Journal of Algebra 453 (2016) 221–248

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Exact pairs of homogeneous zero divisors

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A R T I C L E   I N F O

Article history:
Received 1 April 2013
Available online xxxx
Communicated by Bernd Ulrich

MSC:
13D02
13A02

Keywords:
Compressed level algebra
Determinantal ring
Exact pair of zero divisors
Generic points in projective space
Linear resolution
Matrix factorization
Pfaffians
Segre embedding
Tate resolution
Totally acyclic complex
Totally reflexive module

A B S T R A C T

Let S be a standard graded Artinian algebra over a field k. We identify constraints on the Hilbert function of S which are imposed by the hypothesis that S contains an exact pair of homogeneous zero divisors. As a consequence, we prove that if S is a compressed level algebra, then S does not contain any homogeneous zero divisors.

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1 Supported in part by the National Security Agency and the Simons Foundation.
2 Supported in part by the National Science Foundation.
3 Supported in part by the National Security Agency and the National Science Foundation.

http://dx.doi.org/10.1016/j.jalgebra.2016.01.016
0021-8693/

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In [18], Henriques and Şega defined the pair of elements (a, b) in a commutative ring S to be an *exact pair of zero divisors* if $(0 :_S a) = (b)$ and $(0 :_S b) = (a)$. We take S to be a standard graded Artinian algebra over a field and we identify constraints on the Hilbert function of S which are imposed by the hypothesis that S contains an exact pair (θ_1, θ_2) of homogeneous zero divisors. In Theorem 2.10 we prove that the main numerical constraint depends on the sum $\deg\theta_1 + \deg\theta_2$, but not on the individual numbers $\deg\theta_1$ or $\deg\theta_2$. In other words, the numerical constraint imposed on S by having an exact pair of homogeneous zero divisors of degrees d_1 and d_2 is the same as the constraint imposed by having an exact pair of homogeneous zero divisors of degrees 1 and $d_1 + d_2 - 1$. This result is especially curious because it is possible for S to have an exact pair of homogeneous zero divisors of degrees 2 and 2 without having any homogeneous exact zero divisors of degree 1; see Example 3.1. Our main result is Theorem 2.10.

Theorem 2.10. *Let S be a standard graded Artinian k-algebra. Suppose that (θ_1, θ_2) is an exact pair of homogeneous zero divisors in S. If $D = \deg\theta_1 + \deg\theta_2$, then the Hilbert series of S is divisible by $\frac{t^D-1}{t-1}$.*

In the statement of Theorem 2.10, the algebra S is Artinian, so the Hilbert series, $\mathrm{HS}_S(t)$, of S is a polynomial in $\mathbb{Z}[t]$, the expression $\frac{t^D-1}{t-1}$ is equal to the polynomial $1 + t + t^2 + \cdots + t^{D-1}$ of $\mathbb{Z}[t]$, and

$$\begin{gathered}\text{“the Hilbert series of } S \text{ is divisible by } \tfrac{t^D-1}{t-1}\text{” means that the polynomial}\\ 1 + t + t^2 + \cdots + t^{D-1} \text{ divides the polynomial } \mathrm{HS}_S(t)\\ \text{in the polynomial ring } \mathbb{Z}[t].\end{gathered} \tag{0.1}$$

We apply Theorem 2.10 in Section 3 to obtain a list of conditions on the standard graded k-algebra S, each of which leads to the conclusion that S does not have an exact pair of homogeneous zero divisors. These results are striking due to the connection between the existence of totally reflexive S-modules and the existence of exact zero divisors in S.

Definition 0.2. Let S be a commutative ring. A finitely generated S-module M is called *totally reflexive* if there exists a doubly infinite sequence of finitely generated free S-modules

$$F: \quad \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

such that M is isomorphic to the module $\mathrm{Coker}(F_1 \to F_0)$, and such that both F and the dual sequence $\mathrm{Hom}_S(F, S)$ are exact. The complex F is called *totally acyclic*.

For example, if θ_1 and θ_2 are a pair of exact zero divisors in S, then the complex

$$F: \quad \cdots \xrightarrow{\theta_2} S \xrightarrow{\theta_1} S \xrightarrow{\theta_2} S \xrightarrow{\theta_1} \cdots$$

is totally acyclic and the S-modules $S/(\theta_1)$ and $S/(\theta_2)$ are totally reflexive. Totally reflexive modules were first studied by Auslander and Bridger [2], who proved that S is Gorenstein if and only if every S-module has a totally reflexive syzygy. Over a Gorenstein ring, the totally reflexive modules are precisely the maximal Cohen–Macaulay modules, and these have been studied extensively. A main result of Christensen, Piepmeyer, Striuli, and Takahashi [10, Thm. B], asserts that if S is not Gorenstein, then the existence of one non-free totally reflexive S-module implies the existence of infinitely many non-isomorphic indecomposable totally reflexive S-modules. The proof in [10] is not constructive; however many methods [31,22,9,4] have been found for constructing non-isomorphic indecomposable totally reflexive S-modules. Most of these methods, especially those in [22] and [9], have involved the use of a pair of exact zero divisors. Indeed, one result in [9] gives conditions on S for which the existence of a non-free totally reflexive S-module implies the existence of an exact zero divisor in S. Furthermore, [9, Section 8] reformulates results of Conca [11], Hochster and Laksov [21], and Yoshino [31] to show that, under appropriate hypotheses, a generic standard graded algebra over an infinite field has an exact zero divisor.

All of our notation and conventions are explained in Section 1. Section 2 is dedicated to the proof of Theorem 2.10. Section 3 consists of examples and applications of Theorem 2.10. In Proposition 3.11 we show that, in general, compressed level algebras do not have any homogeneous exact zero divisors. Assorted examples of compressed level algebras are given. These examples include Gorenstein rings with linear resolutions, certain rings defined by Pfaffians, certain determinantal rings, and certain rings arising from sets of generic points in projective space. We also exhibit families of standard graded Artinian k-algebras, which are not compressed level algebras, but which nonetheless do not contain any homogeneous exact zero divisors. These families include rings which arise from Segre embeddings and more determinantal rings.

1. Terminology, notation, and preliminary results

We use gcd as an abbreviation for *greatest common divisor*, $\mathbb{Z}$ to represent the set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$, $\mathbb{N}$ – the set of positive integers $\{1, 2, 3, \dots\}$, and $\mathbb{Q}$ is the field of rational numbers.

If S is a ring, $N \subseteq M$ are S-modules and X is a subset of M, then

$$N :_S X = \{s \in S \mid sx \in N \text{ for all } x \in X\}.$$

Conventions 1.1. Let k be a field. If V is a vector space over k, then $\dim_k V$ is the vector space dimension of V.

(1) A *standard graded k-algebra* is a graded ring $S = \bigoplus_{i\in\mathbb{Z}}[S]_i$, with $[S]_i = 0$ for $i < 0$, $[S]_0 = k$, $\dim_k[S]_1 < \infty$, and S is generated as a k-algebra by $[S]_1$.
(2) For each graded S-module M we use $[M]_i$ to denote the *homogeneous component of M of degree i*.
(3) The *Hilbert function* of M is the function $\mathrm{HF}_S(M, \_)$, from the set $\mathbb{Z}$ to the set $\{0\} \cup \mathbb{N} \cup \{\infty\}$, with $\mathrm{HF}_S(M, i) = \dim_k[M]_i$. We abbreviate $\mathrm{HF}_S(S, i)$ as $\mathrm{HF}_S(i)$.
(4) The *Hilbert series* of a graded, finitely generated, S-module M is the formal generating function $\mathrm{HS}_S(M, t) = \sum_{i\in\mathbb{Z}} \mathrm{HF}_S(M, i)t^i$. We abbreviate $\mathrm{HS}_S(S, t)$ as $\mathrm{HS}_S(t)$; this formal power series is called the *Hilbert series of S*. **If S is Artinian, then the "Hilbert series" of S is actually a polynomial.**
(5) If M is a graded S-module and a is an integer, then $M(a)$ is the graded S-module with $[M(a)]_i = [M]_{a+i}$ for all integers i.
(6) If R is a standard graded k-algebra of dimension d, and $\ell_1, \dots, \ell_d$ is a regular sequence of linear forms in R, then $S = R/(\ell_1, \dots, \ell_d)$ is called an *Artinian reduction* of R.

Definition 1.2. A complex of modules

$$(\mathbb{F}, f) : \cdots \xrightarrow{f_2} \mathbb{F}_1 \xrightarrow{f_1} \mathbb{F}_0 \xrightarrow{f_0} \mathbb{F}_{-1} \xrightarrow{f_{-1}} \cdots$$

over the ring S is *acyclic* if the homology $\mathrm{H}(\mathbb{F})$ is equal to zero. The acyclic complex $\mathbb{F}$ is *totally acyclic* if $\mathbb{F}^* = \mathrm{Hom}_S(\mathbb{F}, S)$ is also acyclic.

Conventions 1.3. Let S be a standard graded algebra over a field k and $\varphi : F \to G$ be a homomorphism of finitely generated free graded S-modules. The homomorphism φ is called *homogeneous of degree d* if, whenever x is a homogeneous element of F, then $\varphi(x)$ is a homogeneous element of G of degree $d + \deg x$. If φ is called a *homogeneous homomorphism* and no degree is specified, then φ is a homogeneous homomorphism of degree 0. The homogeneous homomorphism $\varphi : F \to G$ is a *minimal homogeneous homomorphism* if $\varphi(F) \subseteq S_+G$, where S_+ is the ideal $\sum_{1\le i}[S]_i$ of S.

Proposition 1.4 is our tool for extracting numerical information about the Hilbert function of a standard graded Artinian algebra S from the twists in a minimal homogeneous totally acyclic complex of finitely generated free S-modules. The interesting part of the assertion is that the sum (1.5) is finite.

