for $n \geq 1$. The vector space P_n is a left module over the symmetric group in a natural way, and it is isomorphic to the left regular S_n-module FS_n, and moreover $P_n \cap \mathrm{Id}(A)$ is its submodule. It is more convenient to consider the factor module $P_n(A) = P_n/(P_n \cap \mathrm{Id}(A))$ instead of $P_n \cap \mathrm{Id}(A)$. Following this line one applies the theory of representations of the symmetric group to the study of PI algebras, and in an equivalent form, the representations of the general linear group. Hence it is important to know the decomposition of $P_n(A)$ into irreducible modules, its character, the generators of the irreducible modules and so on. One of the most important numerical invariants of a PI algebra is its *codimension sequence* $c_n(A) = \dim P_n(A)$. Despite its importance the exact computation of the codimensions of an algebra is extremely difficult, and it has been done for very few algebras. That is why one is led to study the asymptotic behaviour of the codimensions. A celebrated theorem of Regev asserts that if A is a PI algebra satisfying an identity of degree d then $c_n(A) \leq (d-1)^{2n}$. Thus the growth of the sequence $c_n(A)$ cannot be very "fast": $\dim P_n = n!$. In the late nineties Giambruno and Zaicev (see for example their monograph [17]) proved that if A is a PI algebra then the limit $\lim_{n\to\infty}(c_n(A)^{1/n})$ exists and is always a non-negative integer called the *exponent* (or PI exponent) of the algebra A, denoted by $\exp(A)$, thus answering in the affirmative a conjecture of Amitsur. The PI exponent of A can be explicitly computed; it is closely related to the structure of A and equals the dimension of certain semisimple algebra related to A.

One may thus classify the PI algebras according to their exponents. Of special interest are the PI algebras with "slow" codimension growth. It is well known that $\exp(A) \leq 1$ if and only if $c_n(A)$ is polynomially bounded. Various descriptions of such algebras were given, the interested reader might want to consult the monographs [17, Chapter 7] for further information about this topic. We recall that a theorem of Kemer [20,21] states that the following conditions are equivalent for the PI algebra A.

(1) The codimension sequence $c_n(A)$ is polynomially bounded.
(2) $\exp(A) \leq 1$.

(3) There is a constant q depending only on A such that the nonzero irreducible S_n-modules appearing in the decomposition of $P_n(A)$ correspond to Young diagrams having at most q boxes below the first row.
(4) Neither the Grassmann algebra G of an infinite dimensional vector space nor the algebra UT_2 of the 2×2 upper triangular matrices lie in the variety of algebras generated by A.

Later on Giambruno and Zaicev [16] proved that a variety of algebras is of polynomial growth if and only if it can be generated by a finite direct sum of finite dimensional algebras A_i such that the Jacobson radical J_i of A_i is of codimension ≤ 1 in A_i for all i. Further results concerning PI algebras with polynomial growth of their codimensions were obtained in [9].

Gradings on algebras and the corresponding graded identities have proven an indispensable tool in the study of PI algebras. In this paper we consider gradings by finite groups only. Recall that the algebra A is G-graded if $A = \oplus_{g\in G} A_g$ is a decomposition of A into a direct sum of its vector subspaces A_g such that $A_g A_h \subseteq A_{gh}$ for all g, $h \in G$. The graded identities of large families of algebras are well understood while their ordinary analogues still remain very far from our knowledge. Recall that the polynomial identities of the matrix algebras $M_n(F)$ are known only for $n \leq 2$, see [30,6,22]. Their graded counterparts have been extensively studied and described under any reasonable grading, see for example [34,3].

One defines graded multilinear elements in analogy with the case of ordinary identities, graded codimensions of an algebra (or variety of G-graded algebras), graded PI exponent and so on, see the precise definitions below. It is known that the graded PI exponent of an algebra exists and is always an integer, see [1]. (Recall that the case when G is abelian was dealt with in [2,10].)

It follows from the theory developed by Kemer that in characteristic 0, every PI algebra A satisfies the same polynomial identities as the Grassmann envelope of an appropriate finite dimensional $\mathbb{Z}_2$-graded algebra. Moreover as recalled above (see [16]) if the codimensions of A are polynomially bounded then A is PI equivalent to a finite dimensional algebra. In the graded case a precise information was obtained in [25]: the graded codimensions of A are polynomially bounded if and only if A satisfies the same graded identities as a finite direct sum of finite dimensional graded algebras B_i such that $\dim B_i/J(B_i) \leq 1$ for all i. Here $J(B_i)$ is as usual the Jacobson radical of B_i. Further detailed descriptions of the varieties of G-graded algebras with polynomial growth of their graded codimensions were given in [13,33,25], see Theorem 1.3 below.

In this paper we study PI algebras graded by a finite group G such that their graded codimensions are of polynomial growth. We deduce that the polynomial growth of the graded codimension sequence of A is equivalent to a condition that several concrete algebras do not belong to the variety generated by A and moreover the multiplicities of the irreducible modules are bounded by a constant, see Theorem 2.1. Later on we prove that in fact the non-zero multiplicities may occur only for multipartitions having bounded quantity of boxes outside the first row of the first partition. In the sequel we transfer a theorem of Gordienko, see [18,19] to the graded case. Namely we prove that these multiplicities are eventually constants depending only on the shape of the multipartition below the first row. To this end we employ some ideas developed in [18,19] adapted to the graded case.

In the sequel we relate the polynomial growth of the graded codimensions to the corresponding colengths (that is the sum of all multiplicities of irreducible modules in the decomposition of the graded multilinear elements). Namely we prove that the colengths must be bounded by a constant, and moreover this condition is equivalent to the polynomial growth of the codimensions.

Assume A is a G-graded algebra of polynomial growth: $c_n^G(A) \leq an^p$ for some positive integer p and positive constant a. We prove that the nonzero irreducible modules correspond to multipartitions having at most p boxes outside the first row of the first partition. In order to make the last statement more precise we introduce some notation (see for details the beginning of Section 2). Let $\lambda(i) \vdash n_i$ be a partition of n_i, $1 \leq i \leq s$, and denote $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ the corresponding multipartition of $n = n_1 + \cdots + n_s$. The statement we discuss here says that $n - \lambda(1)_1 \leq p$. Moreover if the growth of the graded codimensions of A is asymptotically an^p then there must appear in the decomposition into irreducibles at least one module corresponding to a multipartition with $n - \lambda(1)_1 = p$. We describe further the graded algebras A with linear growth of their graded codimensions. We obtain that such an algebra is PI equivalent to a finite direct sum of algebras M_g satisfying the graded identities of UT_2^g, the upper triangular matrices of size 2, and a nilpotent algebra N. Moreover we use the description of the graded subvarieties of UT_2^g given in [25], and obtain a concrete list of possibilities for the algebras M_g.

1. Preliminaries

Throughout the paper F will denote a fixed field of characteristic zero, G a finite group and A an associative G-graded PI-algebra (i.e., A satisfies an ordinary polynomial identity) over F. We do not require the algebras to be unitary. All algebras and vector spaces we consider will be over the field F. Thus if $G = \{1_G = g_1, \dots, g_s\}$ then $A = \bigoplus_{i=1}^s A_{g_i}$ is a direct sum of its vector subspaces A_{g_i} such that $A_{g_i}A_{g_j} \subseteq A_{g_ig_j}$ for all i, $j = 1, \dots, s$.

