



Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Cohomology of Lie algebras of polynomial vector fields on the line over fields of characteristic 2



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ARTICLE INFO

Article history:

Received 28 June 2016

Available online 11 April 2017

Communicated by Vera Serganova

Keywords:

Cohomology

Lie algebras

ABSTRACT

For a field \mathbb{F} , let $L_k(\mathbb{F})$ be the Lie algebra of derivations $f(t) \frac{d}{dt}$ of the polynomial ring $\mathbb{F}[t]$, where $f(t)$ is a polynomial of degree $> k$. For any $k \geq -1$, we present a basis of the space of the cohomology with finite-dimensional support of the Lie algebra $L_k(\mathbb{F})$ with coefficients in the trivial module \mathbb{F} for the case where $\text{char}(\mathbb{F}) = 2$. The main result obtained is an analog of the famous Goncharova's Theorem for the case $\text{char}(\mathbb{F}) = 0$ and $k \geq 1$.

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Introduction

Let $W(\mathbb{F})$ be the vector space of polynomials in the indeterminate t over a field \mathbb{F} ; we consider $W(\mathbb{F})$ as the Lie algebra with commutator

$$[f_1(t), f_2(t)] = f_1(t)f_2'(t) - f_1'(t)f_2(t),$$

where $f'(t)$ denotes the derivative of the polynomial $f(t)$. This algebra is called the *Lie algebra of polynomial vector fields on the line over \mathbb{F}* . The vectors $e_i = t^{i+1}$, where $i \geq -1$, form a basis of $W(\mathbb{F})$, and

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$$[e_a, e_b] = (b - a)e_{a+b}.$$

The Lie algebra $W(\mathbb{F})$ contains a decreasing sequence of the Lie subalgebras

$$W(\mathbb{F}) = L_{-1}(\mathbb{F}) \supset L_0(\mathbb{F}) \supset L_1(\mathbb{F}) \supset L_2(\mathbb{F}) \supset \cdots,$$

where $L_k(\mathbb{F}) \subset W(\mathbb{F})$ is spanned by the vectors e_i with $i \geq k$.

In this article we consider (co)homology of the Lie algebras $L_k(\mathbb{F})$ with coefficients in the trivial module \mathbb{F} . Since $L_k(\mathbb{F})$ is graded by integers $\geq k$ (degrees), the corresponding chain Chevalley–Eilenberg complex decomposes into a direct sum of the finite-dimensional subcomplexes $C_*^{(n)}(L_k(\mathbb{F}))$, enumerated by integers $n \geq k$ (degrees of the chains).

Let us denote by $H_{(n)}^*(L_k(\mathbb{F}))$ the homology space of the dual complex $\text{Hom}(C_*^{(n)}(L_k(\mathbb{F})), \mathbb{F})$ and define the cohomology of $L_k(\mathbb{F})$ by the formula

$$H^*(L_k(\mathbb{F})) := \bigoplus_{n \geq k}^{\infty} H_{(n)}^*(L_k(\mathbb{F})).$$

Since for any $k \geq -1$ the structure constants of the algebra L_k in the basis e_i are integers, then to compute cohomology of $L_k(\mathbb{F})$ over any field \mathbb{F} it suffices to assume that \mathbb{F} is a prime field. That is, $\mathbb{F} = \mathbb{Q}$, the field of rational numbers, if $\text{char}(\mathbb{F}) = 0$, and $\mathbb{F} = \mathbb{Z}_p$, the field with p elements, if $\text{char}(\mathbb{F}) = p > 0$.

It turns out that the dimensions of the spaces $H^q(L_k(\mathbb{Q}))$ are finite. It is easy to find them for $k = -1, 0$. For $k \geq 1$, these dimensions were predicted by D. B. Fuchs. His formula was included in I. M. Gel'fand's talk [3] as a conjecture. It was proved by L. V. Goncharova in [4].

In article [6] (see also [7]) for any $k \geq 1$, a special “filtering” basis in the dual to the chain Chevalley–Eilenberg complex of $L_k(\mathbb{Q})$ was offered. In this basis, the coboundary operator acts simply enough and in particular allows to obtain a transparent proof of Goncharova's result.

For $p > 0$ and $q > 0$ the spaces $H^q(L_k(\mathbb{Z}_p))$ in general are infinite-dimensional. But the above definition implies that each of them is a direct sum of the uniquely defined finite-dimensional subspaces:

$$H^q(L_k(\mathbb{Z}_p)) \simeq \bigoplus_{n \geq qk + \frac{q(q-1)}{2}} H_{(n)}^q(L_k(\mathbb{Z}_p)).$$

The importance of Goncharova's result is explained in the book by D. B. Fuchs [2]. In its Russian edition, as well as in private conversations, D. B. Fuchs repeatedly asked “what is the analog of Goncharova's theorem over fields of characteristic $p > 0$?”. Here I offer an answer for $p = 2$.

In this article, for any $k \geq 1$, I build an analog of “filtering” basis for the Lie algebra $L_k(\mathbb{Z}_2)$ and apply it to the computation of space $H^*(L_k(\mathbb{Z}_2))$. In §5, I formulate

a theorem that explicitly describes a basis of $H^*(L_k(\mathbb{Z}_2))$ and compute the numbers $\dim H_{(n)}^q(L_k(\mathbb{Z}_2))$. A detailed proof is given only in the most interesting case $k = 1$. For $k > 1$, one can prove the theorem from §5 by induction on k , similarly to what was done in [6] for $\mathbb{F} = \mathbb{Q}$.

The construction of a “filtering” basis for $\mathbb{F} = \mathbb{Z}_2$ follows the same scheme as for $\mathbb{F} = \mathbb{Q}$: first, I build a special set of vectors, which spans the space $C^*(L_1(\mathbb{Z}_2))$. Their linear independence follows from a combinatorial formula. When $\mathbb{F} = \mathbb{Q}$ this formula is equivalent to the Sylvester identity in the partitions theory (see [1] or [8]). The corresponding formula in case of $\mathbb{F} = \mathbb{Z}_2$ leads to another interesting identity for partitions, which is established in [8] (see formula (5) below).

