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Finite groups with biprimary Hall subgroups

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ABSTRACT

Let G be a finite group and let r be a prime divisor of the order of G. We prove that if $r \geq 5$ and G has the $E_{\{r,t\}}$-property for all $t \in \pi(G) \backslash \{r\}$, then G is r-solvable. A group G is said to have the E_π-property if G possesses a Hall π-subgroup.

1. Introduction

Let G be a finite group and let $\pi(G)$ be the set of prime divisors of the order of G. Suppose that $\pi \subseteq \pi(G)$. A group G is said to have the E_π-property (G is a E_π-group) whenever G includes a Hall π-subgroup.

The structure of a finite group essentially depends on the existence of Hall subgroups in it. In 1956, P. Hall in [1] advanced the hypothesis on solvability of a finite group that contains biprimary Hall $\{p, q\}$-subgroups for all prime divisors p and q of its order. The

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validity of this hypothesis was proven in 1982 by Z. Arad and M. Ward [2]. In [3], the author has obtained that a finite group G is solvable if it possesses the $E_{\{2,t\}}$-property for all $t \in \pi(G)\backslash\{2\}$. In [4], composition factors of a finite group G with the $E_{\{3,t\}}$-property, where $t \in \pi(G)\backslash\{3\}$ were found. It is natural to establish the structure of a finite group G with the $E_{\{p,t\}}$-property for some fixed $p \in \pi(G)$ and all $t \in \pi(G)\backslash\{p\}$.

In the present paper we complete this direction of research. The following result is proven.

Theorem 1. *Let G be a finite group and let $r \in \pi(G)\backslash\{2,3\}$. If G has the $E_{\{r,t\}}$-property for all $t \in \pi(G)\backslash\{r\}$, then G is r-solvable.*

Using Theorem 1 and taking papers [3] and [4] into account, we obtain the following theorem.

Theorem 2. *Let G be a finite group with $r \in \pi(G)$. Suppose that G has the $E_{\{r,t\}}$-property for all $t \in \pi(G)\backslash\{r\}$. Then the following hold:*

(1) *if $r \neq 3$, then G is r-solvable;*
(2) *if $r = 3$, then all simple nonabelian composition factors of G belong to the list: $PSL_2(7)$; $PSU_3(q)$ for a suitable value of the parameter q; $Sz(2^{2n+1})$.*

From Theorem 1 and the paper [3], we immediately obtain the following criterion for a finite group to be p-nilpotent.

Corollary. *Let G be a finite group, $r \in \pi(G)$, $r \neq 3$, and $R \in Syl_r(G)$. If R normalizes a Sylow t-subgroup for each $t \in \pi(G)\backslash\{r\}$, then G is r-nilpotent.*

Remark. The example of the group $G \cong PSU_3(4)$ shows that there are simple nonabelian groups having 3-nilpotent biprimary Hall $\{3,r\}$-subgroups for all $r \in \pi(G)\backslash\{3\}$.

2. Auxiliary results

We give notation and auxiliary results that will be used to prove the theorem. We use $[A]B$ to denote the semidirect product of A and B, where A is a normal subgroup of $[A]B$, while we use $A \times B$ to denote the direct product of A and B. We write $A < \cdot G$ to denote a maximal subgroup A of a group G. The notation A^l denotes the direct product of l groups isomorphic to A. If a and b are positive integers, then (a,b) is the greatest common divisor of a and b.

Lemma 1. *(See [5], Lemma 3.) Let a, m, n be positive integers. Then $(\frac{a^n-1}{a^{(m,n)}-1}, a^m-1) = (\frac{n}{(m,n)}, a^{(m,n)}-1)$. In particular, $(\frac{a^n-1}{a-1}, a-1) = (n, a-1)$.*

Definition. Let r and p be prime numbers, and let n be a positive integer not smaller than 2. The number r is said to be primitive in relation to the pair $\{p, n\}$ if r divides $p^n - 1$, but does not divide $p^i - 1$ for all $1 \leq i < n$.

Lemma 2. *(See [6].) Let p be a prime number, and let n be a positive integer not smaller than 2. Suppose that $\{p, n\} \neq \{2, 6\}$ or $\{p, n\} \neq \{2, 2^m - 1\}$. Then there exists a prime number r primitive in relation to the pair $\{p, n\}$.*

Lemma 3. *If r is a prime number primitive in relation to the pair $\{p, n\}$, then $r \geq n + 1$.*

Proof. Follows from Fermat's little theorem.

We need the following results which can be found, for example, in [7]. Let Σ be an irreducible root system, and let $a \in \Sigma$. Denote $a^* = 2a/(a, a)$. Then $\Sigma^* = \{a^* | a \in \Sigma\}$ is a root system dual of Σ.

Definition. (See [7], Definition 4.1.) A prime p is called a torsion prime if $L(\Sigma^*)/L({\Sigma_1}^*)$ has a p-torsion for some closed subsystem $\Sigma_1 \subset \Sigma$.