Proposition 1.4. *Let S be a standard graded Artinian algebra over the field k and $(\mathbb{F}, f)$ be a minimal homogeneous totally acyclic complex of finitely generated free S-modules. Write $\mathbb{F}$ in the form*

$$\cdots \xrightarrow{f_{j+1}} \mathbb{F}_j \xrightarrow{f_j} \mathbb{F}_{j-1} \xrightarrow{f_{j-1}} \cdots .$$

Then, for each integer λ, the expression

$$\sum_{p\in\mathbb{Z}} \big(\mathrm{HF}_S(\mathbb{F}_{2p},\lambda) - \mathrm{HF}_S(\mathbb{F}_{2p-1},\lambda)\big) \tag{1.5}$$

is a finite sum and is equal to zero.

Proof. The free S-module $\mathbb{F}_j$ occupies the homological position j in the complex $\mathbb{F}$, where j varies over all of $\mathbb{Z}$. For each integer λ, the homogeneous component $[\mathbb{F}]_\lambda$ of $\mathbb{F}$ of degree λ is $\bigoplus_{j\in\mathbb{Z}}[\mathbb{F}_j]_\lambda$; this component is an exact sequence of finite dimensional vector spaces

$$\cdots \to [\mathbb{F}_{j+1}]_\lambda \to [\mathbb{F}_j]_\lambda \to [\mathbb{F}_{j-1}]_\lambda \to \cdots .$$

We show below that

$$\text{once } \lambda \text{ is fixed, then } [\mathbb{F}_j]_\lambda \text{ is non-zero for only a finite number of } j. \tag{1.6}$$

Vector space dimension is additive on finite exact sequences; thus, for each integer λ,

$$\begin{aligned} 0 = \sum_{j\in\mathbb{Z}} (-1)^j \dim_k[\mathbb{F}_j]_\lambda &= \sum_{p\in\mathbb{Z}} (\dim_k[\mathbb{F}_{2p}]_\lambda - \dim_k[\mathbb{F}_{2p-1}]_\lambda) \\ &= \sum_{p\in\mathbb{Z}} \big(\mathrm{HF}_S(\mathbb{F}_{2p},\lambda) - \mathrm{HF}_S(\mathbb{F}_{2p-1},\lambda)\big). \end{aligned}$$

We complete the argument by establishing (1.6). The graded ring S is Artinian; so $[S]_i$ is zero for all large i. Define e to be the largest integer with $[S]_e \neq 0$. Fix an integer j. The free S-module $\mathbb{F}_j$ is graded and finitely generated. Let r_j be the rank of $\mathbb{F}_j$ and $m_{j,1} \le \cdots \le m_{j,r_j}$ be the degrees of the elements in a homogeneous minimal generating set for $\mathbb{F}_j$. We are particularly interested in $\ell_j = m_{j,1}$ and $h_j = m_{j,r_j}$. Notice that

$$[\mathbb{F}_j]_\lambda = 0 \quad \text{for all } \lambda \text{ with either } \lambda < \ell_j \text{ or } h_j + e < \lambda. \tag{1.7}$$

The key to establishing (1.6) is contained in the inequalities

$$\ell_j + 1 \le \ell_{j+1} \quad \text{and} \quad h_{j-1} \le h_j - 1. \tag{1.8}$$

Roughly speaking, the inequality on the left says that "no column of f_{j+1} can consist entirely of zeros" and the inequality on right says that "no row of f_{j-1} can consist entirely of zeros". To show the equality on the left side of (1.8), we consider a homogeneous minimal generator ξ of $\mathbb{F}_{j+1}$. The complex $\mathbb{F}$ is minimal, homogeneous, and acyclic; thus, $f_{j+1}(\xi)$ is not zero and is not a minimal generator of $\mathbb{F}_j$. Apply (1.7) to conclude, in the first place, that $\ell_j \le \deg f_{j+1}(\xi)$. Every homogeneous element in $\mathbb{F}_j$ of degree ℓ_j is a minimal generator of $\mathbb{F}_j$; so, we also conclude that $\ell_j \ne \deg f_{j+1}(\xi)$; therefore, we have shown that $\ell_j + 1 \le \deg f_{j+1}(\xi)$. The elements ξ and $f_{j+1}(\xi)$ of $\mathbb{F}$ have the same degree and the equality on the left side of (1.8) is established. To prove the inequality on the right side of (1.8), we consider the dual complex

$$\mathbb{F}^* : \quad \cdots \xrightarrow{f_{j-1}^*} \mathbb{F}_{j-1}^* \xrightarrow{f_j^*} \mathbb{F}_j^* \xrightarrow{f_{j+1}^*} \cdots,$$

which is also homogeneous, minimal, and acyclic. The elements of a minimal homogeneous generating set for $\mathbb{F}_j^*$ have degrees

$$-h_j = -m_{j,r_j} \le \cdots \le -m_{j,1} \le -\ell_j.$$

Apply the inequality on the left side of (1.8) to see that $-h_j + 1 \le -h_{j-1}$, which is the inequality on the right side of (1.8).

Suppose that $[\mathbb{F}_j]_\lambda \ne 0$. It follows from (1.7) that $\ell_j \le \lambda \le h_j + e$. In particular, the integers $\lambda - \ell_j$ and $h_j + e - \lambda$ are non-negative. If b is an integer with $\lambda - \ell_j < b$, then (1.8) gives $\lambda < \ell_j + b \le \ell_{j+b}$; hence, (1.7) yields $[\mathbb{F}_{j+b}]_\lambda = 0$. Similarly, if $h_j + e - \lambda < b$, then (1.8) gives $h_{j-b} + e \le h_j - b + e < \lambda$ and (1.7) yields $[\mathbb{F}_{j-b}]_\lambda = 0$. $\square$

2. The proof of the main theorem

Theorem 2.10 is the main theorem of the paper; we describe its proof. Let $d_i = \deg \theta_i$ and for integer i, define

$$\sigma_i = \sum_{j \equiv i \mod D} \mathrm{HF}_S(j),$$

where $\mathrm{HF}_S(\_)$ is the Hilbert function of S. The indexed list $\{\sigma_i\}$ is periodic of period at most D. To prove the result we must show that $\{\sigma_i\}$ has period 1. The proof is carried out as follows. The existence of (θ_1, θ_2) allows us to construct a totally acyclic complex $\mathbb{F}$. Each homogeneous strand of $\mathbb{F}$ is a finite exact sequence of vector spaces. The observation that vector space dimension is additive on finite exact sequences leads to the conclusion that $\{\sigma_i\}$ has period at most d_1; and hence, $\{\sigma_i\}$ has period at most the $\gcd\{d_1, d_2\}$. Henceforth, we assume that d_1 and d_2 have a non-unit factor in common.

One naive approach to proving Theorem 2.10 would involve looking for a homogeneous factorization $\theta_1 = \theta \cdot \check{\theta}$, with $\deg \theta = 1$, and then considering the resulting exact pair of zero divisors $(\theta, \check{\theta} \cdot \theta_2)$. This approach is doomed to fail. However, if one looks for a matrix factorization of θ_1, instead of a factorization of θ_1 in S, then this naive approach does indeed work. In Lemma 2.1, we use the idea of the Tate resolution of the residue field of a hypersurface ring to create a matrix factorization $(M, \check{M})$ of $\theta_1 \cdot \mathrm{id}$. At this point, we have a totally acyclic complex, periodic of period two, whose maps are $M\check{M}$ and $\theta_2 \cdot \mathrm{id}$. We prove in Lemma 2.5 that the maps $\check{M}$ and $\theta_2 \cdot M$ also give rise to a totally acyclic complex, periodic of period two. In Lemma 2.6 we combine Lemmas 2.1 and 2.5 and give a recipe for using an exact pair of zero divisors to build a numerically interesting totally acyclic complex $\mathbb{G}$. By looking at graded strands of $\mathbb{G}$, we obtain the equations

$$\sum_{\ell=0}^{s_1}(-1)^\ell\binom{s_1}{\ell}\sigma_{N-\ell}=0$$

for all integers N, where $s_1 = \mathrm{HS}_S(1)$. The coefficient matrix for this system of linear equations is a circulant matrix; in Lemma 2.7 we show that $\{\sigma_i\}$ has period one is the only solution.

The details in the proof of Theorem 2.10 are given immediately after the statement at the end of the present section. We first produce a matrix factorization of an essentially arbitrary element of a ring. Our technique is inspired by the Tate resolution of the residue field of a hypersurface ring; see, for example, [30, Section 2] or [16, Section 1.2].

Lemma 2.1. *Let be S a standard graded algebra over a field k and θ be a homogeneous element of S of degree $d \geq 2$. Then there exist finitely generated, free, graded S-modules F and G and minimal homogeneous S-module homomorphisms $M : F \to G$ and $\check{M} : G \to F(d)$ such that the compositions $\check{M} \circ M : F \to F(d)$ and $M \circ \check{M} : G \to G(d)$ both are multiplication by θ.*

Proof. Let $x_1, \dots, x_{s_1}$ be a basis for the vector space $[S]_1$. Identify elements $y_1, \dots, y_{s_1}$ in $[S]_{d-1}$ with $\theta = \sum_{i=1}^{s_1} x_i y_i$. Let V be the graded free S-module $\bigoplus_{i=1}^{s_1} S\varepsilon_i$, where each ε_i has degree 1, Θ be a divided power variable of degree 2, and $\mathbb{T}$ be the Graded Divided-power Algebra $\mathbb{T} = (\bigwedge_S^\bullet V)\langle\Theta\rangle$. Define a map $t : \mathbb{T} \to \mathbb{T}$ as follows: $t(\varepsilon_i) = x_i \in S$, the restriction of t to $\bigwedge_S^\bullet V$ is the usual Koszul complex map associated to $t : V \to S$, and if κ is in $\bigwedge_S^i V$ and $1 \leq \ell$, then

$$t(\kappa \otimes \Theta^{(\ell)}) = t(\kappa) \otimes \Theta^{(\ell)} + (-1)^i \kappa \wedge (\textstyle\sum_j y_j \varepsilon_j) \otimes \Theta^{(\ell-1)}.$$