This article is organized as follows. In §1, for any $k \geq 1$, I introduce a complex, for which the cohomology will be computed.

In §2–§6, I consider only the case $k = 1$. The definitions related with integer partitions necessary to formulate the theorem on a “filtering” basis for $k = 1$ are collected in §2.

The “filtering” basis theorem (Theorem 3.3) is formulated and proved in §3.

In §4, I use this result to build a basis of the bigraded space $H^*(L_1(\mathbb{Z}_2))$ and compute the dimensions of its homogeneous components (Theorem 4.7).

In §5, I formulate a theorem describing a basis of the space $H^*(L_k(\mathbb{Z}_2))$ for any $k \geq 1$, thus generalizing the result obtained for $k = 1$ in §4.

In §6, I compute, partly, the multiplication in $H^*(L_1(\mathbb{Z}_2))$. In particular, I show that, as algebra, $H^*(L_1(\mathbb{Z}_2))$ is generated by 1- and 2-dimensional cohomological classes with a non-trivial product. In addition I formulate a conjecture that completely describes the multiplicative structure of $H^*(L_1(\mathbb{Z}_2))$. Observe that the multiplication in the algebra $H^*(L_1(\mathbb{Q}))$ is trivial (see [7], Remark 2.7).

In §7, I use the results of §4 to compute the cohomology of the Lie algebras $L_0(\mathbb{Z}_2)$ and $L_{-1}(\mathbb{Z}_2)$. As an application, I offer an explicit description of the set of equivalence classes of *central extensions of Lie algebra* $W(\mathbb{Z}_2) = L_{-1}(\mathbb{Z}_2)$ *with one-dimensional kernel*, i.e. the set of exact sequences of the Lie algebras

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W}(\mathbb{Z}_2) \longrightarrow W(\mathbb{Z}_2) \longrightarrow 0$$

considered up to the natural isomorphisms between such sequences (see [5], Sec. 7.6).

Notation:

$|M|$ is the cardinality of the finite set M .

$M_1 \sqcup M_2$ is the disjoint union of the sets M_1 and M_2 .

For a finite set of integers I , set $I^- := \min(I)$.

For an integer a , let $\beta(a) := \begin{cases} 0 & \text{if } a \equiv 0 \pmod{2}, \\ 1 & \text{if } a \equiv 1 \pmod{2}. \end{cases}$

The symbol L_k is used as a synonym of $L_k(\mathbb{Z}_2)$.

In what follows all vector spaces are defined over the field \mathbb{Z}_2 .

1. The complex $C^*(L_k)$

The homology of the Lie algebra L_k with coefficients in the trivial module \mathbb{Z}_2 is the homology of complex

$$0 \longleftarrow C_0(L_k) \xleftarrow{d} C_1(L_k) \xleftarrow{d} \cdots \xleftarrow{d} C_{q-1}(L_k) \xleftarrow{d} C_q(L_k) \longleftarrow \cdots,$$

where $C_q(L_k) := \bigwedge^q L_k$ is the restricted (i.e., with finite support of each element) q th exterior power of the space L_k . The vectors of the space $C_q(L_k)$ are called the q -chains of algebra L_k . Set $C_*(L_k) = \bigoplus_{q=0}^{\infty} C_q(L_k)$.

Obviously, the set of q -chains $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}$, where $i_1 < i_2 < \cdots < i_q$, forms a basis of $C_q(L_k)$. The action of the boundary operator d on such a chain is defined by the formula

$$d(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}) = \sum_{1 \leq a < b \leq q} (i_a + i_b) e_{i_a + i_b} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_a} \wedge \cdots \wedge \widehat{e}_{i_b} \wedge \cdots \wedge e_{i_q}.$$

This formula implies that the subspace $C_*^{(n)}(L_k) \subset C_*(L_k)$, spanned by the chains $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}$ with $i_1 + i_2 + \cdots + i_q = n$, is a finite-dimensional subcomplex, and $C_*(L_k) = \bigoplus_{n \geq 0} C_*^{(n)}(L_k)$ is a direct sum of complexes. (By definition, $C_0^{(n)}(L_k) = \{0\}$ for any appropriate integer n .)

Define the space of q -cochains of algebra L_k as

$$C^q(L_k) := \bigoplus_{n \geq 0} C_{(n)}^q(L_k), \quad \text{where} \quad C_{(n)}^q(L_k) := \text{Hom}_{\mathbb{F}}(C_q^{(n)}(L_k), \mathbb{Z}_2).$$

Set $C^*(L_k) = \bigoplus_{q=0}^{\infty} C^q(L_k)$.

On $C_*(L_k)$ let us introduce an Euclidean metric such that the set of chains $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}$ with $i_1 < i_2 < \cdots < i_q$ constitutes an orthonormal basis. For any q and appropriate n this metric defines a unique isomorphism $C_q^{(n)}(L_k) \cong C_{(n)}^q(L_k)$ which allows us to treat the q -chains of the algebra L_k as its q -cochains.

Then the action of the coboundary operator δ_k , i.e. the one dual to d , is expressed by the formulas:

$$\delta_k(e_i) = \sum_{a+b=i, \ k \leq a < b} (a+b) e_a \wedge e_b, \quad (1)$$

$$\delta_k(e_{i_1} \wedge \cdots \wedge e_{i_q}) = \sum_{1 \leq a \leq q} e_{i_1} \wedge \cdots \wedge \delta_k(e_{i_a}) \wedge \cdots \wedge e_{i_q}. \quad (2)$$

Definition 1.1. For any integer $n \geq 0$, the cohomology space of the complex

$$0 \longrightarrow C_{(n)}^1(L_k) \xrightarrow{\delta_k} C_{(n)}^2(L_k) \xrightarrow{\delta_k} \cdots \xrightarrow{\delta_k} C_{(n)}^{q-1}(L_k) \xrightarrow{\delta_k} C_{(n)}^q(L_k) \xrightarrow{\delta_k} \cdots$$

is denoted by $H_{(n)}^*(L_k)$.