We denote by $F\langle X\rangle$ the free associative algebra freely generated over the field F by the countable set of free generators $X = \{x_1, x_2, \dots\}$. It can be given a natural structure of a G-graded algebra in the following way. We write

$$X = \bigcup_{i=1}^{s} X_{g_i}$$

where $X_{g_i} = \{x_{1,g_i}, x_{2,g_i}, \dots\}$ are disjoint infinite sets whose elements are said to be of homogeneous degree g_i. Denote by $\mathcal{F}_{g_i}$ the vector subspace of $F\langle X\rangle$ spanned by all monomials in the variables of X having homogeneous degree g_i. Then $F\langle X\rangle = \bigoplus_{i=1}^{s} \mathcal{F}_{g_i}$ is a G-graded algebra; it is called the *free G-graded algebra* of countable rank over F. We shall denote it by $F\langle X, G\rangle$.

A graded polynomial f of $F\langle X, G\rangle$ is a graded (polynomial) identity of A and we write $f \equiv 0$ in case f vanishes under all graded substitutions $x_{i,g} \to a_g \in A_g$. In other words f is a graded identity of A whenever f lies in the kernels of all homomorphisms of graded algebras $\varphi\colon F\langle X, G\rangle \to A$.

Let $\mathrm{Id}^G(A)$ denote the set of all graded identities of A. It is clear that $\mathrm{Id}^G(A)$ is an ideal of $F\langle X, G\rangle$ which is invariant under all graded endomorphisms of the free algebra; in analogy with the case of algebras satisfying ordinary polynomial identities it is called the T_G-ideal of A.

It is well known that in characteristic zero, every graded identity is equivalent to a system of multilinear graded identities. We denote by

$$P_n^G = \mathrm{span}_F\{x_{\sigma(1),g_{i_1}} \cdots x_{\sigma(n),g_{i_n}} \mid \sigma \in S_n, g_{i_1}, \dots, g_{i_n} \in G\}$$

the vector space of multilinear G-graded polynomials in the variables $x_{1,g_i}, \dots, x_{n,g_i}$, $i = 1, \dots, s$. The study of $\mathrm{Id}^G(A)$ is equivalent to the study of $P_n^G \cap \mathrm{Id}^G(A)$ for all $n \geq 1$ and we denote by

$$c_n^G(A) = \dim_F \frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(A)}, \quad n \geq 1,$$

the n-th G-graded codimension of A.

In order to capture the exponential rate of growth of the above introduced sequence of G-codimensions, in [2,10] for abelian groups and in [1] in the general case, the authors proved that for any associative G-graded algebra A, satisfying an ordinary identity, the limit

$$\exp^G(A) = \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$$

exists and is an integer. It is called the G-exponent of A. Moreover $\exp^G(A)$ can be explicitly computed; it turns out $\exp^G(A)$ equals the dimension of a suitable finite dimensional semisimple G-graded algebra over an algebraically closed field.

In this paper we are interested in algebras having polynomial growth of their G-graded codimensions. The following theorem states that it is equivalent to study G-graded algebras such that $\exp^G(A) \leq 1$.

Theorem 1.1. *For a G-graded algebra A, $\exp^G(A) \leq 1$ if and only if the sequence $c_n^G(A)$, $n = 1, 2, \dots,$ is polynomially bounded.*

Proof. This result follows from the upper and lower bound of the G-exponent obtained in [1] and in [2]. $\square$

Given a variety of G-graded algebras $\mathcal{V}$, the growth of $\mathcal{V}$ is the growth of the sequence of G-codimensions of any algebra A generating $\mathcal{V}$ that is $\mathcal{V} = \mathrm{var}^G(A)$. As a consequence of the characterization of the G-exponent, in [25] the structure of a generating G-graded algebra of a given variety of polynomial growth was described.

We shall need the following definition.

Definition 1.1. Let A and B be G-graded algebras. We say that A is T_G-equivalent to B and we write $A \sim_{T_G} B$ when A and B satisfy the same G-graded identities, that is $\mathrm{Id}^G(A) = \mathrm{Id}^G(B)$.

Theorem 1.2. *(See [25].) Let G be a finite group and A a G-graded algebra over the field F. Then $c_n^G(A)$, $n = 1, 2, \ldots,$ is polynomially bounded if and only if $A \sim_{T_G} B$ where $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \ldots, B_m$ finite dimensional G-graded algebras over F and $\dim B_i/J(B_i) \le 1$ for all $i = 1, \ldots, m$.*

Another characterization of varieties of polynomial growth can be given by means of exhibiting a finite list of G-graded algebras to be excluded from the variety. It consists of the following algebras:

1) E, the infinite dimensional Grassmann algebra with the trivial grading;
2) E^a, the Grassmann algebra with the G-grading induced by its canonical $\mathbb{Z}_2$-grading where $a \in G$ is an element of order 2;
3) UT_2^g, the algebra of 2×2 upper triangular matrices over F with the elementary G-grading induced by $\mathbf{g} = (1_G, g)$, $g \in G$;
4) FC_p^h, the group algebra of the cyclic group $C_p = \langle h \rangle$, with the natural G-grading induced by C_p. Here $h \in G$ is an element of order p where p is a prime.

The following theorem was essentially proved in [33] (see also [13,25]).

Theorem 1.3. *(See [13,33,25].) Let G be a finite group and let A be a G-graded algebra. Then $c_n^G(A)$, $n = 1, 2, \ldots,$ is polynomially bounded if and only if UT_2^g, E, E^a, $FC_p^h \notin \mathrm{var}^G(A)$ for all $g \in G$, $a \in G$ of order 2, and $h \in G$ of prime order p.*

2. Polynomial codimension growth and cocharacters

In this section we shall give characterizations of the varieties of polynomial growth through the behaviour of their sequences of cocharacters.

Let $n \ge 1$ and write $n = n_1 + \cdots + n_s$ as a sum of non-negative integers. We denote by $P_{n_1,\ldots,n_s} \subseteq P_n^G$ the vector space of the multilinear graded polynomials in which the first n_1 variables have homogeneous degree g_1, the next n_2 variables have homogeneous degree g_2, and so on. The group $S_{n_1} \times \cdots \times S_{n_s}$ acts on the left on the vector space $P_{n_1,\ldots,n_s}$ by permuting the variables of the same homogeneous degree. Thus S_{n_1} permutes the variables of homogeneous degree g_1, S_{n_2} those of homogeneous degree g_2, and so on. In this way $P_{n_1,\dots,n_s}$ becomes a module over the group $S_{n_1} \times \cdots \times S_{n_s}$. This action is very useful since T_G-ideals are invariant under renaming of variables of the same homogeneous degree. Moreover the vector space

$$P_{n_1,\dots,n_s}(A) = \frac{P_{n_1,\dots,n_s}}{P_{n_1,\dots,n_s} \cap \mathrm{Id}^G(A)}$$

is an $S_{n_1} \times \cdots \times S_{n_s}$-module with the induced action. We denote by $\chi^G_{n_1,\dots,n_s}(A)$ its character; it is called the $(n_1, \dots, n_s)$-th cocharacter of A.

If λ is a partition of n, we write $\lambda \vdash n$. It is well-known that there is a one-to-one correspondence between partitions of n and irreducible S_n-characters. Hence if $\lambda \vdash n$ we denote by χ_λ the corresponding irreducible S_n-character. If $\lambda(1) \vdash n_1, \dots, \lambda(s) \vdash n_s$ are partitions we write $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_s$.

Since $\operatorname{char} F = 0$, by complete reducibility $\chi^G_{n_1,\dots,n_s}(A)$ can be written as a sum of irreducible characters

$$\chi^G_{n_1,\dots,n_s}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_s)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)} \tag{1}$$

where $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ in $\chi^G_{n_1,\dots,n_s}(A)$.