Notice that $t(t(\Theta)) = \theta$. Fix an integer p with $s_1 + 2 \leq 2p$, then

$$\mathbb{T}_{2p} = \bigoplus_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} \textstyle\bigwedge_S^{2i} V \otimes_S S\Theta^{(p-i)}, \quad \mathbb{T}_{2p-1} = \displaystyle\bigoplus_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} \textstyle\bigwedge_S^{2i+1} V \otimes_S S\Theta^{(p-1-i)}, \quad \text{and}$$

$$(t \circ t)(\kappa \otimes \Theta^{(\ell)}) = \theta \cdot (\kappa \otimes \Theta^{(\ell-1)}) \quad \text{for all } \kappa \otimes \Theta^{(\ell)} \text{ in } \mathbb{T}_{2p} \text{ or } \mathbb{T}_{2p-1}.$$

We see, from the definition of t, that $t : V \to S$ may be written as $S(-1)^{s_1} \to S$ and $t : S\Theta \to V$ may be written as $S(-d)^1 \to S(-1)^{s_1}$. Thus,

$$\begin{aligned}
\mathbb{T}_{2p} &= \bigoplus_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} S(-2i-(p-i)d)^{\binom{s_1}{2i}} \\
&= \bigoplus_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} S(i(d-2)-pd)^{\binom{s_1}{2i}} \quad \text{and} \\
\mathbb{T}_{2p-1} &= \bigoplus_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} S(-2i-1-(p-1-i)d)^{\binom{s_1}{2i+1}} \\
&= \bigoplus_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} S(i(d-2)+d-1-pd)^{\binom{s_1}{2i+1}}. \qquad (2.2)
\end{aligned}$$

Notice that, in the language of (2.2), $\mathbb{T}_{2p-2} = \mathbb{T}_{2p}(d)$, $\mathbb{T}_{2p-3} = \mathbb{T}_{2p-1}(d)$, and $t : \mathbb{T}_{2p-2} \to \mathbb{T}_{2p-3}$ is a shift of $t : \mathbb{T}_{2p} \to \mathbb{T}_{2p-1}$. Define

$$F = \mathbb{T}_{2p}(pd), \quad G = \mathbb{T}_{2p-1}(pd),$$

M to be $t : F \to G$, and $\check{M}$ to be $t : G \to F(d)$. $\square$

Remark 2.3. For future reference we record the fact that the modules F and G of Lemma 2.1 are

$$F = \bigoplus_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} S(i(d-2))^{\binom{s_1}{2i}} \quad \text{and} \quad G = \bigoplus_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} S(i(d-2)+d-1)^{\binom{s_1}{2i+1}}. \qquad (2.4)$$

Suppose that M, $\check{M}$, and M_2 are three maps, which satisfy sufficient commuting relations, and for which there is a totally acyclic complex, periodic of period two, whose maps are $M\check{M}$ and M_2. We prove in Lemma 2.5 that the maps $\check{M}$ and M_2M also give rise to a totally acyclic complex, periodic of period two.

Lemma 2.5. *Let be S a standard graded algebra over a field k, d_1 and d_2 be positive integers, F and G be finitely generated, graded, free S-modules, and $M : F \to G$, $\check{M} : G \to F(d_1)$ be homogeneous S-module homomorphisms. Suppose further that M_2 is a matrix with entries from S such that $M_2 : F \to F(d_2)$ and $M_2 : G \to G(d_2)$ both are homogeneous S-module homomorphisms. Assume that the diagram*

$$\begin{tikzcd}
F \arrow[r, "M"] \arrow[d, "M_2"'] & G \arrow[r, "\check{M}"] \arrow[d, "M_2"'] & F(d_1) \arrow[d, "M_2"'] \\
F(d_2) \arrow[r, "M"] & G(d_2) \arrow[r, "\check{M}"] & F(d_1+d_2)
\end{tikzcd}$$

commutes and that the homomorphisms

$$\begin{aligned}
\mathbb{F}: \quad & \cdots \xrightarrow{\check{M}M} F(-d_2) \xrightarrow{M_2} F \xrightarrow{\check{M}M} F(d_1) \xrightarrow{M_2} F(d_1+d_2) \\
& \xrightarrow{\check{M}M} F(2d_1+d_2) \xrightarrow{M_2} \cdots \quad \textit{and} \\
\check{\mathbb{F}}: \quad & \cdots \xrightarrow{M\check{M}} G(-d_2) \xrightarrow{M_2} G \xrightarrow{M\check{M}} G(d_1) \xrightarrow{M_2} G(d_1+d_2) \\
& \xrightarrow{M\check{M}} G(2d_1+d_2) \xrightarrow{M_2} \cdots
\end{aligned}$$

form acyclic complexes. Then the homomorphisms

$$\begin{aligned}
\mathbb{G}: \quad & \cdots \xrightarrow{MM_2} G(-d_1) \xrightarrow{\check{M}} F \xrightarrow{MM_2} G(d_2) \xrightarrow{\check{M}} F(d_1+d_2) \\
& \xrightarrow{MM_2} G(d_1+2d_2) \xrightarrow{\check{M}} \cdots \quad \textit{and} \\
\check{\mathbb{G}}: \quad & \cdots \xrightarrow{\check{M}M_2} F \xrightarrow{M} G \xrightarrow{\check{M}M_2} F(d_1+d_2) \xrightarrow{M} G(d_1+d_2) \\
& \xrightarrow{\check{M}M_2} F(2d_1+2d_2) \xrightarrow{M} \cdots
\end{aligned}$$

form acyclic complexes. Furthermore, if $\mathbb{F}$ *and* $\check{\mathbb{F}}$ *are totally acyclic complexes, then* $\mathbb{G}$ *and* $\check{\mathbb{G}}$ *also are totally acyclic complexes.*

Proof.

Claim 1. $\operatorname{Ker} M = \operatorname{Im}(\check{M}M_2)$.

The inclusion $\supseteq$ is obvious. We show $\subseteq$. Take $x \in F(a)$, for some integer a, with $M(x) = 0$. It follows that x is in $\operatorname{Ker}(\check{M}M) = \operatorname{Im} M_2$. Thus, there is an element x_1 of $F(a-d_2)$, with $x = M_2x_1$. The hypothesis $x \in \operatorname{Ker} M$ now gives

$$0 = Mx = MM_2x_1.$$

The maps M and M_2 commute; so $Mx_1 \in \operatorname{Ker} M_2 = \operatorname{Im}(M\check{M})$; and

$$Mx_1 = M\check{M}x_2$$

for some x_2 in $G(a-d_1-d_2)$. We see that $x_1 = \check{M}x_2 + x_3$, where x_3 is the element $x_1 - \check{M}x_2$ of $\operatorname{Ker} M$. We have

$$x = M_2x_1 = M_2(\check{M}x_2 + x_3) \in \operatorname{Im}(M_2\check{M}) + M_2(\operatorname{Ker} M).$$

In other words, $\operatorname{Ker} M \subseteq \operatorname{Im}(M_2\check{M}) + M_2 \operatorname{Ker}(M)$. Iterate the above argument to see that $\operatorname{Ker} M \subseteq \operatorname{Im}(M_2\check{M}) + M_2^r \operatorname{Ker}(M)$ for all positive integers r. On the other hand, the maps all are homogeneous; so, for each fixed i,

$$[\operatorname{Ker} M]_i \subseteq [\operatorname{Im}(M_2\check{M})]_i + M_2^r([\operatorname{Ker}(M)]_{i-rd_2}).$$

The module F is finitely generated; thus, when r is sufficiently large, $[F]_{i-rd_2} = 0$. It follows that $[\operatorname{Ker} M]_i \subseteq [\operatorname{Im}(M_2\check{M})]_i$, for all i; hence, $\operatorname{Ker} M \subseteq \operatorname{Im}(M_2\check{M})$ and this completes the proof of Claim 1.

Claim 2. $\operatorname{Ker} \check{M} = \operatorname{Im}(MM_2)$.

One repeats the proof of Claim 1 after reversing the roles of M and $\check{M}$.

Claim 3. $\operatorname{Ker} \check{M}M_2 = \operatorname{Im}(M)$.

The inclusion $\supseteq$ is obvious. We show $\subseteq$. Take $x \in G(a)$, for some integer a, with $x \in \operatorname{Ker} \check{M}M_2$. The hypothesis ensures that $M_2\check{M}x = \check{M}M_2x = 0$. It follows that $\check{M}x$ is in $\operatorname{Ker} M_2 = \operatorname{Im}(\check{M}M)$ and $\check{M}x = \check{M}Mx_1$ for some $x_1 \in F(a)$. Therefore, $x - Mx_1 \in \operatorname{Ker} \check{M}$ which we saw in Claim 2 is equal to $\operatorname{Im}(MM_2)$. We have $x - Mx_1 = MM_2x_2$ for some $x_2 \in F(a - d_2)$ and $x \in \operatorname{Im}(M)$.

The equality $\operatorname{Ker} MM_2 = \operatorname{Im}(\check{M})$ follows from the symmetry of the situation the same way that Claim 2 followed from Claim 1. We conclude that $\mathbb{G}$ and $\check{\mathbb{G}}$ both are acyclic complexes.

Suppose now that $\mathbb{F}$ and $\check{\mathbb{F}}$ are totally acyclic complexes. In this case, $\mathbb{F}^*$ and $\check{\mathbb{F}}^*$ are acyclic complexes, the maps M^* and M_2^* commute, and the maps $\check{M}^*$ and M_2^* commute. One may apply what we have already shown to conclude that the complexes $\mathbb{G}^*$ and $\check{\mathbb{G}}^*$ are acyclic; and therefore, $\mathbb{G}$ and $\check{\mathbb{G}}$ both are totally acyclic complexes. $\square$

In Lemma 2.6 we combine Lemmas 2.1 and 2.5 and give a recipe for using an exact pair of zero divisors to build a numerically interesting totally acyclic complex $\mathbb{G}$.

