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Almost nilpotency of an associative algebra with an almost nilpotent fixed-point subalgebra

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ABSTRACT

Let A be an associative algebra of arbitrary dimension over a field F and G a finite group of automorphisms of A of order n, prime to the characteristic of F. Denote by $A^G = \{a \in A \mid a^g = a \text{ for all } g \in G\}$ the fixed-point subalgebra. By the classical Bergman–Isaacs theorem, if A^G is nilpotent of index d, i.e. $(A^G)^d = 0$, then A is also nilpotent and its nilpotency index is bounded by a function depending only on n and d. We prove, under the additional assumption of solubility of G, that if A^G contains a two-sided nilpotent ideal $I \lhd A^G$ of nilpotency index d and of finite codimension m in A^G, then A contains a nilpotent two-sided ideal $H \lhd A$ of nilpotency index bounded by a function of n and d and of finite codimension bounded by a function of m, n and d. An even stronger result is provided for graded associative algebras: if G is a finite (not necessarily soluble) group of order n and $A = \bigoplus_{g\in G} A_g$ is a G-graded associative algebra over a field F, i.e. $A_gA_h \subset A_{gh}$, such that the identity component A_e has a two-sided nilpotent ideal $I_e \lhd A^G$ of nilpotency index d and of finite codimension m in A_e, then A has a homogeneous nilpotent two-sided ideal $H \lhd A$ of nilpotency index bounded by a function of n and d and of finite codimension bounded by a function of n, d and m.

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1. Introduction

By the classical Bergman–Isaacs theorem [1], if an associative algebra A over a field F admits a finite group of automorphisms G of order $|G| = n$, prime to the characteristic of F, and the fixed-point subalgebra $A^G = \{a \in A \mid a^g = a \text{ for all } g \in G\}$ is nilpotent of index d, i.e. $(A^G)^d = 0$, then A is nilpotent of index bounded by a function of n and d. After this work, a great number of paper deal with properties of an algebra (or a ring) under a finite group action subject to some constraints on the fixed-point subalgebra. In this paper we prove, under the additional assumption of the solubility of the automorphism group, that the “almost nilpotency” of the fixed-point subalgebra implies the “almost nilpotency” of the algebra itself. Namely, the following theorem holds.

Theorem 1.1. *Let A be an associative algebra of arbitrary (possibly infinite) dimension over a field F acted on by a finite soluble group G of order n. Suppose that the characteristic of F does not divide n. If the fixed-point subalgebra A^G has a nilpotent two-sided ideal $I \lhd A^G$ of nilpotency index d and of finite codimension m in A^G, then A has a nilpotent two-sided ideal $H \lhd A$ of nilpotency index bounded by a function of n and d and of finite codimension bounded by a function of m, n and d.*

The restrictions on the order of the automorphism group are unavoidable. There are examples showing that the result is not true either for infinite automorphism groups or for algebras with n-torsion.

Theorem 1.1 follows by induction on the order of G from the Bergman–Isaacs theorem and the following statement on graded associative algebras, in which we do not suppose either G to be soluble or the order of G to be prime to the characteristic of the field.

Theorem 1.2. *Let G be a finite group of order n and let $A = \bigoplus_{g \in G} A_g$ be a G-graded associative algebra over a field F, i.e. $A_g A_h \subset A_{gh}$. If the identity component A_e has a nilpotent two-sided ideal $I_e \lhd A_e$ of nilpotency index d and of finite codimension m in A_e, then A has a homogeneous nilpotent two-sided ideal $H \lhd A$ of nilpotency index bounded by a function on n and d and of finite codimension bounded by a function on n, d and m.*

The proof of Theorem 1.2 is based on the method of generalized centralizers, originally created by Khukhro in [3] for nilpotent groups and Lie algebras with an almost regular automorphism of prime order. In [4,5] the approach was significantly revised and new techniques were introduced to study a more complicated case of an almost regular automorphism of arbitrary (not necessarily prime) finite order. In particular, it was proved that if a Lie algebra L admits an automorphism φ of finite order n with fixed-point subalgebra of finite dimension m, then L has a soluble ideal of derived length bounded by a function of n whose codimension is bounded by a function of m and n. The combinatorial nature of the construction in [5] makes possible to apply it to a wide range of situations. For example, the approach was used to study Lie type algebras (a large class of algebras which includes associative, Lie algebras, color Lie superalgebras) with an almost regular automorphism of finite order in [6].

In the proof of Theorem 1.2 we use virtually the same construction as in [5]. But the strong condition of associativity simplifies the reasoning and allows to provide much stronger results than in the case of Lie algebras. In particular, we do not need to suppose that the automorphism group is cyclic.

We give some definitions and auxiliary lemmas in § 2. In § 3 we prove Theorem 1.2. For this, we set $N = d^2 + 3$ and for each $g \in G \setminus \{e\}$ we construct by induction generalized centralizers $A_g(i)$ of levels $i = 1, 2, \dots N$, which are some subspaces of the homogeneous components A_g. Then we demonstrate that the ideal generated by all the $A_g(N)$, $g \in G \setminus \{e\}$ is the required one. In § 4 we derive the Theorem 1.1 from Theorem 1.2 and the Bergman–Isaacs Theorem by induction on the order of G.

