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Poincaré series of compressed local Artinian rings with odd top socle degree[☆,☆☆]

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ABSTRACT

We define a notion of compressed local Artinian ring that does not require the ring to contain a field. Let $(R, \mathfrak{m})$ be a compressed local Artinian ring with odd top socle degree s, at least five, and $\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s$. We prove that the Poincaré series of all finitely generated modules over R are rational, sharing a common denominator, and that there is a Golod homomorphism from a complete intersection onto R.

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☆☆ Macaulay2 [14] was very valuable to us. We used it to test various conjectures.

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Socle
Trivial Massey operation

1. Introduction

Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local ring and M be a finitely generated R-module. The Poincaré series of M,

$$P_M^R(z) = \sum_{i=0}^{\infty} b_i(M) z^i,$$

is the generating function for the sequence of Betti numbers of M:

$$\begin{aligned} b_i(M) &= \text{ the minimal number of generators of the } i\text{th syzygy of } M \\ &= \dim_{\boldsymbol{k}} \operatorname{Tor}_i^R(M, \boldsymbol{k}). \end{aligned}$$

In the 1950's Kaplansky and Serre [23, pg. 118] asked if the Poincaré series of a local ring is always a rational function. Considerable study was devoted to this question, (see, for example, the survey articles [20,8]), before Anick [1] showed that the answer is no.

Consideration of rational and transcendental Poincaré series has only intensified since the appearance of Anick's example. The example has been simplified, reworked, and reformulated in the language of Algebraic Topology; see the discussion following Problem 4.3.10 in [4] for more details, including references. Roos [21] calls a local ring R *good* if the Poincaré series of all finitely generated modules over R are rational, sharing a common denominator. A list of applications of the hypothesis that a local ring is good may be found in [3].

Nonetheless, at the Introductory Workshop for the special year in Commutative Algebra at the Mathematical Sciences Research Institute in 2012 Irena Peeva observed [19] that "We do not have a feel for which of the following cases holds.

(a) Most Poincaré series are rational, and irrational Poincaré series occur rarely in specially crafted examples.
(b) Most Poincaré series are irrational, and there are some nice classes of rings (for example, Golod rings, complete intersections) where we have rationality.
(c) Both rational and irrational Poincaré series occur widely.

One would like to have results showing whether the Poincaré series are rational generically, or are irrational generically."

A first answer to Peeva's problem is made in the paper by Rossi and Şega, [22], where it is shown that if R is a compressed Artinian Gorenstein local ring with top socle degree not equal to three, then the Poincaré series of all finitely generated modules over R are rational, sharing a common denominator. (In particular, these rings are good, in the sense of Roos.) The Rossi–Şega theorem is a complete answer to the Peeva problem for generic Artinian Gorenstein rings because generic Artinian Gorenstein rings are automatically compressed. Furthermore, it is necessary to avoid top socle degree three because Bøgvad [9] has given examples of compressed Artinian Gorenstein rings with top socle degree three which have transcendental Poincaré series.

In the present paper we carry the Rossi–Şega program further. As in the Gorenstein case, once the relevant parameters are fixed, the set of Artinian standard-graded $\boldsymbol{k}$-algebras is parameterized (now by a non-empty open subset in a chain of relative Grassmannians) and (when $\boldsymbol{k}$ is infinite) the points on a non-empty open subset of this parameter space correspond to compressed algebras. As in [22], we ignore the parameter space and the cardinality of $\boldsymbol{k}$; instead, we prove that compressed local Artinian rings, with odd top socle degree s are also good in the sense of Roos, provided $5 \le s$ and $\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s$. Our result then applies in the "generic case" whenever the "generic case" makes sense. Bøgvad's examples also apply in our situation, so we also are forced to exclude top socle degree equal to three.

Our argument is inspired by the proof in [22]. The key ingredient from local algebra in our proof is Lemma 4.7 which can be interpreted as a statement about the structure of the Koszul homology algebra

$$\mathrm{H}_\bullet(R \otimes_Q K) = \operatorname{Tor}_\bullet^Q(R, \boldsymbol{k}),$$

where $Q \to R$ is a surjection of local rings of the same embedding dimension e, $(Q, \mathfrak{n}, \boldsymbol{k})$ is a regular local ring, $(R, \mathfrak{m}, \boldsymbol{k})$ is a compressed local Artinian ring, and K is the Koszul complex which is a minimal resolution of $\boldsymbol{k}$ by free Q-modules. When the hypotheses of Lemma 4.7 are in effect, then the conclusion may be interpreted to say that there is an element $\bar{g}$ in $\operatorname{Tor}_1^Q(R, \boldsymbol{k})$ with

$$\bar{g} \cdot \operatorname{Tor}_{e-1}^Q(R, \boldsymbol{k}) = \operatorname{Tor}_e^Q(R, \boldsymbol{k}).$$

We use this conclusion to create a Golod homomorphism from a hypersurface ring onto R.

Our reliance on Lemma 4.7 explains the hypotheses in the main theorem about the shape of the socle of R. In particular, when the top socle degree of R is even, it is possible for a non-Golod compressed Artinian standard-graded $\boldsymbol{k}$-algebra R to have $\operatorname{Tor}_1 \cdot \operatorname{Tor}_{e-1} = 0$. (We are writing Tor_i in place of $\operatorname{Tor}_i^Q(R, \boldsymbol{k})$.) For example, if $e = 4$ and $\operatorname{socle}(R)$ is isomorphic to $\boldsymbol{k}(-4)^2$, then according to [10, Conj. 3.13 and 4.1.2], the Betti table for the minimal homogeneous resolution of R by free Q-modules is

$$\begin{array}{rccccc} & 0 & 1 & 2 & 3 & 4 \\ \text{total}: & 1 & 12 & 19 & 10 & 2 \\ 0: & 1 & . & . & . & . \\ 1: & . & . & . & . & . \\ 2: & . & 12 & 15 & . & . \\ 3: & . & . & 4 & 10 & . \\ 4: & . & . & . & . & 2 \end{array} \quad \text{or} \quad \begin{pmatrix} 12 & 15 & 0 \\ 0 & 4 & 10 \end{pmatrix},$$

in the language of Macaulay2 [14] or Boij [10, Notation 3.4], respectively. The numerology alone shows that $\mathrm{Tor}_1 \cdot \mathrm{Tor}_3 = 0$, but the numerology permits $\mathrm{Tor}_2 \cdot \mathrm{Tor}_2$ to be non-zero, and this is precisely what happens. In a similar manner, if the top socle degree of R is three or if $(\mathrm{socle}(R) \cap \mathfrak{m}^{s-1})/\mathfrak{m}^s \neq 0$, then the Betti tables (in the homogeneous case) permit too many non-zero products in Tor. Consequently, if R is compressed and the top socle degree of R is 3, or the top socle degree of R is even, or $(\mathrm{socle}(R) \cap \mathfrak{m}^{s-1})/\mathfrak{m}^s$ is non-zero, then our techniques are not able to determine if the Poincaré series of R is rational. In these cases, the question of Peeva remains wide open.

We prove that the Poincaré series of R is rational by exhibiting a Golod surjection from a complete intersection onto R. It is worth observing that the existence of such a map is an important conclusion in its own right. For example, this hypothesis is used in [18] in the study of the rigidity of the two-step Tate complex, in [5] in study of the non-vanishing of $\mathrm{Tor}_i^R(M,N)$ for infinitely many i, and in [6] in the study of the structure of the set of semi-dualizing modules of a ring R.

Theorem 7.1 is the main result of the paper. To prove this theorem we apply Lemma 5.2, which is established in [22]. Lemma 5.2 is a down-to-earth criteria for proving that a given surjection of local rings is a Golod homomorphism. Massey operations are replaced with calculations involving $\mathrm{Tor}_\bullet^Q(-,\boldsymbol{k})$, where Q is a regular local ring.

A compressed local Artinian ring R exhibits extremal behavior. Such a ring has maximal length among all local Artinian rings with the same embedding dimension and socle polynomial. Extremal objects exhibit special properties and deserve extra study. Indeed, there are many applications of compressed rings and these rings have received much study. However, for thirty years, 1984 – 2014, the notion of "compressed" ring was only defined for rings containing a field. Finally, in 2014, Rossi and Şega [22] proved that the notion of "compressed local Artinian <u>Gorenstein</u> ring" is meaningful, interesting, and works just as well in the non-equicharacteristic case. Furthermore, their theorem about rational Poincaré series is valid in the non-equicharacteristic case.

In sections 3 and 4 we embrace the philosophy of [22] and prove that the phrase "compressed local Artinian ring" is meaningful whether or not the ring contains a field and whether or not the ring is Gorenstein. Furthermore our main theorem, Theorem 7.1, is valid in the context of this enlarged notion of "compressed ring". It is worth noting that although we adopt the philosophy of [22], the techniques about Gorenstein compressed rings in [22] are not relevant in our situation. Our technique is introduced in Section 3 and the proof that the phrase "compressed local Artinian ring" is meaningful is carried out in Section 4. This study of compressed local Artinian rings is an important feature of the present paper.

Section 2 consists of preliminary matters. In Section 3 we introduce our duality technique for studying local Artinian rings. In Section 4 we prove that the notion of compressed ring is meaningful in the non-equicharacteristic case. Section 5 is concerned with Golod homomorphisms and the homological algebra that can be used to prove that a homomorphism is Golod. In Section 6 we explore the consequences of the hypothesis "compressed" on the homological algebra of Section 5. The proof of the main theorem is given in Section 7. In Section 8 it is shown that in the situation of the main theorem, $R/\mathfrak{m}^s$ is a Golod ring and the natural map $R \to R/\mathfrak{m}^s$ is a Golod homomorphism; furthermore, the final statement continues to hold even if $(\operatorname{socle}(R) \cap \mathfrak{m}^{s-1})/\mathfrak{m}^s$ is not zero.

2. Notation, conventions, and preliminary results

In this paper $\boldsymbol{k}$ is always a field.

2.1. Let I be an ideal in a ring A, N be an A-module, and L and M be submodules of N. Then

$$L :_I M = \{x \in I \mid xM \subseteq L\} \quad \text{and} \quad L :_M I = \{m \in M \mid Im \subseteq L\}.$$

If L is the zero module, then we also use "annihilator notation" to describe these "colon modules"; that is,

$$\operatorname{ann}_A M = 0 :_A M \quad \text{and} \quad \operatorname{ann}_N I = 0 :_N I.$$

Any undecorated ":" or "ann" means $:_A$ or ann_A, respectively, where A is the ambient ring.

2.2. If I is an ideal in a ring A, N is an A-module, and L and M are submodules of N with $IL \subseteq M$, then let

$$\operatorname{mult} : I \to \operatorname{Hom}_A(L, M)$$

denote the homomorphism which sends the element θ of I to the homomorphism $\operatorname{mult}_\theta$ of $\operatorname{Hom}_A(L, M)$, where $\operatorname{mult}_\theta(\ell) = \theta\ell$ for all ℓ in L.

2.3. "Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local ring" identifies $\mathfrak{m}$ as the unique maximal ideal of the commutative Noetherian local ring R and $\boldsymbol{k}$ as the residue class field $\boldsymbol{k} = R/\mathfrak{m}$.

(a) The *embedding dimension* of R is $e = \dim_{\boldsymbol{k}}(\mathfrak{m}/\mathfrak{m}^2)$.
(b) The *Hilbert function* of R is the function

$$i \mapsto h_R(i) = \dim_{\boldsymbol{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}).$$

(c) If M is an R-module, then the *socle* of M is the vector space $\operatorname{socle}(M) = 0 :_M \mathfrak{m}$.
(d) If $(R, \mathfrak{m}, \boldsymbol{k})$ is a local Artinian ring, then the *top socle degree* of R is the maximum integer s with $\mathfrak{m}^s \neq 0$ and the *socle polynomial* of R is the formal polynomial $\sum_{i=0}^{s} c_i z^i$, where

$$c_i = \dim_{\boldsymbol{k}} \frac{\operatorname{socle}(R) \cap \mathfrak{m}^i}{\operatorname{socle}(R) \cap \mathfrak{m}^{i+1}}.$$

Further comments about the phrase "top socle degree" may be found in Remark 2.10.

(e) If M is a finitely generated R-module, then $\mu(M)$ denotes the minimal number of generators of M.

(f) The parameter $\mathtt{v}(R)$ is defined by

$$\mathtt{v}(R) = \inf\left\{ i \,\middle|\, \dim_{\boldsymbol{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) < \tbinom{(e-1)+i}{i} \right\},$$

where e is the embedding dimension of R. (This notation is introduced in [22, (4.1.1)].) In particular, if $(Q,\mathfrak{n})$ is a regular local ring with the completion $\hat{R}$ of R equal to Q/I with $I \subseteq \mathfrak{n}^2$, then $\mathtt{v}(R) = \max\{i \mid I \subseteq \mathfrak{n}^i\}$.

Remark. Let $(R,\mathfrak{m},\boldsymbol{k})$ be a local Artinian ring with top socle degree s and socle polynomial equal to $\sum_{i=0}^{s} c_i z^i$. Part of the hypothesis of Theorem 7.1 is that $\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s$. This condition is equivalent to $c_{s-1} = 0$.

Observation 2.4, which follows quickly from the definition of $\mathtt{v}(R)$, gives an idea of the significance of this invariant, and is used in the proof of Lemma 4.5.

Observation 2.4. *If $(R,\mathfrak{m},\boldsymbol{k})$ is a local Artinian ring, x_1 is a minimal generator of $\mathfrak{m}$, and i is an integer with $0 \le i \le \mathtt{v}(R) - 2$, then the linear transformation*

$$x_1 : \mathfrak{m}^i/\mathfrak{m}^{i+1} \to \mathfrak{m}^{i+1}/\mathfrak{m}^{i+2}, \tag{2.4.1}$$

which is given by multiplication by x_1, is an injection. In particular,

$$(\mathfrak{m}^j : \mathfrak{m}) = \mathfrak{m}^{j-1}, \quad \text{for } 1 \le j \le \mathtt{v}(R),$$

and $\operatorname{socle}(R) \subseteq \mathfrak{m}^{\mathtt{v}(R)-1}$.

