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Generalized correspondence functors

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ABSTRACT

A generalized correspondence functor is a functor from the category of finite sets and T-generalized correspondences to the category of all k-modules, where T is a finite distributive lattice and k a commutative ring. We parametrize simple generalized correspondence functors using the notions of T-module and presheaf of posets. As an application, we prove finiteness and stabilization results. In particular, when k is a field, any finitely generated correspondence functor has finite length, and when k is noetherian, any subfunctor of a finitely generated functor is finitely generated.

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1. Introduction

In this article, we study a particular case of the theory of representations of categories: those of generalized correspondences. Our goal is to generalize the work of Serge Bouc and Jacques Thévenaz about correspondence functors in [6–8]. Their methodological point of view, which is also ours, makes crucial the study of simple functors. Its efficiency has been proved by previous works, especially about Mackey functors and biset functors: see for example [21,20,3–5,1].

The literature in this field is flourishing and the points of view are diverse. In particular, we are interested in categories having all finite sets as objects. Similar categories have been studied by various and fruitful methods. This is how Steven V. Sam and Andrew Snowden triumphed of the Lannes–Schwartz artinian conjecture with Gröbner methods in [18]. We can also cite [16,9] among other papers with the same flavor, as well as the more recent works by John D. Wiltshire-Gordon [22,23] and Andrew Gitlin [11].

We first recall in Section 2 fundamental results about representations of categories. In Section 3, we define our category of interest, that is, the category of generalized correspondences, and study its first properties. Sections 4, 5 and 6 are devoted to the study of modules over a finite distributive lattice and presheaves of posets. They play a crucial role in the sequel. In Section 7, we give a parametrization of simple generalized correspondence functors in terms of such presheaves. In accordance with the previously described point of view, it is the core of this paper. In Sections 8 and 9, we illustrate the power of this parametrization by studying some of its consequences, especially finiteness conditions.

Throughout this paper, k denotes a commutative ring, T denotes a finite distributive lattice and $B = \mathrm{Irr}(T)$ denotes the set of join-irreducible elements of T. Additional assumptions about k will always be emphasized.

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2. Representations of categories

In this section, we recall the basics of the representation theory of categories. All of the following results can be found in [6]. We only give proofs when they are scattered in the literature.

Let $\mathscr{C}$ be a category. For all objects X and Y of $\mathscr{C}$, we write $\mathscr{C}(Y, X) = \mathbf{Hom}_{\mathscr{C}}(X, Y)$ for the set of arrows from X to Y in $\mathscr{C}$. We assume that $\mathscr{C}$ has a small skeleton, in order to deal with sets of natural transformations between two functors with source $\mathscr{C}$.

Definition 2.1. The *k-linearization* of $\mathscr{C}$ is the category $k\mathscr{C}$ defined as follows.

- Its objects are those of $\mathscr{C}$.
- If X and Y are two objects of $\mathscr{C}$, the set $k\mathscr{C}(Y,X)$ of arrows from X to Y in $k\mathscr{C}$ is the free k-module with basis $\mathscr{C}(Y,X)$.
- Its composition is the k-bilinear extension of the composition in $\mathscr{C}$.
- Its identity morphisms are those of $\mathscr{C}$.

Definition 2.2. A *k-linear representation* of $\mathscr{C}$ is a k-linear functor from $k\mathscr{C}$ to the category k-**Mod** of all k-modules.

Let F be a k-linear representation of $\mathscr{C}$ and let X, Y be two objects of $\mathscr{C}$. For all $\alpha \in k\mathscr{C}(Y,X)$ and $v \in F(X)$, we write $\alpha \cdot v = F(\alpha)(v)$. Thus $F(X)$ becomes a left $k\mathscr{C}(X,X)$-module.

We write $\mathscr{F}_k(\mathscr{C})$ for the category of k-linear representations of $\mathscr{C}$, the arrows of which are the natural transformations. [2, Proposition 1.4.4] ensures that $\mathscr{F}_k(\mathscr{C})$ is an abelian category. In particular, a sequence of functors $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact if and only if its evaluation $0 \to F_1(X) \to F_2(X) \to F_3(X) \to 0$ is exact for any object X of $\mathscr{C}$. A k-linear representation of $\mathscr{C}$ is called *simple* if it is nonzero and has no proper nonzero subrepresentation.