Lemma 4. *(See [7], 4.3, 4.4, p. 173.) For an arbitrary root system, the torsion primes are:*

(1) *for A_l, C_l — no;*
(2) *for B_l, D_l, G_2 — 2;*
(3) *for E_6, E_7, F_4 — 2, 3;*
(4) *for E_8 — 2, 3, 5.*

Lemma 5. *(See [8], Lemma 1.7.) Let $\overline{G}$ be a connected semisimple algebraic group over the algebraic closure of the field F_q, where $q = p^n$, and let σ be a Frobenius endomorphism of the group $\overline{G}$. Then a direct product $\overline{E} = \overline{Y_1} \times \overline{Y_2} \times \cdots \times \overline{Y_m}$ of cyclic semisimple groups $\overline{Y_i}$ of $\overline{G}_\sigma$ can be embedded into a maximal torus $\overline{T}_\sigma$ of $\overline{G}_\sigma$, if the number of such $|\overline{Y_i}|$ not coprime to all torsion primes of G is at most two. In particular, $N_{\overline{G}_\sigma}(\overline{E})/C_{\overline{G}_\sigma}(\overline{E}) \leq W(\overline{G})$, where $W(\overline{G})$ is a Weyl group of $\overline{G}$.*

Lemma 6. *(See [9], Lemma 3.) Let G be a simple Chevalley group and let $G \notin \{PSL_6(2), PSp_6(2), P\Omega_8^+(2), PSU_4(2)\}$. Then there exists a prime divisor of $|G|$ that does not divide the order of any proper parabolic subgroup of G.*

We consider the following situation. Let $G \in \{PSU(l, q^2), PSU(l + 1, q^2)\}$, where $q = p^n$, $n \geq 3$, and l is an odd number; let r be a prime number primitive in relation to the pair $\{p, 2nl\} \neq \{2, 6\}$. In this notation, the following lemma holds.

Lemma 7. *(See [10], Lemma 18 (a), (b).) Let M be a maximal solvable subgroup of G. Suppose that the order of M is divisible by r. Then the following assertions hold:*

(a) $|M|$ *divides* $l(q^l + 1)(q + 1)^{-1}$ *for* $G \cong PSU(l, q^2)$*;*
(b) $|M|$ *divides* $l(q^l + 1)$ *for* $G \cong PSU(l + 1, q^2)$*.*

3. Proof of Theorem 1

To prove the theorem we use the technique of Chigira's [11] and L.S. Kazarin's [10] works.

Show that there are no simple nonabelian groups satisfying the conditions of the theorem. We consider separately the cases when G is alternating, sporadic and a group of Lie type.

1. $G \cong A_n$ $(n \geq 5)$ is an alternating group.

Since $\pi(A_n) \geq 3$, by hypothesis, A_n has a biprimary Hall subgroup F of odd index. Since $|S_n : A_n| = 2$ for $n \neq 6$, while $|S_6 : A_6| = 4$, it follows that F is a Hall subgroup of S_n. This contradicts Theorem A4 of [1].

2. G is either a sporadic group or the Tits group ${}^2F_4(2)'$.

From Corollary 6.13 in [12] and Theorem 4.1 in [13], we obtain the list of groups containing proper Hall subgroups. All these groups do not satisfy the conditions of Theorem 1.

3. G is a simple group of Lie type over the field $GF(q)$, where $q = p^n$.

Suppose that $r = p$. It follows that for each $t \in \pi(G)\backslash\{r\}$ there is a Hall $\{r, t\}$-subgroup $L = UT$ of G, where $U \in Syl_p(G)$, $T \in Syl_t(G)$. By Theorem 3.3 in [14], L is either parabolic or is contained in a Borel subgroup $B = UH(U \in Syl_p(G)$, H is a Cartan subgroup of G). Suppose first that $L > B$. Since $r = p \geq 5$, it follows that $G \ncong Sz(2^{2l+1})$. So by Corollary 3.4 in [14], $\pi(q + 1) \cup \pi(q - 1) = \{t\}$. Since $r = p \geq 5$, we have $q - 1 \geq 4$ and $(q + 1, q - 1) = 2$. Hence $t = 2$, and $q - 1 = 2^\beta$, $q + 1 = 2^\alpha$, where $\beta \geq 2$. It follows that $2^\beta(2^{\alpha-\beta} - 1) = 2$, this is impossible. Therefore, L is always contained in a Borel subgroup.

It follows that $L \leq M < \cdot G$, where M is any maximal parabolic subgroup of G up to conjugation. Therefore, t divides the order of each own maximal parabolic subgroup of G. Since t is an arbitrary prime divisor of $|G|$, by Lemma 6, we have $G \in \{PSL_6(2), PSp_6(2), P\Omega_8{}^+(2), PSU_4(2)\}$. These groups are defined over the field of characteristic 2. This is a contradiction to the fact that $r \geq 5$.

So $r \neq p$. Consider a biprimary Hall $\{r, p\}$-subgroup $L = UR$, where $U \in Syl_p(G)$, $R \in Syl_r(G)$. By Theorem 3.3 in [14], L is either parabolic or is contained in a Borel subgroup B of G. Suppose that $B < L$. If $G \cong Sz(2^{2l+1})$, then $L \leq B$ because the Lie rank of $Sz(2^{2l+1})$ is 1. Thus $G \not\cong Sz(2^{2l+1})$, and Corollary 3.4 in [14] implies that $\pi(q+1) \cup \pi(q-1) = \{r\}$. If $q-1 > 2$, then $(q+1, q-1) = 2$, and $r = 2$. This contradicts the fact that $r \geq 5$. So $q-1 = 1$, that is $q = 2$. Thus $q+1 = 3 = r \geq 5$, this is impossible. Therefore, L is contained in a Borel subgroup.

Lemma 1.4.3 in [12] implies that $UR \leq B$, $R \leq H$, where B and H are Borel and Cartan subgroups respectively. In particular, R is an abelian group. Moreover, since $R \in Syl_r(G)$, we have $(r, |W(G)|) = 1$, where $W(G)$ is a Weyl group of G.