Proof. Extend the set $\{x_1\}$ to be a minimal generating set $x_1, x_2, \ldots, x_e$ for $\mathfrak{m}$. If d is an arbitrary non-negative integer, then the set of monomials in $x_1, \ldots, x_e$ of degree d represents a generating set for $\mathfrak{m}^d/\mathfrak{m}^{d+1}$. If $d < \mathtt{v}(R)$, then the number of monomials in this set is equal to the dimension of the vector space $\mathfrak{m}^d/\mathfrak{m}^{d+1}$ and hence this set of monomials is a basis for the vector space. The index i satisfies $i + 1 < \mathtt{v}(R)$; consequently, the linear transformation (2.4.1) carries a basis of $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ to part of a basis of $\mathfrak{m}^{i+1}/\mathfrak{m}^{i+2}$; and therefore, this linear transformation is an injection. $\square$

Definition 2.5. Let $(R,\mathfrak{m},\boldsymbol{k})$ be a local Artinian ring of embedding dimension e, top socle degree s, and socle polynomial $\sum_{i=0}^{s} c_i z^i$. If the Hilbert function of R is given by

$$\dim_{\boldsymbol{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \min\left\{\binom{(e-1)+i}{i},\ \sum_{\ell=i}^{s} c_\ell \binom{(e-1)+(\ell-i)}{\ell-i}\right\}, \quad \text{for } 0 \le i \le s,$$

then R is called a *compressed local Artinian ring.*

Alternate definitions of "compressed local Artinian ring" are given in Theorem 4.4 and Remark 4.4.2.

2.6. If S is a ring and M is an S-module, then let $\lambda_S(M)$ denote the length of M as an S-module.

2.7. Let $\boldsymbol{k}$ be a field. A graded ring $R = \bigoplus_{0\le i} R_i$ is called a *standard-graded $\boldsymbol{k}$-algebra*, if $R_0 = \boldsymbol{k}$, R is generated as an R_0-algebra by R_1, and R_1 is finitely generated as an R_0-module.

2.8. If M is a module over a local ring $(R, \mathfrak{m}, \boldsymbol{k})$, then

$$M^{\mathrm{g}} = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i M/\mathfrak{m}^{i+1} M \quad \text{and} \quad R^{\mathrm{g}} = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

are the associated graded objects with respect the maximal ideal: R^{g} is a standard graded $\boldsymbol{k}$-algebra and M^{g} is a graded R-module.

2.9. If V is a graded vector space over the field $\boldsymbol{k}$ with V_i finite dimensional for all i and $V_i = 0$ for all sufficiently small i, then the formal Laurent series

$$\mathrm{HS}_V(z) = \sum_i \dim_{\boldsymbol{k}}(V_i)\, z^i$$

is called the *Hilbert series* of V.

Remark 2.10. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring. There are various reasons that the expression "top socle degree", which is introduced in 2.3.(d), is appropriate.

(a) This is the expression that Iarrobino used when he defined the notion of compressed algebra in [16, Def. 2.2].
(b) The top socle degree of R is the **degree** of the **socle** polynomial of R.
(c) The Hilbert series of the associated graded ring R^{g} of R is a polynomial of **degree** equal to the top socle degree of R.
(d) The top socle degree of R is the **top degree** of the associated graded ring R^{g}.

The following calculation is used in the proof of Lemma 6.3.

Remark 2.11. If $\boldsymbol{k}$ is an infinite field, $(Q, \mathfrak{n}, \boldsymbol{k})$ is a regular local ring of embedding dimension e, t is an integer, and h_0 is an element of $\mathfrak{n}^t \setminus \mathfrak{n}^{t+1}$, then there exists a minimal generating set $X_1, \dots, X_e$ for $\mathfrak{n}$ such that $h_0 - uX_1^t$ is in the ideal $(X_2, \dots, X_e)\mathfrak{n}^{t-1}$, for some unit u in Q. In particular, there is a generator h for the ideal (h_0) of Q such that $h - X_1^t \in (X_2, \dots, X_e)\mathfrak{n}^{t-1}$.

Outline of proof. The ideal $\mathfrak{n}^{t+1}$ is contained in $(X_1^{t+1}) + (X_2, \dots, X_e)\mathfrak{n}^{t-1}$; so, it suffices to show that $h_0 - uX_1^t$ is in the ideal $(X_2, \dots, X_e)\mathfrak{n}^{t-1} + \mathfrak{n}^{t+1}$. To that end, we pass to the associated graded ring Q^{g}. If h_0 is a non-zero homogeneous form of degree t in $\boldsymbol{k}[X_1, \dots, X_e]$, where $\boldsymbol{k}$ is an infinite field, then, there exists a homogeneous change of variables

$$X_1 \mapsto x_1, \quad \text{and} \quad X_i \mapsto a_i x_1 + x_i, \text{ for } 2 \le i \le e,$$

such that, in the new variables, $h_0 = ux_1^t + g$, where g is a homogeneous form of degree t in the ideal $(x_2, \dots, x_e)$ and u is a unit. The proof is clear. Start with

$$h_0 = \sum_{j_1 + \cdots + j_e = t} A_{j_1, \dots, j_e} X_1^{j_1} \cdots X_e^{j_e}.$$

After the change of variables $h_0 = h_0(1, a_2, \dots, a_e)x_1^t + g$, where g is a homogeneous form of degree t in the ideal $(x_2, \dots, x_e)$. The field $\boldsymbol{k}$ is infinite; so, there exists a point $(a_2, \dots, a_e)$ in affine $e-1$ space with $h_0(1, a_2, \dots, a_e) \neq 0$. $\square$

2.12. If $P = \bigoplus_i P_i$ is a graded ring, and $A = \bigoplus_i A_i$ and $B = \bigoplus_i B_i$ are graded P-modules, then the module $\mathrm{Tor}_\bullet^P(A, B)$ is a bi-graded P-module. Indeed, if

$$\mathbb{Y} : \cdots \to Y_1 \to Y_0 \to A$$

is a resolution of A by free P-modules, homogeneous of degree zero, then

$$\mathrm{Tor}_{p,q}^P(A, B) = \frac{\ker[(Y_p \otimes B)_q \to (Y_{p-1} \otimes B)_q]}{\mathrm{im}[(Y_{p+1} \otimes B)_q \to (Y_p \otimes B)_q]}.$$

2.13. If $\mathbb{Y}$ is a complex, then we use $Z_i(\mathbb{Y})$, $B_i(\mathbb{Y})$, and $\mathrm{H}_i(\mathbb{Y})$ to represent the modules of i-cycles, i-boundaries, and ith-homology of $\mathbb{Y}$, respectively. So, in particular, $\mathrm{H}_i(\mathbb{Y}) = Z_i(\mathbb{Y})/B_i(\mathbb{Y})$.

2.14. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local ring of embedding dimension e. The ring R is *Golod* if

$$P_{\boldsymbol{k}}^R(z) = \frac{(1+z)^e}{1 - \sum_{j=1}^e \dim_{\boldsymbol{k}} \mathrm{H}_j(K^R) z^{j+1}},$$

where K^R is the Koszul complex on a minimal set of generators of $\mathfrak{m}$.

3. Homomorphisms from a power of the maximal ideal to the socle

In order to study compressed rings, one must have an appropriate duality theory. Partial derivatives provide the duality for Iarrobino [16]. Fröberg and Laksov [13] and Boij and Laksov [11] pick a vector space V in the polynomial ring $\boldsymbol{k}[x_1,\dots,x_e]$ and use colon ideals to define an ideal I in the polynomial ring with the property that the corresponding quotient ring has socle V. The colon ideals provide the duality in these cases. Rossi and Şega [22] work in a Gorenstein ring and use Gorenstein duality directly. Duality for us is supplied by homomorphisms from a power of the maximum ideal to the socle.

Let $(R,\mathfrak{m},\boldsymbol{k})$ be a local Artinian ring with top socle degree s. If j and k are integers with $0\le j$, $1\le k$, and $j+k\le s+1$, then the R-module homomorphism

$$\text{mult}:\mathfrak{m}^j\cap(0:\mathfrak{m}^k)\to\operatorname{Hom}_R\left(\mathfrak{m}^{k-1},\operatorname{socle}(R)\cap\mathfrak{m}^{j+k-1}\right)$$

of 2.2 induces an injective R-module homomorphism

$$\frac{\mathfrak{m}^j\cap(0:\mathfrak{m}^k)}{\mathfrak{m}^j\cap(0:\mathfrak{m}^{k-1})}\to\operatorname{Hom}_R\left(\frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k},\operatorname{socle}(R)\cap\mathfrak{m}^{j+k-1}\right),\tag{3.0.1}$$

which we also call mult. The injections of (3.0.1) are our main tool for studying compressed rings. The remarkable feature of these injections is that if one of them is a surjection, and all other conditions are favorable, then a whole family of these injections are surjections. Recall the invariant $\mathtt{v}(R)$ from 2.3.(f).

Lemma 3.1. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be a local Artinian ring with embedding dimension e and top socle degree s, and let A and B be non-negative integers with $0\le A+B\le s$. Assume that*

(a) $B\le\mathtt{v}(R)-1$, *and*
(b) $\text{mult}:\mathfrak{m}^A\cap(0:\mathfrak{m}^{B+1})\to\operatorname{Hom}_R(\mathfrak{m}^B,\operatorname{socle}(R)\cap\mathfrak{m}^{A+B})$ *is surjective.*

Then

$$\text{mult}:\mathfrak{m}^{A+\epsilon}\cap(0:\mathfrak{m}^{B+1-\epsilon})\to\operatorname{Hom}_R(\mathfrak{m}^{B-\epsilon},\operatorname{socle}(R)\cap\mathfrak{m}^{A+B})\tag{3.1.1}$$

is surjective for all integers ϵ with $0\le\epsilon\le\min\{B,s-A\}$.

Proof. The proof may be iterated; consequently, it suffices to prove the result for ϵ equal to 1. Let $x_1,\dots,x_e$ be a minimal generating set for $\mathfrak{m}$. For each integer i, let $\binom{x_1,\dots,x_e}{i}$ be the set of monomials of degree i in $x_1,\dots,x_e$. View $\binom{x_1,\dots,x_e}{i}$ as a subset of R. Hypothesis (a) guarantees that $\binom{x_1,\dots,x_e}{i}$ represents a basis for $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ for all i with $0\le i\le B$. Let $c=\dim_{\boldsymbol{k}}(\operatorname{socle}(R)\cap\mathfrak{m}^{A+B})$ and $\sigma_1,\dots,\sigma_c$ be a basis for $\operatorname{socle}(R)\cap\mathfrak{m}^{A+B}$.

If $i \in \{B-1, B\}$, $m \in \binom{x_1,\dots,x_e}{B}$, and γ is an integer with $1 \le \gamma \le c$, then the R-module homomorphism $\phi_{m,\gamma}$, which is defined by

$$\phi_{m,\gamma}(m') = \delta_{m,m'} \cdot \sigma_\gamma, \qquad \text{for } m' \in \binom{x_1,\dots,x_e}{i},$$

is an element of

$$\operatorname{Hom}_R(\mathfrak{m}^i, \operatorname{socle}(R) \cap \mathfrak{m}^{A+B}) \cong \operatorname{Hom}_R(\mathfrak{m}^i/\mathfrak{m}^{i+1}, \operatorname{socle}(R) \cap \mathfrak{m}^{A+B});$$

furthermore,

$$\{\phi_{m,\gamma} \mid m \in \tbinom{x_1,\dots,x_e}{i} \text{ and } 1 \le \gamma \le c\}$$

is a basis for the vector space $\operatorname{Hom}_R(\mathfrak{m}^i, \operatorname{socle}(R) \cap \mathfrak{m}^{A+B})$. In this discussion, "$\delta$" is the Kronecker delta; that is,

$$\delta_{m,m'} = \begin{cases} 1, & \text{if } m = m', \\ 0, & \text{otherwise.} \end{cases}$$

Fix a monomial $m_0 \in \binom{x_1,\dots,x_e}{B-1}$ and an index γ with $1 \le \gamma \le c$. We complete the proof by showing that the basis element $\phi_{m_0,\gamma}$ of $\operatorname{Hom}_R(\mathfrak{m}^{B-1}, \operatorname{socle}(R) \cap \mathfrak{m}^{A+B})$ is in the image of (3.1.1) when $\epsilon = 1$. Observe that $x_1 m_0 \in \binom{x_1,\dots,x_e}{B}$ and $\phi_{x_1 m_0,\gamma}$ is in $\operatorname{Hom}_R(\mathfrak{m}^B, \operatorname{socle}(R) \cap \mathfrak{m}^{A+B})$. The hypothesis guarantees that there is an element $\theta \in \mathfrak{m}^A \cap (0 : \mathfrak{m}^{B+1})$, with $\operatorname{mult}_\theta = \phi_{x_1 m_0,\gamma}$. Observe that

$$x_1\theta \in \mathfrak{m}^{A+1} \cap (0 : \mathfrak{m}^B) \quad \text{and} \quad \operatorname{mult}_{x_1\theta} \in \operatorname{Hom}_R(\mathfrak{m}^{B-1}, \operatorname{socle}(R) \cap \mathfrak{m}^{A+B})$$

with $\operatorname{mult}_{x_1\theta} = \phi_{m_0,\gamma}$. Indeed,

$$\operatorname{mult}_{x_1\theta}(m) = m x_1 \theta = \delta_{m x_1, m_0 x_1} \cdot \sigma_\gamma = \delta_{m,m_0} \cdot \sigma_\gamma = \phi_{m_0,\gamma}(m)$$

for all $m \in \binom{x_1,\dots,x_e}{B-1}$. $\square$

4. Compressed local Artinian rings

A compressed local Artinian ring has maximal length among all local Artinian rings with the same embedding dimension and socle polynomial. Compressed algebras were introduced by Iarrobino [16]. Fröberg and Laksov [13] offer an alternate discussion, essentially from the dual point of view. Traditionally, the concept "compressed" was only defined for equicharacteristic rings. However, the equicharacteristic hypothesis is irrelevant and the proof of our main theorem (Theorem 7.1) holds for arbitrary compressed local Artinian rings.

There are two themes in this section. In Theorem 4.1 and Remark 4.2 we explain the sense in which generic standard-graded Artinian algebras over a field are compressed. A short, self-contained, and direct proof of Theorem 4.1 may be found in [11].

In Theorem 4.4 and Corollary 4.5 we justify the first sentence of the present section and we describe the annihilator of each large power of the maximal ideal of R when R is a compressed local Artinian ring. This information is used heavily in the proofs of Corollary 4.6 and Lemma 4.7. Lemma 4.7 is the key result from local algebra that is used in the second half of the paper about Poincaré series.

Theorem 4.1. *[11, 3.4] Let $\boldsymbol{k}$ be an infinite field, (e,s,c) be integers with $2 \le e$ and*

$$1 \le c < \binom{e+s-1}{s},$$

Q be a standard-graded polynomial ring over $\boldsymbol{k}$ of embedding dimension e, $\mathcal{G}$ be the Grassmannian of subspaces of Q_s of codimension c, and $\mathcal{L}$ be the set of homogeneous ideals I of Q such that Q/I is a standard-graded Artinian $\boldsymbol{k}$-algebra with socle polynomial cz^s. Then the following statements hold.