Remark 1.2. Let $\mathcal{I} \subset L_{-1}$ be an ideal. The adjoint action of L_{-1} on \mathcal{I} induces an action of L_{-1} on $C^*(\mathcal{I})$, which commutes with the coboundary operator. Thus, $H^*(\mathcal{I})$ turns into an L_{-1} -module. In our treatment of the cochains the action of $e_r \in L_{-1}$ on $C^*(\mathcal{I})$ is uniquely defined by the formula

$$e_r(e_{i_1} \wedge \cdots \wedge e_{i_q}) = \sum_{a=1}^q i_a e_{i_1} \wedge \cdots \wedge e_{i_a-r} \wedge \cdots \wedge e_{i_q}. \quad (3)$$

2. Partitions

This section gathers some material pertaining to partitions and is used throughout the article.

Definition 2.1. A *partition* is a finite ordered set of positive integers $\langle i_1, i_2, \dots, i_q \rangle$, referred to the *parts of the partition* such that $i_1 \leq i_2 \leq \dots \leq i_q$. A partition is said to be *strict* if $i_1 < i_2 < \dots < i_q$.

The numbers $\|I\| = i_1 + i_2 + \dots + i_q$ and $|I| = q$ are called the *degree* and *length* of I , respectively.

A *subpartition* of I is an ordered subset $I' \subset I$. The *union* $I_1 \sqcup I_2$ of *partitions* I_1 and I_2 is the partition whose set of parts is the disjoint union of the sets I_1 and I_2 .

Definition 2.2. Let I and I' be distinct partitions such that $\|I\| = \|I'\|$. We write $I' \triangleleft I$ if either $|I'| < |I|$, or

$$|I'| = |I| = q \quad \text{and} \quad i'_1 + \dots + i'_r \leq i_1 + \dots + i_r \quad \text{for any } r \in [1, q].$$

Definition 2.3. A *marked partition* is a pair $\langle I; J \rangle$, where I is a partition and $J \subset I$. The elements of J are called the *marked parts*. We identify each partition I with the marked partition $\langle I; \emptyset \rangle$. The numbers

$$\|\langle I; J \rangle\| := \|I\|, \quad |\langle I; J \rangle| := |I| + |J|, \quad \text{and} \quad |I|$$

are called the *degree*, the *length*, and the *reduced length* of $\langle I; J \rangle$, respectively.

A marked partition $\langle I; J \rangle$ is called *strict* if I is strict.

Define: $\langle I_1; J_1 \rangle \sqcup \langle I_2; J_2 \rangle := \langle I_1 \sqcup I_2; J_1 \sqcup J_2 \rangle$.

Instead of explicitly indicating the set of marked parts, we often underline these parts in I . For example, $\langle 1, 4, 6, 7; 4, 7 \rangle = \langle 1, \underline{4}, 6, \underline{7} \rangle$.

Definition 2.4. Let $\langle I; J \rangle$ and $\langle I'; J' \rangle$ be distinct marked partitions. We write $\langle I'; J' \rangle \triangleleft \langle I; J \rangle$ if either $I' \triangleleft I$, or $I' = I$ and $J' \prec J$, where \prec stands for the lexicographical order.

For example, $\langle \underline{5} \rangle \triangleleft \langle 2, 3 \rangle$, $\langle 3, \underline{9} \rangle \triangleleft \langle 5, \underline{7} \rangle$ and $\langle \underline{3}, 6 \rangle \triangleleft \langle 3, \underline{6} \rangle$.

One can readily show that the \triangleleft is a partial order on the set of marked partitions. Below, we use the following

Lemma 2.5 ([7]). If $\langle I'; J' \rangle \trianglelefteq \langle I; J \rangle$ and $\langle I'_1; J'_1 \rangle \triangleleft \langle I_1; J_1 \rangle$, then $\langle I'; J' \rangle \sqcup \langle I'_1; J'_1 \rangle \triangleleft \langle I; J \rangle \sqcup \langle I_1; J_1 \rangle$.

Definition 2.6. A partition $I = \langle i_1, i_2, \dots, i_q \rangle$ is called *regular* if $i_{m+1} - i_m \geq 2$ for any $m \in [1, q-1]$.

A *dense* partition is a partition of the form $\langle a, a+2, a+4, \dots, a+2(q-1) \rangle$.

A *special* partition is a dense partition of the form $\langle 1, 3, \dots, 2q-1 \rangle$.

An *even* or *odd* partition is a partition all parts of which are either even or odd, respectively.

Definition 2.7. For any regular partition I , there is a unique decomposition $I = I_1 \sqcup I_2 \sqcup \dots \sqcup I_s$, where I_1, I_2, \dots, I_s are the dense subpartitions of the maximal possible length. This decomposition is called the *canonical decomposition* of I . The dense partitions I_1, I_2, \dots, I_s are said to be *simple components* of I .

The minimal parts of the odd non-special simple components are called the *leading parts* of I . The quantity of leading parts is denoted by $\text{ind}(I)$.

Definition 2.8. A marked partition $\langle I; J \rangle$ is called *regular* if I is regular and J is a subset of the set of leading parts of I . Otherwise it is called *singular*.

For a regular marked partition $\langle I; J \rangle$, the decomposition

$$\langle I; J \rangle = \langle I_1; J_1 \rangle \sqcup \langle I_2; J_2 \rangle \sqcup \dots \sqcup \langle I_s; J_s \rangle,$$

where $I = I_1 \sqcup I_2 \sqcup \dots \sqcup I_s$ is the canonical decomposition, is called the *canonical decomposition* of $\langle I; J \rangle$. The marked k -partitions $\langle I_1; J_1 \rangle, \langle I_2; J_2 \rangle, \dots, \langle I_s; J_s \rangle$ are called its *simple components*.