Throughout the paper we will say that a number is "$(a, b, \dots)$-bounded" if it is "bounded above by some function depending only on $a, b, \dots$".

2. Preliminaries

If G is a group of automorphisms of A, then $A^G = \{a \in A \mid a^g = a \text{ for all } g \in G\}$ will denote the fixed-point subalgebra. A two-sided ideal H of A is denoted by $H \lhd A$. If I and J are subspaces of A, IJ will denote the subspace spanned by all products ab with $a \in I$ and $b \in J$, and I^d will denote the d-fold product $\underbrace{I \dots I}_{d}$. We say that an algebra is nilpotent of (nilpotency) index d if the product of any d elements of the algebra A equals zero, i.e. $A^d = 0$. The subalgebra generated by subspaces $B_1, B_2, \dots, B_s$ is denoted by $\langle B_1, B_2, \dots, B_s \rangle$, and the two-sided ideal generated by $B_1, B_2, \dots, B_s$ is denoted by ${}_{\mathrm{id}}\langle B_1, B_2, \dots, B_s \rangle$. If H is an algebra, then $H^{\#}$ will denote the algebra obtained by adjoining 1 to H. The (two-sided) ideal of H generated by a subspace I is sometimes written as $H^{\#} I H^{\#}$.

We now state some facts needed hereinafter.

Lemma 2.1. *(Bergman–Isaacs Theorem [1]). Let G be a finite group of order n of automorphisms of an associative ring (algebra) R. If R has no n-torsion and R^G is nilpotent of index d then R is nilpotent of index at most h^d, where $h = 1 + \prod_{i=0}^{n} (C_n^i + 1)$.*

The following two lemmas are known. We give their proofs for the convenience of readers.

Lemma 2.2. *(Bergman–Isaacs [1, Lemma 1.1]). Let G be a finite group of order n and let $A = \bigoplus_{g \in G} A_g$ be a G-graded associative algebra over a field F, i.e. $A_g A_h \subset A_{gh}$. If the identity component A_e is nilpotent of index d, then A is nilpotent of index at most nd.*

Proof. It suffices to prove that a product $a_{g_1}a_{g_2}\dots a_{g_{nd}}$ in homogeneous elements $a_{g_i} \in A_{g_i}$, $i = 1,\dots,nd$ of length nd is trivial. We consider $nd+1$ products $h_0 = e$, $h_1 = g_1$, $h_i = g_1 \dots g_i$, $i = 1,\dots,nd$. Since the order of G is n, some $d+1$ elements must be equal. If $h_i = h_j$ with $i < j$, then $g_{i+1}\dots g_j = e$. We obtain that $a_{g_1}a_{g_2}\dots a_{g_{nd}}$ can be represented as $P_1Q_1Q_2\dots Q_dP_2$, where P_1 and P_2 are (possibly empty) products in homogeneous elements a_{g_i}, and each Q_i is a non-empty product of the form $Q_i = a_{g_{i+1}}\dots a_{g_j}$ with $g_{i+1}\dots g_j = e$. It follows that $Q_i \in A_e$ for all $i = 1,\dots,d$. Since $(A_e)^d = 0$, we have that $Q_1Q_2\dots Q_d = 0$, and therefore $a_{g_1}a_{g_2}\dots a_{g_{nd}} = 0$. □

Lemma 2.3 *([2, Lemma 1.3.7]). Let A be an associative algebra over a field F acted on by a finite group G of order n. Suppose that the characteristic of F does not divide n. If the fixed-point subalgebra A^G contains a nilpotent ideal $I \lhd A^G$ of nilpotency index d, then A contains a G-invariant nilpotent ideal $J \geq I$ of (n,d)-bounded nilpotency index.*

Proof. Consider the right-sided ideal $B = IA^{\#}$ generated by I (recall that $A^{\#}$ is the algebra obtained from A by joining the unit). Let $b \in B^G$ be an element of B fixed by G, i.e. $b^g = b$ for all $g \in G$. There exist elements $s \in B$, $i_m \in I$, $a_m \in A^{\#}$ such that $b = ns = n\sum_m i_m a_m$. Since $\sum_{g\in G} a_m^g \in A^G$ and I is an ideal in A^G, we have

$$b = ns = \sum_{g\in G} s^g = \sum_m i_m \sum_{g\in G} a_m^g \in I,$$

i.e. $B^G \leq I$ and B^G is nilpotent of index $\leq d$. Applying the Bergman–Isaacs Theorem to the algebra B we obtain that B is nilpotent of index at most h^d, where $h = 1 + \prod_{i=0}^n (C_n^i + 1)$. Finally, two-sided G-invariant ideal $J \geq I$ generated by B is also nilpotent of index at most h^d: $(A^{\#}IA^{\#})^{h^d} = A^{\#}(IA^{\#})^{h^d} = 0$. □

3. Proof of Theorem 1.2

Let G be an arbitrary finite group of order n and $A = \bigoplus_{g\in G} A_g$ be a G-graded associative algebra over a field F, i.e. $A_gA_h \subset A_{gh}$. Suppose that the identity component A_e has a nilpotent ideal I_e of nilpotency index d and $\dim A_e/I_e = m$.