(a) *The set $\mathcal{G}$ parameterizes $\mathcal{L}$.*
(b) *If V is in $\mathcal{G}$, then the corresponding ideal I in $\mathcal{L}$ is generated by*

$$\sum_{i=1}^{s} (V :_{Q_i} Q_{s-i}).$$

(c) *If I is in $\mathcal{L}$, then the corresponding element of $\mathcal{G}$ is I_s.*
(d) *There is a non-empty open subset of $\mathcal{G}$ for which the corresponding quotient Q/I is compressed.*

Remark 4.2. Let $\boldsymbol{k}$ be an infinite field. It is shown in Section 7 of [13], especially Theorem 14, that generic standard-graded Artinian $\boldsymbol{k}$-algebras are compressed for all legal socle polynomials. (Theorem 4.1 deals only with socle polynomials of the form cz^s.) The exact details of the result in [13] are similar to, but more complicated than, the details of Theorem 4.1. There is no need to record the details of the statement of [13] in the present paper. The extra complication arises because

(a) the set of legal socle polynomials is more complicated than

$$\{cz^s \mid 1 \le c < \binom{e+s-1}{s}\},$$

and
(b) the parameterization space for the set of all Artinian $\boldsymbol{k}$-algebras with a given embedding dimension and socle polynomial is more complicated than the Grassmannian of all subspaces of codimension c in a vector space of dimension $\binom{e+s-1}{s}$.

We turn to the second theme in this section which is the notion of compressed local Artinian rings which are not necessarily equicharacteristic. Proposition 4.3 is a very important step in the proof of Theorem 4.4. Ultimately, we use Proposition 4.3 to connect the numerical information given in the definition of compressed rings to the homomorphisms "mult" of Section 3.

Proposition 4.3. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be an Artinian local ring with embedding dimension e, top socle degree s, and socle polynomial $\sum_{i=0}^{s} c_i z^i$. If j is an arbitrary integer, with $0 \le j \le s$, then*

$$\lambda_R(\mathfrak{m}^j) \le \sum_{\ell=j}^{s} c_\ell \binom{e+\ell-j}{\ell-j}.$$

Proof. Observe that

$$\begin{aligned} 0 = \big(\mathfrak{m}^j \cap (0:\mathfrak{m}^0)\big) \subseteq \big(\mathfrak{m}^j \cap (0:\mathfrak{m}^1)\big) \subseteq \big(\mathfrak{m}^j \cap (0:\mathfrak{m}^2)\big) \subseteq \cdots \\ \cdots \subseteq \big(\mathfrak{m}^j \cap (0:\mathfrak{m}^{s-j-1})\big) \subseteq \big(\mathfrak{m}^j \cap (0:\mathfrak{m}^{s-j})\big) \\ \subseteq \big(\mathfrak{m}^j \cap (0:\mathfrak{m}^{s-j+1})\big) = \mathfrak{m}^j \end{aligned} \tag{4.3.1}$$

is a filtration of $\mathfrak{m}^j$. The proof is obtained by exhibiting an injection from each factor of filtration (4.3.1) into a vector space whose dimension is easy to approximate.

If k is an integer with $1 \le k \le s+1-j$, then the R-module injection mult of (3.0.1) yields

$$\lambda_R\left(\frac{\mathfrak{m}^j \cap (0:\mathfrak{m}^k)}{\mathfrak{m}^j \cap (0:\mathfrak{m}^{k-1})}\right) \le \left(\dim_{\boldsymbol{k}}\left(\frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k}\right)\right)\left(\dim_{\boldsymbol{k}}(\operatorname{socle}(R) \cap \mathfrak{m}^{j+k-1})\right).$$

Recall that

$$\dim_{\boldsymbol{k}}\left(\frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k}\right) \le \binom{(e-1)+(k-1)}{k-1}, \tag{4.3.2}$$

because $\mathfrak{m}^{k-1}$ is generated by the set of monomials of degree $k-1$ in any minimal generating set of $\mathfrak{m}$, and

$$\dim_{\boldsymbol{k}}(\operatorname{socle}(R) \cap \mathfrak{m}^{j+k-1}) = \sum_{\ell=j+k-1}^{s} c_\ell,$$

by the definition of socle polynomial. Thus,

$$\lambda_R\left(\frac{\mathfrak{m}^j \cap (0:\mathfrak{m}^k)}{\mathfrak{m}^j \cap (0:\mathfrak{m}^{k-1})}\right) \le \sum_{\ell=j+k-1}^{s} c_\ell \binom{(e-1)+(k-1)}{k-1}, \tag{4.3.3}$$

for $0 \le j \le s$ and $1 \le k \le s-j+1$. Combine (4.3.1) and (4.3.3) to obtain

$$\lambda_R(\mathfrak{m}^j) \le \sum_{k=1}^{s-j+1} \sum_{\ell=j+k-1}^{s} c_\ell \binom{(e-1)+(k-1)}{k-1}.$$

Let $K = k+j-1$; reverse the order of summation; let $\alpha = K-j$; and recall the relationship between the number of monomials of degree at most $\ell - j$ in e variables and the number of monomials of degree equal to $\ell - j$ in $e+1$ variables to conclude

$$\begin{aligned} \lambda_R(\mathfrak{m}^j) &\le \sum_{K=j}^{s} \sum_{\ell=K}^{s} c_\ell \binom{(e-1)+(K-j)}{K-j} \qquad (4.3.4) \\ &= \sum_{\ell=j}^{s} c_\ell \sum_{K=j}^{\ell} \binom{(e-1)+(K-j)}{K-j} \\ &= \sum_{\ell=j}^{s} c_\ell \sum_{\alpha=0}^{\ell-j} \binom{(e-1)+\alpha}{\alpha} = \sum_{\ell=j}^{s} c_\ell \binom{e+\ell-j}{\ell-j}. \quad \square \end{aligned}$$

In Theorem 4.4 we justify the claim that a compressed local Artinian ring has maximal length among all local Artinian rings with the same embedding dimension and socle polynomial. The proof of Theorem 4.4 contains a wealth of information. We mine this information throughout the rest of the section.

Theorem 4.4. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring with embedding dimension e, top socle degree s, and socle polynomial $\sum_{i=0}^{s} c_i z^i$. Then the following statements hold.*

(a) *The length of R satisfies*

$$\lambda_R(R) \le \sum_{i=0}^{s} \min\left\{ \binom{(e-1)+i}{i},\ \sum_{\ell=i}^{s} c_\ell \binom{(e-1)+(\ell-i)}{\ell-i} \right\}. \qquad (4.4.1)$$

(b) *Equality holds in (4.4.1) if and only if R is a compressed local Artinian ring in the sense of Definition 2.5.*

(c) *If R is a compressed local Artinian ring, then the parameter $\mathtt{v}(R)$ from 2.3.(f) satisfies $s \le 2\mathtt{v}(R) - 1$.*

Remark 4.4.2. Equation (4.4.6), the proof of assertion (c), and the identity

$$\sum_{i=0}^{\mathtt{v}(R)-1} \binom{(e-1)+i}{i} = \binom{e+\mathtt{v}(R)-1}{\mathtt{v}(R)-1}$$

show that an alternate version of (4.4.1) is given by

$$\lambda_R(R) \leq \binom{e+\mathtt{v}(R)-1}{\mathtt{v}(R)-1} + \sum_{\ell=\mathtt{v}(R)}^{s} c_\ell \binom{e+\ell-\mathtt{v}(R)}{\ell-\mathtt{v}(R)}. \tag{4.4.3}$$

Once the proof of Theorem 4.4 is complete, then we know that a local Artinian ring R is compressed if and only if equality holds in (4.4.3). This observation provides an effective method for testing if a ring is compressed.

Proof. Define t to be the integer

$$t = \min\left\{ i \,\middle|\, \sum_{\ell=i}^{s} c_\ell \tbinom{(e-1)+(\ell-i)}{\ell-i} < \tbinom{(e-1)+i}{i} \right\}. \tag{4.4.4}$$

Observe that

$$\begin{aligned} \tbinom{(e-1)+i}{i} &\leq \sum_{\ell=i}^{s} c_\ell \tbinom{(e-1)+(\ell-i)}{\ell-i}, && \text{for } 0 \leq i \leq t-1, \text{ and} \\ \sum_{\ell=i}^{s} c_\ell \tbinom{(e-1)+(\ell-i)}{\ell-i} &< \tbinom{(e-1)+i}{i}, && \text{for } t \leq i \leq s; \end{aligned} \tag{4.4.5}$$

hence, the inequality (4.4.1) may be re-written as

$$\lambda_R(R) \leq \sum_{i=0}^{t-1} \tbinom{(e-1)+i}{i} + \sum_{i=t}^{s}\sum_{\ell=i}^{s} c_\ell \tbinom{(e-1)+(\ell-i)}{\ell-i}.$$

Reverse the order of summation, let $\alpha = \ell - i$, and count the number of monomials of degree at most $\ell - t$ in e variables to see that

$$\begin{aligned} \sum_{i=t}^{s}\sum_{\ell=i}^{s} c_\ell \binom{(e-1)+(\ell-i)}{\ell-i} &= \sum_{\ell=t}^{s} c_\ell \sum_{i=t}^{\ell} \binom{(e-1)+(\ell-i)}{\ell-i} \\ = \sum_{\ell=t}^{s} c_\ell \sum_{\alpha=0}^{\ell-t} \binom{(e-1)+\alpha}{\alpha} &= \sum_{\ell=t}^{s} c_\ell \binom{e+\ell-t}{\ell-t}; \end{aligned}$$

and therefore, the inequality (4.4.1) is equivalent to

$$\lambda_R(R) \leq \sum_{i=0}^{t-1} \tbinom{(e-1)+i}{i} + \sum_{\ell=t}^{s} c_\ell \tbinom{e+\ell-t}{\ell-t}. \tag{4.4.6}$$

On the other hand, the inequality (4.4.6) does indeed hold, because

$$\begin{aligned} \lambda_R(R) &= \sum_{i=0}^{t-1} \lambda_R(\mathfrak{m}^i/\mathfrak{m}^{i+1}) + \lambda_R(\mathfrak{m}^t); \\ \lambda_R(\mathfrak{m}^i/\mathfrak{m}^{i+1}) &\leq \binom{(e-1)+i}{i}, \end{aligned} \tag{4.4.7}$$

as described at (4.3.2); and Proposition 4.3 guarantees that

$$\lambda_R(\mathfrak{m}^t) \le \sum_{\ell=t}^{s} c_\ell \binom{e+\ell-t}{\ell-t}. \tag{4.4.8}$$

This completes the proof of (a).

The parameter c_s is at least 1; so, one consequence of the inequality (4.4.5), when $i=t$, is

$$\tbinom{(e-1)+s-t}{s-t} \le c_s\tbinom{(e-1)+s-t}{s-t} \le \sum_{\ell=t}^{s} c_\ell \tbinom{(e-1)+(\ell-t)}{\ell-t} < \tbinom{(e-1)+t}{t}.$$

The binomial coefficient $\binom{e-1+i}{i}$ counts the number of monomials of degree i in a polynomial ring with e variables; therefore, the most recent inequality forces $s-t<t$; and therefore,

$$s \le 2t-1. \tag{4.4.9}$$

(b) It is clear that if R is a compressed local Artinian ring in the sense of Definition 2.5, then equality holds in (4.4.1).

Henceforth, in this proof, we assume that equality holds in (4.4.1). We first prove that R is a compressed local Artinian ring in the sense of Definition 2.5; that is, we prove that

$$\dim_{\boldsymbol{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \begin{cases} \tbinom{e-1+i}{i}, & \text{if } 0 \le i \le t-1, \text{ and} \\ \sum\limits_{\ell=i}^{s} c_\ell \tbinom{(e-1)+(\ell-i)}{\ell-i}, & \text{if } t \le i \le s. \end{cases} \tag{4.4.10}$$

The inequality (4.4.6) is equivalent to (4.4.1); hence equality holds in (4.4.6) and in all of the intermediary inequalities that lead to (4.4.6). In particular,

$$\lambda_R(R) = \sum_{i=0}^{t-1} \tbinom{(e-1)+i}{i} + \sum_{\ell=t}^{s} c_\ell \tbinom{e+\ell-t}{\ell-t} \text{ and} \tag{4.4.11}$$

$$\dim_{\boldsymbol{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \tbinom{e-1+i}{i}, \quad \text{for } 0 \le i \le t-1, \tag{4.4.12}$$

follow from (4.4.7), and

$$\lambda_R(\mathfrak{m}^t) = \sum_{\ell=t}^{s} c_\ell \binom{e+\ell-t}{\ell-t} \tag{4.4.13}$$

follows from (4.4.8). The equality (4.4.13) forces equality to hold in (4.3.4) when $j=t$; hence equality holds in (4.3.3) when $j=t$; and therefore, the injections of (3.0.1) are isomorphisms when $j=t$ and $1 \le k \le s+1-t$.

We apply Lemma 3.1 to each pair (A, B) with $A = t$ and $0 \le B \le s - t$. Recall from (4.4.9) that $s - t \le t - 1$; hence (4.4.12) ensures that hypothesis (a) of Lemma 3.1 is in effect. The isomorphisms of (3.0.1), for $j = t$, ensure that hypothesis (b) is in effect. Conclude that

$$\mathrm{mult} : \mathfrak{m}^j \cap (0 : \mathfrak{m}^k) \to \mathrm{Hom}_R(\mathfrak{m}^{k-1}, \mathrm{socle}(R) \cap \mathfrak{m}^{j+k-1}) \tag{4.4.14}$$

is surjective for all j, k with

$$t \le j \le s \quad \text{and} \quad 1 \le k \le s - j + 1. \tag{4.4.15}$$

Therefore, the injections of (3.0.1) are isomorphisms and equality holds in (4.3.3) when j and k satisfy (4.4.15); that is,

$$\begin{gathered}\mathrm{mult} : \frac{\mathfrak{m}^j \cap (0 : \mathfrak{m}^k)}{\mathfrak{m}^j \cap (0 : \mathfrak{m}^{k-1})} \xrightarrow{\cong} \mathrm{Hom}_R\left(\frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k}, \mathrm{socle}(R) \cap \mathfrak{m}^{j+k-1}\right) \\ \text{for } t \le j \le s \quad \text{and} \quad 1 \le k \le s - j + 1.\end{gathered} \tag{4.4.16}$$

Furthermore, equality holds in (4.3.4) for $t \le j \le s$. In particular,

$$\begin{aligned}\dim_{\boldsymbol{k}} \mathfrak{m}^j/\mathfrak{m}^{j+1} &= \sum_{\ell=j}^{s} c_\ell \binom{e+\ell-j}{\ell-j} - \sum_{\ell=j+1}^{s} c_\ell \binom{e+\ell-j-1}{\ell-j-1} \\ &= \sum_{\ell=j}^{s} c_\ell \binom{e+\ell-j-1}{\ell-j},\end{aligned} \tag{4.4.17}$$

for $t \le j \le s$. Combine (4.4.12) and (4.4.17) to see that (4.4.10) holds. This completes the proof of (b).