Lemma 2.6. *Let S be a standard graded Artinian algebra over the field k. Suppose that (θ_1, θ_2) is an exact pair of homogeneous zero divisors in S, with $d_1 = \deg\theta_1 \geq 2$ and $d_2 = \deg\theta_2$. Let $D = d_1 + d_2$ and $s_1 = \mathrm{HF}_S(1)$. Then there is a homogeneous totally acyclic complex $\mathbb{G}$ of free S-modules of the form*

$$\cdots \xrightarrow{\varphi} \mathbb{G}_{2p+1} \xrightarrow{\psi} \mathbb{G}_{2p} \xrightarrow{\varphi} \mathbb{G}_{2p-1} \xrightarrow{\psi} \cdots, \quad \textit{with}$$

$$\mathbb{G}_{2p} = \bigoplus_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} S(i(d_1-2) - pD)^{\binom{s_1}{2i}} \quad \textit{and}$$

$$\mathbb{G}_{2p-1} = \bigoplus_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} S(i(d_1-2) + d_1 - 1 - pD)^{\binom{s_1}{2i+1}},$$

for all integers p. The matrices φ and ψ are square with 2^{s_1-1} rows and columns.

Proof. Apply Lemma 2.1 to the homogeneous element $\theta_1 \in S$ of degree $d_1 \geq 2$ to obtain finitely generated, free, graded S-modules F and G and minimal homogeneous S-module homomorphisms $M : F \to G$ and $\check{M} : G \to F(d_1)$ such that the compositions $\check{M} \circ M : F \to F(d_1)$ and $M \circ \check{M} : G \to G(d_1)$ both are multiplication by θ_1. Let M_2 be the S-module homomorphism "multiplication by θ_2". Notice that M_2 is a legitimate map from F to $F(d_2)$ and M_2 is also a legitimate map from G to $G(d_2)$. The hypotheses of Lemma 2.5 all are satisfied because (θ_1, θ_2) is an exact pair of zero divisors in S and the composition of M and $\check{M}$, in either order, is multiplication by θ_1. We record the acyclic complex $\mathbb{G}$, from the conclusion of Lemma 2.5, with

$$\cdots \xrightarrow{\varphi} \mathbb{G}_1 = G(-d_1) \xrightarrow{\psi} F = \mathbb{G}_0 \xrightarrow{\varphi} G(d_2) = \mathbb{G}_{-1} \xrightarrow{\psi} \cdots ,$$

$\psi = \check{M}$, $\varphi = MM_2$, and, according to (2.4),

$$\mathbb{G}_{2p} = F(-pD) = \bigoplus_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} S(i(d_1-2)-pD)^{\binom{s_1}{2i}} \quad \text{and}$$

$$\mathbb{G}_{2p-1} = G(-pD) = \bigoplus_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} S(i(d_1-2)+d_1-1-pD)^{\binom{s_1}{2i+1}}. \qquad \square$$

Lemma 2.7 is a straightforward numerical calculation. The assertions about circulant matrices are easy to verify and we verified them. (We learned about these tricks on Wikipedia.) It is gratifying to solve a system of linear equations without making any unpleasant calculations.

Lemma 2.7. *Assume that*

(1) *σ_i is an integer defined for all $i \in \mathbb{Z}$,*
(2) *a is a fixed positive integer with $\sigma_i = \sigma_{i+a}$ for all $i \in \mathbb{Z}$, and*
(3) *b is a fixed positive integer with $\sum_{i=0}^{b} (-1)^i \binom{b}{i} \sigma_{N-i} = 0$ for all $N \in \mathbb{Z}$.*

Then $\sigma_i = \sigma_j$ for all integers i and j.

Proof. Pick an integer B such that $b + 1 \leq B$ and B is a multiple of a. Hypothesis (2) gives

$$\sigma_{i+B} = \sigma_i \text{ for all } i \in \mathbb{Z}. \tag{2.8}$$

Define integers $c_0, \dots, c_{B-1}$ by

$$c_i = \begin{cases} (-1)^i \binom{b}{i} & \text{if } 0 \leq i \leq b \\ 0 & \text{if } b+1 \leq i \leq B-1, \end{cases}$$

and let C be the $B \times B$ circulant matrix

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{B-2} & c_{B-1} \\ c_{B-1} & c_0 & c_1 & \dots & c_{B-3} & c_{B-2} \\ c_{B-2} & c_{B-1} & c_0 & \dots & c_{B-4} & c_{B-3} \\ & & \ddots & \ddots & \ddots & \\ c_2 & c_3 & c_4 & \dots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \dots & c_{B-1} & c_0 \end{bmatrix}$$

and for each integer N, let Σ_N be the column vector

$$\Sigma_N = \begin{bmatrix} \sigma_N \\ \vdots \\ \sigma_{N-B+1} \end{bmatrix}$$

in $\mathbb{Z}^B$. We may read hypothesis (3) to say

$$(\text{row 1 of } C)\Sigma_N = 0, \text{ for all } N \in \mathbb{Z}. \tag{2.9}$$

Equation (2.9) holds when N is replaced by $N+1$ and formula (2.8) yields that $\sigma_{N+1} = \sigma_{N-B+1}$; so (row 2 of C) Σ_N is also zero. One may iterate this procedure to see that $C\Sigma_N$=0. We complete the proof by showing that the kernel of C is generated by the $B \times 1$ column vector with every entry equal to 1.

It is well known that C is a diagonalizable matrix over the complex numbers. Let ω be a primitive Bth root of 1, $p(x)$ be the polynomial $p(x) = \sum_{i=0}^{B-1} c_i x^i \in \mathbb{Z}[x]$, V be the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{B-1} \\ \vdots & & & & \\ 1 & \omega^{B-1} & \omega^{2(B-1)} & \dots & \omega^{(B-1)(B-1)} \end{bmatrix},$$

and $\mathcal{D}$ be the diagonal matrix with diagonal entries $p(1), p(\omega), \dots, p(\omega^{B-1})$. It is clear that $CV = V\mathcal{D}$ and V is invertible; so, the matrices C and $\mathcal{D}$ have the same rank. On the other hand, our choice of the coefficients c_i gives $p(x) = (1-x)^b$. It follows that exactly one of the diagonal entries of $\mathcal{D}$ is zero and the corresponding eigenvector is the first column of V, as we desired. $\square$

Theorem 2.10 is the main result of this paper.

Theorem 2.10. *Let S be a standard graded Artinian algebra over a field k. If (θ_1, θ_2) is an exact pair of homogeneous zero divisors in S and $D = \deg\theta_1 + \deg\theta_2$, then the Hilbert series of S is divisible by $\frac{t^D-1}{t-1}$ in the sense of (0.1).*

Remark 2.11. The set of polynomials in $\mathbb{Z}[t]$ which are divisible by

$$\frac{t^D-1}{t-1}=\sum_{i=0}^{D-1}t^i$$

forms an ideal J and the polynomial $p(t)=\sum c_it^i$ in $\mathbb{Z}[t]$ is in J if and only if

$$\sum_{\ell\in\mathbb{Z}}c_{0+\ell D}=\sum_{\ell\in\mathbb{Z}}c_{1+\ell D}=\cdots=\sum_{\ell\in\mathbb{Z}}c_{D-1+\ell D}. \tag{2.12}$$

View c_i to be zero if i is negative or greater than the degree of $p(t)$. The equations (2.12) often are the easiest way to test if the Hilbert series of a given Artinian algebra satisfy the conditions of Theorem 2.10. When we apply (2.12), we let σ_i denote

$$\sigma_i=\sum_{\ell\in\mathbb{Z}}c_{i+\ell D}.$$

Proof of Theorem 2.10. Let $d_1=\deg\theta_1$ and $d_2=\deg\theta_2$. For each integer i, let $s_i=\mathrm{HF}_S(i)$. (The Hilbert function HF_S and the Hilbert series HS_S are both defined in Conventions 1.1.) Define

$$\sigma_i=\sum_{\ell\in\mathbb{Z}}s_{i+\ell D}, \tag{2.13}$$

for all integers i. The ring S is Artinian; so, each σ_i is a finite integer. The definition of σ_i shows that $\sigma_i=\sigma_{i+D}$ for all integers i. In light of Remark 2.11, it suffices to prove that

$$\sigma_i=\sigma_j \quad \text{for all integers } i \text{ and } j. \tag{2.14}$$

The hypothesis that (θ_1,θ_2) is an exact pair of homogeneous zero divisors in S ensures that there is a homogeneous totally acyclic complex $\mathbb{F}$ of free S-modules of the form

$$\cdots\xrightarrow{\theta_1}\mathbb{F}_{2p+1}\xrightarrow{\theta_2}\mathbb{F}_{2p}\xrightarrow{\theta_1}\mathbb{F}_{2p-1}\xrightarrow{\theta_2}\cdots,$$

with

$$\mathbb{F}_{2p}=S(-pD)\quad\text{and}\quad\mathbb{F}_{2p-1}=S(-pD+d_1).$$

Apply Proposition 1.4 to see that

$$\begin{aligned}0&=\sum_{p\in\mathbb{Z}}(\mathrm{HF}_S(\mathbb{F}_{2p},N)-\mathrm{HF}_S(\mathbb{F}_{2p-1},N))=\sum_{p\in\mathbb{Z}}(s_{N-pD}-s_{N-pD+d_1})\\&=\sigma_N-\sigma_{N+d_1},\end{aligned}$$

for each integer N. It follows by symmetry that $\sigma_{N+d_2} = \sigma_{N+d_2-D} = \sigma_{N-d_1} = \sigma_N$; hence,

$$\sigma_{N+d_2} = \sigma_N = \sigma_{N+d_1} \quad \text{for all } N. \tag{2.15}$$