Index Convention. In what follows, unless otherwise stated, a small letter with an index g will denote an element of the homogeneous component A_g. The index only indicates which component this element belongs to: $x_g \in A_g$. To lighten the notation, we shall not be using numbering indices for elements of the A_g, so that different elements can be denoted by the same symbol. For example, x_g and x_g can be different elements of A_g.

Construction of generalized centralizers and representatives. We fix $N = d^2 + 3$. In each homogeneous component A_g, $g \in G \setminus \{e\}$ we construct by induction a descending chain of subspaces:

$$A_g = A_g(0) \geq A_g(1) \geq \cdots \geq A_g(N).$$

The subspaces $A_g(s)$ are called generalized centralizers of level s. Simultaneously we fixe some homogeneous elements in $A_g(s)$, $s = 0, \dots, N$ which are referred to as representatives of level s. The total number of representatives will be (n, d, m)-bounded.

Definition. For a monomial $a_{g_1} a_{g_2} \dots a_{g_k}$, where $a_{g_i} \in A_{g_i}$, the record $(*_{g_1} *_{g_2} \cdots *_{g_k})$ is called the *pattern* of the monomial. The *length* of a pattern is the degree of the monomial. The monomial is said to be the *value of its pattern* on the given elements.

For example, $a_g a_g a_v$ and $b_g c_g b_v$ are values of the same pattern $(*_g *_g *_v)$. (Under the Index Convention the elements a_g in the first product can be different.)

Definition. Let $g \in G \setminus \{e\}$. For every ordered tuple of elements $\vec{x} = (x_{g_1}, \dots, x_{g_k})$, $x_{g_s} \in A_{g_s}$, such that $g_1 g_2 \dots g_{l-1}\, g\, g_l \dots g_k = e$ for some $l \in \{1, \dots k+1\}$ we define the mapping:

$$\vartheta_{\vec{x},l} :\ A_g \to A_e/I_e;$$
$$\vartheta_{\vec{x},l} :\ y_g \to x_{g_1} x_{g_2} \dots x_{g_{l-1}} y_g x_{g_l} \dots x_{g_k} I_e,$$

where I_e is the nilpotent ideal of A_e of nilpotency index d and of codimension m in A_e. We use index l to distinguish the eventual cases of $g_1 g_2 \dots g_{k-1}\, g\, g_k \dots g_k = e$ and $g_1 g_2 \dots g_{l-1}\, g\, g_l \dots g_k = e$ with $k \neq l$ which lead to different mappings.

By linearity, the mapping $\vartheta_{\vec{x},l}$ is a homomorphism of the subspace A_g into the quotient space A_e/I_e. Since $\dim A_e/I_e \le m$, we have that $\dim A_g/\mathrm{Ker}\, \vartheta_{\vec{x},l} \le m$.

Definition of level 0. We set $A_g(0) = A_g$ for all $g \in G \setminus \{e\}$. To construct the representatives of level 0 we fix some elements $x_e \in A_e$ whose images form a basis of A_e/I_e. These elements are called *representatives of level* 0 and are denoted by $x_e(0)$ (under the Index Convention). In addition we consider a pattern $\mathbf{P} = (*_g *_{g^{-1}})$ of length 2 with $g \in G \setminus \{e\}$. The dimension of the subspace of the quotient space A_e/I_e spanned by all images of the values of $\mathbf{P}$ on homogeneous elements of A_g, $A_{g^{-1}}$ is at most m by hypothesis. Hence we can choose at most m products $c = x_g x_{g^{-1}} \in A_e$ whose images form a basis of this subspace. The elements $x_g, x_{g^{-1}}$ involved in these representations of the elements c are also called *representatives of level* 0 and are denoted by $x_g(0)$, $x_{g^{-1}}(0)$ (under the Index Convention). The same is done for every pattern $\mathbf{P}$ of the form $(*_g *_{g^{-1}})$, $g \in G \setminus \{e\}$.

Since $\dim A_e/I_e \le m$ and the total number of patterns $\mathbf{P}$ is $n-1$, the number of representatives of level 0 is at most $2(n-1)m + m$.

Definition of level 1. Let $W_1 = 2d^3(n-1) + 2$. For each $g \in G \setminus \{e\}$ we set

$$A_g(1) = \bigcap_{\vec{z}} \bigcap_{l} \mathrm{Ker}\, \vartheta_{\vec{z},l},$$

where $\vec{z} = (z_{g_1}(0), \dots, z_{g_k}(0))$ runs over all possible ordered tuples of all lengths $k \le W_1$ consisting of representatives of level 0 such that $g_1 \dots g \dots g_k = e$; if for a fixed tuple $\vec{z} = (z_{g_1}(0), \dots, z_{g_k}(0))$ of length k there are several different integers $l \le k +$

1 such that $g_1 \dots g_{l-1} g g_l \dots g_k = e$, we take the intersection over all such integers l. The subspaces $A_g(1)$ are referred to as *generalized centralizers of level* 1, elements of the $A_g(1)$ are called *centralizers of level* 1 and are denoted by $y_g(1)$ (under the Index Convention).