Theorem 2.1. *Let A be a G-graded algebra, then the following two conditions are equivalent.*

(1) *The sequence of the graded codimensions $c^G_n(A)$, $n = 1, 2, \dots$, is polynomially bounded.*
(2) *The G-graded algebras E, E^a, $FC^h_p \notin \mathrm{var}^G(A)$ and there exists a constant M such that $m_{\langle\lambda\rangle} \leq M$ for all $\langle\lambda\rangle \vdash (n_1, \dots, n_s)$ and for all $n_1 + \cdots + n_s = n$. (Here the gradings on the corresponding algebras are the ones given just before Theorem 1.3).*

Proof. The result follows by applying Theorem 1.3 and Theorem 4.6 in [4]. □

We shall see in the next section that, in case of polynomial growth, condition (2) of the above theorem can be strengthened: the multiplicities $m_{\langle\lambda\rangle}$ are eventually constant.

Now if we set $c_{n_1,\dots,n_s}(A) = \dim_F P_{n_1,\dots,n_s}(A)$ it is immediate to see that

$$c^G_n(A) = \sum_{n_1+\cdots+n_s=n} \binom{n}{n_1, \dots, n_s} c_{n_1,\dots,n_s}(A) \tag{2}$$

where $\binom{n}{n_1,\dots,n_s} = \frac{n!}{n_1! \cdots n_s!}$ stands for the generalized binomial coefficient (also called multinomial coefficient).

Hence the growth of $c_n^G(A)$ is related to the growth of generalized binomial coefficients and of degrees of irreducible characters.

We next state some technical results. Given a partition $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ we denote by d_λ the degree of the corresponding irreducible character. Also if $f(n)$ and $g(n)$ are sequences (or functions $\mathbb{N} \to \mathbb{R}$) we shall use the notation $f \approx g$ whenever there exist positive constants a and b such that $ag(n) \le f(n) \le bg(n)$ for all (large enough) n.

Proposition 2.1. *If $\lambda_1 = n - r$ then $d_\lambda \approx cn^r$ for some constant c.*

Proof. By the hook formula we get immediately that $d_\lambda \le \frac{n!}{(n-r)!} \le an^r$ and also $d_\lambda \ge \frac{n!}{kn(n-1)\cdots(n-(r-1))(n-2r)!} \ge bn^r$ for some constants a, k and b. $\square$

Proposition 2.2. *Let $n = n_1 + \cdots + n_s \ge 1$ and denote by $t = n_2 + \cdots + n_s$. Then there exists a constant c such that $\binom{n}{n_1, \dots, n_s} \approx cn^t$.*

Proof. Notice that

$$\binom{n}{n_1, \dots, n_s} = \frac{n!}{n_1! n_2! \cdots n_s!} \ge \frac{n!}{n_1! n_2! \cdots n_s! (n_2 + \cdots + n_s)!} = k\binom{n}{n_1} \ge an^t$$

and $\binom{n}{n_1, \dots, n_s} = r\frac{n!}{n_1!} = r\frac{n!}{(n-t)!} \le bn^t$ for some constants a, b, k and r. $\square$

Theorem 2.2. *Let A be a G-graded algebra over the field F. The sequence $c_n^G(A)$, $n = 1$, 2, $\dots$, is polynomially bounded if and only if there exists a constant q such that for every n_1, $\dots$, n_s with $n_1 + \cdots + n_s = n$ it holds*

$$\chi^G_{n_1,\dots,n_s}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_s) \\ n - \lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}.$$

If A is finite dimensional then the constant q is such that $J(A)^q = 0$.

Proof. Notice that neither the decomposition of $\chi^G_{n_1,\dots,n_s}(A)$ into irreducible characters nor $c_n^G(A)$ change under extensions of the base field. This fact can be proved following word by word the proof given in [17, Theorem 4.1.9] for the ordinary case. It relies on the well known fact that the irreducible representations of the symmetric group over $\mathbb{Q}$ are absolutely irreducible (that is do not change under extension of the base field). Also if $\bar{F}$ is the algebraic closure of F and $J(A)^q = 0$ then $J(A \otimes_F \bar{F})^q = 0$. Therefore we may assume, without loss of generality, that F is algebraically closed.

Suppose first that $c_n^G(A)$, $n = 1, 2, \dots$, is polynomially bounded. According to Theorem 1.2 we have that $A \sim_{T_G} B$ where B is a finite dimensional algebra. Therefore we assume that A is a finite dimensional algebra. By the Wedderburn–Malcev theorem [5,31], we can write

$$A = A' + J$$

where A' is a maximal semisimple graded subalgebra of A and $J = J(A)$ is its Jacobson radical. Recall that J is a graded (or homogeneous) ideal. Also the latter is a direct sum of vector subspaces of A. We can write

$$A' = A_1 \oplus \cdots \oplus A_k$$

where $A_1, \ldots, A_k$ are G-graded simple algebras.

Since $c_n^G(A)$ is polynomially bounded then $\exp^G(A) \leq 1$ and by the characterization of the G-exponent we must have $A_i J A_j = 0$ for all $i \neq j$. Moreover $A_i \cong F$ (otherwise one would have $\exp^G(A_i) > 1$ and consequently $\exp^G(A) > 1$) and A_i is endowed with the trivial grading.

Hence $\bigoplus_{g \in G, g \neq 1_G} A_g \subseteq J$ and, if q is the least positive integer such that $J^q = 0$ then $\bigoplus_{g \in G, g \neq 1_G} A_g$ generates a nilpotent ideal of A of index of nilpotence $\leq q$. Let $\langle \lambda \rangle = (\lambda(1), \ldots, \lambda(s))$ be a multipartition of $n = n_1 + \cdots + n_s$ such that $n - \lambda(1)_1 \geq q$. We claim that every multilinear G-graded polynomial $f = e_{T_{\langle\lambda\rangle}} f_0$ corresponding to the multitableau $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \ldots, T_{\lambda(s)})$ vanishes on A. Here $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(s)}}$ is an essential idempotent of $F(S_{n_1} \times \cdots \times S_{n_s})$ corresponding to $T_{\langle\lambda\rangle}$. Write $f = f(X_{g_1}, \ldots, X_{g_s})$ and let $\lambda(1)' = (\lambda(1)'_1, \ldots, \lambda(1)'_d)$ be the conjugate (also called transpose) partition of $\lambda(1)$. As in the proof of [10, Lemma 4], if $\lambda(1)'_1 > 1$ then there exists a subset Y_{1_G} of X_{1_G} such that

$$Y_{1_G} = Y^1 \cup \cdots \cup Y^d, \tag{3}$$

$|Y^i| = \lambda(1)'_i$ and for some $r \in F(S_{n_1} \times \cdots \times S_{n_s})$, the element $rf \neq 0$ is alternating on Y^i for all $1 \leq i \leq d$.

Notice that f generates an irreducible left $S_{n_1} \times \cdots \times S_{n_s}$-module. Therefore

$$F(S_{n_1} \times \cdots \times S_{n_s})f = F(S_{n_1} \times \cdots \times S_{n_s})rf.$$

Thus in order to prove that $f \in \mathrm{Id}^G(A)$ it is enough to prove that $rf \in \mathrm{Id}^G(A)$.

Since rf is alternating on each set Y^i, in order to get a non-zero value, no two variables of Y^i can take values in the same $A_i \cong F$. But $A_i A_j = 0$ whenever $i \neq j$, hence in order to get a non-zero evaluation, we must substitute, in rf, at least $n_1 - \lambda(1)_1$ variables for elements of the radical J. Therefore as $\bigoplus_{g \in G, g \neq 1_G} A_g \subseteq J$, at least $n - \lambda(1)_1 \geq q$ variables must be evaluated on J, and since $J^q = 0$ we get that rf vanishes on A.

In this way all the irreducible characters appearing in $\chi^G_{n_1,\ldots,n_s}(A)$ with non-zero multiplicities correspond to multipartitions $\langle\lambda\rangle$ with $n - \lambda(1)_1 < q$.