We need the following facts about algebraic groups. Let $\overline{G}$ be a simple connected algebraic group. Surjective endomorphism σ of $\overline{G}$ is called the Frobenius map, if its group of fixed points $\overline{G}_\sigma = C_{\overline{G}}(\sigma)$ is finite. There is a connected algebraic group $\overline{G}$ with Frobenius map σ such that $\overline{G}_\sigma / Z(\overline{G}_\sigma) \cong G$, where G is a finite simple group of Lie type except the case when $G \cong {}^2F_4(2)'$. Denote by $\overline{B}_\sigma$ a Borel subgroup of $\overline{G}_\sigma$. Then $\overline{B}_\sigma = \overline{U}_\sigma \overline{H}_\sigma$, where $\overline{U}_\sigma \in Syl_p(\overline{G}_\sigma)$, $\overline{H}_\sigma$ is a Cartan subgroup. Note that if G contains a biprimary Hall π-subgroup for $\pi \subset \pi(G)$, then $\overline{G}_\sigma$ also contains a biprimary Hall π-subgroup and vice versa, if $\overline{G}_\sigma$ contains a biprimary Hall π-subgroup for $\pi \subset \pi(\overline{G}_\sigma)$, then G contains a biprimary Hall π-subgroup. At the same time for $\overline{R} \in Syl_r(\overline{G}_\sigma)$ we see that $\overline{R} \subseteq \overline{H}_\sigma$, $\overline{R}$ is an abelian group, $(r, |W(\overline{G}_\sigma)|) = 1$.

Consider consecutively all cases.

(1) $G \cong A_l(q)$, $l \geq 1$, $q = p^n$; $|\overline{G}_\sigma| = q^{l(l+1)/2} \prod_{i=2}^{l+1}(q^i - 1)$.

In this case, $\overline{H}_\sigma \cong Z_{q-1}^l$ and $\overline{R} \leq \overline{H}_\sigma$. Wherein r divides $q-1$, and $(r, |W(\overline{G}_\sigma)|) = 1$. Since $r \geq 5$, we have $q \geq 11$. $\overline{G}_\sigma$ has a maximal tor $\overline{T}_\sigma$ of order $(q^{l+1}-1)/(q-1)$ by [15]. If $G \cong A_5(2)$, then $q-1 = 1$. This is a contradiction to the fact that r divides $q-1$. Let $G \cong A_1(q)$, where $q = p^n = 2^k - 1$. By Dixon's theorem (II.8.27 [16]), we obtain that G does not have the $E_{\{2,r\}}$-property. So by Lemma 2, there is a prime m such that m divides $q^{l+1} - 1$, but does not divide $q^i - 1$ for all $i \in \{1, \ldots, l\}$. By Lemma 3, we get that $m \geq l+2$. Let $\overline{M} \leq \overline{T}_\sigma$ for some abelian m-subgroup $\overline{M} \in Syl_m(\overline{G}_\sigma)$. Consider the Hall subgroup $\overline{L} = \overline{M}\ \overline{R}$ of $\overline{G}_\sigma$. Since $\overline{M}$ and $\overline{R}$ are abelian, we deduce that either $\overline{L} = [\overline{M}]\overline{R}$ or $\overline{L} = [\overline{R}]\overline{M}$. Let $\overline{L} = [\overline{M}]\overline{R}$. By Lemma 5, $N_{\overline{G}_\sigma}(\overline{M})/C_{\overline{G}_\sigma}(\overline{M}) \leq W(\overline{G})$. Since $|W(\overline{G})| = |W(\overline{G}_\sigma)|$ and $(r, |W(\overline{G}_\sigma)|) = 1$, we have $\overline{L} = \overline{M} \times \overline{R}$. If $\overline{L} = [\overline{R}]\overline{M}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. Since $|W(\overline{G}_\sigma)| = (l+1)!$, and $m \geq l+2$, then we again get that $\overline{L} = \overline{M} \times \overline{R}$.

The group $\overline{G}_\sigma$ has the only maximal torus $\overline{T}_\sigma$ such that $|\overline{T}_\sigma|$ is divisible by m, so $\overline{L} \leq \overline{T}_\sigma$. It follows that $r \in \pi(\frac{q^{l+1}-1}{q-1}, q-1)$. By Lemma 1, $(\frac{q^{l+1}-1}{q-1}, q-1) = (l+1, q-1)$, hence r divides $l+1$. But $|W(\overline{G}_\sigma)| = (l+1)!$. This contradicts the fact that $(r, |W(\overline{G}_\sigma)|) = 1$.

(2) $G \cong B_l(q)$, $l \geq 2$, $q = p^n$; $|\overline{G}_\sigma| = q^{l^2} \prod_{i=1}^{l}(q^{2i}-1)$.