For any object X of $\mathscr{C}$, Yoneda's lemma implies that the representable functor $k\mathscr{C}(-,X)$ is a projective functor. Its evaluation $k\mathscr{C}(Y,X)$ at Y has a structure of $(k\mathscr{C}(Y,Y), k\mathscr{C}(X,X))$-bimodule given by composition on both sides. If V is a left $k\mathscr{C}(X,X)$-module, we then write $L_{X,V} = k\mathscr{C}(-,X) \otimes_{k\mathscr{C}(X,X)} V$. It is a k-linear representation of $\mathscr{C}$.

Proposition 2.3. *Let X be an object of $\mathscr{C}$. The functor*

$$\begin{array}{ccc} k\mathscr{C}(X,X)\text{-}\mathbf{Mod} & \longrightarrow & \mathscr{F}_k(\mathscr{C}) \\ V & \longmapsto & L_{X,V} \end{array}$$

is left adjoint to the evaluation functor

$$\begin{array}{ccc} \mathscr{F}_k(\mathscr{C}) & \longrightarrow & k\mathscr{C}(X,X)\text{-}\mathbf{Mod} \\ F & \longmapsto & F(X). \end{array}$$

Proof. Apply the tensor-hom adjunction, then Yoneda's lemma. □

Let X be an object of $\mathscr{C}$ and let V be a left $k\mathscr{C}(X,X)$-module. For any object Y of $\mathscr{C}$, we define

$$J_{X,V}(Y) = \left\{ \sum_{i=1}^{n} \alpha_i \otimes v_i \in L_{X,V}(Y), \forall \beta \in k\mathscr{C}(X,Y), \sum_{i=1}^{n} (\beta\alpha_i) \cdot v_i = 0 \right\}.$$

Proposition 2.4.

1. *$J_{X,V}$ is the largest subrepresentation of $L_{X,V}$ that vanishes at X.*
2. *If V is a simple $k\mathscr{C}(X,X)$-module, then $J_{X,V}$ is the largest proper subrepresentation of $L_{X,V}$.*

Proof. First notice that $J_{X,V}(X) = 0$, because if $\alpha \otimes v \in J_{X,V}(X)$. Then $\alpha \cdot v = (\mathrm{Id}_X\, \alpha) \cdot v = 0$, whence $\alpha \otimes v = \mathrm{Id}_X \otimes (\alpha \cdot v) = 0$.

Moreover, the definition of $J_{X,V}(Y)$ can immediately be rewritten as

$$J_{X,V}(Y) = \bigcap_{\beta \in k\mathscr{C}(X,Y)} \operatorname{Ker} L_{X,V}(\beta).$$

This implies that $J_{X,V}$ is a subrepresentation of $L_{X,V}$.

Assume that F is a subrepresentation of $L_{X,V}$ that vanishes at X. This implies that for all $\beta \in \mathscr{C}(X,Y)$, one has $F(Y) = \operatorname{Ker} F(\beta) \subseteq \operatorname{Ker} L_{X,V}(\beta)$. So $F(Y) \subseteq J_{X,V}(Y)$. This proves the first assertion of the proposition.

Now assume that V is a simple $k\mathscr{C}(X,X)$-module and let F be a subrepresentation of $L_{X,V}$. Then $F(X)$ is a $k\mathscr{C}(X,X)$-submodule of $L_{X,V}(X) \cong V$. So $F(X)$ is either 0 or $L_{X,V}(X)$. If $F(X) = L_{X,V}(X)$, then $F = L_{X,V}$. Indeed, for all $\alpha \otimes v \in L_{X,V}(Y)$, one has

$$\alpha \otimes v = L_{X,V}(\alpha)(\mathrm{Id}_X \otimes v) \in L_{X,V}(\alpha)(L_{X,V}(X)) = L_{X,V}(\alpha)(F(X)) \subseteq F(Y),$$

since F is a subrepresentation of $L_{X,V}$.

So any proper subrepresentation of $L_{X,V}$ vanishes at X. Thus the second assertion of the proposition is a consequence of the first one. □

We write $S_{X,V} = L_{X,V}/J_{X,V}$, so that $S_{X,V}$ is a simple k-linear representation when V is a simple $k\mathscr{C}(X,X)$-module.