Now suppose that

$$\chi^G_{n_1,\ldots,n_s}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\ldots,n_s) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}.$$

Thus if $m_{\langle\lambda\rangle} \neq 0$, for some multipartition $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$, we have $t = n_1 - \lambda(1)_1 < q$. By Proposition 2.1 we obtain that

$$\deg \chi_{\lambda(1)} \leq an^t \leq an^q.$$

Moreover $\deg \chi_{\lambda(i)} \leq q!$ for any $i = 2, \dots, s$, and $\binom{n}{n_1 \cdots n_s} < n^q$ since $n_2 + \cdots + n_s < q$ (see Proposition 2.2).

Since the multiplicities are polynomially bounded by Lemma 2.1 of [1] and since there are finitely many multipartitions $\langle\lambda\rangle \vdash (n_1, \dots, n_s)$ satisfying the condition $n - \lambda(1)_1 < q$, by (2) we have that the codimensions are polynomially bounded. $\square$

We draw the reader's attention to the fact that in case G is abelian, a weaker result was proved in [14].

3. The multiplicities are eventually constant

In this section we show that if A is a G-graded algebra of polynomial growth then the multiplicities of the irreducible characters in the decomposition of the G-graded cocharacter of A are eventually constants depending only on the shape of the multipartitions $\langle\lambda\rangle$. We follow the scheme of proof of Gordienko, see [18,19] where the same fact was proved for the ordinary cocharacter instead of the G-graded one. We keep the notation introduced above.

Theorem 3.1. *Let G be a finite group and let A be G-graded algebra over the field F of characteristic 0. Assume that $\exp^G(A) \leq 1$ and that $\chi^G_{n_1,\dots,n_s}(A) = \sum m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$. Here the summation runs over all multipartitions $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s))$ of n such that $n - \lambda(1)_1 < q$.*

There exists a positive integer n_0 such that for every $n \geq n_0$ and for every multipartitions $\langle\lambda\rangle$ of n as above, and $\langle\mu\rangle = (\mu(1), \lambda(2), \dots, \lambda(s))$ of $n+1$ where $|\mu(1)_1| = |\lambda(1)_1| + 1$, $|\mu(1)_2| = |\lambda(1)_2|$, $|\mu(1)_3| = |\lambda(1)_3|$, $\dots$, it holds $m_{\langle\lambda\rangle} = m_{\langle\mu\rangle}$.

We shall divide the proof of the theorem into several steps. We keep the notation introduced in Theorem 2.2 and its proof. First we make a number of reductions and remarks.

(1) As we mentioned in the beginning of the proof of Theorem 2.2, we consider the field F algebraically closed. Since $\exp^G(A) \leq 1$ then we may assume $\dim_F A < \infty$, and moreover $A = Fa_1 \oplus \cdots \oplus Fa_k + J$ where $J = J(A)$ is the Jacobson radical of A. Furthermore $a_1, \dots, a_k \in A$ are orthogonal idempotents. As it was done in the proof of Theorem 2.2 we may suppose $J^{q-1} \neq 0$ and $J^q = 0$.
Once again by $\exp^G(A) \leq 1$ we have $a_i J a_j = 0$ whenever $i \neq j$ and, since the theorem is true when the algebra A is nilpotent, we may assume that there exists $a_i = a$ such that $B = Fa + J$ is a non-nilpotent subalgebra of A. Here $a^2 = a$.

(2) If $n \geq (2r+1)^2$ and m is a multilinear monomial of degree n in the variables $x_1, \ldots, x_n$ then by the pigeonhole principle there exists a submonomial m' of m, $\deg m' \geq 2r+1$ such that no variable from the set $\{x_1, \ldots, x_{2r}\}$ appears in m'.

(3) Let $Q_n = P_n \cap \mathrm{Id}^G(B)$, and suppose that in the decomposition of the module $P_n(B) = P_n/Q_n$ into irreducibles the module corresponding to the multipartition $\langle\lambda\rangle$ appears with multiplicity $m_{\langle\lambda\rangle}$. There are exactly $m_{\langle\lambda\rangle}$ linearly independent elements of the type $e_{T_{\langle\lambda\rangle}} h$ in P_n/Q_n. Here $e_{T_{\langle\lambda\rangle}}$ is the standard essential idempotent corresponding to the multidiagram $T_{\langle\lambda\rangle}$ obtained from the multipartition $\langle\lambda\rangle$ by filling it with the integers from 1 to n in some (fixed) way. Thus it suffices to prove that the multiplicities $m_{\langle\lambda\rangle}$ are constant whenever n is sufficiently large.

(4) The vector space $P_{n_1,\ldots,n_s}$ has a basis consisting of all multilinear monomials such that exactly n_i variables are of G-degree g_i, $i = 1, \ldots, s$. We denote by $R_{n_1,\ldots,n_s}$ the set of all multilinear polynomials where for $g_i \in G$ there are exactly n_i variables of degree g_i and, moreover, there is a distinguished variable y of G-degree $g_1 = 1_G$ (thus $n_1 \geq 1$).

(5) Suppose that $f_1 = f|_{y=x_i}$ and $f_2 = f|_{y=x_i x_{n+1}\cdots x_{n+v}}$ are graded multilinear polynomials of degrees n and $n+v$, respectively, obtained from the same polynomial $f \in R_{n_1,\ldots,n_s}$, $n_1 + \cdots + n_s = n$, by substituting the variable y with x_i and with $x_i x_{n+1} \cdots x_{n+v}$, correspondingly. Then the evaluations of f_1 and of f_2 on A where one substitutes x_i, and all of $x_{n+1}, \ldots, x_{n+v}$, by the element $a \in B$ coincide, since $a^2 = a$.

(6) We fill the multi-diagram $T_{\langle\lambda\rangle}$ with the integers $\{1, 2, \ldots, n\}$ as follows. The integers $\{1, 2, \ldots, 2q\}$ fill in the leftmost columns of $\lambda(1)$, and fill in the whole of $\lambda(2), \ldots, \lambda(s)$. As we shall add boxes to the first row of $\lambda(1)$ the corresponding variables will be of G-degree $g_1 = 1_G$. Moreover the polynomial that corresponds to $e_{T_{\langle\lambda\rangle}}$ is symmetric in $\{x_r \mid r > 2q\}$, and all of these variables are of G-degree g_1.

(7) In order to evaluate such a graded polynomial on B one has to substitute $x_1, \ldots, x_{2q}$ by some homogeneous elements in B (respecting the grading), and the x_r, $r > 2q$, by either elements a or by elements from J. Suppose one chooses ν elements from the radical, then one may suppose $\nu < q$ (otherwise the polynomial vanishes on B and on A). As our polynomials are multilinear one may consider for such substitutions only elements from some fixed bases of the vector spaces B and J. But $\dim B < \infty$ hence one has finitely many such substitutions to consider, as $a^2 = a$.

Let these substitutions be $\sigma_1, \ldots, \sigma_z$ where σ_i is defined by the ordered sets $(\beta_1, \ldots, \beta_{2q})$, $\beta_j \in B$, and $(\gamma_1, \ldots, \gamma_\nu)$, $\gamma_j \in J$.