The subspace $A_g(1)$ has (n,d,m)-bounded codimension in A_g since the intersection here is taken over an (n,d,m)-bounded number of subspaces of m-bounded codimension in A_g.

The representatives of level 1 are constructed in two different ways. First, for each $g \in G \setminus \{e\}$ we fix some elements of A_g whose images form a basis of the quotient space $A_g/A_g(1)$. These elements are called *b-representatives of level* 1 and are denoted by $b_g(1) \in A_g$ (under the Index Convention). Since the dimensions of $A_g/A_g(1)$ are (n,d,m)-bounded for all $g \in G \setminus \{e\}$, the total number of b-representatives of level 1 is (n,d,m)-bounded.

Second, for each pattern $\mathbf{P} = (*_g *_{g^{-1}})$ of length 2 with indices $g, g^{-1} \in G \setminus \{e\}$ we consider the subspace of the quotient space A_e/I_e spanned by all images of the values of $\mathbf{P}$ on homogeneous elements of $A_g(1)$, $A_{g^{-1}}(1)$. Since $\dim A_e/I_e \le m$, we can choose at most m products $c = y_g(1) y_{g^{-1}}(1) \in A_e$ whose images form a basis of this subspace in A_e/I_e and fix the elements $y_g(1)$, $y_{g^{-1}}(1)$ involved in these representations. These elements are called *x-representatives of level* 1 and are denoted by $x_g(1)$ (under the Index Condition). Since the number of patterns under consideration is equal to $n-1$, the total number of x-representatives of level 1 is at most $2(n-1)m$.

By construction, if $g_1 \dots g_{t-1}\, g\, g_t \dots g_k = e$, for some $t \le k+1$ and $k \le W_1$, a centralizer $y_g(1)$ has the following property with respect to representatives $x_{g_j}(0)$ of level 0:

$$x_{g_1}(0) \, \dots \, x_{g_{t-1}}(0)\, y_g(1)\, x_{g_t}(0) \, \dots \, x_{g_k}(0) \in I_e. \tag{1}$$

Definition of level $s > 0$. Suppose that we have already fixed representatives of level $< s$, which are either x-representatives or b-representatives and its number is (m,n,d)-bounded. We now define the *generalized centralizers of level s*. Let $W_s = W_{s-1} + 1 = 2d^3(n-1) + 1 + s$. For each $g \in G \setminus \{e\}$ we set

$$A_g(s) = \bigcap_{\vec{z}} \bigcap_{l} \operatorname{Ker} \vartheta_{\vec{z},l},$$

where $\vec{z} = (z_{g_1}(\varepsilon_1), \dots, z_{g_k}(\varepsilon_k))$ runs over all possible ordered tuples of all lengths $k \le W_s$ consisting of representatives of (possibly different) levels $< s$ (i.e., $z_{g_u}(\varepsilon_u)$ denote elements of the form $x_{g_u}(\varepsilon_u)$ or $b_{g_u}(\varepsilon_u)$, $\varepsilon_u < s$, in any combination) such that

$$g_1 \cdots g \cdots g_k = e;$$

if for a fixed tuple $\vec{z} = (z_{g_1}(\varepsilon_1), \dots, z_{g_k}(\varepsilon_k))$ of length k there are several different integers $l \le k+1$ such that $g_1 \dots g_{l-1} g g_l \dots g_k = e$, we take the intersection over all such integers l. Elements of the $A_g(s)$ are also called *centralizers of level* s and are denoted by $y_g(s)$ (under the Index Convention).

The intersection here is taken over an (n,d,m)-bounded number of subspaces of m-bounded codimension in A_g, since the number of representatives of all levels $<s$ is (n,d,m)-bounded and $\dim A_g/\operatorname{Ker}\vartheta_{\vec{z},l} \leq m$ for all $\vec{z}$. Hence $A_g(s)$ also has (n,d,m)-bounded codimension in the subspace A_g.

We now fix representatives of level s. First, for each $g \in G \setminus \{e\}$ we fix some elements of A_g whose images form a basis of the quotient space $A_g/A_g(s)$. These elements are denoted by $b_g(s) \in A_g$ (under the Index Convention) and are called *b-representatives of level* s. The total number of b-representatives of level s is (n,d,m)-bounded, since the dimensions of $A_g/A_g(s)$ are (n,d,m)-bounded for all $g \in G \setminus \{e\}$.