Proposition 2.5. *Let S be a simple k-linear representation of $\mathscr{C}$ and let X be an object of $\mathscr{C}$ such that $S(X) \neq 0$.*

1. *S is generated by $S(X)$, that is, $S(Y) = k\mathscr{C}(Y,X) \cdot S(X)$ for any object Y of $\mathscr{C}$. More precisely, $S(Y) = k\mathscr{C}(Y,X) \cdot v$ for all $v \in S(X) \smallsetminus \{0\}$.*
2. *$S(X)$ is a simple $k\mathscr{C}(X,X)$-module.*
3. *S is isomorphic to $S_{X,S(X)}$.*

Proof. See [6, Proposition 2.7]. □

Lemma 2.6. *Let S be a simple k-linear representation of $\mathscr{C}$, let X be an object of $\mathscr{C}$ such that $S(X) \neq 0$ and let F be any k-linear representation of $\mathscr{C}$. Finally, let $V_1 \subseteq F(X)$ be a k-submodule of $F(X)$. The following properties are equivalent.*

(i) There exists a k-submodule $V_2 \subseteq V_1$ such that $S(X)$ is isomorphic to V_1/V_2.
(ii) There exist subrepresentations $F_2 \subseteq F_1 \subseteq F$ such that $F_1(X) = V_1$ and S is isomorphic to F_1/F_2.

Proof. See [6, Proposition 2.8] or, for more details, [15, Lemma 3.1]. □

3. Generalized correspondences

We now move to a particular case of the general setting studied in Section 2. We first define what will be the arrows in our category of interest.

Definition 3.1. Let X and Y be two finite sets. A *T-generalized correspondence* between Y and X is a map $Y \times X \to T$. We write $\mathscr{C}_T(Y,X) = T^{Y\times X}$ for the set of T-generalized correspondences between Y and X.

When $T = \{0,1\}$ is the two-element lattice, maps $Y \times X \to \{0,1\}$ can be identified with subsets of $Y \times X$. So a $\{0,1\}$-generalized correspondence is just what is called a correspondence in [6].

Let X, Y and Z be three finite sets. For all T-generalized correspondences $R \in \mathscr{C}_T(Y,X)$ and $S \in \mathscr{C}_T(Z,Y)$, we define their product $SR \in \mathscr{C}_T(Z,X)$ by

$$SR(z,x) = \bigvee_{y\in Y} S(z,y) \wedge R(y,x)$$

for all $z \in Z$ and $x \in X$. We also define a diagonal correspondence $\Delta_X \in \mathscr{C}_T(X,X)$ by

$$\Delta_X(x,y) = \begin{cases} 1_T & \text{if } x = y \\ 0_T & \text{otherwise} \end{cases}$$

for all $x, y \in X$.

Lemma 3.2.

1. *The product of T-generalized correspondences is associative. That is, for all finite sets W, X, Y, Z and for all T-generalized correspondences $Q \in \mathscr{C}_T(X,W)$, $R \in \mathscr{C}_T(Y,X)$ and $S \in \mathscr{C}_T(Z,Y)$, one has $(SR)Q = S(RQ)$.*
2. *The diagonal correspondences are identities. That is, for all finite sets X, Y and for any T-generalized correspondence $R \in \mathscr{C}_T(Y,X)$, one has $\Delta_Y R = R = R\Delta_X$.*

Proof. It is a straightforward computation. □

Thus we can consider the category $\mathscr{C}_T$ of T-generalized correspondences. Its objects are all finite sets and the set of arrows from X to Y is, in accordance with our notation, the set $\mathscr{C}_T(Y,X)$ of T-generalized correspondences between Y and X.

A k-linear representation of the category $\mathscr{C}_T$ is called a *T-generalized correspondence functor over k*. If we need not emphasize k or T, we may simply talk about generalized correspondence functors.

We now give a series of easy consequences of the following lemma. All of them are direct generalizations of the results of [6, Section 3].

Lemma 3.3. *Let X and Y be finite sets such that $|X| \leqslant |Y|$. There exist $i_* \in \mathscr{C}_T(Y, X)$ and $i^* \in \mathscr{C}_T(X, Y)$ such that $i^* i_* = \Delta_X$.*

Proof. Since $|X| \leqslant |Y|$, there exists an injection $i\colon X \to Y$. Consider the T-generalized correspondences $i_* \in \mathscr{C}_T(Y, X)$ and $i^* \in \mathscr{C}_T(X, Y)$ defined by

$$i_*(y, x) = \begin{cases} 1_T & \text{if } y = i(x) \\ 0_T & \text{otherwise} \end{cases}$$

and

$$i^*(x, y) = \begin{cases} 1_T & \text{if } y = i(x) \\ 0_T & \text{otherwise} \end{cases}$$

for all $x \in X$ and $y \in Y$. It is then easy to check that $i^* i_* = \Delta_X$. □