3. A basis of e -monomials of the complex $C^*(L_1)$

Definition 3.1. Given $\langle I; J \rangle$, where $I = \langle i_1, i_2, \dots, i_q \rangle$, define

$$e_{\langle I; J \rangle} = \widehat{e}_{i_1} \wedge \widehat{e}_{i_2} \wedge \dots \wedge \widehat{e}_{i_q} \in C^*(L_1), \quad \text{where} \quad \widehat{e}_{i_a} = \begin{cases} e_{i_a} & \text{if } i_a \notin J, \\ \delta_1(e_{i_a}) & \text{if } i_a \in J. \end{cases}$$

A nonzero cochain of the form $e_{\langle I; J \rangle} \in C^*(L_k)$ is called an *e -monomial*.

For example, $e_{\langle I; J \rangle}$ is an e -monomial for any regular marked partition $\langle I; J \rangle$.

Keeping in mind the correspondence between marked partitions and e -monomials, we apply the notions related to such partitions (degree, length, regularity, order \triangleleft , etc.), to e -monomials. Lemma 2.5 implies the following Lemma:

Lemma 3.2. If $e_{\langle I;J \rangle} \trianglelefteq e_{\langle I';J' \rangle}$, $e_{\langle I_1;J_1 \rangle} \triangleleft e_{\langle I'_1;J'_1 \rangle}$, and $e_{\langle I';J' \rangle} \wedge e_{\langle I'_1;J'_1 \rangle} \neq 0$, then

$$e_{\langle I;J \rangle} \wedge e_{\langle I_1;J_1 \rangle} \triangleleft e_{\langle I';J' \rangle} \wedge e_{\langle I'_1;J'_1 \rangle}.$$

The next claim is the main result of this section:

Theorem 3.3. The set of regular e -monomials forms a basis of the space $C^*(L_1)$. The decomposition of any singular e -monomial $e_{\langle I;J \rangle}$ in this basis has the form

$$e_{\langle I;J \rangle} = \sum_{\langle I';J' \rangle} e_{\langle I';J' \rangle}, \quad (4)$$

where $\langle I';J' \rangle \triangleleft \langle I;J \rangle$ for any $\langle I';J' \rangle$ included in this sum.

Proof. Let $D(n, q)$ be the set of partitions of degree n and length q , and let $M(n, q)$ be the set of regular marked partitions of degree n and length q . Since $\dim C^q_{(n)}(L_1) = |D(n, q)|$, and assuming the existence of decomposition (4), linear independence of the regular e -monomials follows from the combinatorial identity

$$|D(n, q)| = |M(n, q)|. \quad (5)$$

This identity is proved in [8] (Theorem 2.1 for $\lambda = 2$).

Therefore, it is sufficient to establish only the existence of the decomposition (4). The set of singular e -monomials of reduced length 2 is exhausted by the following e -monomials

$$e_i \wedge e_{i+1}, \quad e_{2i} \wedge \delta_1(e_{2i+1}), \quad e_{2i-1} \wedge \delta_1(e_{2i+1}), \quad \delta_1(e_{2i-1}) \wedge \delta_1(e_{2i+1}),$$

where $i \geq 1$ is an appropriate integer. For this set of e -monomials of degree n , the decomposition (4) directly follows from the easily checked identities

$$\sum_{a+b=n, 1 \leq a < b} e_a \wedge e_b = \delta_1(e_n) \quad \text{if } n \equiv 1 \pmod{2}, \quad (6)$$

$$\sum_{a+b=n, a, b \geq 1} e_a \wedge \delta_1(e_b) = 0, \quad (7)$$

$$\sum_{a+b=n, 1 \leq a < b} \delta_1(e_a) \wedge \delta_1(e_b) = 0 \quad \text{if } n \not\equiv 2 \pmod{4}. \quad (8)$$

Using these identities we can express any singular submonomial of reduced length ≤ 2 in $e_{\langle I;J \rangle}$ as a linear combination of the regular ones. As a result, we present the monomial $e_{\langle I;J \rangle}$ as a linear combination of e -monomials $e_{\langle I';J' \rangle}$ such that $e_{\langle I';J' \rangle} \triangleleft e_{\langle I;J \rangle}$ as it follows from Lemma 3.2.

Let us now apply the same procedure to each singular e -monomial $e_{\langle I';J' \rangle}$ of the obtained linear combination. Since the set of e -monomials of fixed degree is finite, a finite number of such iterations leads to the decomposition (4). \square

4. Computing the space $H^*(L_1)$

Let M_0 be the set of marked regular partitions. For $r \geq 1$, define

$$M_r := M_{r-1} \setminus \{\text{the set of maximal partitions (with respect to order } \triangleleft \text{) in } M_{r-1}\}.$$

In view of [Theorem 3.3](#), the sequence of partially ordered sets $M_0 \supset M_1 \supset M_2 \supset \dots$ induces a filtration of the vector spaces

$$C^*(L_1) = C^*(M_0) \supset C^*(M_1) \supset C^*(M_2) \supset \dots, \quad (9)$$

where $C^*(M_r) \subset C^*(L_1)$ is the subspace spanned by regular e -monomials $e_{\langle I; J \rangle}$ such that $\langle I; J \rangle \in M_r$.

For the canonical decomposition $\langle I; J \rangle = \langle I_1; J_1 \rangle \sqcup \dots \sqcup \langle I_s; J_s \rangle$, formula [\(2\)](#) implies that

$$\delta_1(e_{\langle I; J \rangle}) = \sum_{1 \leq a \leq s} e_{\langle I_1; J_1 \rangle} \wedge \dots \wedge \delta_1(e_{\langle I_a; J_a \rangle}) \wedge \dots \wedge e_{\langle I_s; J_s \rangle}. \quad (10)$$

This expression, [Theorem 3.3](#), and [Lemma 3.2](#) show that [\(9\)](#) is a filtration of the complex $C^*(L_1)$.