The Cartan subgroup $\overline{H}_\sigma \cong Z_{q-1}^l$, and r divides $q-1$. In particular, $q \geq 11$. $\overline{G}_\sigma$ has a maximal torus $\overline{T}_\sigma$ of order q^l+1 by [15]. The case $2l = 2$, $q = 2^a - 1$ is not fulfilled since $l \geq 2$, the case $2l = 6$, $q = 2$ also is not fulfilled since $q \geq 11$. Thus there is a prime m such that m divides $q^{2l}-1$, but does not divide $q^i - 1$ for all $i \in \{1, 2, \dots, 2l-1\}$. By Lemma 3, we obtain that $m \geq 2l+1$. Since $q^{2l}-1 = (q^l-1)(q^l+1)$, we see that m divides q^l+1. Denote by $\overline{M}_\sigma \leq \overline{T}_\sigma$ some abelian m-subgroup $\overline{M} \in Syl_m(\overline{G}_\sigma)$. Consider the Hall subgroup $\overline{L} = \overline{M}\,\overline{R}$ of $\overline{G}_\sigma$. Since $\overline{M}$ and $\overline{R}$ are abelian, then either $\overline{L} = [\overline{M}]\overline{R}$ or $\overline{L} = [\overline{R}]\overline{M}$. Suppose first that $\overline{L} = [\overline{M}]\overline{R}$. For the root system B_l, the torsion prime is 2. Then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{M})/C_{\overline{G}_\sigma}(\overline{M}) \leq W(\overline{G})$. Since $|W(\overline{G})| = |W(\overline{G}_\sigma)|$, and $(r, |W(\overline{G}_\sigma)|) = 1$ we have $\overline{L} = \overline{M} \times \overline{R}$. If $\overline{L} = [\overline{R}]\overline{M}$, then by Lemma 5 $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. Since $|W(\overline{G}_\sigma)| = 2^{l-1}l!$ and $m \geq 2l+1$, we conclude that $(m, |W(\overline{G}_\sigma)|) = 1$, and $\overline{L} = \overline{R} \times \overline{M}$.

Since $\overline{G}_\sigma$ has the only maximal torus $\overline{T}_\sigma$ such that $|\overline{T}_\sigma|$ is divisible by m, we have $\overline{L} \leq \overline{T}_\sigma$. It follows that $r \in \pi(q^l+1, q-1) \subseteq \pi(q^l+1, q^l-1) \subseteq \{1, 2\}$. This contradicts the fact that $r \geq 5$.

(3) $G \cong C_l(q)$, $l \geq 2$, $q = p^n$.

This case is considered exactly the same as (2).

(4) $G \cong D_l(q)$, $l \geq 4$, $q = p^n$; $|\overline{G}_\sigma| = q^{l(l-1)}(q^l-1)\prod_{i=1}^{l-1}(q^{2i}-1)$.

The Cartan subgroup $\overline{H}_\sigma \cong Z_{q-1}^l$, and r divides $q-1$. In particular, $q \geq 11$. $\overline{G}_\sigma$ has a maximal torus $\overline{T}_\sigma$ of order $(q^{l-1}+1)(q+1)$ by [15]. Since $q \geq 11$ and $l \geq 4$, we see that the cases $2(l-1) = 6$, $q = 2$, and $2(l-1) = 2$, $q = 2^k - 1$ are not fulfilled. By Lemma 2, there is a prime m which divides $q^{2(l-1)}-1$, but does not divide $q^i - 1$ for all i such that $1 \leq i \leq 2l-3$. By Lemma 2, we get that $m \geq 2l-1$. Let $\overline{M} \leq \overline{T}_\sigma$ be an abelian Sylow m-subgroup of $\overline{G}_\sigma$. Since $m \geq 2l-1$, we have $m \neq 2$. By Lemma 4, the torsion prime for type D_l is 2. Consider the Hall subgroup $\overline{L} = \overline{M}\,\overline{R}$ of $\overline{G}_\sigma$. Since $\overline{M}$ and $\overline{R}$ are abelian, then either $\overline{L} = [\overline{M}]\overline{R}$ or $\overline{L} = [\overline{R}]\overline{M}$. Let $\overline{L} = [\overline{M}]\overline{R}$. By Lemma 5, $N_{\overline{G}_\sigma}(\overline{M})/C_{\overline{G}_\sigma}(\overline{M}) \leq W(\overline{G})$. Since $|W(\overline{G}_\sigma)| = |W(\overline{G})|$ and $(r, |W(\overline{G}_\sigma)|) = 1$, we have $\overline{L} = \overline{M} \times \overline{R}$. If $\overline{L} = [\overline{R}]\overline{M}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. Since $|W(\overline{G}_\sigma)| = 2^{l-1} \cdot l!$ and $m \geq 2l-1$, we have $\overline{L} = \overline{M} \times \overline{R}$.

The group $\overline{G}_\sigma$ has the only maximal torus $\overline{T}_\sigma$ such that $|\overline{T}_\sigma|$ is divisible by m, so $\overline{L} \leq \overline{T}_\sigma$. It follows that r divides $((q^{l-1}+1)(q+1), q-1)$. Since $r \geq 5$ and $(q+1, q-1) \in \{1, 2\}$, we conclude that r divides $(q^{l-1}+1, q-1)$, and thus r divides $(\frac{q^{2(l-1)}-1}{q-1}, q-1)$. By Lemma 1, r divides $l-1$. Since $(|W(\overline{G}_\sigma)|, r) = 1$ and $|W(\overline{G}_\sigma)| = 2^{l-1} \cdot l!$, we deduce that r does not divide $l-1$. We have a contradiction.

(5) $G \cong E_6(q)$, $q = p^n$; $|\overline{G}_\sigma| = q^{36}(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)$.