(c) The inequalities of (4.4.5) and (4.4.9) hold because of the definition of t which is given in (4.4.4). We assume equality holds in (4.4.1); so (4.4.10) holds. We conclude that $t = \mathtt{v}(R)$ and $s \le 2\mathtt{v}(R) - 1$. $\square$

In Corollary 4.5 we describe the annihilator of each large power of the maximal ideal of R, when R is a compressed local Artinian ring. This information is used heavily in the proofs of Corollary 4.6 and Lemma 4.7.

Corollary 4.5. *If $(R, \mathfrak{m}, \boldsymbol{k})$ is a compressed local Artinian ring with top socle degree s, then the following statements hold.*

(a) *If $\mathtt{v}(R) \le j \le s$, then $(0 : \mathfrak{m}^j) = \mathfrak{m}^{s-j+1}$.*
(b) *If $1 \le j \le s + 1$, then $\mathfrak{m}^j : \mathfrak{m} = \mathfrak{m}^{j-1} + \mathrm{socle}(R)$.*

(c) *If R is also a level ring* (*that is, if* $\operatorname{socle}(R) = \mathfrak{m}^s$), *then*

$$(0 : \mathfrak{m}^j) = \mathfrak{m}^{s-j+1} \qquad \textit{for } 0 \le j \le s+1.$$

Proof. The ring R is compressed, local, and Artinian; consequently, R satisfies all of the statements in the proof of Theorem 4.4 and, according to Theorem 4.4.(c), the parameter $\mathrm{v}(R)$ is equal to the "t" of (4.4.4).

(a) Apply (4.4.14) with $t \le j \le s$ and $k = s - j + 1$ to see that

$$\text{mult} : \mathfrak{m}^j \to \operatorname{Hom}_R\left(\frac{\mathfrak{m}^{s-j}}{\mathfrak{m}^{s-j+1}}, \mathfrak{m}^s\right)$$

is a surjection. It follows that

$$\frac{\mathfrak{m}^{s-j} \cap (0 : \mathfrak{m}^j)}{\mathfrak{m}^{s-j+1}} \subseteq \bigcap \ker f = 0,$$

as f roams over $\operatorname{Hom}_R\left(\frac{\mathfrak{m}^{s-j}}{\mathfrak{m}^{s-j+1}}, \mathfrak{m}^s\right)$. The final equality is a statement about homomorphisms of vector spaces. Thus,

$$\mathfrak{m}^{s-j} \cap (0 : \mathfrak{m}^j) \subseteq \mathfrak{m}^{s-j+1};$$

hence,

$$\begin{aligned} &\mathfrak{m}^{s-j+1} \subseteq \mathfrak{m}^{s-j} \cap (0 : \mathfrak{m}^j) \subseteq \mathfrak{m}^{s-j+1} \quad \text{and} \\ &t \le j \le s \Longrightarrow \mathfrak{m}^{s-j} \cap (0 : \mathfrak{m}^j) = \mathfrak{m}^{s-j+1}. \end{aligned} \tag{4.5.1}$$

Apply descending induction on j to see that

$$t \le j \le s \Longrightarrow (0 : \mathfrak{m}^j) = \mathfrak{m}^{s-j+1}. \tag{4.5.2}$$

Indeed, (4.5.2) holds when $j = s$. Assume that (4.5.2) holds when $j+1$. We prove that (4.5.2) holds when j. Observe that

$$(0 : \mathfrak{m}^j) \subseteq (0 : \mathfrak{m}^{j+1}) = \mathfrak{m}^{s-j}. \tag{4.5.3}$$

(The final equality is due to the induction hypothesis.) Thus,

$$(0 : \mathfrak{m}^j) = (0 : \mathfrak{m}^j) \cap \mathfrak{m}^{s-j} = \mathfrak{m}^{s-j+1}.$$

The equality on the left is due to (4.5.3) and the equality on the right is due to (4.5.1).

(b) We saw in Observation 2.4 that

$$(\mathfrak{m}^j : \mathfrak{m}) = \mathfrak{m}^{j-1} = \mathfrak{m}^{j-1} + \operatorname{socle}(R),$$

for $1 \le j \le \mathtt{v}(R)$. Also, the assertion of (b) is obvious at $j = s+1$. The parameter $\mathtt{v}(R)$ continues to equal to the "t" of (4.4.4). We prove that if $t+1 \le j \le s$, then

$$\mathfrak{m}^j : \mathfrak{m} = \mathfrak{m}^{j-1} + \operatorname{socle}(R).$$

It suffices to prove the inclusion "$\subseteq$". To do this, it suffices to prove the following claim.

Claim. *If $2 \le a \le s-j+2$ and $\theta \in (\mathfrak{m}^j : \mathfrak{m}) \cap (0 : \mathfrak{m}^a)$, then there exists an element θ' in $\mathfrak{m}^{j-1} \cap (0 : \mathfrak{m}^a)$ with $\theta - \theta' \in (0 : \mathfrak{m}^{a-1})$.*

We prove the claim. Observe that multiplication by θ is an element of

$$\operatorname{Hom}_R(\mathfrak{m}^{a-1}/\mathfrak{m}^a, \operatorname{socle}(R) \cap \mathfrak{m}^{a+j-2}).$$

Of course, we know from (4.4.16), that there is an element $\theta' \in \mathfrak{m}^{j-1} \cap (0 : \mathfrak{m}^a)$ with multiplication by θ' equal to multiplication by θ on $\mathfrak{m}^{a-1}$.

(c) One direction of assertion (c) is obvious. We prove the other direction. The special hypothesis $\operatorname{socle}(R) = \mathfrak{m}^s$ of (c) guarantees that $\operatorname{socle}(R) \subseteq \mathfrak{m}^a$ for $0 \le a \le s$; and therefore, under this special hypothesis, assertion (b) becomes

$$1 \le j \le s+1 \Longrightarrow \mathfrak{m}^j : \mathfrak{m} = \mathfrak{m}^{j-1}. \tag{4.5.4}$$

Fix an element x in R and an integer i with $0 \le i \le s$ and $x\mathfrak{m}^i = 0$. We use descending induction to prove that

$$0 \le a \le i \Longrightarrow x\mathfrak{m}^a \subseteq \mathfrak{m}^{s+a+1-i}. \tag{4.5.5}$$

It is clear that (4.5.5) holds at $a = i$. Suppose $1 \le a \le i$ and (4.5.5) holds at a. Then

$$x\mathfrak{m}^{a-1} \subseteq (\mathfrak{m}^{s+a+1-i} : \mathfrak{m}) = \mathfrak{m}^{s+a-i}.$$

(Use the induction hypothesis (4.5.5) for the inclusion and (4.5.4) for the equality.) Thus (4.5.5) holds at $a = i-1$ and the induction is complete. Apply (4.5.5) at $a = 0$ to see that $x \in \mathfrak{m}^{s+1-i}$. $\square$

Associated graded objects are discussed in 2.8. The following result is shown by Iarrobino [16, Cor. 3.8] in the equicharacteristic case. This result is confirmation that Definition 2.5 (or equivalently, equality in (4.4.1), or equivalently, equality in (4.4.3)) is the correct intrinsic definition of a compressed local Artinian ring.

Corollary 4.6. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring. Then R is a compressed ring if and only if the associated graded ring R^{g} is a compressed ring and R and R^{g} have the same socle polynomial.*

Proof. ($\Leftarrow$) This direction is obvious. Indeed, the Hilbert function of R is always equal to the Hilbert function of R^{g} and the hypothesis asserts that the relationship of Definition 2.5 holds between $h_{R^{\mathrm{g}}}$ and the socle polynomial of R^{g}.

($\Rightarrow$) As described above, it suffices to show R and R^{g} have the same socle polynomial. The isomorphism theorem $I/(I \cap J) \cong (I + J)/J$ ensures that

$$\frac{\operatorname{socle}(R) \cap \mathfrak{m}^i}{\operatorname{socle}(R) \cap \mathfrak{m}^{i+1}} \cong \frac{(\operatorname{socle}(R) \cap \mathfrak{m}^i) + \mathfrak{m}^{i+1}}{\mathfrak{m}^{i+1}};$$

hence the socle polynomial R, defined in 2.3.(d), is also equal to

$$\sum_{i=0}^{s} \dim_{\boldsymbol{k}} \frac{(\operatorname{socle}(R) \cap \mathfrak{m}^i) + \mathfrak{m}^{i+1}}{\mathfrak{m}^{i+1}} z^i,$$

where s is the top socle degree of R. On the other hand, the socle polynomial of the graded local ring R^{g} is

$$\sum_{i=0}^{s} \dim_{\boldsymbol{k}} \frac{\mathfrak{m}^i \cap (\mathfrak{m}^{i+2} : \mathfrak{m})}{\mathfrak{m}^{i+1}} z^i.$$

The ring R is compressed; hence Corollary 4.5.(b) guarantees that

$$\mathfrak{m}^j : \mathfrak{m} = \mathfrak{m}^{j-1} + \operatorname{socle}(R) \quad \text{for } 1 \le j \le s$$

and the two socle polynomials are equal. $\square$

The statement of the main result, Theorem 7.1, depends on the relationship between s and the invariant $\mathrm{v}(R)$. Recall from Theorem 4.4.(c) that $s \le 2\mathrm{v}(R) - 1$. We show in Observation 5.6 that the critical situation is $s = 2\mathrm{v}(R) - 1$. The first step in the critical situation is taken in Lemma 4.7. This step is the main ingredient in the proof of Lemma 6.3.

Lemma 4.7. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a compressed local Artinian ring with embedding dimension e and top socle degree s. Assume that s is odd and that $s = 2\mathrm{v}(R) - 1$. Decompose the maximal ideal $\mathfrak{m}$ as the sum of two subideals $\mathfrak{m} = (x_1) + \mathfrak{m}'$ with x_1 a minimal generator of $\mathfrak{m}$ and $\mu(\mathfrak{m}') = e - 1$. Then*

$$x_1^{\frac{s-1}{2}} [\operatorname{ann}_R(\mathfrak{m}') \cap \mathfrak{m}^{\frac{s+1}{2}}] = \mathfrak{m}^s.$$

Proof. Let t denote $\mathtt{v}(R)$, which by hypothesis is equal to $(s+1)/2$. It is clear that

$$x_1^{t-1}[\operatorname{ann}_R(\mathfrak{m}') \cap \mathfrak{m}^t] \subseteq \mathfrak{m}^s.$$

For the other direction, let σ be an element of $\mathfrak{m}^s$. We will construct an element Θ of $\operatorname{ann}_R(\mathfrak{m}') \cap \mathfrak{m}^t$ such that $x_1^{t-1}\Theta = \sigma$. We build Θ as $\theta_0 + \cdots + \theta_{t-2}$, where, for each i,

$$\begin{cases} \theta_i \in \mathfrak{m}^t \cap (0 : \mathfrak{m}^{t-i}), \\ (\theta_0 + \cdots + \theta_i)x_1^{t-1} = \sigma, & \text{and} \\ (\theta_0 + \cdots + \theta_i)\mathfrak{m}'\mathfrak{m}^{t-i-2} = 0. \end{cases} \tag{4.7.1}$$

We first build θ_0. Consider the homomorphism

$$\phi_0 \in \operatorname{Hom}_R(\mathfrak{m}^{t-1}/\mathfrak{m}^t, \operatorname{socle}(R) \cap \mathfrak{m}^s),$$

which is given by

$$\phi_0(\overline{\mathfrak{m}'\mathfrak{m}^{t-2}}) = 0 \quad \text{and} \quad \phi_0(\overline{x_1^{t-1}}) = \sigma.$$

(Keep in mind that $\mathfrak{m}^{t-1}/\mathfrak{m}^t$ and $\overline{\mathfrak{m}'\mathfrak{m}^{t-2}} \oplus \boldsymbol{k}\overline{x_1^{t-1}}$ are isomorphic as R-modules. At this point $^-$ means mod $\mathfrak{m}^t$.) Apply (4.4.16), with $j = k = t$ to obtain an element $\theta_0 \in \mathfrak{m}^t \cap \operatorname{ann}(\mathfrak{m}^t)$ with $x_1^{t-1}\theta_0 = \sigma$ and $\theta_0\mathfrak{m}'\mathfrak{m}^{t-2} = 0$.

Suppose $0 \le i \le t-3$ and elements $\theta_0, \dots \theta_i$, which satisfy (4.7.1), have been identified. We now build θ_{i+1}. Consider the homomorphism

$$\phi_{i+1} \in \operatorname{Hom}_R(\mathfrak{m}^{t-i-2}/\mathfrak{m}^{t-i-1}, \operatorname{socle}(R) \cap \mathfrak{m}^{s-i-1}),$$

which is given by

$$\phi_{i+1}(\bar{u}) = -(\theta_0 + \cdots + \theta_i)u, \quad \text{for } u \in \mathfrak{m}'\mathfrak{m}^{t-i-3}, \quad \text{and} \quad \phi_{i+1}(\overline{x_1^{t-i-2}}) = 0.$$

(At this point $^-$ means mod $\mathfrak{m}^{t-i-1}$. We have taken advantage of a direct sum decomposition of $\mathfrak{m}^{t-i-2}/\mathfrak{m}^{t-i-1}$ to define ϕ_{i+1}. The image of ϕ_{i+1} is contained in the socle of R because of the properties of the earlier θ's as described in (4.7.1).) Apply (4.4.16), with $j = t$ and $k = t - i - 1$ to obtain an element

$$\begin{cases} \theta_{i+1} \in \mathfrak{m}^t \cap \operatorname{ann} \mathfrak{m}^{t-i-1} & \text{with} \\ (\theta_0 + \cdots + \theta_{i+1})x_1^{t-1} = \sigma & \text{and} \\ (\theta_0 + \cdots + \theta_{i+1})(\mathfrak{m}'\mathfrak{m}^{t-i-3}) = 0. \end{cases}$$

Iterate this procedure to find θ_{t-2} and thereby complete the proof. $\square$

5. Golod homomorphisms

In this paper we exhibit a Golod homomorphism from a complete intersection onto a compressed local Artinian ring R and then use facts about Golod homomorphisms to draw conclusions about the Poincaré series of R-modules. The present section is mainly concerned with techniques from homological algebra that can be used to prove that a homomorphism is Golod. The hypothesis "compressed" is not used anywhere in the present section.