Corollary 3.4. *Let X and Y be finite sets such that $|X| \leqslant |Y|$. The functor $k\mathscr{C}_T(-, X)$ is isomorphic to a direct summand of $k\mathscr{C}_T(-, Y)$.*

Proof. Right multiplication by i^* defines a natural transformation $k\mathscr{C}_T(-, X) \to k\mathscr{C}_T(-, Y)$. Similarly, right multiplication by i_* defines a natural transformation $k\mathscr{C}_T(-, Y) \to k\mathscr{C}_T(-, X)$. Their composition is the identity of $k\mathscr{C}_T(-, X)$, thanks to Lemma 3.3. □

Corollary 3.5. *Let X and Y be finite sets such that $|X| \leqslant |Y|$. The left $k\mathscr{C}_T(Y, Y)$-module $k\mathscr{C}_T(Y, X)$ is projective.*

Proof. Corollary 3.4 implies that $k\mathscr{C}_T(Y, X)$ is isomorphic to a direct summand of $k\mathscr{C}_T(Y, Y)$, which is a free $k\mathscr{C}_T(Y, Y)$-module. □

Corollary 3.6. *Let X and Y be finite sets such that $|X| \leqslant |Y|$. If F is a generalized correspondence functor such that $F(Y) = 0$, then $F(X) = 0$.*

Proof. For all $v \in F(X)$, one has $v = i^* i_* \cdot v$. But $i_* \cdot v \in F(Y) = 0$, so $v = 0$. □

We now make a first study of finitely generated generalized correspondence functors. It will be completed in Sections 8 and 9. The following definition and results are a direct generalization of [6, Section 6]

Definition 3.7. Let $(X_i)_{i\in I}$ be a family of finite sets. For all $i \in I$, let $v_i \in X_i$. A generalized correspondence functor F is *generated* by $(v_i)_{i\in I}$ if for any finite set Y and for all $v \in F(Y)$, there exist a finite subset $J \subseteq I$ and elements $\alpha_j \in k\mathscr{C}_T(Y, X_j)$ such that

$$v = \sum_{j\in J} \alpha_j \cdot v_j.$$

If the set I is finite, F is said *finitely generated.*

For example, the representable functor $k\mathscr{C}_T(-, X)$ is generated by the single element Δ_X.

Lemma 3.8. *Let F be a finitely generated generalized correspondence functor. For any finite set X, the k-module $F(X)$ is finitely generated.*

Proof. Let $(v_i)_{1\leqslant i\leqslant n}$ be a finite family generating F, with $v_i \in F(X_i)$. Let $A = \{R \cdot v_i, R \in \mathscr{C}_T(X, X_i), 1 \leqslant i \leqslant n\}$. Then any element of $F(X)$ is a k-linear combination of elements in A. The latter being a finite set, $F(X)$ is indeed finitely generated. □

Proposition 3.9. *Let F be a generalized correspondence functor. The following properties are equivalent.*

(i) *F is finitely generated.*
(ii) *There exists a finite family $(X_i)_{1\leqslant i\leqslant n}$ of finite sets such that F is isomorphic to a quotient of*

$$\bigoplus_{i=1}^{n} k\mathscr{C}_T(-, X_i).$$

(iii) *There exist a finite set X and an integer $n \in \mathbb{N}$ such that F is isomorphic to a quotient of $k\mathscr{C}_T(-, X)^{\oplus n}$.*
(iv) *There exist a finite set X and a finite subset $A \subseteq F(X)$ such that F is generated by A.*

Proof. (i) implies (ii): assume that F is generated by the finite family $(v_i)_{1\leqslant i\leqslant n}$, with $v_i \in X_i$. By Yoneda's lemma, there exist morphisms $k\mathscr{C}_T(-, X_i) \to F$ mapping Δ_{X_i} to v_i. Their sum is a morphism

$$\bigoplus_{i=1}^{n} k\mathscr{C}_T(-, X_i) \longrightarrow F.$$

This morphism is surjective because F is generated by the family $(v_i)_{1\leqslant i\leqslant n}$.