The Cartan subgroup $\overline{H}_\sigma \cong Z^6_{q-1}$, and r divides $q-1$. $\overline{G}_\sigma$ has a maximal torus $\overline{T}_\sigma$ of order $(q^4-q^2+1)(q^2+q+1)$ by [15]. By Lemma 2, there exists a prime m that divides $q^{12}-1$ but does not divide q^i-1 for all $i \in \{1, 2, \dots, 11\}$. By Lemma 3, $m \geq 13$. Have the equality: $q^{12}-1 = (q^6-1)(q^2+1)(q^4-q^2+1)$. Hence it is easy to deduce that m divides q^4-q^2+1. Consider a group $\overline{M} \in Syl_m(\overline{G}_\sigma)$ such that $\overline{M} \leq \overline{T}_\sigma$. Let $\overline{L} = \overline{M}\,\overline{R}$. Torsion primes for the root system of type E_6 are 2 and 3. If $\overline{L} = [\overline{M}]\overline{R}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{M})/C_{\overline{G}_\sigma}(\overline{M}) \leq W(\overline{G})$. Since $|W(\overline{G})| = |W(\overline{G}_\sigma)|$ and $(r, |W(\overline{G}_\sigma)|) = 1$, we have $\overline{L} = \overline{M} \times \overline{R}$. If $\overline{L} = [\overline{R}]\overline{M}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. Since $|W(\overline{G}_\sigma)| = 2^7 \cdot 3^4 \cdot 5$ and $m \geq 13$, we have $(m, |W(\overline{G}_\sigma)|) = 1$ and $\overline{L} = \overline{R} \times \overline{M}$.

From the uniqueness of the maximal torus $\overline{T}_\sigma$ whose order is divisible by m, we conclude that $\overline{L} \leq \overline{T}_\sigma$. It follows that r divides $((q^4-q^2+1)(q^2+q+1), q-1) = ((q^2(q-1)^2+1)((q-1)^2+3q), q-1) \in \{1, 3\}$. This contradicts the fact that $r \geq 5$.

(6) $G \cong E_7(q)$, $q = p^n$; $|\overline{G}_\sigma| = q^{63}(q^2-1)(q^6-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)$.

The Cartan subgroup $\overline{H}_\sigma \cong Z^7_{q-1}$, and r divides $q-1$. $\overline{G}_\sigma$ has a torus $\overline{T}_\sigma$ of order $(q^6-q^3+1)(q+1)$ by [15]. By Lemma 2, there is a prime m which divides $q^{18}-1$, but does not divide q^i-1 for all $i \in \{1, 2, \dots, 17\}$. By Lemma 3, $m \geq 19$. The following equality holds: $q^{18}-1 = (q^9-1)(q^3+1)(q^6-q^3+1)$. It is easy to see that m divides q^6-q^3+1. Let $\overline{M} \in Syl_m(\overline{G}_\sigma)$ and $\overline{M} \leq \overline{T}_\sigma$. Torsion primes for the root system of type E_7 are 2 and 3. Denote $\overline{L} = \overline{M}\,\overline{R}$. If $\overline{L} = [\overline{M}]\overline{R}$, then by Lemma 5, we have $N_{\overline{G}_\sigma}(\overline{M})/C_{\overline{G}_\sigma}(\overline{M}) \leq W(\overline{G})$. Since $|W(\overline{G})| = |W(\overline{G}_\sigma)|$, and $(r, |W(\overline{G}_\sigma)|) = 1$, we get $\overline{L} = \overline{M} \times \overline{R}$. If $\overline{L} = [\overline{R}]\overline{M}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. Since $|W(\overline{G}_\sigma)| = 2^4 \cdot 3^5 \cdot 5^2 \cdot 7$ and $m \geq 19$, then $(m, |W(\overline{G}_\sigma)|) = 1$ and $\overline{L} = \overline{R} \times \overline{M}$.

From the uniqueness of the maximal torus $\overline{T}_\sigma$ whose order is divisible by m, we conclude that $\overline{L} \leq \overline{T}_\sigma$. So r divides $((q^6-q^3+1)(q+1), q-1) \in \{1, 2\}$. This contradicts the fact that $r \geq 5$.

(7) $G \cong E_8(q)$, $q = p^n$; $|\overline{G}_\sigma| = q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)$.

The Cartan subgroup $\overline{H}_\sigma \cong Z^8_{q-1}$, and r divides $q-1$. $\overline{G}_\sigma$ has a torus $\overline{T}_\sigma$ of order $q^8+q^7-q^5-q^4-q^3+q+1$ by [15]. Lemma 2 implies that there is a prime m that divides $q^{30}-1$, but does not divide q^i-1 for all $i \in \{1, 2, \dots, 29\}$. By Lemma 3, $m \geq 31$. Have the equality: $q^{30}-1 = (q^{15}-1)(q^5+1)(q^2+q+1)(q^8+q^7-q^5-q^4-q^3+q+1)$ which implies that m divides $q^8+q^7-q^5-q^4-q^3+q+1$. Let $\overline{M} \in Syl_m(\overline{G}_\sigma)$ and $\overline{M} \leq \overline{T}_\sigma$. Torsion primes for the root system of type E_8 are 2, 3 and 5. Since $|W(\overline{G}_\sigma)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$, we have $r \geq 11$. Denote $\overline{L} = \overline{M}\,\overline{R}$. If $\overline{L} = [\overline{M}]\overline{R}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{M})/C_{\overline{G}_\sigma}(\overline{M}) \leq W(\overline{G})$. Since $|W(\overline{G})| = |W(\overline{G}_\sigma)|$ and $(r, |W(\overline{G}_\sigma)|) = 1$, we deduce that $\overline{L} = \overline{M} \times \overline{R}$. If $\overline{L} = [\overline{R}]\overline{M}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. Since $|W(\overline{G}_\sigma)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ and $m \geq 31$, we have $(m, |W(\overline{G}_\sigma)|) = 1$ and $\overline{L} = \overline{R} \times \overline{M}$.