There are numerous definitions of Golod homomorphism (see for example [2]); we give the version involving trivial Massey operations, found, for example, in [15]. In Lemma 5.2 we record a result from [22] which shows how to use homological algebra to prove that trivial Massey operations exist. Most of the section is about homological algebra. Indeed, in Lemmas 5.4 and 5.5 we prove that various maps of Tor are zero. Lemma 5.5 is used in Observation 5.6 to show that if the top socle degree of a local Artinian ring R is small compared to the invariant $\mathtt{v}(R)$ of 2.3.(f), then R is a Golod ring. Lemmas 5.8 and 5.9 are a short study of the effect on Tor associated to taking a hypersurface section. The section concludes with Theorem 5.10 which is a well-known result that exhibits the common denominator for all Poincaré series $P_M^R(z)$ when there is a Golod homomorphism from a local hypersurface ring onto R and M roams over all finitely generated R-modules.

Definition 5.1. [15] (see also, [22, 1.1]) Let $\kappa : (P, \mathfrak{p}, \boldsymbol{k}) \to (R, \mathfrak{m}, \boldsymbol{k})$ be a surjective homomorphism of local rings and $\mathcal{D}$ be an associative, graded-commutative, Differential Graded (DG) algebra with divided powers, which is also a homogeneous minimal resolution of $\boldsymbol{k}$ by free P-modules. Let $(\mathcal{A}, \partial)$ denote $\mathcal{D} \otimes_P R$. If x is a homogeneous element in $\mathcal{A}_\ell$, then let $|x|$ denote the degree ℓ of x and $\bar{x}$ denote $(-1)^{|x|+1}x$. Let $\mathbf{h} = \{h_i\}_{i\geq 1}$ be a homogeneous basis of the graded $\boldsymbol{k}$-vector space $\mathrm{H}_{\geq 1}(\mathcal{A})$. The homomorphism $\kappa : P \to R$ is *Golod* if there is a function $\mu : \bigsqcup_{n=1}^{\infty} \mathbf{h}^n \to \mathcal{A}$ which satisfies:

(a) $\mu(h)$ is a cycle in the homology class of h for each $h \in \mathbf{h}$,
(b) $\partial\mu(h_1, \dots, h_n) = \sum_{i=1}^{n-1} \overline{\mu(h_1, \dots, h_i)}\mu(h_{i+1}, \dots, h_n)$ for each n with $2 \leq n$, and
(c) $\mu(\mathbf{h}^n) \subseteq \mathfrak{m}\mathcal{A}$ for each positive n.

Lemma 5.2 is our main tool for proving that a homomorphism is Golod.

Lemma 5.2. *[22, Lem. 1.2] Let $\kappa : (P, \mathfrak{p}, \boldsymbol{k}) \to (R, \mathfrak{m}, \boldsymbol{k})$ be a surjective homomorphism of local rings. If there exists a positive integer a such that:*

(a) *the map $\mathrm{Tor}_i^P(R, \boldsymbol{k}) \to \mathrm{Tor}_i^P(R/\mathfrak{m}^a, \boldsymbol{k})$, induced by the canonical quotient map $R \to R/\mathfrak{m}^a$, is zero for all positive i, and*
(b) *the map $\mathrm{Tor}_i^P(\mathfrak{m}^{2a}, \boldsymbol{k}) \to \mathrm{Tor}_i^P(\mathfrak{m}^a, \boldsymbol{k})$, induced by the inclusion $\mathfrak{m}^{2a} \subseteq \mathfrak{m}^a$, is zero for all non-negative integers i,*

then κ is a Golod homomorphism.

It is convenient to name the following family of maps of Tor.

Definition 5.3. [22, 1.3.1] If M is a module over the local ring $(R, \mathfrak{m}, \boldsymbol{k})$, then let $\nu_i^R(M)$ represent the R-module homomorphism

$$\nu_i^R(M) : \operatorname{Tor}_i^R(\mathfrak{m}M, \boldsymbol{k}) \to \operatorname{Tor}_i^R(M, \boldsymbol{k}),$$

which is induced by the inclusion $\mathfrak{m}M \subseteq M$.

We use Lemma 5.4 to calculate ν_i. Associated graded objects are discussed in 2.8.

Lemma 5.4. *Let $(Q, \mathfrak{n}, \boldsymbol{k})$ be a regular local ring, $(R, \mathfrak{m}, \boldsymbol{k})$ be the local ring $R = Q/I$ for some ideal I of Q, and i and ℓ be two integers. If $\operatorname{Tor}_{i,j}^{Q^{\mathrm{g}}}(R^{\mathrm{g}}, \boldsymbol{k}) = 0$ for all j with $\ell + 1 + i \leq j$, then the map*

$$\nu_i^Q(\mathfrak{m}^\ell) \colon \operatorname{Tor}_i^Q(\mathfrak{m}^{\ell+1}, \boldsymbol{k}) \to \operatorname{Tor}_i^Q(\mathfrak{m}^\ell, \boldsymbol{k})$$

is identically zero.

Proof. Let K^R denote the Koszul complex over R on a minimal generating set $x_1, \dots, x_e$ of $\mathfrak{m}$. We identify $\nu_i^Q(\mathfrak{m}^\ell)$ with the map $\mathrm{H}_i(\mathfrak{m}^{\ell+1}K^R) \to \mathrm{H}_i(\mathfrak{m}^\ell K^R)$ induced by the inclusion

$$\mathfrak{m}^{\ell+1}K^R \subseteq \mathfrak{m}^\ell K^R.$$

Let Z denote the module of cycles in degree i of $\mathfrak{m}^{\ell+1}K^R$ and B denote the module of boundaries of degree i in $\mathfrak{m}^\ell K^R$. Note that $B \subseteq Z$. To show that $\nu_i^Q(\mathfrak{m}^\ell)$ is zero, we need to show that $Z \subseteq B$. We will show that $Z \subseteq B + \mathfrak{m}^j K_i^R$ for all j with $\ell + 2 \leq j$.

For each j, let x_j^* denote the image the element x_j in $\mathfrak{m}/\mathfrak{m}^2 = (R^{\mathrm{g}})_1$. Let L denote the graded Koszul complex over R^{g} on $x_1^*, \dots, x_e^*$. When writing $L_{p,q}$, the index p stands for the homological degree and the index q for the internal degree. Note that L can be thought of as the associated graded complex of K^R, with respect to the standard $\mathfrak{m}$-adic filtration of K^R. In particular, $L_p = ((K_p^R)^{\mathrm{g}})(-p)$ for each p, and the differential d_L of L is induced from the differential d_{K^R} of K^R as follows: If $y \in \mathfrak{m}^q K_p^R$ and y^* is the image of y in $\mathfrak{m}^q K_p^R/\mathfrak{m}^{q+1}K_p^R = L_{p,p+q}$, then $d_L(y^*)$ is equal to the image of $d_{K^R}(y)$ in $\mathfrak{m}^{q+1}K_{p-1}^R/\mathfrak{m}^{q+2}K_{p-1}^R = L_{p-1,p+q}$. We identify $\operatorname{Tor}^{Q^{\mathrm{g}}}(R^{\mathrm{g}}, \boldsymbol{k})$ with the homology of the complex L.

Fix an integer p with $\ell + 1 \leq p$ and let $z \in Z \cap \mathfrak{m}^p K_i^R$. In particular, $d_{K^R}(z) = 0$. We consider z^* to be the image of z in $\mathfrak{m}^p K_i^R/\mathfrak{m}^{p+1}K_i^R = L_{i,p+i}$ and note that $d_L(z^*) = 0$ because $d_{K^R}(z) = 0$. The hypothesis that $\operatorname{Tor}_{i,p+i}^{Q^{\mathrm{g}}}(R^{\mathrm{g}}, \boldsymbol{k}) = 0$ implies that $z^* = d_L(y^*)$ where $y^* \in \mathfrak{m}^{p-1}K_{i+1}^R/\mathfrak{m}^p K_{i+1}^R = L_{i+1,p+i}$ is the image of an element $y \in \mathfrak{m}^{p-1}K_{i+1}^R$. It follows that $z - d_{K^R}(y) \in \mathfrak{m}^{p+1}K_i^R$, and we conclude that $z \in B + \mathfrak{m}^{p+1}K_i^R$. It follows that

$$Z \cap \mathfrak{m}^p K_i^R \subseteq Z \cap (B + \mathfrak{m}^{p+1} K_i^R) = B + Z \cap \mathfrak{m}^{p+1} K_i^R. \tag{5.4.1}$$

Since $Z = Z \cap \mathfrak{m}^{\ell+1} K_i^R$, we conclude inductively, using (5.4.1), that $Z \subseteq B + \mathfrak{m}^j K_i^R$ for all j with $\ell + 2 \le j$, hence $Z \subseteq B$ by the Krull intersection Theorem. □

Lemma 5.5 is a straightforward consequence of Lemma 5.4 and also appears as [22, 1.4]. This result is used in the proof of Observation 5.6; furthermore, a consequence of Lemma 5.5 is restated as 7.2.1.

Lemma 5.5. *[22, 1.4] Let $(Q, \mathfrak{n}, \boldsymbol{k})$ be a regular local ring and $(R, \mathfrak{m}, \boldsymbol{k})$ be the local ring $R = Q/I$ for some ideal I of Q. Then the maps*

$$\operatorname{Tor}_i^Q(R/\mathfrak{m}^{\ell+1}, \boldsymbol{k}) \to \operatorname{Tor}_i^Q(R/\mathfrak{m}^{\ell}, \boldsymbol{k}) \tag{5.5.1}$$

and

$$\operatorname{Tor}_i^Q(R, \boldsymbol{k}) \to \operatorname{Tor}_i^Q(R/\mathfrak{m}^{\ell}, \boldsymbol{k}), \tag{5.5.2}$$

are each the zero map for all (i, ℓ) with $1 \le i$ and $1 \le \ell \le \mathtt{v}(R) - 1$. The map of (5.5.1) is induced by the natural quotient map $R/\mathfrak{m}^{\ell+1} \to R/\mathfrak{m}^{\ell}$ and the map of (5.5.2) is induced by the natural quotient map $R \to R/\mathfrak{m}^{\ell}$.

Observation 5.6 takes care of the "easy case" in the proof of the main theorem, which is Theorem 7.1.

Observation 5.6. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring with top socle degree s. If $s \le 2\mathtt{v}(R) - 3$, then R is a Golod ring.*

Proof. Let t denote $\mathtt{v}(R)$. The ring R is complete and local; so the Cohen structure theorem guarantees that there is a regular local ring $(Q, \mathfrak{n}, \boldsymbol{k})$ with $R = Q/I$ and $I \subseteq \mathfrak{n}^2$. We apply Lemma 5.2, with $a = t - 1$, to show that the canonical quotient map $Q \to Q/I = R$ is a Golod homomorphism. It follows that R is a Golod ring. It suffices to show that

(i) the map

$$\operatorname{Tor}_i^Q(R, \boldsymbol{k}) \to \operatorname{Tor}_i^Q(R/\mathfrak{m}^{t-1}, \boldsymbol{k}),$$

induced by the quotient map $R \to R/\mathfrak{m}^{t-1}$, is zero for all positive i, and

(ii) the map

$$\operatorname{Tor}_i^Q(\mathfrak{m}^{2t-2}, \boldsymbol{k}) \to \operatorname{Tor}_i^Q(\mathfrak{m}^{t-1}, \boldsymbol{k}),$$

induced by the inclusion $\mathfrak{m}^{2t-2} \to \mathfrak{m}^{t-1}$ is zero for all non-negative i.

Condition (i) is established in Lemma 5.5 and (ii) obviously holds. Indeed, by hypothesis, the top socle degree s of R satisfies $s \le 2t-3$. It follows that $\mathfrak{m}^{2t-2} = 0$. $\square$

The following two results are proven in [22]; but in each case the statement given in [22] is slightly different than the statement given here.

Set up 5.7. Let $(Q, \mathfrak{n}, \boldsymbol{k})$ and $(P, \mathfrak{p}, \boldsymbol{k})$ be local rings with $P = Q/(h)$ for some element h in $\mathfrak{n}^t$ with h not a zerodivisor on Q and $2 \le t$. Let $N \subseteq M$ be finitely generated P-modules, $\mathrm{incl} : N \to M$ represent the inclusion map, and $\varphi : Q \to P$ represent the natural quotient map. For any P-module X, let $\varphi_i^X : \mathrm{Tor}_i^Q(X, \boldsymbol{k}) \to \mathrm{Tor}_i^P(X, \boldsymbol{k})$ be the map on Tor induced by the change of rings $\varphi : Q \to P$. For either ring $A = P$ or $A = Q$, let $\mathrm{incl}_i^A : \mathrm{Tor}_i^A(N, \boldsymbol{k}) \to \mathrm{Tor}_i^A(M, \boldsymbol{k})$ be the map on Tor induced by the A-module homomorphism $\mathrm{incl} : N \to M$.

Lemma 5.8. [22, Lem. 2.4] *Adopt the notation of 5.7. If $\mathfrak{n}^{t-1}(M/N)$ is zero, then*

$$\ker\left(\varphi_i^M : \mathrm{Tor}_i^Q(M, \boldsymbol{k}) \to \mathrm{Tor}_i^P(M, \boldsymbol{k})\right) \subseteq \mathrm{im}\left(\mathrm{incl}_i^Q : \mathrm{Tor}_i^Q(N, \boldsymbol{k}) \to \mathrm{Tor}_i^Q(M, \boldsymbol{k})\right)$$

for all i.