Second, for each pattern $\mathbf{P} = (*_g *_{g^{-1}})$ of length 2 with indices $g \in G \setminus \{e\}$ we consider the subspace of the quotient space A_e/I_e spanned by all images of the values of $\mathbf{P}$ on homogeneous elements of $A_g(s)$, $A_{g^{-1}}(s)$. Since $\dim A_e/I_e \leq m$, we can choose at most m products $c = y_g(s)y_{g^{-1}}(s) \in A_e$ whose images form a basis of this subspace in A_e/I_e and fix the elements $y_g(s)$, $y_{g^{-1}}(s)$ involved in these representations. These fixed elements are called *x-representatives of level* s and are denoted by $x_g(s)$ (under the Index Condition). The total number of x-representatives of level s is at most $2(n-1)m$.

Note that x-representatives of level s, elements $x_g(s)$, are also centralizers of level s.

It is clear from the construction that

$$A_g(k+1) \leq A_g(k) \tag{2}$$

for all $g \in G \setminus \{e\}$ and any k.

By definition, if $g_1 \dots g_{t-1}\, g\, g_t \dots g_k = e$, for some $t \leq k+1$ and $k \leq W_s$, then a centralizer $y_g(s)$ has the following property with respect to representatives of lower levels:

$$z_{g_1}(\varepsilon_1)\, \dots\, z_{g_{t-1}}(\varepsilon_{t-1})\, y_g(s)\, z_{g_t}(\varepsilon_t)\, \dots\, z_{g_k}(\varepsilon_k) \in I_e, \tag{3}$$

where the elements $z_{g_j}(\varepsilon_j)$ are representatives (that is, either $b_{g_j}(\varepsilon_j)$ or $x_{g_j}(\varepsilon_j)$, in any combination) of any (possible different) levels $\varepsilon_l < s$.

The following lemmas are direct consequences of the inclusions (2), (3) and the definitions of representatives.

Lemma 3.1. *Let $g \in G \setminus \{e\}$. Then*

1) *every homogeneous element $a_e \in A_e$ can be represented modulo I_e as a linear combination of representatives $x_e(0)$ of level* 0.

2) *every product $a_g\, b_{g^{-1}}$ in homogeneous elements can be represented modulo I_e as a linear combination of products of the same pattern in representatives of level* 0.

3) *every product $y_g(k_1)y_{g^{-1}}(k_2)$ in centralizers of levels k_1, k_2 can be represented modulo I_e as a linear combination of products $x_g(s)x_{g^{-1}}(s)$ of the same pattern in x-representatives of any level s satisfying $0 \leq s \leq \min\{k_1, k_2\}$.*

Lemma 3.2. *Let $y_g(l+1)$ be a centralizer of level $l+1$, $b_h(l)$ be b-representative of level l with $g, h, gh \in G \setminus \{e\}$. Then elements of the form $u_{gh} = y_g(l+1)b_h(l)$ or $v_{hg} = b_h(l)y_g(l+1)$ are centralizers of level l.*

Proof. The proof follows directly from (3) and the definitions of W_i. □

Lemma 3.3. *Any product of the form $a_{g^{-1}}\, y_g(k+1)$ or $y_g(k+1)\, a_{g^{-1}}$, where $y_g(k+1)$ is a centralizer of level $k+1$, is equal modulo I_e to a product of the form $y_{g^{-1}}(k)\, y_g(k)$ or accordingly $y_g(k)\, y_{g^{-1}}(k)$, where $y_{g^{-1}}(k)$, $y_g(k)$ are centralizers of level k.*

Proof. We represent $a_{g^{-1}}$ as a sum of a centralizer $y_{g^{-1}}(k)$ of level k and a linear combination of b-representatives $b_{g^{-1}}(k)$ of level k and substitute this sum into the product $a_{g^{-1}}\, y_g(k+1)$. We obtain a sum of the element $y_{g^{-1}}(k)\, y_g(k+1)$ and a linear combination of elements of the form $b_{g^{-1}}(k)\, y_g(k+1)$. By (3) the product $b_{g^{-1}}(k)\, y_g(k+1)$ belongs to I_e. Hence $a_{g^{-1}}\, y_g(k+1) = y_{g^{-1}}(k)\, y_g(k+1) \pmod{I_e}$. Similarly, $y_{g^{-1}}(k+1)\, a_g = y_{g^{-1}}(k+1)\, y_g(k) \pmod{I_e}$. Since $A_g(k) \geq A_g(k+1)$, both products have the required form. □

Construction of the nilpotent ideal. Recall that $N = d^2 + 3$ is the fixed notation for the highest level. We have constructed the generalized centralizers $A_g(N)$ for $g \in G \setminus \{e\}$. Let $G \setminus \{e\} = \{g_1, \ldots, g_{n-1}\}$ We set

$$Z =_{\mathrm{id}} \left\langle A_{g_1}(N),\, A_{g_2}(N), \ldots, A_{g_{n-1}}(N), I_e \right\rangle.$$

This ideal has (n, d, m)-bounded codimension in A, since each subspace $A_h(N)$, $h \in G \setminus \{e\}$, has (n, d, m)-bounded codimension in A_h, while the dimension of A_e/I_e is at most m by hypothesis. To prove Theorem 1.2 we show that the ideal Z is nilpotent of (n, d)-bounded nilpotency index.