If the greatest common divisor of d_1 and d_2 is 1, then there is nothing more to prove. Henceforth, we may assume that d_1 and d_2 have a non-unit factor in common. In particular, we may assume that $d_1 \geq 2$. The fact that S contains an exact pair of homogeneous zero divisors of degrees d_1 and d_2, with $d_1 \geq 2$, allows us to create the homogeneous totally acyclic complex $\mathbb{G}$ of Lemma 2.6. Apply Proposition 1.4, together with (2.13) and (2.15), to see that

$$\begin{aligned}
0 &= \sum_{p\in\mathbb{Z}} (\mathrm{HF}_S(\mathbb{G}_{2p}, N) - \mathrm{HF}_S(\mathbb{G}_{2p-1}, N)) \\
&= \sum_{p\in\mathbb{Z}} \left(\sum_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} \binom{s_1}{2i} s_{N+i(d_1-2)-pD} - \sum_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} \binom{s_1}{2i+1} s_{N+i(d_1-2)+d_1-1-pD} \right) \\
&= \sum_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} \binom{s_1}{2i} \left(\sum_{p\in\mathbb{Z}} s_{N+i(d_1-2)-pD} \right) - \sum_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} \binom{s_1}{2i+1} \left(\sum_{p\in\mathbb{Z}} s_{N+i(d_1-2)+d_1-1-pD} \right) \\
&= \sum_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} \binom{s_1}{2i} \sigma_{N+i(d_1-2)} - \sum_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} \binom{s_1}{2i+1} \sigma_{N+i(d_1-2)+d_1-1} \\
&= \sum_{i=0}^{\lfloor \frac{s_1}{2} \rfloor} \binom{s_1}{2i} \sigma_{N-2i} - \sum_{i=0}^{\lfloor \frac{s_1-1}{2} \rfloor} \binom{s_1}{2i+1} \sigma_{N-2i-1} \\
&= \sum_{\ell=0}^{s_1} (-1)^\ell \binom{s_1}{\ell} \sigma_{N-\ell},
\end{aligned}$$

for each integer N. Apply Lemma 2.7 to conclude that $\sigma_i = \sigma_j$ for all integers i and j. Now that (2.14) has been established, the proof is complete. □

3. Examples

In Example 3.1 we exhibit a standard graded Artinian k-algebra S which has an exact pair of homogeneous zero divisors (θ_1, θ_2) of degrees 2 and 2 without having any homogeneous exact zero divisors of degree 1. This example is striking because the numerical result of Theorem 2.10 depends on the sum $\deg\theta_1 + \deg\theta_2 = 2 + 2$; but not on the particular summands $\deg\theta_1$ and $\deg\theta_2$.

The majority of the section consists of a list of standard graded Artinian k-algebras which do not contain any homogeneous exact zero divisors. To obtain these rings, we apply Theorem 2.10 in combination with a result from [25] (see Proposition 3.2 below). The combined result is called Corollary 3.3. Examples 3.4 and 3.5 are very explicit. In Proposition 3.11 we show that, in general, compressed level algebras do not have any homogeneous exact zero divisors. Assorted examples of compressed level algebras are given in Examples 3.17 (Gorenstein rings with linear resolutions), 3.18 (rings defined by Pfaffians), 3.20 (determinantal rings), and 3.22 (rings corresponding to generic points in projective space). In Proposition 3.23 and Example 3.30, we exhibit families of standard graded Artinian k-algebras, which are not compressed level algebras, which nonetheless do not contain any homogeneous exact zero divisors. These families arise from Segre embeddings and more determinantal rings.

Example 3.1. If

$$S = \mathbb{Q}[x, y, z, w, t]/(x^4, y^4, z^4, w^4, x^2y^2, z^2w^2, y^2w^2, xt, zt, wt, t^2),$$

then S has an exact pair of homogeneous zero divisors, both of degree two, but S does not have any exact zero divisors of degree one. One may check, using, for example, Macaulay2, that $\theta_1 = x^2 + y^2 - z^2 - w^2$ and $\theta_2 = x^2 + y^2 + z^2 + w^2$ is an exact pair of zero divisors. One may also check that the ideal $0 :_S \ell$ is not principal for any ℓ of the form $\ell = a_1x + b_2y + a_3z + a_4w + a_5t$, with $a_i \in \{0, 1\}$. If L is an arbitrary linear form from S, then there is a $\mathbb{Q}$-algebra automorphism of S which carries L to one of the ℓ's that have already been tested. The ideal $0 :_S \ell$ is not principal; hence the ideal $0 :_S L$ is also not principal. The Hilbert function of S is

i	0	1	2	3	4	5	6	7	8	9
$\mathrm{HF}_S(i)$	1	5	11	21	29	28	22	12	3	0.

Notice that $\sigma_0 = \sigma_1 = \sigma_2 = \sigma_3 = 33$, as promised by Theorem 2.10 and Remark 2.11.

Retain the notation of Example 3.1 and let

$$\Theta_1 = x^2 + y^2 - z^2 - w^2 \quad \text{and} \quad \Theta_2 = x^2 + y^2 + z^2 + w^2$$

be pre-images, in $P = \mathbb{Q}[x, y, z, w, t]$, of the exact zero divisors θ_1 and θ_2. Observe that the product $\Theta_1\Theta_2 = x^4 + y^4 - z^4 - w^4 + 2x^2y^2 - 2z^2w^2$ is a minimal generator of the defining ideal of S. It is shown in [25] that this is a general property of exact zero divisors.

Proposition 3.2. *Let P be a standard graded polynomial ring over the field k, I be a homogeneous ideal of P which is primary to the maximal homogeneous ideal of P, and Θ_1 and Θ_2 be homogeneous elements of P whose images in $S = P/I$ form an exact pair of homogeneous zero divisors. Then $\Theta_1 \cdot \Theta_2$ is a minimal generator of I.*

Throughout the rest of this section we apply Proposition 3.2 in conjunction with Theorem 2.10 in order to prove that various standard graded Artinian k-algebras do not contain any homogeneous exact zero divisors. The combined result is the following.

Corollary 3.3. *Let P be a standard graded polynomial ring over the field k and I be a homogeneous ideal of P which is primary to the maximal homogeneous ideal of P. Assume that $[I]_1 = 0$, $2 \leq \dim_k [P]_1$, and $S = P/I$ contains at least one homogeneous exact zero divisor. Then the following statements hold.*

(1) *If I is minimally generated by homogeneous forms of degree n, then $\frac{t^n-1}{t-1}$ divides the Hilbert series $\mathrm{HS}_S(t)$ of S in the sense of (0.1).*
(2) *If n and e are integers with $[I]_i = 0$, for all i with $i < n$, and $[I]_i = [P]_i$, for all i with $e < i$, then there is an integer D with $n \leq D \leq e$ such that $\frac{t^D-1}{t-1}$ divides the Hilbert series $\mathrm{HS}_S(t)$ of S in the sense of (0.1).*

A comment about the proof. In the situation of (2), the minimal homogeneous generators of I have degree between n and $e+1$; so a direct application of Theorem 2.10, combined with Proposition 3.2, yields that $\frac{t^D-1}{t-1}$ divides $\mathrm{HS}_S(t)$ for some D with $n \leq D \leq e+1$. However, the ambient hypotheses guarantee that $\dim_k [S]_0 \neq \dim_k [S]_1$ and therefore $\frac{t^{e+1}-1}{t-1}$ cannot possibly divide $\mathrm{HS}_S(t)$. $\square$

Examples 3.4 and 3.5 involve determinantal rings. We use Corollary 1 in [12] to determine the Hilbert functions of these rings.

Example 3.4. Let k be a field, X be a 4×5 matrix of indeterminates, and $I_3(X)$ be the ideal generated by the 3×3 minors of X. Let

$$S = \frac{k[X]}{I_3(X) + (\ell_1, \ldots, \ell_{14})}$$

where $\ell_1, \ldots, \ell_{14} \in k[X]$ are linear forms such that their images in $k[X]/I_3(X)$ form a system of parameters. Then the Hilbert function of S is

i	0	1	2	3	4	5
$\mathrm{HF}_S(i)$	1	6	21	16	6	0.

If S has an exact pair of homogeneous zero divisors, then the degrees of the exact zero divisors would have to add up to three according to Proposition 3.2. But note that when $D = 3$, we have $\sigma_0 = 17$, $\sigma_1 = 12$, $\sigma_2 = 21$. Therefore, Theorem 2.10, by way of Remark 2.11, yields that S does not have an exact pair of homogeneous zero divisors.

Example 3.5. Let k be a field, X be a 5×5 matrix of indeterminates, and $I_4(X)$ be the ideal generated by the 4×4 minors of X. Let

$$S = \frac{k[X]}{I_4(X) + (\ell_1, \dots, \ell_d)}$$

where $\ell_1, \dots, \ell_d \in k[X]$ are linear forms such that their images in $k[X]/I_4(X)$ form a system of parameters. Then the Hilbert function of S is

i	0	1	2	3	4	5	6	7
$\mathrm{HF}_S(i)$	1	4	10	20	10	4	1	0.

If S has an exact pair of homogeneous zero divisors, then the degrees of the exact zero divisors would have to add up to four by Proposition 3.2. But note that when $D = 4$, we have $\sigma_0 = 11$, $\sigma_1 = 8$, $\sigma_2 = 11$, and $\sigma_3 = 20$. Therefore S does not have an exact pair of homogeneous zero divisors. (This example is a special case of Proposition 3.11; see Example 3.20.) In the language of Conventions 1.1, S an Artinian reduction of $\frac{k[X]}{I_4(X)}$.

Example 3.6. Let S be a standard graded Artinian algebra such that $[S]_{e+1} = 0$. Let i be such that $\mathrm{HF}_S(i) \neq \mathrm{HF}_S(i+1)$ and let $D > \max\{i+1, e-i\}$. Then S cannot have any exact pair of homogeneous zero divisors with degrees adding up to D, since the inequality satisfied by D implies that $\sigma_i = \mathrm{HF}_S(i)$ and $\sigma_{i+1} = \mathrm{HF}_S(i+1)$.

In Propositions 3.11 and 3.23 and Example 3.30 we give families of standard graded Artinian k-algebras which do not contain any homogeneous exact zero divisors. The algebras of Proposition 3.11 are compressed level algebras. One introduction to this topic may be found in [6]. The following data is in effect.