Definition 4.1. For any $x, y \in C^*(M_r)$, we write $x \approx y$ if $x - y \in C^*(M_{r+1})$.

Lemma 4.2. Let I be a dense partition. Then

$$\delta_1(e_I) \approx \begin{cases} 0 & \text{if } I \text{ is special or even,} \\ \beta(|I|) e_{\langle I; I^- \rangle} & \text{otherwise.} \end{cases}$$

Proof. For any special or even partition I , it is clear since $\delta_1(e_I) = 0$ for such a partition.

Formula [\(7\)](#) implies that $e_{a+2r} \wedge \delta_1(e_{a+2(r+1)}) \approx \delta_1(e_{a+2r}) \wedge e_{a+2(r+1)}$ for any $r \geq 0$. Therefore, for the remaining partitions, [Lemma 4.2](#) follows from formula [\(2\)](#) and [Lemma 3.2](#). \square

Lemma 4.3. Let I be a dense odd non-special partition of even length. Then there exist cocycles ε_I and $\varepsilon_{\langle I; I^- \rangle}$ of the complex $C^*(L_1)$ such that

$$\varepsilon_I \approx e_I \quad \text{and} \quad \varepsilon_{\langle I; I^- \rangle} \approx e_{\langle I; I^- \rangle}.$$

Proof. Formulas [\(7\)](#) and [\(8\)](#) imply that, for $a \geq 3$ odd, the cochains

$$\varepsilon_{\langle a, a+2 \rangle} := \sum_{r=0}^{\frac{a-1}{2}} e_{a-2r} \wedge e_{a+2r+2}, \quad \varepsilon_{\langle \underline{a}, a+2 \rangle} := \sum_{r=0}^{\frac{a-1}{2}} \delta_1(e_{a-2r}) \wedge e_{a+2r+2} \quad (11)$$

are cocycles of the complex $C^*(L_1)$. Therefore, for $I = \langle a, a+2, \dots, a+2(q-1) \rangle$, q even, and $a \geq 3$ odd, formula (10) implies that the following cochains are cocycles as well:

$$\begin{aligned}\varepsilon_I &:= \varepsilon_{\langle a, a+2 \rangle} \wedge \varepsilon_{\langle a+4, a+6 \rangle} \wedge \dots \wedge \varepsilon_{\langle a+2(q-2), a+2(q-1) \rangle}, \\ \varepsilon_{\langle I; I^- \rangle} &:= \varepsilon_{\langle \underline{a}, a+2 \rangle} \wedge \varepsilon_{\langle a+4, a+6 \rangle} \wedge \dots \wedge \varepsilon_{\langle a+2(q-2), a+2(q-1) \rangle}.\end{aligned}$$

The formulas $\varepsilon_I \approx e_I$ and $\varepsilon_{\langle I; I^- \rangle} \approx e_{\langle I; I^- \rangle}$ follow from Theorem 3.3 and Lemma 3.2. \square

For dense partitions I not mentioned in Lemma 4.3, define

$$\varepsilon_I := e_I, \quad \text{and} \quad \varepsilon_{\langle I; I^- \rangle} := \delta_1(e_I) \quad \text{if } I \text{ is odd, non-special, and } \beta(|I|) = 1.$$

These formulas, together with the formulas from Lemma 4.3, define cochains ε_I and $\varepsilon_{\langle I; I^- \rangle}$ for any dense regular partition I . Cochains of such a form are said to be the *simple ε -monomials*.

Definition 4.4. Let $\langle I; J \rangle = \langle I_1; J_1 \rangle \sqcup \dots \sqcup \langle I_s; J_s \rangle$ be the canonical decomposition of a regular partition. Define $\varepsilon_{\langle I; J \rangle} := \varepsilon_{\langle I_1; J_1 \rangle} \wedge \dots \wedge \varepsilon_{\langle I_s; J_s \rangle}$. The cochain $\varepsilon_{\langle I; J \rangle}$ is called the ε -monomial corresponding to $\langle I; J \rangle$. Its simple \wedge -factors are called the *simple components* of the ε -monomial $\varepsilon_{\langle I; J \rangle}$.

Since the matrix of passage from the set of ε -monomials to the basis of regular e -monomials is a square lower triangle matrix with units on the main diagonal, Theorem 3.3 and Lemmas 3.2, 4.2, and 4.3 imply the following result:

Theorem 4.5. *The set of ε -monomials is a basis of the complex $C^*(L_1)$. For a simple ε -monomial $\varepsilon_{\langle I; J \rangle}$, we have:*

$$\delta_1(\varepsilon_{\langle I; J \rangle}) = \begin{cases} \varepsilon_{\langle I; I^- \rangle} & \text{if } I \text{ is an odd non-special partition, } J = \emptyset, \text{ and } \beta(|I|) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

From this theorem, formula (10), and the definition of the tensor product of complexes, we obtain

Corollary 4.6. *For a regular partition I , let $T^*(I) \subset C^*(L_1)$ be the linear span of the set of ε -monomials $\varepsilon_{\langle I; J \rangle}$. Then $T^*(I)$ is a subcomplex in $C^*(L_1)$ and*

$$C^*(L_1) = \bigoplus_I T^*(I),$$

where I runs over the set of regular partitions. For the canonical decomposition $I = I_1 \sqcup I_2 \sqcup \dots \sqcup I_s$, we have an isomorphism of complexes

$$T^*(I) \simeq T^*(I_1) \otimes T^*(I_2) \otimes \dots \otimes T^*(I_s).$$

Theorem 4.7. Let $R(n)$ be the set of the regular partitions of degree n , for which each simple component is either special, or has an even degree.