Definition. For every $g \in G$ we set $Z_g = Z \cap A_g$.

Lemma 3.4. *The subspace Z_e is contained modulo I_e in the subspace spanned by products of the form $y_{h^{-1}}(N-2)\, y_h(N-2)$ and by products of the form $a_{g^{-1}}\, i_e\, a_g$, where $y_{h^{-1}}(N-2)$, $y_h(N-2)$ are centralizers of level $N-2$, $a_{g^{-1}} \in A_{g^{-1}}$, $a_g \in A_g$, $i_e \in I_e$, $h, g \in G \setminus \{e\}$.*

Proof. An element of Z_e is modulo I_e a linear combination of products of the forms:

$$a_{g^{-1}}\, i_e\, a_g, \text{ where } g \neq e,\ a_{g^{-1}} \in A_g,\ i_e \in I_e,\ a_g \in A_g, \tag{4}$$

$$a_{g^{-1}}\, y_g(N), \text{ where } g \neq e,\ a_{g^{-1}} \in A_{g^{-1}},\ y_g(N) \in A_g(N), \tag{5}$$

$$y_g(N)\, a_{g^{-1}}, \text{ where } g \neq e,\ y_g(N) \in A_g(N),\ a_{g^{-1}} \in A_{g^{-1}}, \tag{6}$$

$$a_{g_1}\, y_g(N)\, a_{g_2}, \text{ where } g \neq e,\ g_1 g g_2 = e\ a_{g_1} \in A_{g_1},\ a_{g_2} \in A_{g_2},\ y_g(N) \in A_g(N). \tag{7}$$

The product (4) is already of the required form. By Lemma 3.3 the products $y_g(N)\, a_{g^{-1}}$ and $a_{g^{-1}}\, y_g(N)$ can be represented modulo I_e as linear combinations of products of the form $y_g(N-1)\, y_{g^{-1}}(N-1)$ and therefore have also the required representation since $A_g(N-1) \leq A_g(N-2)$.

Consider the product (7). Since $g_1 g g_2 = e$ and $g \neq e$, at least one g_i, $i = 1, 2$ is not equal to e. Let, for example, $g_1 \neq e$. We represent a_{g_1} as a sum of a centralizer $y_{g^{-1}}(N-1)$ of level $N-1$ and a linear combination of b-representatives $b_{g^{-1}}(N-1)$ of level $N-1$ and insert this expression into (7). We obtain a linear combination of products of the following two forms

$$y_{g_1}(N-1)\, y_g(N)\, a_{g_2} \tag{8}$$

and

$$b_{g_1}(N-1)\, y_g(N)\, a_{g_2}. \tag{9}$$

In (8) we set $a_{g_1^{-1}} := y_g(N)\, a_{g_2}$. Applying Lemma 3.3 and inclusions (2) to $y_{g_1}(N-1) a_{g_1^{-1}}$ we obtain that (8) is equal modulo I_e to a product of the required form $y_{g_1}(N-2)\, y_{g_1^{-1}}(N-2)$.

Let us now consider the product (9). If $g_2 = e$, then $g_1 g = e$ and $b_{g_1}(N-1)\, y_g(N) \in I_e$ by (3). Since I_e is an ideal of A_e and $g_2 = e$,

$$b_{g_1}(N-1)\, y_g(N)\, a_e \in I_e.$$

If $g_2 \neq e$, then $g_1 g \neq e$ and $b_{g_1}(N-1)\, y_g(N)$ is a centralizer of level $N-1$ by Lemma 3.2:

$$b_{g_1}(N-1)\, y_g(N)\, a_{g_2} = y_{g_1 g}(N-1)\, a_{g_2}.$$

Again by Lemma 3.3 the product $y_{g_1 g}(N-1)\, a_{g_2}$ is equal modulo I_e to the product of the required form $y_{g_1 g}(N-2)\, y_{g_2}(N-2)$. The case where $g_1 = e, g_2 \neq e$ in (7) can be treated in the same manner. $\square$

Proof of Theorem 1.2. We set $H = d^2 + 1$, $T = d(H-1) + 1 = d^3 + 1$, $S = (T-1) \times (n-1) + 1 = d^3(n-1) + 1$, $U = d(n-1)$ and $Q = (U+d)(S-1) + 1 = (d(n-1) + d)d^3(n-1) = d^4(n-1)^2 + d^4(n-1)$. By Lemma 2.2 it suffices to show that $(Z_e)^Q = 0$.