We choose a homogeneous basis of graded polynomials $f_1, \ldots, f_\rho$ in the vector space $P_{n_1,\ldots,n_s}(B)$, and write the corresponding elements $e_{T_{\langle\lambda\rangle}} f_i$. Evaluate them on the substitutions σ_j and write the resulting elements in B as linear combinations of a fixed basis b_τ of B, as in [18, Lemma 2]:

$$e_{T_{\langle\lambda\rangle}} f_i|_{\sigma_j} = \sum_\tau H([j,\tau]; i) b_\tau.$$

Let $H = (H([j,\tau];i))$ be the matrix whose rows are indexed by all ordered pairs $[j,\tau]$, the latter ordered in some way, and whose columns are indexed by the i's.
Thus $H = H(n)$ has $z \dim B$ rows and its rank satisfies $r(H) \le z \dim B$. While the number of rows of H is constant the number of columns varies with n.
We denote by $r_n = r(H)$ the rank of H. Suppose $r(N)$ is the largest value in the sequence r_n, $n \ge n_0$ for some large enough n_0. Here $n = n_1 + \cdots + n_s$. Then in $H(N)$ there exist $r(N)$ linearly independent rows and also $r(N)$ linearly independent columns whose intersection produces an invertible matrix of order $r(N)$. Without loss of generality we suppose these are the leftmost $r(N)$ columns of $H(N)$ (this can be achieved by reordering the basic vectors). Therefore the linearly independent columns come from the polynomials $e_{T_{\langle\lambda\rangle}} f_i$, $1 \le i \le r(N)$.

(8) The polynomials $\{f_1, \dots, f_{r(N)}\}$ can be obtained by some $\{\varphi_1, \dots, \varphi_{r(N)}\} \subseteq R_{n'}$ for some n', by means of substituting the variable y by some product of variables of homogeneous degree $g_1 = 1_G$, according to Step 2. We suppose that f_i is obtained by φ_i substituting the variable y by some monomial m_i.
Substitute, in φ_i, the variable y by the monomial $m_i x_{N+1} \cdots x_{N+v}$ where $v \ge 1$ and denote the resulting polynomials by h_i. Clearly $\deg h_i = N+v$ and h_i are multilinear. Setting $M = N+v$, as in [18] one writes down the matrix $H(M)$; its columns coming from the h_i consist of entries of the type $H([j,\tau];i) = \sum_\kappa \pi_\kappa(M) G_\kappa([j,\tau];i)$.
Here κ runs over all triples (m, ℓ, k) (following the notation of [18]) where m, ℓ, k depend on the substitution σ:
- m stands for the quantity of radical elements that we substitute for variables from the first row of $\lambda(1)$ plus the quantity of all boxes in $\langle\lambda\rangle$ outside the first row of $\lambda(1)$ (which equals $M - \lambda(1)_1$). Clearly $m < 2r$.
- ℓ stands for the quantity of variables in the first row of $\lambda(1)$, *not* coming from the monomials m_i substituted for the variable y, and that are evaluated on the element a. Clearly ℓ is bounded by above.
- k is the quantity of nontrivial segments in the monomial m_i that are substituted by the element a. Since two such segments are separated by an element from the radical, one has $k \le r$.

Therefore we have to compute the coefficients $\pi_\kappa(M)$. We draw the readers' attention that $G_\kappa([j,\tau];i)$ does not depend on M.

(9) The coefficient $\pi_\kappa(M)$ counts the possibilities to place $M - m$ symbols a in the first row of $\lambda(1)$. As these can be inside the m_i as well as outside it one has, repeating the argument from the proof of Lemma 2 in [18], that

$$\pi_\kappa(M) = (M-m)! \binom{M - m - \ell}{k-1}.$$

Continuing as in [18] we cancel out the common multiplier from the chosen rows of $H(M)$ (all of these contain a multiplier $(M - M_1)!$ where $M_1 = \max\{m\}$). Then the remaining entries will be polynomials in M, and the determinant of order r_M is a polynomial in M as well. If this polynomial is identically 0 then the multiplicity will always be 0. If it is nonzero we take M large enough so that the determinant is always nonzero.

But this means one has at least r_M independent columns and $r(H) \geq r_M$. As the opposite inequality is obvious we have the equality, and the theorem is proved.

4. Polynomial codimension growth and colengths

One of the known characterizations of the ordinary varieties of polynomial growth involves the sequence of colengths; these must be bounded by some constant [28]. A similar result also holds for varieties of graded algebras. We start with recalling a definition.

Definition 4.1. For $n \geq 1$ we define the n-th graded colength of A as:

$$l_n^G(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_s) \\ n_1+\cdots+n_s=n}} m_{\langle\lambda\rangle},$$

where $m_{\langle\lambda\rangle} \geq 0$ is defined as in (1).

For every $n = n_1 + \cdots + n_s \geq 1$ the $(n_1, \dots, n_s)$-cocharacter of the algebras UT_2^g, E, E^a were computed (see [4,13,28,29]) and it turned out that the corresponding sequences of colengths are not bounded by any constant.

In the next theorem we give the decomposition of the $(n_1, \dots, n_s)$-th cocharacter and of the n-th graded colength of FC_p^h. (Recall that these algebras were defined just after Theorem 1.2.)

We observe that FC_p^h has the structure of C_p-graded algebra and moreover $\mathrm{var}^G(FC_p^h) = \mathrm{var}^{C_p}(FC_p^h)$. In order to simplify the notation, we assume it in the following theorem.

A sequence of non-negative integers $\mu = (\mu_1, \dots, \mu_p)$ is called a composition of n into p parts if $\sum_{i=1}^{p} \mu_i = n$.

Theorem 4.1. *For all $n = n_1 + \cdots + n_p$ one has*

$$\chi_{n_1,\dots,n_p}(FC_p^h) = \chi_{(n_1)} \otimes \cdots \otimes \chi_{(n_p)} \quad \textit{and} \quad l_n^G(FC_p^h) = c(n),$$

where $c(n)$ denotes the number of all compositions of n into p parts.

Here G stands for the group C_p.

Proof. It follows from [33] that $\mathrm{Id}^G(FC_p^h) = \langle [x_1, x_2] \rangle_{T_G}$ where $x_1, x_2 \in X_{h^i}$, $i = 1, \dots, p$, and for every $n_1, \dots, n_p$ the vector space $P_{n_1,\dots,n_p}$ is generated modulo $P_{n_1,\dots,n_p} \cap \mathrm{Id}^G(FC_p^h)$ by the monomial

$$x_{1,1_G} \cdots x_{n_1,1_G} x_{1,h} \cdots x_{n_2,h} \cdots x_{1,h^{p-1}} \cdots x_{n_p,h^{p-1}}. \tag{4}$$

Since such a monomial does not vanish on FC_p^h we get that

$$\dim P_{n_1,\dots,n_p}/(P_{n_1,\dots,n_p}\cap \mathrm{Id}^G(FC_p^h)) = 1, \quad \chi^G_{n_1,\dots,n_p}(A) = \chi_{(n_1)}\otimes\cdots\otimes\chi_{(n_p)}.$$

Therefore $l_n^G(A) = c(n)$ where $c(n)$ denotes the number of all compositions of n into p parts. □

As a consequence we also have that the sequence of the graded colengths of FC_p^h is not bounded by any constant.

Now we are in a position to prove the following theorem.

Theorem 4.2. *Let A be a G-graded algebra. Then $c_n^G(A)$, $n = 1, 2, \dots,$ is polynomially bounded if and only if there exists a constant k such that $l_n^G(A) \le k$ for all $n \ge 1$.*

Proof. Assume first that $c_n^G(A)$, $n = 1, 2, \dots,$ is polynomially bounded. By Theorem 2.2 we obtain that

$$\chi^G_{n_1,\dots,n_s}(A) = \sum_{\substack{\langle\lambda\rangle\vdash(n_1,\dots,n_s)\\ n-\lambda(1)_1<q}} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)}$$

for some constant q. Moreover, by Theorem 2.1, there exists a constant M such that $m_{\langle\lambda\rangle} \le M$ for all $\langle\lambda\rangle \vdash (n_1,\dots,n_s)$ and for all $n_1+\cdots+n_s = n$. On the other hand there are finitely many multipartitions $\langle\lambda\rangle$ satisfying the condition $n-\lambda(1)_1 < q$. Hence it follows that

$$l_n^G(A) = \sum_{\substack{\langle\lambda\rangle\vdash(n_1,\dots,n_s)\\ n_1+\cdots+n_s=n}} m_{\langle\lambda\rangle} \le k$$

for some constant k.