(ii) implies (iii): assume that F is isomorphic to a quotient of

$$\bigoplus_{i=1}^{n} k\mathscr{C}_T(-, X_i).$$

Fix a finite set X such that $|X_i| \leqslant |X|$ for all $i \in [\![1, n]\!]$. By Corollary 3.4, each $k\mathscr{C}_T(-, X_i)$ is isomorphic to a direct summand of $k\mathscr{C}_T(-, X)$. So F is isomorphic to a quotient of $k\mathscr{C}_T(-, X)^{\oplus n}$.

(iii) implies (iv): assume that F is isomorphic to a quotient of $k\mathscr{C}_T(-, X)^{\oplus n}$. For all $i \in [\![1, n]\!]$, let $b_i \in k\mathscr{C}_T(X, X)^{\oplus n}$ the element the i-th coordinate of which is Δ_X, the others being 0. Since $k\mathscr{C}_T(-, X)$ is generated by Δ_X, then F is generated by the images of the elements b_i.

It is obvious that (iv) implies (i). □

Corollary 3.10. *For any finite set X and for any finitely generated $k\mathscr{C}_T(X, X)$-module V, the functors $L_{X,V}$ and $S_{X,V}$ are finitely generated.*

Proof. Let $(v_i)_{1 \leqslant i \leqslant n}$ be a finite family generating V. By Yoneda's lemma, there exist morphisms $k\mathscr{C}_T(-, X_i) \to L_{X,V}$ mapping Δ_X to v_i. Their sum is a surjective morphism $k\mathscr{C}_T(-, X)^{\oplus n} \to L_{X,V}$. Proposition 3.9 then ensures that $L_{X,V}$ is finitely generated. Thus $S_{X,V}$ is finitely generated too because it is a quotient of $L_{X,V}$. □

4. Modules over a distributive lattice

This section and the two following ones are devoted to the study of modules over a distributive lattice. These objects will play a crucial role in describing simple generalized correspondence functors.

Recall that T is a finite distributive lattice. It is a semiring in the sense of [12], with its join and meet as semiring operations. Hence we can consider semimodules over them. Here we add an order relation to this semimodule structure.

Definition 4.1. A *T-module* is a join-semilattice U equipped with an action of T

$$\begin{array}{ccc} T \times U & \longrightarrow & U \\ (t, u) & \longmapsto & t \cdot u \end{array}$$

fulfilling the following axioms.

- For all $t \in T$ and $u, v \in U$, one has $t \cdot (u \vee v) = (t \cdot u) \vee (t \cdot v)$.
- For all $s, t \in T$ and $u \in U$, one has $(s \vee t) \cdot u = (s \cdot u) \vee (t \cdot u)$.
- For all $s, t \in T$ and $u \in U$, one has $(s \wedge t) \cdot u = s \cdot (t \cdot u)$.
- For all $u \in U$, one has $1_T \cdot u = u$.
- There exists $0_U \in U$ such that for all $t \in T$ and $u \in U$, one has $t \cdot 0_U = 0_U = 0_T \cdot u$.

Lemma 4.2. *Let U be a T-module. For all $s, t \in T$ and $u \in U$, the following properties hold.*

1. $0_U \leqslant u$.
2. $s \cdot u \leqslant u$.
3. *If* $s \leqslant t$, *then* $s \cdot u \leqslant t \cdot u$.
4. *If* $u \leqslant v$, *then* $t \cdot u \leqslant t \cdot v$.

Proof. We prove Assertion 3: since $s \leqslant t$, one has $t = s \vee t$. So $t \cdot u = (s \vee t) \cdot u = (s \cdot u) \vee (t \cdot u)$, that is, $s \cdot u \leqslant t \cdot u$. A similar argument proves Assertion 4. Assertions 1 and 2 are a obtained from Assertion 3 by taking $(s,t) = (0_T, 1_T)$ and $t = 1_T$ respectively. $\square$

Lemma 4.3. *Any finite T-module is a lattice.*

Proof. It is an immediate consequence of the following classical lemma. $\square$

Lemma 4.4. *Any finite join-semilattice having a smallest element is a lattice.*

Now that we have a notion of linearity, we can define linear maps: they are maps between T-modules that preserve the join and the action of T. We can similarly define linear isomorphisms. Here we give without proof a list of easy properties of T-linear maps.

Lemma 4.5.