Lemma 5.9. [22, Lem. 2.3] *Adopt the notation of 5.7. If the modules $\mathfrak{n}^{t-1}N$ and $\mathfrak{n}^{t-1}(M/N)$ are both zero, then the following statements are equivalent*:

(a) *the map $\mathrm{Tor}_i^P(N, \boldsymbol{k}) \xrightarrow{\mathrm{incl}_i^P} \mathrm{Tor}_i^P(M, \boldsymbol{k})$ is identically zero for all i, and*
(b) *the composition $\mathrm{Tor}_i^Q(N, \boldsymbol{k}) \xrightarrow{\mathrm{incl}_i^Q} \mathrm{Tor}_i^Q(M, \boldsymbol{k}) \xrightarrow{\varphi_i^M} \mathrm{Tor}_i^P(M, \boldsymbol{k})$ is identically zero for all i.*

Remark. To prove these results, in each case start with the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

and follow the argument given in [22]. Keep in mind that the hypothesis that $\mathfrak{n}^{t-1}(M/N)$ is zero ensures that

$$\varphi_i^{M/N} : \mathrm{Tor}_i^Q(M/N, \boldsymbol{k}) \to \mathrm{Tor}_i^P(M/N, \boldsymbol{k})$$

is injective for all i, see [22, line 3 on page 427]. In particular, the conclusion we have drawn in Lemma 5.8 does not require $\mathrm{incl}_i^Q : \mathrm{Tor}_i^Q(N, \boldsymbol{k}) \to \mathrm{Tor}_i^Q(M, \boldsymbol{k})$ to be the zero map; that is, Lemma 5.8 follows from the proof, but not the statement, of [22, Lem.24].

It is worth noting that the change of rings involved in constructing

$$\varphi_i^M : \mathrm{Tor}_i^Q(M, \boldsymbol{k}) \to \mathrm{Tor}_i^P(M, \boldsymbol{k})$$

is fairly subtle; see [4, Thm. 3.1.3] for details. The original construction was due to Shamash [24]; this construction planted a seed that evolved into the Eisenbud operators.

We conclude this section with a result which exhibits the common denominator for all Poincaré series $P_M^R(z)$ when there is a Golod homomorphism from a local hypersurface ring onto R and M roams over all finitely generated R-modules.

Theorem 5.10. *Let $(Q, \mathfrak{n}, \boldsymbol{k})$ be a regular local ring of embedding dimension e, $(P, \mathfrak{p}, \boldsymbol{k})$ be a local ring with $P = Q/(h)$ for some $h \in \mathfrak{n}^2$, $(R, \mathfrak{m}, \boldsymbol{k})$ be a local ring, $\kappa : P \to R$ be a surjective Golod homomorphism, $\varphi_\bullet^R : \mathrm{Tor}_\bullet^Q(R, \boldsymbol{k}) \to \mathrm{Tor}_\bullet^P(R, \boldsymbol{k})$ be the map induced by the natural quotient map $Q \to P$, and $d_R(z)$ be the polynomial*

$$d_R(z) = 1 - z(P_R^Q(z) - 1) + (z + z^2) \cdot \left(\mathrm{HS}_{\ker \varphi_\bullet^R}(z) - z\right) \in \mathbb{Z}[z].$$

Then, for every finitely generated R-module M, there exists a polynomial $p_M(z)$ in $\mathbb{Z}[z]$ with

$$P_M^R(z) d_R(z) = p_M(z).$$

In particular, $p_{\boldsymbol{k}}(z) = (1 + z)^e$.

Proof. Results of Levin, see for example [7, Prop. 5.18], give all of the conclusions, except for the formula for $d_R(z)$. The denominator $d_R(z)$ is calculated in [22]; although the exact form given above is not explicitly identified there. Most of the steps are well known. One starts with the equation

$$d_R(z) = \frac{(1 + z)^e}{P_{\boldsymbol{k}}^R(z)};$$

so it suffices to calculate $P_{\boldsymbol{k}(z)}^R$. The homomorphism κ is Golod; hence the equation

$$P_{\boldsymbol{k}}^R(z) = \frac{P_{\boldsymbol{k}}^P(z)}{1 - z(P_R^P(z) - 1)}$$

holds; see [15, Prop. 1]. The key new step is taken in [22, 2.2.1] where it is shown that

$$P_X^P(z) = \frac{P_X^Q(z) - (1 + z) \cdot \mathrm{HS}_{\ker \varphi_\bullet^X}(z)}{1 - z^2}, \tag{5.10.1}$$

for all finitely generated P-modules X. (The calculation (5.10.1) is valid whenever the hypotheses of 5.7 are satisfied.) In the present calculation, one takes X to be R. The ring P is a hypersurface; consequently, the Poincaré series

$$P_{\boldsymbol{k}}^P(z) = \frac{(1 + z)^e}{1 - z^2}$$

is well known. (Indeed the resolution of $\boldsymbol{k}$ by free P-modules is known.) Combine everything to obtain the formula for $d_R(z)$. $\square$

6. Homological consequences of the hypothesis that R is compressed

We deduce three homological consequences of the hypothesis that local Artinian ring R is compressed. These Lemmas (6.1, 6.3, and 6.4) play a major role in the proof of the main result, Theorem 7.1.

Lemma 6.1. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring and $(R, \mathfrak{m}, k)$ be the local ring $R = Q/I$ for some ideal I of Q. Assume that R is a compressed local Artinian ring of embedding dimension e. If $\mathtt{v}(R) \le \ell$, then the map $\nu_i^Q(\mathfrak{m}^\ell)$ of Definition 5.3 is zero for $i < e$.*

Proof. Apply Corollary 4.6 to see that R^{g} is a standard-graded compressed Artinian $\boldsymbol{k}$-algebra with the top socle degree of R^{g} equal to the top socle degree of R and $\mathtt{v}(R^{\mathrm{g}})$ equal to $\mathtt{v}(R)$; and therefore, [13, Prop. 16] guarantees that $\mathrm{Tor}_i^{Q^{\mathrm{g}}}(R^{\mathrm{g}}, \boldsymbol{k})$ is concentrated in degrees $\mathtt{v}(R) + i - 1$ and $\mathtt{v}(R) + i$, for $1 \le i \le e - 1$. Of course, $\mathrm{Tor}_0^{Q^{\mathrm{g}}}(R^{\mathrm{g}}, \boldsymbol{k})$ is concentrated in degree 0. Lemma 5.4 ensures that $\nu_i^Q(\mathfrak{m}^\ell)$ is identically zero for all pairs (i, ℓ) with $i < e$ and $\mathtt{v}(R) \le \ell$. $\square$

Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local Artinian ring. This ring is complete and local; hence the Cohen structure theorem guarantees that R is the quotient of a regular local ring. We often use information from Data 6.2. This information all automatically exists as soon as the local Artinian ring $(R, \mathfrak{m}, \boldsymbol{k})$ is chosen. Observe that the parameter t of Data 6.2 is equal to the invariant $\mathtt{v}(R)$ of 2.3.(f).

Data 6.2. Let $(Q, \mathfrak{n}, \boldsymbol{k})$ be a regular local ring and $(R, \mathfrak{m}, \boldsymbol{k})$ be the local Artinian ring $R = Q/I$, where I is an ideal of Q with $I \subseteq \mathfrak{n}^2$. Define t to be the largest integer with $I \subseteq \mathfrak{n}^t$. Let $(P, \mathfrak{p}, \boldsymbol{k})$ be the local hypersurface ring $P = Q/L$, where L is the principal ideal of Q generated by a non-zero element of I which is not in $\mathfrak{n}^{t+1}$, (K, ∂) be the Koszul complex which is a minimal resolution of $\boldsymbol{k}$ by free Q-modules, and $\pi : Q \to R$ and $\kappa : P \to R$ be the natural quotient homomorphisms.

Lemma 6.3. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a compressed local Artinian ring of embedding dimension e and top socle degree s. Adopt Data 6.2. Assume that the field $\boldsymbol{k}$ is infinite and that $s = 2t - 1$. Then there exists $G \in \mathfrak{n}^{t-1} K_1$ such that $\partial(G)$ generates L and*

$$Z_e(\mathfrak{m}^s \otimes_Q K) \subseteq \bar{g} Z_{e-1}(\mathfrak{m}^t \otimes_Q K),$$

where g denotes the image of G in $P \otimes_Q K$ and $\bar{g}$ is the image of G in $R \otimes_Q K$.

Proof. The field $\boldsymbol{k}$ is infinite; therefore we may apply Remark 2.11 and decompose $\mathfrak{n}$ into subideals $(X_1) + \mathfrak{n}'$ with X_1 a minimal generator of $\mathfrak{n}$, $\mu(\mathfrak{n}') = e - 1$, and $h - X_1^t$ in the ideal $\mathfrak{n}'\mathfrak{n}^{t-1}$ of Q, for some generator h of L. The decomposition $\mathfrak{n} = X_1Q + \mathfrak{n}'$ induces a decomposition $\mathfrak{m} = x_1R + \mathfrak{m}'$ with x_1 equal to the image of X_1 and $\mathfrak{m}'$ equal to the image of $\mathfrak{n}'$. Let $\mathfrak{q}$ be the ideal $\operatorname{ann}_R(\mathfrak{m}') \cap \mathfrak{m}^t$ of R. We proved in Lemma 4.7 that

$$x_1^{t-1}\mathfrak{q} = \mathfrak{m}^s. \tag{6.3.1}$$

Let $X_2, \dots, X_e$ be a minimal generating set for $\mathfrak{n}'$ and $T_1, \dots, T_e$ be a basis for K_1 with $\partial(T_i) = X_i$. Recall that h has the property that $h - X_1^t \in (X_2, \dots, X_e)\mathfrak{n}^{t-1}$. It follows that there is an element G in K_1 of the form

$$G = X_1^{t-1}T_1 + \sum_{i=2}^{e} \alpha_i T_i, \tag{6.3.2}$$

for some $\alpha_i \in \mathfrak{n}^{t-1}$, with $\partial(G) = h$. The image of G in $R \otimes_Q K$, denoted by $\bar{g}$, is a cycle in $Z_1(\mathfrak{m}^{t-1} \otimes_Q K)$. Observe that

$$\begin{aligned}
Z_e(\mathfrak{m}^s \otimes_Q K) &= \mathfrak{m}^s \cdot T_1 \cdots T_e \\
&= (x_1^{t-1}\mathfrak{q}) \cdot T_1 \cdots T_e && \text{by (6.3.1)} \\
&= \mathfrak{q}(x_1^{t-1}T_1) \cdots T_e \\
&= \mathfrak{q}\Big(\bar{g} - \sum_{i=2}^{e} \alpha_i T_i\Big) T_2 \cdots T_e && \text{by (6.3.2)} \\
&= \mathfrak{q}\bar{g}T_2 \cdots T_e \\
&= \bar{g}\mathfrak{q}T_2 \cdots T_e \\
&\subseteq \bar{g}Z_{e-1}(\mathfrak{q} \otimes_Q K) && \text{since } \partial(T_i)\mathfrak{q} = 0 \text{ for } i \le 2 \le e. \\
&\subseteq \bar{g}Z_{e-1}(\mathfrak{m}^t \otimes_Q K) && \text{since } \mathfrak{q} \subseteq \mathfrak{m}^t. \quad \square
\end{aligned}$$

Lemma 6.4 is the third of three Lemmas in the section. These Lemmas are used in the proof of the main result. The proof of Lemma 6.4 is a continuation of the proof of Lemma 6.3.

Lemma 6.4. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a compressed local Artinian ring of embedding dimension e and top socle degree s. Adopt Data 6.2 with $s = 2t - 1$. The following statements hold.*

(a) *If j is an integer which satisfies*

$$t + 1 \le j \le s \quad \text{and} \quad \operatorname{socle}(R) \cap \mathfrak{m}^j = \mathfrak{m}^s,$$

then the maps

$$\nu_i^P : \operatorname{Tor}_i^P(\mathfrak{m}^j, \boldsymbol{k}) \to \operatorname{Tor}_i^P(\mathfrak{m}^t, \boldsymbol{k}),$$

induced by the inclusion $\mathfrak{m}^j \subseteq \mathfrak{m}^t$, *are zero for all* i.

(b) *The maps* $\operatorname{Tor}_i^R(\mathfrak{m}^s, \boldsymbol{k}) \to \operatorname{Tor}_i^R(\mathfrak{m}^t, \boldsymbol{k})$, *induced by the inclusion* $\mathfrak{m}^s \subseteq \mathfrak{m}^t$, *are zero for all* i.

Proof. Without loss of generality, we may assume that $\boldsymbol{k}$ is infinite. Indeed, if $\boldsymbol{k}' = \boldsymbol{k}(y)$, $Q' = Q[y]_{\mathfrak{n}Q[y]}$, $P' = P[y]_{\mathfrak{p}P[y]}$, $R' = R[y]_{\mathfrak{m}R[y]}$, and $\mathfrak{m}' = \mathfrak{m}R'$, then the extensions $Q \to Q'$, $P \to P'$, and $R \to R'$ are faithfully flat, and therefore, $\nu_i^P = 0$ if and only if $\nu_i^{P'} = 0$, and

$$\operatorname{Tor}_i^R(\mathfrak{m}^s, \boldsymbol{k}) \to \operatorname{Tor}_i^R(\mathfrak{m}^t, \boldsymbol{k}) \text{ is zero} \iff \operatorname{Tor}_i^{R'}(\mathfrak{m}'^s, \boldsymbol{k}') \to \operatorname{Tor}_i^{R'}(\mathfrak{m}'^t, \boldsymbol{k}') \text{ is zero.}$$

(a) Let $\nu_i^Q\colon \operatorname{Tor}_i^Q(\mathfrak{m}^j, \boldsymbol{k}) \to \operatorname{Tor}_i^Q(\mathfrak{m}^t, \boldsymbol{k})$ denote the map induced by the inclusion $\mathfrak{m}^j \subseteq \mathfrak{m}^t$. Apply Lemma 5.9 to the inclusion $\mathfrak{m}^j \subseteq \mathfrak{m}^t$. Observe that $\mathfrak{n}^{t-1}$ annihilates $\mathfrak{m}^j$ and $\mathfrak{m}^t/\mathfrak{m}^j$. Observe also that the map incl_i^A of 5.9 is now denoted ν_i^A for $A = P$ or $A = Q$. We conclude that assertion (a) is equivalent to the assertion

$$\varphi_i^{\mathfrak{m}^t} \circ \nu_i^Q = 0, \quad \text{for all } i, \tag{6.4.1}$$

where $\varphi_i^{\mathfrak{m}^t}\colon \operatorname{Tor}_i^Q(\mathfrak{m}^t, \boldsymbol{k}) \to \operatorname{Tor}_i^P(\mathfrak{m}^t, \boldsymbol{k})$ is the map induced by the natural quotient map $Q \to P$. If $0 \le i \le e-1$, then Lemma 6.1 yields that the map $\nu_i^Q(\mathfrak{m}^t)$ of Definition 5.3 is identically zero. The map ν_i^Q factors through $\nu_i^Q(\mathfrak{m}^t)$; therefore, $\nu_i^Q = 0$ and (6.4.1) holds for $0 \le i \le e-1$.