Conversely, assume that $l_n^G(A) \le k$ is bounded by a constant k. In this case UT_2^g, E, E^a, $FC_p^h \notin \mathrm{var}^G(A)$ for all $g \in G$, $a \in G$ of order 2, and $h \in G$ of order a prime p (see [4,13,28,29] and Theorem 4.1). By Theorem 1.3 this implies that $\mathrm{var}^G(A)$ is of polynomial growth. □

Remark 4.1. Actually, if $c_n^G(A)$, $n = 1, 2, \dots,$ is polynomially bounded, by Theorem 3.1 the multiplicities are eventually constant and so are the colengths.

The following theorem collects results about graded varieties of polynomial growth.

Theorem 4.3. *For a G-graded algebra A the following conditions are equivalent:*

1) *$c_n^G(A)$ is polynomially bounded;*
2) *$\exp^G(A) \le 1$;*

3) *$A \sim_{T_G} B$ where $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \ldots, B_m$ finite dimensional G-graded algebras over F, and $\dim B_i/J(B_i) \leq 1$ for all $i = 1, \ldots, m$;*
4) *UT_2^g, E, E^a, $FC_p^h \notin var^G(A)$ for all $g \in G$, $a \in G$ of order 2 and $h \in G$ of order a prime p;*
5) *there exists a constant q such that for every $n_1, \ldots, n_s$ with $n_1 + \cdots + n_s = n$ it holds*

$$\chi^G_{n_1,\ldots,n_s}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\ldots,n_s) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}.$$

If A is a finite dimensional algebra then q is such that $J(A)^q = 0$;

6) *E, E^a, $FC_p^h \notin var^G(A)$ and there exists a constant M such that $m_{\langle\lambda\rangle} \leq M$ for all $\langle\lambda\rangle \vdash (n_1, \ldots, n_s)$ and for all $n_1 + \cdots + n_s = n$;*
7) *there exists a constant k such that $l_n^G(A) \leq k$.*

Let G be a group. Then the group $G \wr S_n$ acts on the space P_n^G (see [15]) and, since T_G-ideals are invariant under this action, $P_n^G/(P_n^G \cap Id^G(A))$ becomes a $G \wr S_n$-module whose character, denoted by $\chi_n^G(A)$ is called the n-th graded cocharacter of A. Now assume that G is an abelian group.

By complete reducibility we write

$$\chi_n^G(A) = \sum_{\langle\lambda\rangle \vdash n} m'_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle} \tag{5}$$

where $\chi_{\langle\lambda\rangle}$ is the irreducible $G \wr S_n$-character associated to the multipartition $\langle\lambda\rangle$, and $m'_{\langle\lambda\rangle}$ is the corresponding multiplicity. If $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash (n_1, \ldots, n_s)$ is a multipartition of n then in (1) and (5) one has that $m_{\langle\lambda\rangle} = m'_{\langle\lambda\rangle}$ [8].

Thus the statement (5) in the previous theorem can be rewritten in the following way:

5) There exists a constant q such that

$$\chi_n^G(A) = \sum_{\substack{\langle\lambda\rangle = (\lambda(1),\ldots,\lambda(s)) \vdash n \\ |\lambda(1)|+\cdots+|\lambda(s)| - \lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle}.$$

If A is a finite dimensional algebra then q is such that $J(A)^q = 0$.

5. Classifying varieties of slow growth

In this section we present a classification, up to T_G-equivalence, of the G-graded algebras that generate varieties of at most linear growth.

First we show that the growth of the codimensions is strictly related to the number of some boxes of the multipartitions corresponding to irreducible characters which appear with non-zero multiplicities.

Theorem 5.1. *Let A be a G-graded algebra. Then $c_n^G(A) \le an^p$ for some constants a and p if and only if for every $n_1 + \cdots + n_s = n$ it holds*

$$\chi^G_{n_1,\dots,n_s}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_s)\\ n-\lambda(1)_1 \le p}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}.$$

The summation runs over all multipartitions $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ such that $n - \lambda(1)_1 \le p$, $n = n_1 + \cdots + n_s$.

Proof. Suppose first that $c_n^G(A) \le an^p$. According to Theorem 2.2, since $c_n^G(A)$ is polynomially bounded, for all $n_1 + \cdots + n_s = n$ one has

$$\chi^G_{n_1,\dots,n_s}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_s)\\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)},$$

for some q.

Moreover, since $c_n^G(A) \le an^p$, by (2) it follows that

$$\binom{n}{n_1, \dots, n_s} c_{n_1,\dots,n_s} \le an^p.$$

Hence if $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ is a multipartition we must have

$$\binom{n}{n_1, \dots, n_s} m_{\langle\lambda\rangle} \deg \chi_{\lambda(1)} \cdots \deg \chi_{\lambda(s)} \le an^p.$$

But the multiplicities $m_{\langle\lambda\rangle}$ are bounded by a constant, and $\deg \chi_{\lambda(i)}$ is a constant for $i = 2, \dots, s$. This implies $\binom{n}{n_1,\dots,n_s} \deg \chi_{\lambda(1)} \le a' n^p$ for some constant a'. By Propositions 2.1 and 2.2, we have that

$$bn^{t+r} \le \binom{n}{n_1, \dots, n_s} \deg \chi_{\lambda(1)} \le cn^{t+r}$$

where r is the number of boxes under the first row of $\lambda(1)$ and $t = n_2 + \cdots + n_s$. Hence we must have $t + r = n - \lambda(1)_1 \le p$ and this implies

$$\chi^G_{n_1,\dots,n_s}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_s)\\ n-\lambda(1)_1 \le p}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}.$$

The converse is deduced by proceeding backward through the proof. $\square$

As a consequence we obtain the following corollaries.

Corollary 5.1. *Let A be a G-graded algebra. Then $c_n^G(A) \approx an^p$ if and only if there exists n_0 such that for every $n \geq n_0$*

a) $\chi^G_{n_1,\dots,n_s}(A) = \sum_{\substack{\langle\lambda\rangle\vdash(n_1,\dots,n_s)\\ n-\lambda(1)_1\leq p}} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)}$ *for every* $n_1+\cdots+n_s=n$;

b) *there exist* $n'_1+\cdots+n'_s=n$ *and a multipartition* $\langle\mu\rangle=(\mu(1),\dots,\mu(s))\vdash(n'_1,\dots,n'_s)$ *such that* $n-\mu(1)_1=p$, *and the corresponding multiplicity* $m_{\langle\mu\rangle}\neq 0$.

Corollary 5.2. *Let A be a G-graded algebra. Then $c_n^G(A) \leq an$ for some constant a if and only if for every $n_1+\cdots+n_s=n$ it holds*

$$\chi^G_{n_1,\dots,n_s}(A) = \sum_{\langle\lambda\rangle\vdash(n_1,\dots,n_s)} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)},$$

where $\langle\lambda\rangle=(\lambda(1),\dots,\lambda(s))$ is such that either

$$\lambda(1)=(n-1,1) \text{ and } \lambda(i)=\emptyset,\ i=2,\dots,s$$

or

$$\lambda(1)=(n) \text{ and } \lambda(i)=\emptyset,\ i=2,\dots,s$$

or

$$\lambda(1)=(n-1) \text{ and } \lambda(i)=(1) \text{ for some } i\in\{2,\dots,s\} \text{and } \lambda(j)=\emptyset, \text{for all} j\neq i.$$

Proof. Both corollaries follow directly from Theorem 5.1. □

As a consequence of the previous corollary we get the following.