1. *Any T-linear map maps* 0 *to* 0.
2. *Any T-linear map is order-preserving.*
3. *The inverse of a bijective T-linear map is also T-linear, so it is a T-linear isomorphism.*
4. *Any T-linear isomorphism between finite T-modules is a lattice isomorphism.*
5. *Any T-linear isomorphism between finite T-modules maps join-irreducible elements to join-irreducible elements.*

Let E be a finite set and let $\bullet$ be a set of cardinality 1. We identify $\bullet \times E$ with E, hence $\mathscr{C}_T(\bullet, E)$ with T^E. We write $\mathscr{C}_T(E)$ for the latter. The product of the T-generalized correspondences $\varphi \in \mathscr{C}_T(E)$ and $e \in \mathscr{C}_T(E, E)$ is the element $\varphi e \in \mathscr{C}_T(E)$ given by

$$\varphi e(\alpha) = \bigvee_{\beta \in E} \varphi(\beta) \wedge e(\beta, \alpha)$$

for all $\alpha \in E$.

We write $\mathscr{C}_T(E)e = \{\varphi e, \varphi \in \mathscr{C}_T(E)\}$. From now on, we assume that e is idempotent, that is, $e^2 = e$. In this case, $\mathscr{C}_T(E)e = \{\varphi \in \mathscr{C}_T(E), \varphi e = \varphi\}$. Notice that $\mathscr{C}_T(E)\Delta_E = \mathscr{C}_T(E)$, so all of the statements below are in particular true for the set $\mathscr{C}_T(E)$.

The elements of $\mathscr{C}_T(E)e$ are maps with values in a poset, so $\mathscr{C}_T(E)e$ it is actually a poset itself. For all $\varphi, \psi \in \mathscr{C}_T(E)e$, the function $\varphi \vee \psi \in \mathscr{C}_T(E)$ is defined by $(\varphi \vee \psi)(\alpha) = \varphi(\alpha) \vee \psi(\alpha)$ for all $\alpha \in E$. Then, one can check that $(\varphi \vee \psi)e = \varphi e \vee \psi e = \varphi \vee \psi$, so that the function $\varphi \vee \psi$ is an element of $\mathscr{C}_T(E)e$. Hence, it is the join of φ and ψ in $\mathscr{C}_T(E)e$, so $\mathscr{C}_T(E)e$ is a join-semilattice.

It is also equipped with an action of T defined by $(t \cdot \varphi)(\alpha) = t \wedge \varphi(\alpha)$ for all $\alpha \in E$. Indeed, for all $\alpha \in E$, one has

$$\begin{aligned}(t \cdot \varphi)e(\alpha) &= \bigvee_{\beta \in E} (t \cdot \varphi)(\beta) \wedge e(\beta, \alpha) = \bigvee_{\beta \in E} t \wedge \varphi(\beta) \wedge e(\beta, \alpha) \\ &= t \wedge \bigvee_{\beta \in E} \varphi(\beta) \wedge e(\beta, \alpha) = t \wedge \varphi e(\alpha) = t \wedge \varphi(\alpha) = (t \cdot \varphi)(\alpha),\end{aligned}$$

so that $t \cdot \varphi \in \mathscr{C}_T(E)e$. The third equality above is a consequence of the distributivity of T.

Lemma 4.6. *For any finite set E and for any idempotent $e \in \mathscr{C}_T(E, E)$, the set $\mathscr{C}_T(E)e$ is a T-module.*

Proof. The first two axioms defining a T-module are a consequence of the distributivity of T. The other ones are immediate to check, the smallest element of $\mathscr{C}_T(E)e$ being the constant function equal to 0_T, which is indeed an element of $\mathscr{C}_T(E)e$. $\square$

Since $\mathscr{C}_T(E)e$ is a finite T-module, it is a lattice. An important question is to determine its join-irreducible elements. Given $\alpha \in E$ and $t \in T$, we consider maps of type

$$\begin{array}{rccc} \delta_\alpha^t \colon & E & \longrightarrow & T \\ & \beta & \longmapsto & t \wedge \Delta_E(\alpha, \beta). \end{array}$$

In the sequel, we will simply write δ_α instead of $\delta_\alpha^{1_T}$. Recall that B is the set of join-irreducible elements of T.