We now prove (6.4.1) for $i = e$. Recall the Koszul complex (K, ∂) of Data 6.2 which is a resolution of $\boldsymbol{k}$ by free Q-modules. We identify the functors

$$\mathrm{H}_\bullet(- \otimes_Q K) \quad \text{and} \quad \operatorname{Tor}_\bullet^Q(-, \boldsymbol{k}). \tag{6.4.2}$$

Observe that

$$\operatorname{Tor}_e^Q(\mathfrak{m}^j, \boldsymbol{k}) = \mathrm{H}_e(\mathfrak{m}^j \otimes_Q K) = \operatorname{socle}(\mathfrak{m}^j) \otimes_Q K_e \tag{6.4.3}$$

and

$$\operatorname{Tor}_e^Q(\mathfrak{m}^t, \boldsymbol{k}) = \mathrm{H}_e(\mathfrak{m}^t \otimes_Q K) = \operatorname{socle}(\mathfrak{m}^t) \otimes_Q K_e = (\operatorname{socle}(R) \cap \mathfrak{m}^t) \otimes_Q K_e. \tag{6.4.4}$$

The hypothesis that $\operatorname{socle}(R) \cap \mathfrak{m}^j = \mathfrak{m}^s$ yields $\operatorname{socle}(\mathfrak{m}^j) \otimes_Q K_e = \mathfrak{m}^s \otimes_Q K_e$. Thus, $\operatorname{im} \nu_e^Q$ is equal to the submodule $\mathfrak{m}^s \otimes_Q K_e$ of $(\operatorname{socle}(R) \cap \mathfrak{m}^t) \otimes_Q K_e$.

We compute $\varphi_e^{\mathfrak{m}^t}(\mathfrak{m}^s \otimes_Q K_e)$. Let G be as in Lemma 6.3. The image of G in $P \otimes_Q K$, denoted by g, is a cycle and the minimal resolution of $\boldsymbol{k}$ by free P-modules is the Tate complex $T = (P \otimes_Q K)\langle Y \rangle$, with

$$\partial(Y) = g. \tag{6.4.5}$$

The homomorphism $\varphi_e^{\mathfrak{m}^t}$ is induced by the natural map

$$\mathfrak{m}^t \otimes_Q K \longrightarrow \mathfrak{m}^t \otimes_P T = \mathfrak{m}^t \otimes_P (P \otimes_Q K)\langle Y\rangle = (\mathfrak{m}^t \otimes_Q K)\langle Y\rangle;$$

hence $\varphi_e^{\mathfrak{m}^t}$ is the natural map

$$\varphi_e^{\mathfrak{m}^t} \colon (\mathrm{socle}(R) \cap \mathfrak{m}^t) \otimes_Q K_e \to \mathrm{H}_e\left((\mathfrak{m}^t \otimes_Q K)\langle Y\rangle\right).$$

Let $z \in \mathfrak{m}^s \otimes_Q K_e$. According to Lemma 6.3, $z = \bar{g}z'$ for some z' in $Z_{e-1}(\mathfrak{m}^t \otimes_Q K)$, where $\bar{g}$ is the image of g in $R \otimes_Q K$. The defining property of Y, given in (6.4.5), together with the graded product rule yields

$$z = \bar{g}z' = \partial(Y)z' = \partial(Yz') - Y\partial(z') = \partial(Yz'), \tag{6.4.6}$$

which establishes that the image of z under the map $\varphi_e^{\mathfrak{m}^t}$ is represented by a boundary in $(\mathfrak{m}^t \otimes_Q K)\langle Y\rangle$; and therefore is zero in $\mathrm{H}_e\left((\mathfrak{m}^t \otimes_Q K)\langle Y\rangle\right) = \mathrm{Tor}_e^P(\mathfrak{m}^t, \boldsymbol{k})$. This finishes the proof of (6.4.1) and hence the proof of (a).

(b) Apply Theorem 6.5 with $b = t$, $\tau = t-1$, $K^R = R \otimes_Q K$, and $z_1 = \bar{g}$. Recall that $\bar{g} \in Z_1(\mathfrak{m}^{t-1} \otimes_Q K)$. It is clear that the one-cycle $\bar{g}$ squares to zero. We verify that hypothesis (6.5.1) is satisfied. On the one hand, Lemma 6.3 yields that

$$\mathfrak{m}^s \otimes_Q K_e \subseteq \bar{g}Z_{e-1}(\mathfrak{m}^t \otimes_Q K)$$

and, on the other hand, Lemma 6.1 yields that $\mathrm{Tor}_i^Q(\mathfrak{m}^s \otimes_Q K) \to \mathrm{Tor}_i^Q(\mathfrak{m}^{s-1} \otimes_Q K)$ is the zero map for $i < e$. It follows that $\mathfrak{m}^s \otimes_Q K_i \subset B(\mathfrak{m}^{s-1} \otimes_Q K)$ for $i < e$. $\square$

The following Theorem is a special case of [12, Thm. 3.1]. This result was used in the proof of Lemma 6.4.

Theorem 6.5. [12] *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be an Artinian local ring with top socle degree s, K^R be the Koszul complex on a minimal generating set of $\mathfrak{m}$, and τ and b be integers with $s - \tau \le b \le s-1$ and $2 \le \tau + 1 \le \mathtt{v}(R)$. If there exists a cycle z_1 in $Z(\mathfrak{m}^\tau K^R)$ with $z_1^2 = 0$ and*

$$\mathfrak{m}^s K^R \subseteq z_1 \cdot Z(\mathfrak{m}^b K^R) + B(\mathfrak{m}^{s-1}K^R), \tag{6.5.1}$$

then the maps $\mathrm{Tor}_i^R(\mathfrak{m}^s, \boldsymbol{k}) \to \mathrm{Tor}_i^R(\mathfrak{m}^b, \boldsymbol{k})$, induced by the inclusion $\mathfrak{m}^s \subseteq \mathfrak{m}^b$, are zero for all i.

7. Proof of the main result

In this section we prove Theorem 7.1, which is the main result of the paper. The short version of the statement is "If R is a compressed local Artinian ring with top socle degree

s, with s odd, $5 \le s$, and $\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s$, then the Poincaré series of all finitely generated modules over R are rational, sharing a common denominator, and there is a Golod homomorphism from a complete intersection onto R." Recall that the data of 6.2 is constructed from R.

Theorem 7.1. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a compressed local Artinian ring of embedding dimension e and top socle degree s. Assume that s is odd, $5 \le s$, and*

$$\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s.$$

Adopt Data 6.2. Then $s \le 2t-1$ and the following statements hold:

$$\begin{cases} \kappa : P \to R \text{ is a Golod homomorphism}, & \text{if } s = 2t-1, \text{ and} \\ \pi : Q \to R \text{ is a Golod homomorphism}, & \text{if } s < 2t-1. \end{cases}$$

Furthermore, if $d_R(z)$ is the polynomial

$$d_R(z) = \begin{cases} 1 - z(P_R^Q(z) - 1) + c_s z^{e+1}(1+z), & \text{if } s = 2t-1, \text{ and} \\ 1 - z(P_R^Q(z) - 1), & \text{if } s < 2t-1, \end{cases}$$

where $c_s = \dim_{\boldsymbol{k}}(\mathfrak{m}^s)$ then, for every finitely generated R-module M, there exists a polynomial $p_M(z)$ in $\mathbb{Z}[z]$ with

$$P_M^R(z) d_R(z) = p_M(z).$$

In particular, $p_{\boldsymbol{k}}(z) = (1+z)^e$.

Proof. It is shown in Theorem 4.4.(c) that $s \le 2t-1$. If $s < 2t-1$, then it is shown in the proof and statement of Observation 5.6 that π is a Golod homomorphism and R is a Golod ring. The statement about the common denominator $d_R(z)$ is due to Lescot [17], see also [4, Thm. 5.3.2].

Henceforth, we assume $s = 2t-1$. The following two conditions hold:

7.1.1. the map $\operatorname{Tor}_i^P(R, \boldsymbol{k}) \to \operatorname{Tor}_i^P(R/\mathfrak{m}^{t-1}, \boldsymbol{k})$, induced by the canonical quotient map $R \to R/\mathfrak{m}^{t-1}$, is zero for all positive i, and

7.1.2. the map

$$\nu_i^P : \operatorname{Tor}_i^P(\mathfrak{m}^{2t-2}, \boldsymbol{k}) \to \operatorname{Tor}_i^P(\mathfrak{m}^t, \boldsymbol{k}),$$

induced by the inclusion $\mathfrak{m}^{2t-2} \subseteq \mathfrak{m}^t$, is zero for all non-negative integers i.

Indeed, assertion 7.1.1 follows from [22, Lemma 1.4], whose proof is similar to the proof of Lemma 5.5 and assertion 7.1.2 is established in Lemma 6.4.(a) with $j = s - 1$. The hypothesis

$$\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s$$

of the present result is used to verify the critical hypothesis $\operatorname{socle}(R) \cap \mathfrak{m}^j = \mathfrak{m}^s$ of Lemma 6.4.

Now that 7.1.2 holds, the map $\operatorname{Tor}_i^P(\mathfrak{m}^{2t-2}, \boldsymbol{k}) \to \operatorname{Tor}_i^P(\mathfrak{m}^{t-1}, \boldsymbol{k})$ is also zero, and Lemma 5.2 can be applied with $a = t - 1$ to conclude that κ is Golod.

Apply Theorem 5.10 to finish the proof. It remains to prove that the Hilbert series of the kernel of

$$\varphi_\bullet^R : \operatorname{Tor}_\bullet^Q(R, \boldsymbol{k}) \to \operatorname{Tor}_\bullet^P(R, \boldsymbol{k})$$

is $\operatorname{HS}_{\ker(\varphi_\bullet^R)}(z) = z + c_s z^e$. It suffices to prove that

$$\dim_{\boldsymbol{k}} \ker(\varphi_i^R) = \begin{cases} 0, & \text{if } i = 0 \text{ or } 2 \le i \le e-1, \\ 1, & \text{if } i = 1, \text{ and} \\ \dim_{\boldsymbol{k}} \mathfrak{m}^s, & \text{if } i = e. \end{cases} \tag{7.1.3}$$

Observe that $\varphi_0^R : \operatorname{Tor}_0^Q(R, \boldsymbol{k}) \to \operatorname{Tor}_0^P(R, \boldsymbol{k})$ is the isomorphism $\boldsymbol{k} \to \boldsymbol{k}$. It follows that $\dim_{\boldsymbol{k}} \ker(\varphi_0^R) = 0$. Observe that $\varphi_1^R : \operatorname{Tor}_1^Q(R, \boldsymbol{k}) \to \operatorname{Tor}_1^P(R, \boldsymbol{k})$ is the natural map

$$\frac{\ker \pi}{\mathfrak{n} \ker \pi} \to \frac{\ker \pi}{\mathfrak{n} \ker \pi + L}.$$

The kernel of this map has dimension 1 because one of the minimal generators of $\ker \pi$ has been sent to zero. It is shown in Lemma 7.2 that $\ker(\varphi_e^R) \cong \mathfrak{m}^s$. We complete the proof of (7.1.3), hence the proof of the Theorem, by showing that

$$\varphi_i^R \text{ is injective for } 2 \le i \le e-1. \tag{7.1.4}$$

Fix i with $2 \le i \le e - 1$. The hypothesis

$$5 \le s = 2t - 1$$

ensures that $3 \le t$; hence,

$$\mathfrak{m}^{2t-2} \subseteq \mathfrak{m}^{t+1} \subseteq \mathfrak{m}^t \subseteq \mathfrak{m}^{t-1}$$

and

$$\operatorname{Tor}_i^Q(\mathfrak{m}^{2t-2}, \boldsymbol{k}) \xrightarrow{\text{incl}_i} \operatorname{Tor}_i^Q(\mathfrak{m}^{t-1}, \boldsymbol{k}) \tag{7.1.5}$$

factors through

$$\operatorname{Tor}_i^Q(\mathfrak{m}^{t+1}, \boldsymbol{k}) \xrightarrow{\text{incl}_i} \operatorname{Tor}_i^Q(\mathfrak{m}^{t}, \boldsymbol{k}). \tag{7.1.6}$$

Lemma 6.1 yields that (7.1.6) is the zero map; hence, (7.1.5) is also the zero map. Apply Lemma 5.8, together with the fact that (7.1.5) is the zero map, to the inclusion $\mathfrak{m}^{2t-2} \subseteq \mathfrak{m}^{t-1}$. Observe that $\mathfrak{n}^{t-1}$ annihilates $\mathfrak{m}^{t-1}/\mathfrak{m}^{2t-2}$. Conclude that

$$\varphi_i^{\mathfrak{m}^{t-1}} : \operatorname{Tor}_i^Q(\mathfrak{m}^{t-1}, \boldsymbol{k}) \to \operatorname{Tor}_i^P(\mathfrak{m}^{t-1}, \boldsymbol{k})$$

is injective. One can now employ the commutative diagram in proof of Claim 2 in the proof of [22, Lem. 3.4] to complete the proof of (7.1.4). □

The following calculation is used in the proof of Theorem 7.1.

Lemma 7.2. *Adopt the notation and hypotheses of Theorem 7.1 with $s = 2t - 1$. Let $\varphi_e^R\colon \operatorname{Tor}_e^Q(R, k) \to \operatorname{Tor}_e^P(R, k)$ be the map induced by the natural quotient map $Q \to P$ and let K be the Koszul complex which is a minimal resolution of $\boldsymbol{k}$ by free Q-modules. Then*

$$\ker(\varphi_e^R) = \mathfrak{m}^s \otimes_Q K_e.$$

Proof. As described at the beginning of the proof of Lemma 6.4, it does no harm to assume that $\boldsymbol{k}$ is infinite. The following consequence of Lemma 5.5 is used repeatedly in this proof.

7.2.1. The homomorphism $\operatorname{Tor}_i^Q(R/\mathfrak{m}^t, \boldsymbol{k}) \to \operatorname{Tor}_i^Q(R/\mathfrak{m}^{t-1}, \boldsymbol{k})$, which is induced by the natural quotient map $R/\mathfrak{m}^t \to R/\mathfrak{m}^{t-1}$, is zero for $1 \le i$.

We continue the identification of the functors

$$\mathrm{H}_\bullet(- \otimes_Q K) \quad \text{and} \quad \operatorname{Tor}_\bullet^Q(-, \boldsymbol{k})$$

which was begun in (6.4.2). In other words, we take

$$\operatorname{Tor}_e^Q(R, \boldsymbol{k}) \text{ to be } \operatorname{socle}(R) \otimes_Q K_e \quad \text{and} \quad \operatorname{Tor}_e^P(R, k) \text{ to be } \mathrm{H}_e((R \otimes_Q K)\langle Y\rangle);$$

furthermore, φ_e^R carries the cycle z in $\operatorname{socle}(R) \otimes_Q K_e$ to the homology class of z in $(R \otimes_Q K)\langle Y\rangle$. The argument (6.4.6) shows that if $z \in \mathfrak{m}^s \otimes_Q K_e$, then the image of z in $(\mathfrak{m}^t \otimes_Q K)\langle Y\rangle$ is a boundary; hence the image of z in $(R \otimes_Q K)\langle Y\rangle$ is a boundary. Thus, $\mathfrak{m}^s \otimes_Q K_e \subseteq \ker(\varphi_e^R)$. We prove the other direction.