Then any ε -monomial $\varepsilon_{\langle I; J \rangle}$, where $I \in R(n)$, is a nonzero cocycle of the complex $C^*(L_1)$. Cohomological classes of these cocycles form a basis of the space $H_{(n)}^*(L_1)$. Moreover, the classes with $|I| + |J| = q$ form a basis of the space $H_{(n)}^q(L_1)$. In particular,

$$\sum_{q=1}^{\infty} \dim H_{(n)}^q(L_1) t^q = \sum_{I \in R(n)} (1+t)^{\text{ind}(I)} t^{|I|}.$$

Proof. Theorem 4.5 implies that, for a dense partition I , we have

$$H^*(T^*(I)) = \begin{cases} 0 & \text{if } I \text{ is odd non-special partition and } \beta(|I|) = 1, \\ T^*(I) & \text{otherwise.} \end{cases}$$

Therefore Theorem 4.7 follows from Corollary 4.6 and the Künneth isomorphism

$$H^*(T^*(I)) \simeq H^*(T^*(I_1)) \otimes H^*(T^*(I_2)) \otimes \cdots \otimes H^*(T^*(I_s)). \quad \square$$

For example, $R(12) = \{\langle 12 \rangle, \langle 2, 10 \rangle, \langle 4, 8 \rangle, \langle 5, 7 \rangle, \langle 1, 3, 8 \rangle, \langle 2, 4, 6 \rangle\}$. If $I \in R(12)$ and $I \neq \langle 5, 7 \rangle$, then $\text{ind}(I) = 0$. Since $\text{ind}(\langle 5, 7 \rangle) = 1$, we have

$$\sum_{q=1}^{\infty} \dim H_{(12)}^q(L_1) t^q = t + 3t^2 + 3t^3.$$

5. A basis of the space $H^*(L_k)$ for $k \geq 1$

Definition 5.1. A k -partition is a pair $(k; I)$, where I is a partition and $I^- \geq k$.

We write k -partitions as usual partitions, emphasizing that we only consider k -partitions. For example, one may consider $\langle 2, 7 \rangle$ as either a 1-partition or a 2-partition; these objects are different.

Definition 5.2. A regular k -partition $I = \langle i_1, i_2, \dots, i_q \rangle$ is called *special* if $i_q < 2(k+q-1)$.

It is easy to check that the quantity of the special k -partitions of length q is equal to $\binom{q+k-1}{k-1}$.

Definition 5.3. A regular k -partition is *simple* if it is either special, or dense.

Definition 5.4. For any regular k -partition I , there exists a unique decomposition $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_s$, where I_1, I_2, \dots, I_s are simple k -partitions of the maximal possible length. This decomposition is called the *canonical decomposition* of I . The partitions I_1, I_2, \dots, I_s are called the *simple components* of I .

The minimal parts of the odd non-special simple components of I are called the *leading parts of I* . The quantity of them is denoted by $\text{ind}_k(I)$.

For example, for $I = \langle 3, 5, 9, 13, 15, 18 \rangle$ and $k = 1, 2, 3$, the canonical decompositions are

$$I = \begin{cases} \langle 3, 5 \rangle \sqcup \langle 9 \rangle \sqcup \langle 13, 15 \rangle \sqcup \langle 18 \rangle & \text{if } k = 1, \\ \langle 3, 5 \rangle \sqcup \langle 9 \rangle \sqcup \langle 13, 15 \rangle \sqcup \langle 18 \rangle & \text{if } k = 2, \\ \langle 3, 5, 9 \rangle \sqcup \langle 13, 15 \rangle \sqcup \langle 18 \rangle & \text{if } k = 3. \end{cases}$$

Observe that decompositions for $k = 1$ and $k = 2$ are distinct as k -partitions (see Definition 5.1). The definition of $\text{ind}_k(I)$ implies that $\text{ind}_1(I) = 3$, $\text{ind}_2(I) = 2$, and $\text{ind}_3(I) = 1$, respectively.

Definition 5.5. We say that $\langle I; J \rangle$ is a k -partition if I is a k -partition. A marked k -partition $\langle I; J \rangle$ is called *regular* if I is regular and J is a subset of the set of the leading parts of I .

Theorem 5.6. Let $R_k(n)$ be the set of k -regular partitions of degree n , each simple component of which is either special, or of even degree.

For any regular marked k -partition $\langle I; J \rangle$, where $I \in R_k(n)$, one can uniquely define a nonzero cocycle $\varepsilon_{\langle I; J \rangle} \in C_{(n)}^*(L_k)$. Cohomological classes of these cocycles form a basis of the space $H_{(n)}^*(L_k)$. Moreover, the classes with $|I| + |J| = q$ form a basis of the space $H_{(n)}^q(L_k)$. In particular,

$$\sum_{q=1}^{\infty} \dim H_{(n)}^q(L_k) t^q = \sum_{I \in R_k(n)} (1+t)^{\text{ind}_k(I)} t^{|I|}.$$

6. On the multiplicative structure of the algebra $H^*(L_1)$

The exterior product of cochains in the complex $C^*(L_1)$ induces a multiplication that turns $H^*(L_1)$ into an algebra. Theorem 4.7 and formulas (11) imply that ε -monomials e_1 and

$$\begin{aligned} x(i) &:= e_{2i}, & y(i) &:= \sum_{r=0}^{i-1} e_{2i-2r-1} \wedge e_{2i+2r+1}, & \text{where } i \geq 1, \\ z(i) &:= \sum_{r=0}^{i-2} \delta_1(e_{2i-2r-1}) \wedge e_{2i+2r+1}, & & \text{where } i \geq 2, \end{aligned}$$

are cocycles which represent nonzero cohomological classes of L_1 . Let us denote these classes by e , x_i , y_i , and z_i , respectively. Theorem 4.7 implies that they multiplicatively generate the algebra $H^*(L_1)$.