We consider an arbitrary product of length Q in elements c_i from Z_e:

$$c_1\, c_2 \ldots c_Q, \tag{10}$$

(here the indices are numbering). By Lemma 3.4 we can represent modulo I_e every c_k as a linear combination of products of the form $y_{h^{-1}}(N-2)\, y_h(N-2)$, where $y_{h^{-1}}(N-2)$, $y_h(N-2)$ are centralizers of level $N-2$ and of products of the form $a_{g^{-1}}\, i_e\, a_g$, where

$a_{g^{-1}} \in A_{g^{-1}}$, $a_g \in A_g$, $i_e \in I_e$. Substituting these expressions into (10) we obtain a linear combination of elements

$$z_1\, z_2\, \dots z_Q, \tag{11}$$

where the z_k (here the indices are numbering) are either elements $i_e \in I_e$ or products $c_e = a_{g^{-1}}\, w_e\, a_g \in A_e$, $w_e \in I_e$, or products $v_e = y_{g_k^{-1}}(N-2)\, y_{g_k}(N-2) \in A_e$ in centralizers $y_{g_k^{-1}}(N-2)$, $y_{g_k}(N-2)$ of level $N-2$.

If in (11) among z_k there are at least d occurrences of elements $i_e \in I_e$, the summand is trivial since I_e is an ideal of A_e and $(I_e)^d = 0$.

Suppose now that in (11) there are at least $(d-1)n+1$ entries of products $c_e = a_{g^{-1}}\, i_e\, a_g$. Among them we can choose d products $c_e = a_{g_k^{-1}}\, i_e\, a_{g_k} \in A_e$ with the same pair of indices g_k^{-1}, g_k:

$$z_1\, \dots z_{l_1}\, \underbrace{a_{g_k^{-1}}\, i_e\, a_{g_k}}\, z_{l_1+1} \dots z_{l_2}\, \underbrace{a_{g_k^{-1}}\, i_e\, a_{g_k}}\, z_{l_2} \dots z_{l_k}\, \underbrace{a_{g_k^{-1}}\, i_e\, a_{g_k}}\, z_{l_k+1} \dots z_Q.$$

Since the products $a_{g_k}\, z_{l_s+1} \dots z_{l_{s+1}}\, a_{g_k^{-1}}$ between the elements i_e belong to A_e, I_e is an ideal in A_e and $(I_e)^d = 0$, then the product (11) is equal to 0.

Consider the case where the number of i_e-occurrences in (11) is at most $d-1$ and the number of c_e-occurrences is at most $U = d(n-1)$. Since $Q = (U+d)(S-1)+1$, the product (11) has at least one subproduct consisting of S elements v_e (going one after another):

$$\big(y_{g_1^{-1}}(N-2)\, y_{g_1}(N-2)\big)\ \big(y_{g_2^{-1}}(N-2)\, y_{g_2}(N-2)\big) \dots \big(y_{g_S^{-1}}(N-2)\, y_{g_S}(N-2)\big), \tag{12}$$

where $y_{g_i}(N-2) \in A_{g_i}(N-2)$, $y_{g_i^{-1}}(N-2) \in A_{g_i^{-1}}(N-2)$ are (possibly different) centralizers of level $N-2$. Since $S = (T-1)(n-1)+1$ in (12) there are at least T entries of products $y_{g_i^{-1}}(N-2)y_{g_i}(N-2)$ with the same pair of indices, say, g_k^{-1}, g_k. We choose any T such products and represent modulo I_e all the other pairs as linear combinations of products in representatives of level 0 by Lemma 3.1:

$$w_e \dots w_e\big(y_{g_k^{-1}}(N-2)\, y_{g_k}(N-2)\big)\, w_e \dots w_e\big(y_{g_k^{-1}}(N-2)\, y_{g_k}(N-2)\big) \dots, \tag{13}$$

where there are T occurrences of (possibly different) products $y_{g_k^{-1}}(N-2)$, $y_{g_k}(N-2)$ with the same pair of indices g_k^{-1}, g_k, the w_e are possibly different elements of A_e: either $i_e \in I_e$ or representatives $x_e(0)$. If in (13) among w_e there are at least d occurrences of elements of I_e, the summand is trivial since I_e is an ideal of A_e and $(I_e)^d = 0$. In the opposite case, as $T = d(H-1)+1$, there is a subproduct of the form

$$\big(y_{g_k^{-1}}(N-2)\, y_{g_k}(N-2)\big)\, x_e(0) \dots x_e(0)\big(y_{g_k^{-1}}(N-2)\, y_{g_k}(N-2)\big) \dots,$$

with $H = d^2+1$ occurrences of products $y_{g_k^{-1}}(N-2)\, y_{g_k}(N-2)$ between which there are only x-representative of level 0 and no elements from I_e. By Lemma 3.1 we represent modulo I_e the first entry $y_{g_k^{-1}}(N-2)\, y_{g_k}(N-2)$ as a linear combination of the products of the same pattern in representatives in level 1, the second — in level 2, and so on, the last one — in level H. We obtain a linear combination

$$\big(x_{g_k^{-1}}(1)\, x_{g_k}(1) + i_e\big)\, x_e(0) \dots x_e(0) \big(x_{g_k^{-1}}(2)\, x_{g_k}(2) + i_e\big) \dots \big(x_{g_k^{-1}}(H)\, x_{g_k}(H) + i_e\big).$$