Let w be an element of $\operatorname{socle}(R) \otimes_Q K_e$ which is an element of the kernel of $\ker \varphi_e^R$. It follows that w is a boundary in $(R \otimes_Q K)\langle Y\rangle$; therefore,

$$\begin{aligned} w &= \partial\big(a_0 + Ya_1 + Y^{(2)}a_2 + \cdots + Y^{(m)}a_m\big) \\ &= \big(\partial(a_0) + \bar{g}a_1\big) + Y\big(\partial(a_1) + \bar{g}a_2\big) + \cdots + Y^{(m-1)}\big(\partial(a_{m-1}) + \bar{g}a_m\big) + Y^{(m)}\partial(a_m), \end{aligned}$$

for some $a_i \in R \otimes_Q K_{e+1-2i}$, with $1 \le i \le \lfloor \frac{e+1}{2} \rfloor$. The module K_{e+1} is zero; consequently, $a_0 = 0$. The $(R \otimes_Q K)$-module $(R \otimes_Q K)\langle Y\rangle$ is free, with basis $\{Y^{(i)}\}$, and therefore

$$\begin{aligned} &w = \bar{g}a_1, \quad \partial(a_1) + \bar{g}a_2 = 0, \quad \ldots, \\ &\partial(a_{m-1}) + \bar{g}a_m = 0, \quad \text{and} \quad \partial(a_m) = 0. \end{aligned} \tag{7.2.2}$$

It is possible that $m = (e+1)/2$ and $a_m \in R \otimes_Q K_0 = R$. Observe that, in this case, $a_m \in \mathfrak{m}$. Indeed, if a_m were a unit, then the equation $\partial(a_{m-1}) + \bar{g}a_m = 0$ of (7.2.2) would yield that $\bar{g}$ is a boundary in $R \otimes_Q K$ and it would follow from Lemma 6.3 that $\mathfrak{m}^s \otimes_Q K_e \subseteq \partial(R \otimes_Q K_{e+1}) = 0$. The most recent statement is impossible because R has top socle degree s.

We claim that for each i, there exists $b_i \in R \otimes_Q K_{e+2-2i}$, $c_i \in \mathfrak{m}^{t-1} \otimes_Q K_{e+1-2i}$, and $d_i \in R \otimes_Q K_{e-2i}$ such that

$$a_i = \partial(b_i) + c_i + \bar{g}d_i. \tag{7.2.3}$$

We prove (7.2.3) by descending induction.

If $m < (e+1)/2$, then a_m is a $(e+1-2m)$-cycle in $R \otimes_Q K$. (Of course, a_m is also a cycle in $R/\mathfrak{m}^t \otimes_Q K$.) Apply (7.2.1) to find $b_m \in R \otimes_Q K_{e+2-2m}$ and c_m in $\mathfrak{m}^{t-1} \otimes_Q K_{e+1-2m}$ with $a_m = \partial(b_m) + c_m$. If $m = (e+1)/2$, then $a_m \in \mathfrak{m}$ and $a_m = \partial(b_m)$ for some $b_m \in R \otimes_Q K_1$.

In any event, (7.2.3) holds for $i = m$. Suppose, by induction, that (7.2.3) holds at i, for some i with $2 \le i \le m$. We will establish (7.2.3) at $i-1$. Apply (7.2.2), the induction hypothesis (7.2.3), the fact that $\bar{g}$ is a cycle in $R \otimes_Q K$, and the fact that $\bar{g} \in (R \otimes_Q K)_1$ in order to see that

$$\begin{aligned} 0 = \partial(a_{i-1}) + \bar{g}a_i &= \partial(a_{i-1}) + \bar{g}\left(\partial(b_i) + c_i + \bar{g}d_i\right) \\ &= \partial\big(a_{i-1} - (\bar{g}b_i)\big) + \bar{g}c_i. \end{aligned} \tag{7.2.4}$$

The product $\bar{g}c_i$ is in $\mathfrak{m}^{2t-2} \otimes K_{e+2-2i}$; and therefore, equation (7.2.4) exhibits $a_{i-1} - (\bar{g}b_i)$ as a cycle in $R/\mathfrak{m}^t \otimes_Q K$. Apply (7.2.1) to find b_{i-1} in $R \otimes_Q K_{e+4-2i}$ and $c_{i-1} \in \mathfrak{m}^{t-1} \otimes_Q K_{e+3-2i}$ with

$$a_{i-1} - (\bar{g}b_i) = \partial(b_{i-1}) + c_{i-1}.$$

Thus, (7.2.3) holds at $i-1$.

By induction, (7.2.3) holds at $i = 1$ and

$$\begin{aligned} w = \bar{g}a_1 &= \bar{g}\big(\partial(b_1) + c_1 + \bar{g}d_1\big) \\ &= -\partial(\bar{g}b_1) + \bar{g}c_1 = \bar{g}c_1, \end{aligned}$$

for some $b_1 \in R \otimes_Q K_e$, $c_1 \in \mathfrak{m}^{t-1} \otimes_Q K_{e-1}$, and $d_1 \in R \otimes_Q K_{e-2}$. We used the fact that $\bar{g}b_1 \in R \otimes_Q K_{e+1} = 0$. Thus,

$$w = \bar{g}c_1 \in (\operatorname{socle}(R) \cap \mathfrak{m}^{2t-2}) \otimes_Q K_e.$$

We have assumed that $s = 2t - 1$ and that

$$\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s.$$

It follows that $w \in \mathfrak{m}^s \otimes_Q K_e$, and the proof is complete. $\square$

8. Factoring out the highest power of the maximal ideal

The hypotheses $s = 2t - 1$, $5 \le s$, and $\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s$ all are in effect in the interesting case of the main theorem, Theorem 7.1. If we only assume $s = 2t - 1$, then we are not able to make any claim about the Poincaré series $P_{\boldsymbol{k}}^R$; nonetheless, in Corollary 8.1, we prove that the homomorphism $R \to R/\mathfrak{m}^s$ is Golod. As a consequence, when all of the hypotheses of the interesting case of Theorem 7.1 are reimposed, we prove, in Corollary 8.3, that $R/\mathfrak{m}^s$ is a Golod ring.

Corollary 8.1. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a compressed local Artinian ring with top socle degree s. If $s = 2\mathtt{v}(R) - 1$, then the natural quotient homomorphism $\rho : R \to R/\mathfrak{m}^s$ is Golod.*

Proof. Let $t = \mathtt{v}(R)$. It is shown in Lemma 6.4.(b) that the maps

$$\operatorname{Tor}_i^R(\mathfrak{m}^s, \boldsymbol{k}) \to \operatorname{Tor}_i^R(\mathfrak{m}^t, \boldsymbol{k}),$$

induced by the inclusion $\mathfrak{m}^s \subseteq \mathfrak{m}^t$, are zero for all i. It follows that the maps

$$\operatorname{Tor}_i^R(R/\mathfrak{m}^s, \boldsymbol{k}) \to \operatorname{Tor}_i^R(R/\mathfrak{m}^t, \boldsymbol{k}), \tag{8.1.1}$$

induced by the natural quotient homomorphism $R/\mathfrak{m}^s \to R/\mathfrak{m}^t$, are zero for all positive i. Apply Lemma 5.2 with $P = R$, R replaced by $R/\mathfrak{m}^s$, and $a = t$. Condition (a) of Lemma 5.2 is satisfied by (8.1.1). Condition (b) of Lemma 5.2 holds because $\mathfrak{m}^{2t} = 0$. Conclude that ρ is a Golod homomorphism. $\square$

The next result describes how to use a mapping cone to obtain a minimal resolution of the Q-module $R/\mathfrak{m}^s$ if one already knows the minimal resolution of R.

Lemma 8.2. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a compressed local Artinian ring of embedding dimension e and top socle degree s, $(Q, \mathfrak{n}, \boldsymbol{k})$ be a regular local ring of embedding dimension e with $R = Q/I$ for some ideal I of Q, and c_s be $\dim_{\boldsymbol{k}} \mathfrak{m}^s$. If $\mathtt{v}(R) + 1 \le s$, then*

$$P^Q_{R/\mathfrak{m}^s}(z) = P^Q_R(z) + c_s z(1+z)^e - c_s z^e(1+z).$$

Proof. Observe that the inclusion $\mathfrak{m}^s \subseteq R$ induces the following statements:

$$\begin{cases} \operatorname{Tor}^Q_i(\mathfrak{m}^s, \boldsymbol{k}) \to \operatorname{Tor}^Q_i(R, \boldsymbol{k}) \text{ is zero} & \text{for } 0 \le i \le e-1, \text{ and} \\ \operatorname{Tor}^Q_i(\mathfrak{m}^s, \boldsymbol{k}) \to \operatorname{Tor}^Q_i(R, \boldsymbol{k}) \text{ is an injection} & \text{for } i = e. \end{cases} \tag{8.2.1}$$

Before establishing (8.2.1); we draw consequences from these statements. One combines (8.2.1) and the short exact sequence

$$0 \to \mathfrak{m}^s \to R \to R/\mathfrak{m}^s \to 0 \tag{8.2.2}$$

to relate the Betti numbers (denoted $b_i(M)$) of the Q-modules $M = R/\mathfrak{m}^s$, $M = R$, and $M = \mathfrak{m}^s$. Keep in mind that $\mathfrak{m}^s$ is isomorphic to the direct sum of c_s copies of $\boldsymbol{k}$. The Betti numbers are related by

$$b_i(R/\mathfrak{m}^s) = \begin{cases} b_0(R), & \text{if } i = 0, \\ b_i(R) + c_s b_{i-1}(\boldsymbol{k}), & \text{if } 1 \le i \le e-1, \text{ and} \\ b_e(R) - c_s + c_s b_{e-1}(\boldsymbol{k}), & \text{if } i = e. \end{cases}$$

It follows that

$$P^Q_{R/\mathfrak{m}^s}(z) = P^Q_R(z) + c_s P^Q_{\boldsymbol{k}}(z) - c_s z^e - c_s z^{e+1},$$

as claimed.

Now we prove (8.2.1). The long exact sequence of homology that is associated to (8.2.2) ends with

$$0 \to \operatorname{Tor}^Q_e(\mathfrak{m}^s, \boldsymbol{k}) \to \operatorname{Tor}^Q_e(R, \boldsymbol{k});$$

hence, the lower line in (8.2.1) holds. On the other hand, if $0 \le i \le e-1$, then Lemma 6.1 guarantees that the inclusion $\mathfrak{m}^{\ell+1} \subseteq \mathfrak{m}^\ell$ induces the zero map

$$\operatorname{Tor}^Q_i(\mathfrak{m}^{\ell+1}, \boldsymbol{k}) \to \operatorname{Tor}^Q_i(\mathfrak{m}^\ell, \boldsymbol{k})$$

for all ℓ with $\mathtt{v}(R) \le \ell$. The hypothesis ensures that $\mathtt{v}(R) \le s-1$; hence,

$$\operatorname{Tor}^Q_i(\mathfrak{m}^s, \boldsymbol{k}) \to \operatorname{Tor}^Q_i(\mathfrak{m}^{s-1}, \boldsymbol{k})$$

is the zero map for $i < e$. The top line of (8.2.1) holds because the inclusion $\mathfrak{m}^s \subseteq R$ factors through the inclusion $\mathfrak{m}^s \subseteq \mathfrak{m}^{s-1}$. $\square$

In the interesting case of the main theorem, the ring $R/\mathfrak{m}^s$ is Golod.

Corollary 8.3. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a compressed local Artinian ring with top socle degree s. If $s = 2\mathrm{v}(R) - 1$, $5 \le s$, and $\operatorname{socle}(R) \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s$, then the ring $R/\mathfrak{m}^s$ is Golod.*

Proof. Let e be the embedding dimension of R, c_s be $\dim_{\boldsymbol{k}} \mathfrak{m}^s$, and $(Q, \mathfrak{n}, \boldsymbol{k})$ be a regular local ring with $R = Q/I$ for some ideal $I \subseteq \mathfrak{n}^2$. Recall (see, for example, [4, (5.0.1)] or 2.14) that $R/\mathfrak{m}^s$ is Golod if and only if

$$P_{\boldsymbol{k}}^{R/\mathfrak{m}^s}(z) = \frac{P_{\boldsymbol{k}}^Q(z)}{1 - z(P_{R/\mathfrak{m}^s}^Q(z) - 1)}. \tag{8.3.1}$$

We calculate both sides of (8.3.1), verify the equality, and thereby prove the result. Observe first that

$$P_{R/\mathfrak{m}^s}^R(z) = 1 + c_s z P_{\boldsymbol{k}}^R(z). \tag{8.3.2}$$

Indeed, the exact sequence

$$0 \to \mathfrak{m}^s \to R \to R/\mathfrak{m}^s \to 0$$

is the beginning of the minimal resolution of $R/\mathfrak{m}^s$ by free R-modules and $\mathfrak{m}^s$ is isomorphic to $\bigoplus_{c_s} \boldsymbol{k}$.

The hypotheses of Theorem 7.1 are in effect; and therefore,

$$P_{\boldsymbol{k}}^R(z) = \frac{(1+z)^e}{1 - z(P_R^Q(z) - 1) + c_s z^{e+1}(1+z)}. \tag{8.3.3}$$

The map $R \to R/\mathfrak{m}^s$ is Golod by Corollary 8.1; and therefore,

$$P_{\boldsymbol{k}}^{R/\mathfrak{m}^s}(z) = \frac{P_{\boldsymbol{k}}^R(z)}{1 - z(P_{R/\mathfrak{m}^s}^R(z) - 1)}; \tag{8.3.4}$$

see, for example [15, Prop. 1] or [4, Prop. 3.3.2]. Use (8.3.4), (8.3.2), (8.3.3), and then Lemma 8.2 to calculate

$$P_k^{R/\mathfrak{m}^s}(z) = \frac{(1+z)^e}{1 - z\big(P_{R/\mathfrak{m}^s}^Q(z) - 1\big)}.$$

Apply (8.3.1) to conclude that $R/\mathfrak{m}^s$ is a Golod ring. $\square$

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