From the uniqueness of the maximal torus $\overline{T}_\sigma$ whose order is divisible by m, we conclude that $\overline{L} \leq \overline{T}_\sigma$. Therefore, r divides $(q^8 + q^7 - q^5 - q^4 - q^3 + q + 1, q - 1) = 1$. A contradiction.

(8) $G \cong F_4(q)$, $q = p^n$; $|\overline{G}_\sigma| = q^{24}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{12} - 1)$.

The Cartan subgroup $\overline{H}_\sigma \cong Z^4_{q-1}$, and r divides $q-1$. $\overline{G}_\sigma$ has a maximal torus $\overline{T}_\sigma$ of order $q^4 - q^2 + 1$ by [15]. By Lemma 2, there is a prime m that divides $q^{12} - 1$, but does not divide $q^i - 1$ for all $i \in \{1, 2, \ldots, 11\}$. By Lemma 3, $m \geq 13$. The following equality holds: $q^{12} - 1 = (q^6 - 1)(q^2 + 1)(q^4 - q^2 + 1)$. So m divides $q^4 - q^2 + 1$. Consider the group $\overline{M} \leq Syl_m(\overline{G}_\sigma)$ such that $\overline{M} \leq \overline{T}_\sigma$. Let $\overline{L} = \overline{M}\,\overline{R}$. Torsion primes for the root system of type F_4 are 2 and 3. If $\overline{L} = [\overline{M}]\overline{R}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{M})/C_{\overline{G}_\sigma}(\overline{M}) \leq W(\overline{G})$. Since $|W(\overline{G})| = |W(\overline{G}_\sigma)|$ and $(r, |W(\overline{G}_\sigma)|) = 1$, we have $\overline{L} = \overline{M} \times \overline{R}$. If $\overline{L} = [\overline{R}]\overline{M}$, then by Lemma 5, $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. Since $|W(\overline{G}_\sigma)| = 2^7 \cdot 3^2$ and $m \geq 13$, we have $(m, |W(\overline{G}_\sigma)|) = 1$ and $\overline{L} = \overline{R} \times \overline{M}$.

From the uniqueness of the maximal torus $\overline{T}_\sigma$ whose order is divisible by m, we conclude that $\overline{L} \leq \overline{T}_\sigma$. So r divides $(q^4 - q^2 + 1, q - 1) = 1$. This is a contradiction.

(9) $G \cong G_2(q)$, $q = p^n$; $|\overline{G}_\sigma| = q^6(q^2 - 1)(q^6 - 1)$.

In this case, $\overline{H}_\sigma \cong Z^2_{q-1}$ and r divides $q - 1$. $\overline{G}_\sigma$ has a maximal torus $\overline{T}_\sigma$ of order $q^2 - q + 1$ by [15]. Since $q \geq 11$, the case $q = 2$ is impossible. By Lemma 2, there is a prime m that divides $q^6 - 1$, but does not divide $q^i - 1$ for all $i \in \{1, 2, \ldots, 5\}$. By Lemma 3, $m \geq 13$. Have the equality: $q^6 - 1 = (q^3 - 1)(q + 1)(q^2 - q + 1)$, so m divides $q^2 - q + 1$. Consider the group $\overline{L} = \overline{M}\,\overline{R}$. The torsion prime for the root system of type G_2 is 2. As in the preceding paragraphs we show that $\overline{L} = \overline{M} \times \overline{R}$.

From the uniqueness of the maximal torus $\overline{T}_\sigma$ whose order is divisible by m, we conclude that $\overline{L} \leq \overline{T}_\sigma$. So r divides $(q^2 - q + 1, q - 1) = 1$, it is impossible.

(10) $G \cong {}^2A_{l-1}(q)$, $l \geq 3$, $q = p^n$.

Suppose first that l is odd. The order of a Cartan subgroup is $|H| = \frac{1}{(l,q+1)}(q - 1)^{(l-1)/2}(q + 1)^{(l-1)/2}$. Since $\frac{1}{(l,q+1)}(q + 1)^{l-1}$ divides $|G|$ and $r \geq 5$, it follows that r divides $q - 1$. The group ${}^2A_{l-1}(q) \cong PSU(l, q)$. The equality $\{2, 6\} = \{p, 2nl\}$ holds if $p = 2$ and $l = 3$. Since $PSU(3, 2)$ is solvable, this case is impossible. By Lemma 2, there exists a prime s primitive in relation to the pair $\{p, 2nl\}$. Consider a biprimary Hall $\{r, s\}$-subgroup M of G. By Lemma 7, $|M|$ divides $l(q^l + 1)(q + 1)^{-1}$. Since $|M| = s \cdot r^{(l-1)/2} \cdot f$, where f is an integer, we see that $\frac{l(q^l+1)}{(q+1)\cdot s\cdot r^{(l-1)/2}\cdot f}$ is an integer. Since

$(q^l+1, q+1) \in \{1, 2\}$, and r divides $q-1$, we conclude that $\frac{l}{r^{(l-1)/2}}$ is an integer. But since $r \geq 5$, it is impossible.

Let l be an even number, and let $l \geq 4$. The order of a Cartan subgroup $|H| = \frac{1}{(l,q+1)}(q-1)^{l/2}(q+1)^{(l-1)/2}$. Since $\frac{1}{(l,q+1)}(q+1)^{l-1}$ divides $|G|$, we see that r divides $q-1$. By Lemma 2, there is a prime s primitive in relation to the pair $\{p, 2nl\}$. Consider a Hall $\{r, s\}$-subgroup M of G. By Lemma 7, $|M|$ divides $l(q^l+1)$. Since $|M| = s \cdot r^{l/2} \cdot f$, where f is an integer, we obtain that $\frac{l(q^l+1)}{s \cdot r^{l/2} \cdot f}$ is an integer. It follows that $\frac{l}{r^{l/2}}$ is an integer. But since $r \geq 5$ and $l \geq 4$, it is impossible.