Lemma 4.7. *The join-irreducible elements of $\mathscr{C}_T(E)e$ are exactly the maps $\delta_\alpha^b e$ with $\alpha \in E$ and $b \in B$ such that $b \leqslant e(\alpha, \alpha)$.*

Proof. Let $\varphi \in \operatorname{Irr}(\mathscr{C}_T(E)e)$. One has

$$\varphi = \bigvee_{\alpha \in E} \delta_\alpha^{\varphi(\alpha)}.$$

For all $\alpha \in E$, the element $\varphi(\alpha) \in T$ is the join of all elements $b \in B$ with $b \leqslant \varphi(\alpha)$. So

$$\delta_\alpha^{\varphi(\alpha)} = \bigvee_{\substack{b\in B\\ b\leqslant\varphi(\alpha)}} \delta_\alpha^b.$$

Hence

$$\varphi = \bigvee_{\alpha\in E} \bigvee_{\substack{b\in B\\ b\leqslant\varphi(\alpha)}} \delta_\alpha^b.$$

Since $\varphi \in \mathscr{C}_T(E)e$, one has moreover

$$\varphi = \varphi e = \left(\bigvee_{\substack{b\in B\\ b\leqslant\varphi(\alpha)}} \delta_\alpha^b \right) e = \bigvee_{\substack{b\in B\\ b\leqslant\varphi(\alpha)}} \delta_\alpha^b e.$$

Every function of type $\delta_\alpha^b e$ in an element of $\mathscr{C}_T(E)e$. Since φ is join-irreducible in $\mathscr{C}_T(E)e$, there exist $\alpha \in E$ and $b \in B$ with $b \leqslant \varphi(\alpha)$ such that $\varphi = \delta_\alpha^b e$. So

$$b \leqslant \varphi(\alpha) = \delta_\alpha^b e(\alpha) = \bigvee_{\beta\in E} \delta_\alpha^b(\beta) \wedge e(\beta,\alpha) = b \wedge e(\alpha,\alpha) \leqslant e(\alpha,\alpha).$$

So any join-irreducible element of $\mathscr{C}_T(E)e$ is of type $\delta_\alpha^b e$ for some $\alpha \in E$ and $b \in B$ such that $b \leqslant e(\alpha,\alpha)$.

Conversely, let $\alpha \in E$ and $b \in B$ be such that $b \leqslant e(\alpha,\alpha)$ and assume that there exist $\varphi, \psi \in \mathscr{C}_T(E)e$ such that $\delta_\alpha^b e = \varphi \vee \psi$. Then $b = b \wedge e(\alpha,\alpha) = \delta_\alpha^b e(\alpha) = \varphi(\alpha) \vee \psi(\alpha)$. Since b is join-irreducible, one has $b \in \{\varphi(\alpha), \psi(\alpha)\}$, say $b = \varphi(\alpha)$. Then for all $\beta \in E$, one has

$$\varphi(\beta) = \varphi e(\beta) \geqslant \varphi(\alpha) \wedge e(\alpha,\beta) = b \wedge e(\alpha,\beta) = \delta_\alpha^b e(\beta) = \varphi(\beta) \vee \psi(\beta).$$

So $\varphi \vee \psi = \varphi$, that is, $\delta_\alpha^b e = \varphi$. Hence $\delta_\alpha^b e$ is join-irreducible. $\square$

Lemma 4.8. *For all $\varphi \in \mathscr{C}_T(E)e$, one has*

$$\bigvee_{\gamma\in E} \varphi(\gamma) \cdot \delta_\gamma e = \varphi.$$

Proof. For all $\alpha \in E$, one has

$$\left(\bigvee_{\gamma\in E} \varphi(\gamma) \cdot \delta_\gamma e \right)(\alpha) = \bigvee_{\gamma\in E} \varphi(\gamma) \wedge \delta_\gamma e(\alpha).$$

Now

$$\delta_\gamma e(\alpha) = \bigvee_{\beta \in E} \delta_\gamma(\beta) \wedge e(\beta, \alpha) = e(\gamma, \alpha),$$

so

$$\bigvee_{\gamma \in E} \varphi(\gamma) \wedge \delta_\gamma e(\alpha) = \bigvee_{\gamma \in E} \varphi(\gamma) \wedge e(\gamma, \alpha) = \varphi e(\alpha) = \varphi(\alpha)$$

since $\varphi \in \mathscr{C}_T(E)e$. Hence

$$\bigvee_{\gamma \in E} \varphi(\gamma) \cdot \delta_\gamma e = \varphi. \quad \square$$

The end of this section is devoted to the study of T-modules of type $\mathscr{C}_T(E)e$. We will see that their properties generalize those of finite distributive lattices in a sense that will be made explicit later.

Definition 4.9. A T-module is *regular* if it is isomorphic to some $\mathscr{C}_T(E)e$ for an idempotent $e \in \mathscr{C}_T(E, E)$.