Expanding this expression we get a linear combination of products of the form

$$c_1\, x_e(0) \dots x_e(0)\, c_2\, x_e(0) \dots x_e(0)\, \dots c_H,$$

(here the indices are numbering) where the c_k are either elements $i_e \in I_e$ or products $x_{g_k^{-1}}(n_k)\, x_{g_k}(n_k)$ of different levels with one and the same pair of indices g_k^{-1}, $g_k \in G$. If in a summand there are at least d entries of $i_e \in I_e$ it is trivial by assumptions. In the summands with less than d entries of the i_e, we can find an interval long enough without i_e-entries. More precisely, since $H = d^2 + 1$ there is a subproduct of the form

$$\big(x_{g_k^{-1}}(s)\, x_{g_k}(s)\big)\, x_e(0) \dots x_e(0) \big(x_{g_k^{-1}}(s+1)\, x_{g_k}(s+1)\big) \dots \Big(x_{g_k^{-1}}(s+d)\, x_{g_k}(s+d)\Big), \quad (14)$$

where there are $d+1$ products $y_{g_k^{-1}}(l)\, y_{g_k}(l)$ of different levels $l = s, s+1, \dots, s+d$. For each $t = 0, \dots, d-1$, the product

$$x_{g_k}(s+t)\, x_e(0) \dots x_e(0)\, x_{g_k^{-1}}(s+t+1)$$

includes exactly one centralizer of level $s+t+1$, all the other elements are representatives of lower levels, and the weight of the product is at most $2S = 2d^3(n-1) + 2 = W_1 \le W_{s+t+1}$. By (3)

$$x_{g_k}(s+t)\, x_e(0) \dots x_e(0)\, x_{g_k^{-1}}(s+t+1) \in I_e$$

for all $t = 0, \dots, d-1$. It follows that (14) is equal to the product

$$x_{g_k^{-1}}(s)\, \underbrace{(i_e\, i_e\, \dots\, i_e)}_{d}\, x_{g_k}(s+d) = 0,$$

which is trivial, since $(I_e)^d = 0$. $\square$

4. Proof of Theorem 1.1

Recall that we are given an associative algebra A over a field F that admits a finite soluble automorphism group G of order n prime to the characteristic of F such that the fixed-point subalgebra A^G has a two-sided nilpotent ideal $I \lhd A^G$ of nilpotency index d and of finite codimension m in A^G. The aim is to find a nilpotent ideal in A of (n, d)-bounded nilpotency index and of finite (n, d, m)-bounded codimension.

Proof of Theorem 1.1. First, we consider the case where G is a cyclic group of prime order p. Let g be a generator of G. Then g induces an automorphism of the algebra $A \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$, where ω is a primitive p-th root of unity. The fixed-point subalgebra of $A \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ with respect to this automorphism (denoted by the same letter g) has the same dimension m. It suffices to prove Theorem 1.1 for the algebra $A \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. Hence in what follows we can assume that the ground field F contains ω. We define the homogeneous *components* A_k for $k = 0, \dots, p-1$ as the subspaces

$$A_k = \left\{ a \in A \mid a^g = \omega^k a \right\}.$$

Since the characteristic of F does not divide p, we have

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{p-1}.$$

This decomposition determines a grading on A by a cyclic group of prime order p, with $A_0 = A^G$ in view of the obvious inclusions

$$A_s A_t \subseteq A_{s+t},$$

where $s + t$ is computed modulo p. Hence the case $|G| = p$ in Theorem 1.1 follows from Theorem 1.2.

Let now G be any finite soluble group of automorphisms of the algebra A, and suppose that its order $n = |G|$ is not divisible by the characteristic of F. We use induction on n. We may assume that n is not a prime number. This means that G contains a non-trivial normal subgroup H. We consider its fixed-point subalgebra $C = A^H$. Since $H \lhd G$, we have $C^g \leqslant C$ for any $g \in G$. The subalgebra C admits a finite solvable group of automorphisms of order $\leq |G/H|$ which is strictly less than $|G|$ and not divisible by the characteristic of F. By induction C has a nilpotent ideal $J \lhd C$ of $(|G/H|, d, m)$-bounded codimension $t = t(|G/H|, d, m)$ and of $(|G/H|, d)$-bounded nilpotency index $h = h(|G/H|, d)$. By Lemma 2.3 there exists a nilpotent G-invariant ideal $K \geq J$ in A of nilpotency index $h_1 = h_1(|H|, h)$ bounded by $|H|$ and by the nilpotency index of J. The subgroup H acts on the factor-algebra $\bar{A} = A/K$ and subalgebra of fixed points $\bar{A}^H$ has dimension at most t. We apply induction hypothesis to the algebra $\bar{A}$ and the automorphism group H of $\operatorname{Aut} \bar{A}$ whose order is strictly less than $|G|$. The algebra $\bar{A}$ has a nilpotent ideal Z of $(|H|, t)$-bounded codimension and of $|H|$-bounded nilpotent index $h_2 = h_2(|H|)$. The image of Z in A is a required ideal since its nilpotency index is at most $h_1 h_2$, which is an (n, d)-bounded number, and the codimension is (n, d, m)-bounded. $\square$

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