Lemma 6.1. *In the algebra $H^*(L_1)$, we have $e^2 = 0$, $x_i^2 = y_i^2 = 0$ for all $i \geq 1$ and*

$$\begin{aligned} e \cdot x_1 &= e \cdot y_1 = 0, \\ z_i &= \sum_{a=1}^{i-1} x_{2a} \cdot y_{i-a} \quad \text{for } i \geq 2, \\ \sum_{a=0}^{i-1} x_{2a+1} \cdot y_{i-a} &= 0, \quad \sum_{a=0}^{i-1} y_{i-a} \cdot y_{i+a+1} = 0 \quad \text{for } i \geq 1. \end{aligned}$$

In particular, the algebra $H^(L_1)$ is multiplicatively generated by the classes e , x_i , and y_i for all $i \geq 1$.*

Proof. Clearly, $e \wedge x_1 = e \wedge y_1 = 0$. The remaining formulas follow from the directly checked relations

$$\begin{aligned} z(i) &= \sum_{a=1}^{i-1} x(2a) \wedge y(i-a) + \delta_1 \left(\sum_{m=0}^{\lfloor \frac{i-2}{2} \rfloor} e_{2i-4m-3} \wedge e_{2i+4m+3} \right) \quad \text{for } i \geq 2, \\ \sum_{a=0}^{i-1} x(2a+1) \wedge y(i-a) &= \delta_1 \left(\sum_{m=1}^{\lfloor \frac{i+1}{2} \rfloor} e_{4m-1-2\beta(i)} \wedge e_{4(i-m)+3+2\beta(i)} \right) \quad \text{for } i \geq 1, \\ \sum_{a=0}^{i-1} y(i-a) \wedge y(i+a+1) &= 0 \quad \text{for } i \geq 1. \end{aligned}$$

In addition, [Lemma 3.2](#) implies that

$$e_1 \wedge e_3 \wedge \dots \wedge e_{2q-1} \approx \begin{cases} y(1) \wedge y(3) \wedge \dots \wedge y(q-1) & \text{if } \beta(q) = 0, \\ e_1 \wedge y(2) \wedge y(4) \wedge \dots \wedge y(q-1) & \text{if } \beta(q) = 1. \end{cases}$$

To finish the proof, it remains to apply [Theorems 4.7 and 3.3](#). \square

Let $\mathbb{P}[E, X, Y]$ be the exterior algebra generated over \mathbb{Z}_2 by E , X_i , and Y_i for all $i \geq 1$. Consider $\mathbb{P}[E, X, Y]$ as a bigraded algebra having defined the bigrading as follows:

$$\deg(E) = (1, 1), \quad \deg(X_i) = (1, 2i), \quad \deg(Y_i) = (2, 4i).$$

Conjecture 6.2. *Let A be a homogeneous ideal in $\mathbb{P}[E, X, Y]$ with generators*

$$E \wedge X_1, \quad E \wedge Y_1, \quad \sum_{a=0}^{i-1} X_{2a+1} \wedge Y_{i-a}, \quad \sum_{a=0}^{i-1} Y_{i-a} \wedge Y_{i+a+1}, \quad \text{where } i \geq 1.$$

Then we have the exact sequence of bigraded algebras

$$0 \longrightarrow A \longrightarrow \mathbb{P}[E, X, Y] \xrightarrow{\pi} H^*(L_1) \longrightarrow 0,$$

where $\pi(E) = e$, $\pi(X_i) = x_i$, $\pi(Y_i) = y_i$ for all $i \geq 1$.

Remark 6.3. Thanks to [Theorem 4.7](#) one can reduce this conjecture to a combinatorial question about integer partitions. Namely, let $P(n, q)$ be the set of pairs (K, L) of partitions such that

- (1) $K = \langle k_1, k_2, \dots, k_a \rangle$ is a strict partition and $L = \langle l_1, l_2, \dots, l_b \rangle$ is a regular partition.
- (2) $a + 2b = q$ and $2k_1 + 2k_2 + \dots + 2k_a + 4l_1 + 4l_2 + \dots + 4l_b = n$.
- (3) $|k_i - l_j| \geq 2$ for all $i \in [1, a]$ such that $\beta(k_i) = 1$, and for all $j \in [1, b]$.

On the other hand, let $M_0(n, q)$ be the set of marked regular partitions $\langle I; J \rangle$ of degree n , length q , and such that the degree of any simple component of I is even. Then it is easy to see that [Conjecture 6.2](#) follows from the conjectural identity $|P(n, q)| = |M_0(n, q)|$.

7. Computing the spaces $H^*(L_0)$ and $H^*(L_{-1})$

In this section, $k = 0$ or $k = -1$. Since the cohomology of L_1 are now known and in both the cases $L_1 \subset L_k$ is an ideal, to compute $H^*(L_k)$ one could use the corresponding Hochschild–Serre spectral sequence. But we prefer a direct computation, which agrees with the computations from [§4](#).

Let $A_{(n)}^q(k)$ be the vector subspace of $C_{(n)}^q(L_k)$ spanned by the cochains $c = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_q}$ such that $e_0 \wedge c \neq 0$. Obviously,

$$A_{(n)}^q(0) = C_{(n)}^q(L_1), \quad A_{(n)}^q(-1) = \begin{cases} \mathbb{Z}_2 e_n & \text{if } q = 1 \text{ and } n \neq 0, \\ e_{-1} \wedge C_{(n+1)}^{q-1}(L_1) \oplus C_{(n)}^q(L_1) & \text{if } q > 1 \text{ and } n \geq 0. \end{cases}$$

Then

$$C_{(n)}^q(L_k) = \begin{cases} \mathbb{Z}_2 e_n & \text{if } q = 1, \\ e_0 \wedge A_{(n)}^{q-1}(k) \oplus A_{(n)}^q(k) & \text{if } q > 1. \end{cases}$$