Corollary 5.3. *Any G-graded algebra A such that $c_n^G(A) \leq an$, for some constant a, satisfies the polynomial identities $x_{1,g}x_{2,h}\equiv 0$ for all $g,\ h\in G\setminus\{1_G\}$.*

In what follows we denote by $y_1,\ y_2,\ \dots,$ graded variables from X_{1_G} and by $z_1^g,\ z_2^g,\ \dots,$ graded variables from X_g, and by $t_1^{\hat g},\ t_2^{\hat g},\ \dots,$ graded variables from $X\setminus\{X_{1_G}\cup X_g\}$.

Recall that if $A=F+J$ is a finite dimensional G-graded algebra over F where $J=J(A)$ is its Jacobson radical, then J is a graded ideal which can be decomposed into the direct sum of graded F-bimodules

$$J=J_{00}\oplus J_{01}\oplus J_{10}\oplus J_{11}.$$

Here for $i\in\{0,1\}$, J_{ik} is a left faithful module or a 0-left module according as $i=1$ or $i=0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, one has $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$ where δ_{kl} is the Kronecker delta.

Lemma 5.1. *Let $A = F + J$ be a G-graded algebra satisfying the polynomial identities $x_{1,g}x_{2,h} \equiv 0$ for all $g, h \in G \setminus \{1_G\}$. Then $A \sim_{T_G} \bigoplus_{g\in G} B_g$ where $B_g = F \oplus J_g$.*

Proof. Notice that for each $g \in G$ the vector space B_g is a graded subalgebra of A. Hence $\mathrm{Id}^G(A) \subseteq \mathrm{Id}^G(\bigoplus_{g\in G} B_g)$. Conversely, let $f \in \mathrm{Id}^G(\bigoplus_{g\in G} B_g)$ be a multilinear polynomial of degree n. By multihomogeneity of T_G-ideals we may assume that, modulo $\langle x_{1,g}x_{2,h} \mid g, h \neq 1_G\rangle_{T_G}$, either

$$f = \sum_{\sigma\in S_n} \alpha_\sigma y_{\sigma(1)} \cdots y_{\sigma(n)}$$

or

$$f = \sum_{\substack{i=1,\dots,n \\ \sigma\in S_n}} \alpha_{i,\sigma} y_{\sigma(1)} \cdots z^g_{\sigma(i)} \cdots y_{\sigma(n)},\ g \in G. \tag{6}$$

If f is of the second type, in order to get a non-zero value, we should evaluate f on B_g. Since by hypothesis $f \in \mathrm{Id}^G(B_g)$, we get that $f \equiv 0$ on A. In a similar manner if f is of the first type, we get that f vanishes on A. Hence $\mathrm{Id}^G(\bigoplus_{g\in G} B_g) \subseteq \mathrm{Id}^G(A)$ and the proof of the lemma is complete. □

Lemma 5.2. *Let $g \in G$ and let $A = F + J_g$ be a G-graded algebra such that $c^G_n(A) \leq an$ for some constant a. Then $A \sim_{T_G} B \oplus N$ where $B \in \mathrm{var}(UT^g_2)$ and N is a nilpotent G-graded algebra.*

Proof. If $g \neq 1_G$ the result is obvious since $[y_1, y_2] \equiv 0$, $z^g_1 z^g_2 \equiv 0$ and $t^{\hat{g}}_1 \equiv 0$ are identities for A. Therefore $A \in \mathrm{var}(UT^g_2)$ [32].

Suppose now $g = 1_G$. Then A is endowed with the trivial grading and we write $A = F + J_{10} + J_{01} + J_{11} + J_{00}$. We must have $J_{10}J_{01} = J_{01}J_{10} = J_{10}J_{00} = J_{00}J_{01} = 0$.

Suppose that $J_{10}J_{01} \neq 0$ and let $a \in J_{10}$ and $b \in J_{01}$ such that $ab \neq 0$. Let

$$f_{(n-2,1,1)} = \sum_{\sigma\in S_3} (sgn\sigma) y_{\sigma(1)} y_1^{n-3} y_{\sigma(2)} y_{\sigma(3)}$$

be a highest weight vector corresponding to $\lambda = (n - 2, 1, 1)$ (see [7, Chapter 12, Theorem 12.4.12]). By making the evaluation $y_1 = 1_F$, $y_2 = a$, and $y_3 = b$ we get $ab + ba \neq 0$ since $ab \in J_{11}$ and $ba \in J_{00}$. Thus the polynomial $f_{(n-2,1,1)}$ is not an identity of A. Therefore $\chi_{(n-2,1,1)}$ appears with non-zero multiplicity in the decomposition of $P_n/(P_n \cap \mathrm{Id}(A)$ into irreducible characters which is a contradiction to Corollary 5.2.

If $J_{01}J_{10} \neq 0$ or $J_{10}J_{00} \neq 0$ then, as above, we reach a contradiction since the same polynomial $f_{(n-2,1,1)}$ is not an identity for A.

Finally suppose $J_{00}J_{01} \neq 0$. Then $f_{(n-2,1,1)} = \sum_{\sigma \in S_3}(sgn\sigma)y_{\sigma(1)}y_{\sigma(2)}y_1^{n-3}y_{\sigma(3)}$ is a highest weight vector corresponding to $\lambda = (n-2,1,1)$ which is not an identity, a contradiction. Then clearly one has

$$A = (F + J_{10} + J_{01} + J_{11}) \oplus J_{00} \sim_{T_G} A_1 \oplus A_2 \oplus N$$

where $A_1 = (F + J_{11} + J_{10})$, $A_2 = (F + J_{11} + J_{01})$, and N is a nilpotent algebra. Now we claim that $[J_{11}, J_{11}] = 0$. If not let a, $b \in J_{11}$ be such that $ab \neq ba$ and let $f_{(n-2,1,1)} = \sum_{\sigma \in S_3}(sgn\sigma)y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}y_1^{n-3}$ be a highest weight vector corresponding to $\lambda = (n-2,1,1)$. By evaluating $y_1 = 1_F$, $y_2 = a$, and $y_3 = b$ we get $f_{(n-2,1,1)} = ab-ba \neq 0$ which is a contradiction.

Now it is immediate to see that $[y_1, y_2][y_3, y_4] \equiv 0$ is an identity of A_1 and A_2 and the proof is complete [32]. $\square$

Lemma 5.3. *Let $A = F \oplus J$ be a G-graded algebra such that $c_n^G(A) \leq an$, for some constant a. Then $A \sim_{T_G} M_{g_1} \oplus \cdots \oplus M_{g_s} \oplus N$ where $M_{g_i} \in \mathrm{var}(UT_2^{g_i})$, $i = 1, \ldots, s$ and N is a nilpotent G-graded algebra.*

Proof. Since $c_n^G(A) \leq an$, by Corollary 5.3 A satisfies the polynomial identities $x_{1,g}x_{2,h} \equiv 0$ for all g, $h \in G \setminus \{1_G\}$. Hence by Lemma 5.1, $A \sim_{T_G} \bigoplus_{g \in G} B_g$ where $B_g = F \oplus J_g$. By applying Lemma 5.2 we get the desired conclusion. $\square$

Theorem 5.2. *Let A be a G-graded algebra such that $c_n^G(A) \leq an$ for some constant a. Then*

$$A \sim_{T_G} M_{g_1} \oplus \cdots \oplus M_{g_s} \oplus N$$

where $M_{g_i} \in \mathrm{var}(UT_2^{g_i})$, $i = 1, \ldots, s$, and N is a nilpotent G-graded algebra.

Proof. By Theorem 1.2 we may assume that

$$A = A_1 \oplus \cdots \oplus A_m$$

where $A_1, \ldots, A_m$ are finite dimensional G-graded algebras with $\dim A_i/J(A_i) \leq 1$, $1 \leq i \leq m$. Notice that this says that either $A_i \cong F + J(A_i)$ or $A_i = J(A_i)$ is a nilpotent algebra. Since $c_n^G(A_i) \leq c_n^G(A)$ then $c_n^G(A_i) \leq an$ for all $i = 1, \ldots, s$. Now the result follows by applying Lemma 5.3 to each non-nilpotent A_i. $\square$

Here we recall that the algebras inside the graded varieties $\mathrm{var}(UT_2^g)$, for all $g \in G$, have been classified up to T_G-equivalence (see [23,24,26,27,25]).