Data 3.7. Let $S = P/I$ be a standard graded Artinian algebra over the field k, where P is a standard graded polynomial ring, with c variables, over k, and I is generated by homogeneous forms of degree at least two. The parameter c is the *codimension* of S in the sense that c is equal to the Krull dimension of P minus the Krull dimension of S.

Language 3.8. Retain Data 3.7. Recall that the *socle* of S is the vector space $0 :_S S_+$ and the *type* of S is $\dim_k \operatorname{socle} S$. The algebra S is called *Gorenstein* if it has type 1 and S is called *level* if its socle is concentrated in one degree. (Notice that a Gorenstein algebra is automatically a level algebra.) Let

$$\mathbb{F}: \quad 0 \to \mathbb{F}_c \xrightarrow{f_c} \mathbb{F}_{c-1} \xrightarrow{f_{c-1}} \cdots \xrightarrow{f_2} \mathbb{F}_1 \xrightarrow{f_1} \mathbb{F}_0 \tag{3.9}$$

be a minimal homogeneous resolution of S by free P-modules. If the ring S is not Gorenstein, then the resolution $\mathbb{F}$ is called *linear* if the entries of f_i are linear forms for $2 \leq i$ and the entries of f_1 are homogeneous forms of the same degree $n \geq 2$. If S is a Gorenstein ring, then the resolution $\mathbb{F}$ must be symmetric; and therefore, $\mathbb{F}$ is called a *linear resolution* if the entries of f_i are linear forms for $2 \leq i \leq c-1$ and the entries of f_1 and f_c are homogeneous forms of the same degree $n \geq 2$.

Definition 3.10. Retain Data 3.7 with S a level algebra of socle degree e and type r. If the Hilbert function is given by

$$\mathrm{HF}_S(i) = \min\{\dim_k[P]_i,\ r\cdot \dim_k[P]_{e-i}\},\quad \text{for } 0 \le i \le e,$$

then S is called a *compressed level* algebra.

Notice that once the codimension, socle degree, and type of a level algebra are fixed, then the Hilbert function of a compressed level algebra is as large as possible in each degree. Compressed algebras were first defined (in a more general context than level algebras) by Iarrobino [23]; where he proved that for all pairs (e, r) there exists a non-empty open set of parameters which correspond to a compressed level algebra of socle degree e and type r. Fröberg and Laksov [14] offer alternate proofs some of Iarrobino's results. Zanello [32,33] has generalized the concept to arbitrary Artinian algebras.

Proposition 3.11. *Adopt Data 3.7 with S a compressed level algebra with socle degree e, type r, and codimension c. If either one the following conditions hold*

$$r = 1,\ 2 \le e \text{ with } e \ne 3, \text{ and } 3 \le c, \text{ or} \tag{3.12}$$

$$c,\ e,\ \text{and } r \text{ are all at least } 2 \text{ and } (c, e, r) \ne (c, 2, c-1), \tag{3.13}$$

then S does not have any homogeneous exact zero divisors.

Remarks 3.14. (1) The condition $e \ne 3$ is necessary (3.12) because $S = \frac{k[x,y,z]}{(x^2,y^2,z^2)}$ is a compressed Gorenstein algebra with Hilbert function

i	0	1	2	3	4
$\mathrm{HF}_S(i)$	1	3	3	1	0

and socle degree $e = 3$. The element x of S is a homogeneous exact zero divisor.

(2) The condition $3 \le c$ is necessary in (3.12) because

$$S = \begin{cases} \dfrac{k[x,y]}{(x^{\frac{e}{2}+1}, y^{\frac{e}{2}+1})} & \text{if } e \text{ is even} \\ \dfrac{k[x,y]}{(x^{\frac{e+1}{2}}, y^{\frac{e+3}{2}})} & \text{if } e \text{ is odd} \end{cases}$$

is a compressed Gorenstein algebra of socle degree e. The element x of S is a homogeneous exact zero divisor.

(3) The condition $(c, e, r) \ne (c, 2, c-1)$ is necessary in (3.13) because

$$S = \frac{k[x_1, \dots, x_c]}{(x_1, \dots, x_{c-1})^2 + (x_c^2)}$$

is a compressed level algebra with Hilbert function

$$\begin{array}{ccccc} i & 0 & 1 & 2 & 3 \\ \mathrm{HF}_S(i) & 1 & c & c-1 & 0, \end{array}$$

codimension c, socle degree $e = 2$, and type $c-1$. The element x_c of S is a homogeneous exact zero divisor.

Proof of Proposition 3.11. Let n be the minimal generator degree of I. We apply Corollary 3.3. It suffices to show that $\frac{t^D-1}{t-1}$ does not divide $\mathrm{HS}_S(t)$, in the sense of (0.1), for any integer D with $n \le D \le e$. For each relevant D, we will exhibit two subscripts a and b with $\sigma_a \neq \sigma_b$ for σ as defined in (2.13).

Observe that $\lceil \frac{e+1}{2} \rceil \le n$. Indeed, if $i < \lceil \frac{e+1}{2} \rceil$, then $i \le e-i$ and

$$\dim_k[S]_i = \mathrm{HF}_S(i) = \min\{\dim_k[P]_i,\ r\cdot \dim_k[P]_{e-i}\} = \dim_k[P]_i.$$

If the hypotheses of (3.13) are in effect with $e = 2$, then $n = D = e = 2$, the Hilbert function of S is

$$\begin{array}{ccccc} i & 0 & 1 & 2 & 3 \\ \mathrm{HF}_S(i) & 1 & c & r & 0, \end{array}$$

with $\sigma_0 = 1 + r \neq c = \sigma_1$ and there is nothing more to prove. Henceforth, when the hypotheses of (3.13) are in effect we will assume that $3 \le e$.

We separate the proof into four cases. In every calculation we consider all D with

$$\lceil \tfrac{e+1}{2} \rceil \le n \le D \le e.$$

Case 1. Take $2 \le r$, $2 \le \dim_k[P]_1$, and $4 \le e$, with e even. If $\frac{e}{2}+1 < D$, then

$$\sigma_{\frac{e}{2}-1} = \mathrm{HF}_S(\tfrac{e}{2}-1) = \dim_k[P]_{\frac{e}{2}-1} < \dim_k[P]_{\frac{e}{2}} = \mathrm{HF}_S(\tfrac{e}{2}) = \sigma_{\frac{e}{2}},$$

and if $\frac{e}{2}+1 = n = D$, then

$$\begin{aligned} \sigma_{\frac{e}{2}-1} &= \mathrm{HF}_S(e) + \mathrm{HF}_S(\tfrac{e}{2}-1) = r + \dim_k[P]_{\frac{e}{2}-1} < 1 + r\dim_k[P]_{\frac{e}{2}-1} \\ &= \mathrm{HF}_S(0) + \mathrm{HF}_S(\tfrac{e}{2}+1) = \sigma_{\frac{e}{2}+1}. \end{aligned}$$

The critical inequality holds because $2 \le r$ and $2 \le \dim_k[P]_{\frac{e}{2}-1}$; hence

$$0 < (r-1)(\dim_k[P]_{\frac{e}{2}-1} - 1).$$

Case 2. Take $2 \le r$, $2 \le \dim_k[P]_1$, and $3 \le e$, with e odd. If $\frac{e+1}{2} < D$, then

$$\begin{aligned} \sigma_{\frac{e-1}{2}} &= \mathrm{HF}_S(\tfrac{e-1}{2}) = \dim_k[P]_{\frac{e-1}{2}} < \min\{\dim_k[P]_{\frac{e+1}{2}},\ r\cdot\dim_k[P]_{\frac{e-1}{2}}\} \\ &= \mathrm{HF}_S(\tfrac{e+1}{2}) = \sigma_{\frac{e+1}{2}}, \end{aligned}$$

and if $\frac{e+1}{2} = n = D$, then

$$\begin{aligned}\sigma_{\frac{e-1}{2}} &= \mathrm{HF}_S(\tfrac{e-1}{2}) + \mathrm{HF}_S(e) = \dim_k[P]_{\frac{e-1}{2}} + r < 1 + r \cdot \dim_k[P]_{\frac{e-1}{2}} \\ &= \mathrm{HF}_S(0) + \mathrm{HF}_S(\tfrac{e+1}{2}) = \sigma_{\frac{e+1}{2}}.\end{aligned}$$

Case 3. Take $r = 1$, $3 \le \dim_k[P]_1$, and $2 \le e$, with e even. If $\frac{e}{2} + 1 < D \le e$, then

$$\sigma_{\frac{e}{2}+1} = \mathrm{HF}_S(\tfrac{e}{2}+1) = \dim_k[P]_{\frac{e}{2}-1} < \dim_k[P]_{\frac{e}{2}} = \mathrm{HF}_S(\tfrac{e}{2}) = \sigma_{\frac{e}{2}},$$

and if $\frac{e}{2} + 1 = D$, then

$$\sigma_{\frac{e}{2}+1} = \mathrm{HF}_S(0) + \mathrm{HF}_S(\tfrac{e}{2}+1) = 1 + \dim_k[P]_{\frac{e}{2}-1} < \dim_k[P]_{\frac{e}{2}} = \mathrm{HF}_S(\tfrac{e}{2}) = \sigma_{\frac{e}{2}}.$$

Case 4. Take $r = 1$, $3 \le \dim_k[P]_1$, and $5 \le e$, with e odd. Observe first that if A and B are integers with $1 \le A$ and $2 \le B$,

$$\tbinom{A+1}{1} \le \tbinom{A+B}{B} \tag{3.15}$$

because

$$A + 1 < \tfrac{A+1}{1} \cdot \tfrac{A+2}{2} \cdots \tfrac{A+B}{B} = \tbinom{A+B}{B}.$$

The polynomial ring P has c variables with $3 \le c$. It follows that

$$\dim_k[P]_1 + \dim_k[P]_{B-1} < \dim_k[P]_0 + \dim_k[P]_B \quad \text{for all integers } B \text{ with } 2 \le B. \tag{3.16}$$