Since e_0 is a cocycle of the complex $C^*(L_k)$, the space $e_0 \wedge A_{(n)}^*(k)$ is a subcomplex of $C_{(n)}^*(L_k)$. Using the natural projection, let us identify the space $A_{(n)}^*(k)$ with the space of the factor-complex $C_{(n)}^*(L_k) / (e_0 \wedge A_{(n)}^*(k))$ and transfer its differential, denoted by δ , to $A_{(n)}^*(k)$. Then

$$H^q(e_0 \wedge A_{(n)}^*(k)) \cong e_0 \wedge H^{q-1}(A_{(n)}^*(k)).$$

The exact sequence of complexes

$$0 \longrightarrow e_0 \wedge A_{(n)}^*(k) \longrightarrow C_{(n)}^*(L_k) \longrightarrow A_{(n)}^*(k) \longrightarrow 0,$$

induces the exact cohomology sequence

$$\begin{aligned} \cdots \longrightarrow H_{(n)}^{q-1}(A^*(k)) &\xrightarrow{b} e_0 \wedge H_{(n)}^{q-1}(A^*(k)) \longrightarrow H_{(n)}^q(L_k) \longrightarrow H_{(n)}^q(A^*(k)) \\ &\xrightarrow{b} e_0 \wedge H_{(n)}^q(A^*(k)) \longrightarrow \cdots, \end{aligned}$$

where b is the Bockstein homomorphism. Its definition implies that, for any $h \in H_{(n)}^*(A^*(k))$, we have

$$b(h) = ne_0 \wedge h$$

Therefore, for integer $l \geq 0$ if $n = 2l + 1$ then $H_{(n)}^q(L_k) = \{0\}$, whereas

$$H_{(2l)}^q(L_k) \cong \begin{cases} \mathbb{Z}_2 e_{2l} & \text{if } q = 1, \\ e_0 \wedge H_{(2l)}^{q-1}(A_{(2l)}^*(k)) \oplus H^q(A_{(2l)}^*(k)) & \text{if } q > 1. \end{cases} \quad (12)$$

For $k = 0$, this implies the following result:

Theorem 7.1. *We have, $H_{(2l+1)}^q(L_0) = \{0\}$, whereas*

$$H_{(2l)}^q(L_0) \cong \begin{cases} \mathbb{Z}_2 e_{2l} & \text{if } q = 1, \\ e_0 \wedge H_{(2l)}^{q-1}(L_1) \oplus H_{(2l)}^q(L_1) & \text{if } q > 1. \end{cases}$$

Let now $k = -1$. Since $\delta(e_{-1}) = 0$, we see that $e_{-1} \wedge C_{(n+1)}^*(L_1) \subset A_{(n)}^*(-1)$ is a subcomplex and $H^q(e_{-1} \wedge C_{(n+1)}^*(L_1)) \cong e_{-1} \wedge H_{(n+1)}^q(L_1)$. The exact sequence of complexes

$$0 \longrightarrow e_{-1} \wedge C_{(n+1)}^*(L_1) \longrightarrow A_{(n)}^*(-1) \longrightarrow C_{(n)}^*(L_1) \longrightarrow 0,$$

induces the exact cohomology sequence

$$\begin{aligned} \cdots \longrightarrow H_{(n)}^{q-1}(L_1) &\xrightarrow{b} e_{-1} \wedge H_{(n+1)}^{q-1}(L_1) \longrightarrow H^q(A_{(n)}^*(-1)) \longrightarrow H_{(n)}^q(L_1) \\ &\xrightarrow{b} e_{-1} \wedge H_{(n+1)}^q(L_1) \longrightarrow \cdots. \end{aligned}$$

The definition of b implies that for any $h \in H_{(n)}^*(L_1)$ we have

$$b(h) = e_{-1} \wedge e_{-1}(h),$$

where $e_{-1}(h)$ denotes the action of e_{-1} on $h \in H_{(n)}^*(L_1)$ (see formula (3)). It is subject to a direct verification that for $a \geq 1$ odd, we have

$$e_{-1}(\varepsilon_{a,a+2}) = \delta_1(e_{2a+3}) \quad \text{and} \quad e_{-1}(\varepsilon_{\underline{a},a+2}) = \delta_1(e_a \wedge e_{a+3}).$$

Since in addition $e_{-1}(e_i) = 0$ for any $i \geq 2$ even, Theorem 4.7 implies that $b(h) = 0$. Thus,

$$H^q(A_{(n)}^*(-1)) \cong e_{-1} \wedge H_{(n+1)}^{q-1}(L_1) \bigoplus H_{(n)}^q(L_1).$$

Therefore, the isomorphism (12) implies the following result:

Theorem 7.2. *We have $H_{(2l+1)}^q(L_{-1}) = \{0\}$, whereas*

$$H_{(2l)}^q(L_{-1}) \cong \begin{cases} \mathbb{Z}_2 e_{2l} & \text{if } q = 1, \\ e_{-1} \wedge e_0 \wedge H_{(2l+1)}^{q-2}(L_1) \bigoplus e_0 \wedge H_{(2l)}^{q-1}(L_1) \bigoplus e_{-1} \\ \quad \wedge H_{(2l+1)}^{q-1}(L_1) \bigoplus H_{(2l)}^q(L_1) & \text{if } q > 1. \end{cases}$$

Corollary 7.3. *The cohomological classes of the following cocycles in $C_{(n)}^*(L_{-1})$:*

$$u_{a,b}(n) := e_{2a} \wedge e_{2b}, \quad \text{where } 2a + 2b = n \text{ for integers } 0 \leq a < b,$$

$$v(n) := \sum_{r=0}^{\frac{n}{4}} e_{\frac{n}{2}-2r-1} \wedge e_{\frac{n}{2}+2r+1} \quad \text{if } n \equiv 0 \pmod{4}$$

constitute a basis of the space $H_{(n)}^2(L_{-1})$. In particular,

$$\dim H_{(n)}^2(L_{-1}) = \left\lfloor \frac{n}{4} \right\rfloor + 1.$$

Thanks to a well-known interpretation of the 2-dimensional cohomology of Lie algebras with trivial coefficients (see [5], Sec. 7.6), the cocycles $u_{a,b}(n)$ and $v(n)$ explicitly describe the basis of the space of equivalence classes of the central extensions with one-dimensional kernel of the Lie algebra $W(\mathbb{Z}_2) = L_{-1}$.

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