(11) $G \cong {}^2D_l(q)$, $l \geq 4$, $q = p^n$; $|\overline{G}_\sigma| = q^{l(l-1)}(q^l+1)\prod_{i=1}^{l-1}(q^{2i}-1)$.

In the group $\overline{G}_\sigma$, the order of a Cartan subgroup $|\overline{H}_\sigma| = (q-1)^{l-1}(q+1)$, and r divides $q-1$. The group $\overline{G}_\sigma$ has a maximal torus $\overline{T}_\sigma$ of order q^l+1. Cases $2l = 2$, $q = 2^a - 1$ and $2l = 6$, $q = 2$ obviously are not fulfilled. Therefore, by Lemma 2, there is a prime m, which divides $q^{2l}-1$, but does not divide $q^i - 1$ for all $i \in \{1, 2, \ldots, 2l-1\}$. By Lemma 3, we obtain that $m \geq 2l+1 \geq 11$. Let $\overline{M} \leq \overline{T}_\sigma$ for an abelian m-subgroup $\overline{M} \in Syl_m(\overline{G}_\sigma)$. Consider a biprimary Hall $\{r, m\}$-subgroup $\overline{L}$ of $\overline{G}_\sigma$. Assume first that $\overline{L} = [\overline{M}]\overline{R}$. Note that $W(\overline{G}_\sigma) \neq W(\overline{G})$. Since $l \geq 4$, there is a pair of commuting elements $\bar{x} \in \overline{M}$ and $\bar{y} \in \overline{R}$. Denote $\bar{z} = \bar{x}\,\bar{y}$. By Lemma 5, there is maximal torus $\overline{T}$ containing $\bar{z}$. Since there is the only type of maximal tori whose order is divisible by m, then $\overline{T} = \overline{T}_\sigma$. Thus r divides $(q^l+1, q-1) \in \{1, 2\}$. This contradicts the fact that $r \geq 5$.

Let $\overline{L} = [\overline{R}]\overline{M}$. By Lemma 5, $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. Since the algebraic group $\overline{G}$ corresponding to $\overline{G}_\sigma$ is of type D_l, we have $|W(\overline{G})| = 2^{l-1} \cdot l!$. Since $m \geq 2l+1$, it follows that $(m, |W(\overline{G})|) = 1$ and $\overline{L} = \overline{R} \times \overline{M}$. In the group $\overline{G}_\sigma$, there is the only maximal torus $\overline{T}_\sigma$ whose order is divisible by m, so $\overline{L} \leq \overline{T}_\sigma$. It follows that r divides $(q^l+1, q-1)$, which is 1 or 2. Since $r \geq 5$, it is impossible.

(12) $G \cong {}^2E_6(q)$, $q = p^n$; $|\overline{G}_\sigma| = q^{36}(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)$.

In the group $\overline{G}_\sigma$, the Cartan subgroup has order $|\overline{H}_\sigma| = (q-1)^4(q+1)^2$, and r divides $q-1$. There is a maximal torus $\overline{T}_\sigma$ of order $(q^4-q^2+1)(q^2-q+1)$. By Lemma 2, there is a prime m, which divides $q^{12}-1$, but does not divide q^i-1 for all $i \in \{1, 2, \ldots, 11\}$. By Lemma 3, $m \geq 13$. The decomposition $q^{12}-1 = (q^6-1)(q^2+1)(q^4-q^2+1)$ implies that m divides q^4-q^2+1. Let $\overline{M} \leq \overline{T}_\sigma$ for a m-subgroup $\overline{M} \in Syl_m(\overline{G}_\sigma)$. Denote by $\overline{L}$ a Hall $\{r, m\}$-subgroup of $\overline{G}_\sigma$. Let $\overline{L} = [\overline{M}]\overline{R}$. There are $\bar{x} \in \overline{M}$ and $\bar{y} \in \overline{R}$ such that $[\bar{x}, \bar{y}] = 1$. By Lemma 5, there exists a maximal torus $\overline{T}$ containing $\bar{z} = \bar{x}\,\bar{y}$. Since there is the only type of maximal tori whose order is divisible by m, we see that $\overline{T} = \overline{T}_\sigma$, and r divides $((q^4-q^2+1)(q^2-q+1), q-1) \in \{1, 2, 4\}$. Since $r \geq 5$, we get a contradiction.

Let $\overline{L} = [\overline{R}]\overline{M}$. By Lemma 5, $N_{\overline{G}_\sigma}(\overline{R})/C_{\overline{G}_\sigma}(\overline{R}) \leq W(\overline{G})$. The algebraic group $\overline{G}$, corresponding to the group $\overline{G}_\sigma$ is of type E_6, so $|W(\overline{G})| = 2^7 \cdot 3^4 \cdot 5$. Since $m \geq 13$, we have $(m, |W(\overline{G})|) = 1$ and $\overline{L} = \overline{R} \times \overline{M}$. From the uniqueness of the maximal torus whose order is divisible by m, we conclude that r divides $((q^4-q^2+1)(q^2-q+1), q-1) \in \{1, 2, 4\}$, a contradiction.

(13) $G \cong {}^2F_4(q)$, $q = 2^{2n+1} > 2$; $|\overline{G}_\sigma| = q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)$.