Indeed,

$$\begin{aligned}(3.16) \text{ holds} &\iff \dim_k[P]_1 - \dim_k[P]_0 < \dim_k[P]_B - \dim_k[P]_{B-1} \\ &\iff c - 1 < \tbinom{c-1+B}{B} - \tbinom{c-2+B}{B-1} \\ &\iff \tbinom{(c-2)+1}{1} < \tbinom{(c-2)+B}{B},\end{aligned}$$

and this follows from (3.15). If $\frac{e+1}{2} \le D \le e - 2$, then apply (3.16) to see that

$$\begin{aligned}\sigma_1 &= \mathrm{HF}_S(1) + \mathrm{HF}_S(D+1) \\ &= \dim_k[P]_1 + \dim_k[P]_{e-D-1} < \dim_k[P]_0 + \dim_k[P]_{e-D} \\ &= \mathrm{HF}_S(0) + \mathrm{HF}_S(D) = \sigma_0.\end{aligned}$$

If $D = e - 1$, then

$$\begin{aligned}\sigma_0 &= \mathrm{HF}_S(0) + \mathrm{HF}_S(e-1) = \dim_k[P]_0 + \dim_k[P]_1 = c + 1 < \tbinom{c+1}{2} \\ &= \dim_k[P]_2 = \mathrm{HF}_S(2) = \sigma_2,\end{aligned}$$

and if $D = e$, then

$$\sigma_0 = \mathrm{HF}_S(0) + \mathrm{HF}_S(e) = 2 < c = \mathrm{HF}_S(1) = \sigma_1. \qquad \square$$

Example 3.17. Adopt Data 3.7. If S is Gorenstein and the minimal homogeneous resolution of S by free P-modules is linear, then S is a compressed level algebra. In particular, if (c, e, r) satisfy (3.12), then S does not have any homogeneous exact zero divisors.

Proof. The entries in the matrices f_j from (3.9) are homogeneous forms of degree

$$\begin{cases} 1 & \text{for } 2 \le j \le c-1 \\ n & \text{for } j = 1 \text{ and } j = c, \end{cases}$$

for some fixed integer n, with $2 \le n$. In particular,

$$\mathbb{F}_j = \begin{cases} P & \text{for } j = 0 \\ P(-n-j+1)^{\beta_j} & \text{for } 1 \le j \le c-1 \\ P(-2n-c+2) & \text{for } j = c. \end{cases}$$

One can use the Herzog–Kühl formulas [19] to produce the Betti numbers β_j, although the present argument will not use these betti numbers. One can read the socle degree $e = 2n-2$ of S from the back twist in $\mathbb{F}$; see, for example [26, Cor. 1.7]. At this point the Hilbert function of S is known. The hypothesis that the generators of I have degree n means that $\mathrm{HF}_S(i) = \dim_k [P]_i$, for $0 \le i \le n-1$. The Hilbert function of S is symmetric (because S is Gorenstein) and we have already described half of the Hilbert function; therefore, the other half is known by symmetry:

$$\mathrm{HF}_S(i) = \begin{cases} \dim_k [P]_i, & \text{for } 0 \le i \le n-1 \\ \dim_k [P]_{2n-2-i}, & \text{for } n \le i \le 2n-2, \end{cases}$$

and S is a compressed level algebra as described in Definition 3.10. $\square$

Example 3.18. Let n be an integer, X be a $2n+1 \times 2n+1$ alternating matrix whose entries are linear forms from $P = k[x, y, z]$, and I be the ideal in P generated by the maximal order Pfaffians of X. If I is primary to the maximal homogeneous ideal (x, y, z) of P, then $S = P/I$ is a compressed level algebra. In particular, if $2 \le n$, then S does not have any homogeneous exact zero divisors.

Proof. The resolution of S by free P-modules is known [8] to be linear. Apply Example 3.17. $\square$

Remark 3.19. There are plenty of compressed Gorenstein algebras which do not have linear resolutions. For example, if S is described in Data 3.7 with $P = k[x, y, z]$ and (3.9) given by

$$0 \to P(-8) \to \begin{matrix} P(-4) \\ \oplus \\ P(-5)^4 \end{matrix} \to \begin{matrix} P(-3)^4 \\ \oplus \\ P(-4) \end{matrix} \to P,$$

then the Hilbert function of S is

$$\begin{array}{cccccccc} i & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathrm{HF}_S(i) & 1 & 3 & 6 & 6 & 3 & 1 & 0; \end{array}$$

hence S is a compressed Gorenstein algebra which does not contain any homogeneous exact zero divisors; but the resolution of S is not linear. One such algebra $S = P/I$ is defined by $I = (x^2y, x^2z, y^3, z^3, x^4 + y^2z^2)$; this ideal plays an important role in [6].

The ring of Example 3.5 is an excellent representative of a family of compressed Gorenstein algebras which do not contain any homogeneous exact zero divisors.

Example 3.20. Let $n \geq 3$ be a positive integer, X be a $(n+1) \times (n+1)$ matrix whose entries are linear forms from $P = k[x, y, z, w]$, and I be the ideal in P generated by the $n \times n$ minors of X. If I is primary to the maximal homogeneous ideal (x, y, z, w) of P, then $S = P/I$ is a compressed Gorenstein algebra, of socle degree $2n - 2$, which does not contain any homogeneous exact zero divisors.

Proof. The resolution of S by free P-modules is known [17] to be linear. Apply Example 3.17. $\square$

We have drawn many consequences from Proposition 3.11 when the type is one. Remark 3.21, and Example 3.22, are analogous to Example 3.17, and Remark 3.19, respectively, when the type is greater than 1.

Remark 3.21. Adopt Data 3.7. If S is not Gorenstein and the minimal resolution of S by free P-modules is linear, then S is a compressed level algebra. Indeed, the free modules of (3.9) have the form

$$\mathbb{F}_j = \begin{cases} P & \text{if } j = 0 \\ P(-n-j+1)^{\beta_j} & \text{if } 1 \leq j \leq c, \end{cases}$$

where the generators of I are homogeneous forms of degree n. The Herzog–Kühl formulas [19] give $\beta_1 = \binom{n+c-1}{c-1} = \dim_k[P]_n$. It follows that $I = ([P]_n)$, S has socle degree $e = n-1$ and type $r = \dim_k[P]_{n-1}$, and

$$\mathrm{HS}_S(i) = \dim_k[P]_i = \min\{\dim_k[P]_i,\ r \cdot \dim_k[P]_{n-1-i}\}, \quad \text{for } 0 \leq i \leq n-1.$$

Thus, S is a compressed level algebra. One could conclude that $S = \frac{k[x_1,\dots,x_c]}{(x_1,\dots,x_c)^n}$ does not contain any homogeneous exact zero divisors when c and n are at least two. Of course, one could also employ a direct argument to reach the same conclusion. Furthermore,

S is a Golod ring [13, Prop. 1.9], of minimal multiplicity; so all of the totally reflexive S-modules are free; see, [3, (3.5)] or [31, Cor. 2.5]. (This provides a third argument that S does not contain any homogeneous exact zero divisors.)

Example 3.22. Let (c, e, r) be a triple of positive integers which satisfy

$$\tfrac{1}{c}\tbinom{c+e-2}{c-1} \le r \le \tbinom{c+e-1}{c-1}.$$

If $n = \binom{c+e-1}{c} + r$ and $S = P/I$, as described in Data 3.7, is the Artinian reduction of the coordinate ring of n generic points in $\mathbb{P}^c$, then, according to [5, Cor. 3.22], S is a compressed level algebra of codimension c, socle degree e, and type r; in particular, if (c, e, r) also satisfy (3.12) or (3.13), then S does not contain any homogeneous exact zero divisors.

In Proposition 3.23 and Example 3.30 we again apply Corollary 3.3 to produce families of standard graded Artinian k-algebras which do not contain any homogeneous exact zero divisors. These families have nothing to do with compressed algebras.

Proposition 3.23. *Let $s \ge 3$ be an odd integer and let $\mathfrak{S}$ be the homogeneous coordinate ring of the Segre embedding of s copies of $\mathbb{P}^1$ into projective space. If ℓ is a linear system of parameters in $\mathfrak{S}$ and $S = \mathfrak{S}/(\ell)$, then S is a standard graded Artinian Gorenstein k-algebra and S does not have any homogeneous exact zero divisors.*

Proof. The Segre embedding takes $(\mathbb{P}^1)^s = \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{s}$ into $\mathbb{P}^{2^s-1}$. We take

$$P = k[\{W_{(j_1,\dots,j_s)} \mid j_i \in \{0,1\}\}] \quad \text{and} \quad T = k[\{X_{i,j} \mid 1 \le i \le s \text{ and } j \in \{0,1\}\}]$$

to be the coordinate rings of $\mathbb{P}^{2^s-1}$ and $(\mathbb{P}^1)^s$, respectively. The ring $\mathfrak{S}$ is equal to P/I for I equal to the kernel of the ring homomorphism $\varphi : P \to T$ with $\varphi(W_{j_1,\dots,j_s}) = \prod_{i=1}^{s} X_{i,j_i}$.

The following properties of $\mathfrak{S}$ are well-known; we refer to [27], and this reference uses results from [28] and [20]:

$$\text{The ideal } I \text{ is generated by quadratic polynomials.} \tag{3.24}$$

$$\text{The ring } \mathfrak{S} \text{ is Gorenstein of dimension } s+1. \tag{3.25}$$

$$\text{The Hilbert series of } \mathfrak{S} \text{ is } \mathrm{HS}_{\mathfrak{S}}(t) = \sum_{i \ge 0} (i+1)^s t^i. \tag{3.26}$$

Assertion (3.24) holds because I is the “cut ideal of a tree”; see [28, Ex. 2.3]. The cut ideal of a graph is defined in [28]. Furthermore, in [28], the defining equations of the cut ideal of a graph are described in terms of the defining equations of the cut ideals of smaller graphs provided the graph G may be decomposed in an appropriate manner as the "clique sum" of smaller graphs. Trees have the appropriate decomposition. Assertion (3.26) is an immediate consequence of the fact that the Hilbert **function** of a Segre product is the product of the Hilbert functions. Assertion (3.25) is established in [15, Thms. 4.2.3, 4.4.4, 4.3.3]. In particular, $\mathfrak{S}$ is Gorenstein because it is the Segre product of rings with the same a-invariant.