We shall give the list of these algebras since we will need them in order to get a complete classification of the algebras, up to T_G-equivalence, of linear codimension growth.

Let $\mathbf{g} = (g_1, \dots, g_k) \in G^k$ be a k-tuple of elements of G. One defines a G-grading on the algebra of the $k \times k$ upper triangular matrices, UT_k, by setting $(UT_k)_g = \operatorname{span}\{e_{ij} \mid g_i^{-1}g_j = g\}$ for all $g \in G$. Such a grading is called the elementary G-grading defined by the k-tuple $\mathbf{g}$. If A is a graded subalgebra of UT_k the induced grading on A is also called elementary.

Let $k \geq 2$ and denote

$$N_k = \operatorname{span}\{U, U_1, U_1^2, \dots, U_1^{k-2}; e_{12}, e_{13}, \dots, e_{1k}\} \subseteq UT_k$$

where U denotes the $k \times k$ identity matrix, $U_1 = \sum_{i=1}^{k-1} e_{i,i+1}$ and the e_{ij}'s are the usual matrix units.

Also let

$$A_k = A_k(F) = \operatorname{span}\{e_{11}, U_1, U_1^2, \dots, U_1^{k-2}; e_{12}, e_{13}, \dots, e_{1k}\} \subseteq UT_k$$

and let A_k^* be the graded subalgebra of UT_k obtained by flipping A_k along its secondary diagonal.

Fixed $g \in G$ let N_k^g and A_k^g denote the algebra N_k and A_k, respectively, with the elementary grading induced by $\mathbf{g} = (1_G, g, \dots, g) \in G^k$.

Therefore $(A_k^g)^*$ is a G-graded algebra with $((A_k^g)^*)_h = ((A_k^g)_h)^*$ for all $h \in G$. The following result characterizes the graded identities and the graded codimensions of N_k (see [12] and [26]).

Theorem 5.3. *The T_G-ideal $\operatorname{Id}^G(N_k^g)$ is generated by the polynomials*

$$[y_1, y_2], \quad [z_1^g, y_2, \dots, y_k], \quad z_1^g z_2^g, \quad t_1^{\hat{g}}$$

in case $g \neq 1_G$, and by

$$[y_1, y_2, \dots, y_k], \quad [y_1, y_2][y_3, y_4], \quad t_1^{\hat{1}_G}$$

in case $g = 1_G$.

Moreover $c_n^G(N_k) = 1 + \sum_{j=2}^{k-1}(j-1)\binom{n}{j} \approx \frac{k-2}{(k-1)!}\, n^{k-1}$ and, whenever $g \neq 1_G$, $c_n^G(N_k^g) = 1 + \sum_{j=1}^{k-1}\binom{n}{j} j \approx \frac{1}{(k-2)!}\, n^{k-1}$.

Theorem 5.4. *The T_G-ideal $\operatorname{Id}^G(A_k^g)$ is generated by the polynomials*

$$[y_1, y_2], \quad z_1^g y_2 \cdots y_k, \quad z_1^g z_2^g, \quad t_1^{\hat{g}}$$

in case $g \neq 1_G$, and by

$$[y_1, y_2][y_3, y_4], \quad [y_1, y_2] y_3 \dots y_{k+1}, \quad t_1^{\hat{1}_G}$$

in case $g = 1_G$.

Moreover $c_n^G(A_k) = \sum_{l=0}^{k-2} \binom{n}{l}(n-l-1)+1 \approx qn^{k-1}$, *and for every* $g \neq 1_G$, $c_n^G(A_k^g) = \sum_{l=0}^{k-2} \binom{n}{l}(n-l)+1 \approx q'n^{k-1}$. *Here* q, $q' \in \mathbb{Q}$ *are non-zero constants.*

In order to describe the generators of the T_G-ideal $\mathrm{Id}^G((A_k^g)^*)$ it is enough to reverse the order of the variables in each monomial of the generators of the T_G-ideal $\mathrm{Id}^G(A_k^g)$. Clearly for each $n \geq 1$ one has $c_n^G((A_k^g)^*) = c_n^G(A_k^g)$.

The following lemma gives the classification of the algebras with at most linear codimension growth.

Lemma 5.4. *(See [25].) Let A be a G-graded algebra such that $A \in \mathrm{var}(UT_2^g)$ for some $g \in G$, and $c_n^G(A) \leq an$ for some constant a. Then A is T_G-equivalent to one of the following algebras:*

$$C \oplus N, \quad A_2^g \oplus N, \quad (A_2^g)^* \oplus N, \quad A_2^g \oplus (A_2^g)^* \oplus N,$$
$$N, \quad N_2^g \oplus N, \quad A_2^g \oplus N_2^g \oplus N, \quad (A_2^g)^* \oplus N_2^g \oplus N, \quad A_2^g \oplus (A_2^g)^* \oplus N_2^g \oplus N$$

where C is a commutative algebra equipped with the trivial grading and N is a nilpotent G-graded algebra.

Notice that if $g = 1_G$ and A is an algebra satisfying the statement of the lemma then A satisfies the same graded identities as one of the algebras in the second row. This follows from the fact that $N_2^{1_G} \sim_{T_G} F$.

The previous lemma allows us to get a finer classification of the algebras of linear codimension growth. This finer classification is given in the following theorem.

Theorem 5.5. *Let A be a G-graded algebra such that $c_n^G(A) \leq an$ for some constant a. Then*

$$A \sim_{T_G} M_{g_1} \oplus \cdots \oplus M_{g_s} \oplus N$$

where N is a nilpotent G-graded algebra and M_{g_i} is T_G-equivalent to one of the following algebras:

$$C \oplus N_i, \quad A_2^{g_i} \oplus N_i, \quad (A_2^{g_i})^* \oplus N_i, \quad A_2^{g_i} \oplus (A_2^{g_i})^* \oplus N_i,$$
$$N_i, \quad N_2^{g_i} \oplus N_i, \quad A_2^{g_i} \oplus N_2^{g_i} \oplus N_i, \quad (A_2^{g_i})^* \oplus N_2^{g_i} \oplus N_i, \quad A_2^{g_i} \oplus (A_2^{g_i})^* \oplus N_2^{g_i} \oplus N_i$$

where C is a commutative algebra with the trivial grading and N_i is a nilpotent G-graded algebra.

Notice that we could have obtained this last result directly from Theorem 1.2, considering that every graded algebra $B_g = F + J_g$ in Lemma 5.1 has a structure of a superalgebra whose graded codimension sequence is linearly bounded and thus it must be T_G-equivalent to one of the algebras in Lemma 5.4 (see [9] and [11]).

Recall that a variety $\mathcal{V}$ is called *minimal of polynomial growth* if $c_n^G(\mathcal{V}) \approx qn^k$ for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneqq \mathcal{V}$ we have that $c_n^G(\mathcal{U}) \approx q'n^t$ with $t < k$.

As a consequence of the previous theorem and Theorem 4.5 in [25] we obtain the following corollary.

Corollary 5.4. *A G-graded algebra A generates a minimal variety of linear growth if and only if either $A \sim_{T_G} N_2^h$ or $A \sim_{T_G} A_2^g$ or else $A \sim_{T_G} (A_2^g)^*$, for some $g, h \in G$ with $h \neq 1_G$.*

Hence all the minimal varieties of linear growth are inside $\mathrm{var}^G(UT_2^g)$, $g \in G$.

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