The Cartan subgroup of $\overline{G}_\sigma$ has the order $|\overline{H_\sigma}| = (q-1)^2$, so r divides $q-1$. By Lemma 2, there is a prime m that divides $q^{12}-1$, but does not divide q^i-1 for all $i \in \{1, 2, \ldots, 11\}$. By Lemma 3, $m \geq 13$. Have the equality: $q^{12}-1 = (q^6-1)(q^2+1)(q^2+q\sqrt{2q}+q+\sqrt{2q}+1)(q^2-q\sqrt{2q}+q-\sqrt{2q}+1)$. Therefore, m divides the order of one of the maximal tori: $\overline{T}_1$ of order $q^2+q\sqrt{2q}+q+\sqrt{2q}+1$ or $\overline{T}_2$ of order $q^2-q\sqrt{2q}+q-\sqrt{2q}+1$. Suppose that m divides $|\overline{T}_1|$. Let $\overline{M} \leq \overline{T}_1$ for a m-subgroup $\overline{M} \in Syl_m(\overline{G}_\sigma)$. Since $r \geq 5$, $m \geq 13$ and $|W(\overline{G})| = 2^7 \cdot 3^2$, it follows by Lemma 5, that the group $\overline{G}_\sigma$ has a Hall $\{m, r\}$-subgroup $\overline{L} = \overline{M} \times \overline{R}$. $\overline{T}_1$ is the only maximal torus of $\overline{G}_\sigma$ whose order is divisible by m. So r divides $(q^2+q\sqrt{2q}+q+\sqrt{2q}+1, q-1)$, and hence r divides $(q^4-q^2+1, q-1) = 1$, that is impossible. The case when m divides $|\overline{T}_2|$ considered similarly.

(14) $G \cong {}^3D_4(q)$, $q = p^n$; $|\overline{G}_\sigma| = q^{12}(q^8+q^4+1)(q^2-1)(q^6-1)$.

The Cartan subgroup of $\overline{G}_\sigma$ has the order $|\overline{H}_\sigma| = (q^3-1)(q-1)$, and r divides $q-1$. The group $\overline{G}_\sigma$ has a maximal torus $\overline{T}_\sigma$ of order q^4-q^2+1. By Lemma 2, there exists a prime m that divides $q^{12}-1$, but does not divide q^i-1 for all $i \in \{1, 2, \ldots, 11\}$. By Lemma 3, $m \geq 13$. The decomposition $q^{12}-1 = (q^6-1)(q^2+1)(q^4-q^2+1)$ implies that m divides $|\overline{T}_\sigma|$. Suppose that a Sylow m-subgroup $\overline{M}$ of $\overline{G}_\sigma$ is contained in $\overline{T}_\sigma$. Since $r \geq 5$, $m \geq 13$ and $|W(\overline{G})| = 2^6 \cdot 3$, it follows by Lemma 5, that $\overline{G}_\sigma$ has a Hall $\{r, m\}$-subgroup $\overline{L} = \overline{R} \times \overline{M}$. From the uniqueness of the maximal torus $\overline{T}_\sigma$ whose order is divisible by m, we conclude that r divides $(q^4-q^2+1, q-1) = 1$, a contradiction.

(15) $G \cong {}^2B_2(q)$, $q = 2^{2n+1}$; $|{}^2B_2(q)| = q^2(q^2+1)(q-1)$.

The order of the Cartan subgroup of ${}^2B_2(q)$ is $q-1$, and r divides $q-1$. By Lemma 2, there is a prime m that divides q^4-1, but does not divide q^i-1 for all $i \in \{1, 2, 3\}$. By Lemma 3, $m \geq 5$. From [17] we deduce that every subgroup of odd order of ${}^2B_2(q)$ is contained in one of maximal tori which have pairwise coprime orders $q-1$, $q+\sqrt{2q}+1$ and $q-\sqrt{2q}+1$. Then m divides either $q+\sqrt{2q}+1$ or $q-\sqrt{2q}+1$. In both cases we obtain a contradiction to the fact that G has a Hall $\{r, m\}$-subgroup.

(16) $G \cong {}^2G_2(q)$, $q = 3^{2n+1}$; $|{}^2G_2(q)| = q^3(q^3+1)(q-1)$.

The order of the Cartan group of ${}^2G_2(q)$ is q^2-1, and r divides q^2-1. By Lemma 2, there exists a prime m that divides q^6-1, but does not divide q^i-1 for all $i \in \{1, 2, \ldots 5\}$. By Lemma 3, $m \geq 7$. Using [18], we conclude that any subgroup of odd order of ${}^2G_2(q)$, which is not divisible by 3, is contained in one of maximal tori with pairwise coprime orders $q^2 - 1$, $q + \sqrt{3q} + 1$ and $q - \sqrt{3q} + 1$. Arguments from the end of the previous paragraph leads to a contradiction.

Thus, there is no simple nonabelian group satisfying the conditions of Theorem 1.

Let G be a minimal counterexample to Theorem 1, and let M be a proper normal subgroup of G. Show that M is r-solvable. If M is either a r-group or a r'-group, then it is r-solvable. Therefore, $r \in \pi(M)$ and $|\pi(M)| \geq 2$. So M has the $E_{\{r,t\}}$-property for all $t \in \pi(M) \backslash \{r\}$. Since G is a minimal counterexample to the theorem, it follows that M is r-solvable.

Similarly it is shown that G/M is r-solvable. It follows that the group G is r-solvable. The theorem is proved.

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