On the other hand, the generating function $\sum_{i\geq 0}(i+1)^s t^i$ for $\mathrm{HS}_{\mathfrak{S}}(t)$, given in (3.26), is a formal power series of historical interest in Combinatorics. Indeed, if $A_s(t)$ is the polynomial defined by

$$\sum_{i\geq 0} i^s t^i = \frac{A_s(t)}{(1-t)^{s+1}}, \tag{3.27}$$

then $A_s(t)$ is called the sth *Eulerian polynomial* and, if s is positive, then

$$A_s(t) = \sum_{k=1}^{s} A(s,k)t^k,$$

for positive integers $A(s,k)$, which are called *Eulerian numbers*; see Proposition 1.4.4 and display (1.36) in [29]. The first few Eulerian polynomials are also given in [29]:

$$\begin{aligned}
A_0(t) &= 1\\
A_1(t) &= t\\
A_2(t) &= t + t^2\\
A_3(t) &= t + 4t^2 + t^3\\
A_4(t) &= t + 11t^2 + 11t^3 + t^4\\
A_5(t) &= t + 26t^2 + 66t^3 + 26t^4 + t^5.
\end{aligned}$$

The value

$$A_s(-1) = \begin{cases} (-1)^{(s+1)/2}E_s, & \text{if } s \text{ is odd}\\ 0, & \text{if } s \text{is even and positive}\end{cases}$$

may be found in Exercise 135, at the end of Chapter 1 of [29], where E_s is the sth *Euler number* as defined in section 1.6.1 of [29]. The Euler numbers are positive; they satisfy the recurrence relation

$$2E_{s+1} = \sum_{k=0}^{s} \binom{s}{k} E_k E_{s-k} \text{ for } 1 \leq s, \text{ and } E_0 = E_1 = 1;$$

and they are related to the coefficients in the Maclaurin series for $\sec x + \tan x$ in the sense that

$$\sec x+\tan x=\sum_{s\geq 0}E_s\frac{x^s}{s!}.$$

We apply Corollary 3.3 to prove that S does not have any homogeneous exact zero divisors when $s\geq 3$ is an odd integer. The ideal I, which defines S, is generated by forms of degree 2 according to (3.24); so it suffices to show that $\frac{t^2-1}{t-1}=t+1$ does not divide the Hilbert series $\mathrm{HS}_S(t)$, in the sense of (0.1); that is, it suffices to show that $\mathrm{HS}_S(-1)\neq 0$. On the other hand, $\mathfrak{S}$ is a Cohen–Macaulay ring of dimension $s+1$ and $S=\mathfrak{S}/(\ell)$, where ℓ is a linear system of parameters in $\mathfrak{S}$. It follows that

$$\mathrm{HS}_{\mathfrak{S}}(t)=\frac{\mathrm{HS}_S(t)}{(1-t)^{\dim\mathfrak{S}}}=\frac{\mathrm{HS}_S(t)}{(1-t)^{s+1}}. \tag{3.28}$$

Combine (3.28), (3.26), and (3.27) to see that

$$\frac{\mathrm{HS}_S(t)}{(1-t)^{s+1}}=\mathrm{HS}_{\mathfrak{S}}(t)=\sum_{i\geq 0}(i+1)^s t^i=\frac{\sum\limits_{i\geq 0} i^s t^i}{t}=\frac{\frac{A_s(t)}{t}}{(1-t)^{s+1}};$$

thus, $\mathrm{HS}_S(t)=\frac{A_s(t)}{t}$ and $\mathrm{HS}_S(-1)=-A_s(-1)=(-1)^{(s-1)/2}E_s\neq 0$. $\square$

Observation 3.29. *If $s=3$, then the ring S of Proposition 3.23 is also studied in Proposition 3.11; however, if $s\geq 5$, then the ring S of Proposition 3.23 is not studied in Proposition 3.11.*

Proof. If $s=3$, then the ring S of Proposition 3.23 is defined by the 2×2 minors of the unit cube with W_{i_0,i_1,i_2} placed on the vertex (i_0,i_1,i_2), with $i_j\in\{0,1\}$. The defining ideal of S has nine minimal generators: one for each of the six faces of the unit cube and one for the intersection of the unit cube with each of the planes $x=y$, $x=z$, and $y=z$. The resolution of the Gorenstein ring S over $P=k[\{W_{i_0,i_1,i_2}\}]$ is linear:

$$0\to P(-6)\to P(-4)^9\to P(-3)^{16}\to P(-2)^9\to P\to S,$$

and S is studied in Example 3.17; hence also in Proposition 3.11. (One can use Macaulay2 for this calculation.)

If $S=P/I$, as described in Data 3.7, is a compressed Gorenstein algebra with the minimal generator degree of I equal to 2 and socle degree equal to e, then

$$\min\{\dim_k[P]_2,\ \dim_k[P]_{e-2}\}=\mathrm{HF}_S(2)<\dim_k[P]_2;$$

and therefore, $e\leq 3$. The algebras S of Theorem 3.23 are defined by ideals generated in degree 2 and have socle degree $s-1$ because

$$\sum_{i\geq 0}\dim[S]_i t^i=\mathrm{HS}_S(t)=\frac{A_s(t)}{t}=\sum_{i=0}^{s-1}A(s,i+1)t^i,$$

with $A(s,s) = 1$. (The number $A(s,k)$ counts the permutations of $\{1,\dots,s\}$ with exactly $k-1$ descents; see [29, (1.36)]; in particular, $A(s,s) = 1$.) If $5 \le s$, then $3 < s-1$; hence, the socle degree of S is more than 3 and S is not a compressed Gorenstein algebra. □

Example 3.30. Let k be a field, r and c be integers with $2 \le r \le c$, X be a $r \times c$ matrix of indeterminates, $I_2(X)$ be the ideal of $k[X]$ generated by the 2×2 minors of X, and S be an Artinian reduction of $k[X]/I_2(X)$, as described in Conventions 1.1. If any of the following conditions hold:

(1) $r = c$ and this common number is odd, or
(2) $r < c \le r+3$, or
(3) $c-1$ does not divide $(r-1)!$,

then S does not have any homogeneous exact zero divisors.

Remark. The constraint $c \le r+3$ in condition (2) has been artificially imposed. We do not know any real upper constraint on c; indeed condition (3) applies to all large c, once r is fixed.

Proof. Let R be the ring $k[X]/I_2(X)$ and d be the Krull dimension of R. It is well-known; see, for example, [7, Cor. 4], [12, Cor. 1], [24], or [1], that

$$\mathrm{HS}_R(t) = \frac{\sum\limits_{i=0}^{r-1} \binom{r-1}{i}\binom{c-1}{i} t^i}{(1-t)^d}.$$

The ring S is equal to $R/(\underline{\ell})$, where $\underline{\ell}$ is a regular sequence $\ell_1, \dots, \ell_d$ of homogeneous linear forms on R. It follows that $\mathrm{HS}_S(t) = \sum\limits_{i=0}^{r-1} \binom{r-1}{i}\binom{c-1}{i} t^i$. We apply Corollary 3.3. The defining ideal for S is generated by homogeneous forms of degree 2. It suffices to show that $\frac{t^2-1}{t-1} = t+1$ does not divide $\mathrm{HS}_S(t)$, in the sense of (0.1). In other words, it suffices to show that $\mathrm{HS}_S(-1) \neq 0$.

For positive integers $a \le b$, define $N_{a,b}$ to be the integer

$$N_{a,b} = \left| \sum_{i=0}^{a} (-1)^i \binom{a}{i}\binom{b}{i} \right|.$$

We see that $|\,\mathrm{HS}_S(-1)| = N_{r-1,c-1}$. Note that $N_{a,b}$ is equal (up to sign) to the coefficient of x^a in $(x-1)^a(x+1)^b$, since the latter can be found as

$$\sum_{i+j=a} (-1)^{a-i} \binom{a}{i}\binom{b}{j} = \sum_{j=0}^{a} (-1)^j \binom{a}{a-j}\binom{b}{j}.$$

It follows from $(x-1)^a(x+1)^b = (x^2-1)^a(x+1)^{b-a}$ that

$$N_{a,a} = \begin{cases} 0 & \text{when } a \text{ is odd} \\ \binom{a}{\frac{a}{2}} & \text{when } a \text{ is even,} \end{cases}$$

$$N_{a,a+1} = \begin{cases} \binom{a}{\frac{a-1}{2}} & \text{when } a \text{ is odd} \\ \binom{a}{\frac{a}{2}} & \text{when } a \text{ is even,} \end{cases}$$

$$N_{a,a+2} = \begin{cases} 2\binom{a}{\frac{a-1}{2}} & \text{when } a \text{ is odd} \\ \binom{a}{\frac{a}{2}} - \binom{a}{\frac{a-2}{2}} & \text{when } a \text{ is even, and} \end{cases}$$

$$N_{a,a+3} = \begin{cases} 3\binom{a}{\frac{a-1}{2}} - \binom{a}{\frac{a-3}{2}} & \text{when } a \text{ is odd} \\ |\binom{a}{\frac{a}{2}} - 3\binom{a}{\frac{a-2}{2}}| & \text{when } a \text{is even.} \end{cases}$$

Note that in all of the above cases (with the exception of $a = b = \text{odd}$) we have $N_{a,b} \neq 0$.

Also note that if $N_{a,b} = 0$, then b must divide $a!$. In order to see this, we view $N_{a,b}$ as a polynomial of degree a with rational coefficients in the variable b. After clearing the denominators, the constant term is $a!$ and the coefficient of b^a is $(-1)^a$. $\square$

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