Definition 4.10. Two idempotents $e \in \mathscr{C}_T(E, E)$ and $f \in \mathscr{C}_T(F, F)$ are *equivalent* if there exist $x \in e\mathscr{C}_T(E, F)f$ and $y \in f\mathscr{C}_T(F, E)e$ such that $e = xy$ and $f = yx$.

Lemma 4.11. *Let $e \in \mathscr{C}_T(E, E)$ and $f \in \mathscr{C}_T(F, F)$ be two idempotents. The following properties are equivalent.*

(i) e and f are equivalent.
(ii) There exist $x, b \in e\mathscr{C}_T(E, F)f$ and $y, a \in f\mathscr{C}_T(F, E)e$ such that $e = xy$ and $f = ab$.

Proof. The arguments are given in the proofs of [13, Theorem 3] and [17, Proposition A.1.15], with an easy generalization due to the fact that e and f need not lie in the same monoid. $\square$

Proposition 4.12. *The T-modules $\mathscr{C}_T(E)e$ and $\mathscr{C}_T(F)f$ are isomorphic if and only if the idempotents e and f are equivalent.*

Proof. Let $\Theta\colon \mathscr{C}_T(E)e \to \mathscr{C}_T(F)f$ be any T-linear map. Then we define a T-generalized correspondence $R_\Theta \in \mathscr{C}_T(E, F)$ by $R_\Theta(\alpha, \beta) = \Theta(\delta_\alpha e)(\beta)$ for all $\alpha \in E$ and $\beta \in F$. Then

$$\begin{aligned} R_\Theta f(\alpha, \beta) &= \bigvee_{\gamma \in F} R_\Theta(\alpha, \gamma) \wedge f(\gamma, \beta) = \bigvee_{\gamma \in F} \Theta(\delta_\alpha e)(\gamma) \wedge f(\gamma, \beta) \\ &= \Theta(\delta_\alpha e) f(\beta) = \Theta(\delta_\alpha e)(\beta) = R_\Theta(\alpha, \beta), \end{aligned}$$

the fourth equality coming from the fact that $\Theta(\delta_\alpha e) \in \mathscr{C}_T(F)f$.

On the other hand,

$$eR_\Theta(\alpha,\beta) = \bigvee_{\gamma\in E} e(\alpha,\gamma)\wedge R_\Theta(\gamma,\beta) = \bigvee_{\gamma\in E} e(\alpha,\gamma)\wedge\Theta(\delta_\gamma e)(\beta) = \Theta\left(\bigvee_{\gamma\in E} e(\alpha,\gamma)\cdot\delta_\gamma e\right)(\beta)$$

since Θ is T-linear. Since e is idempotent, the function $e(\alpha,-)$ is an element of $\mathscr{C}_T(E)e$. Then Lemma 4.8 ensures that

$$\bigvee_{\gamma\in E} e(\alpha,\gamma)\cdot\delta_\gamma e = e(\alpha,-) = \delta_\alpha e.$$

Hence $eR_\Theta(\alpha,\beta) = \Theta(\delta_\alpha e)(\beta) = R_\Theta(\alpha,\beta)$. So one has $R_\Theta \in e\mathscr{C}_T(E,F)f$.

Now if Θ is an isomorphism, one has similarly $R_{\Theta^{-1}} \in f\mathscr{C}_T(F,E)e$. Moreover, for all $\alpha,\beta\in F$, one has

$$\begin{aligned} R_{\Theta^{-1}}R_\Theta(\alpha,\beta) &= \bigvee_{\gamma\in E} R_{\Theta^{-1}}(\alpha,\gamma)\wedge R_\Theta(\gamma,\beta) = \bigvee_{\gamma\in E}\Theta^{-1}(\delta_\alpha f)(\gamma)\wedge\Theta(\delta_\gamma e)(\beta)\\ &= \Theta\left(\bigvee_{\gamma\in F}\Theta^{-1}(\delta_\alpha f)(\gamma)\cdot\delta_\gamma e\right)(\beta) = \Theta(\Theta^{-1}(\delta_\alpha f))(\beta)\\ &= \delta_\alpha f(\beta) = f(\alpha,\beta). \end{aligned}$$

Here the third equality comes from the T-linearity of Θ, while the fourth one is again a consequence of Lemma 4.8. Hence $R_{\Theta^{-1}}R_\Theta = f$.

One has similarly $R_\Theta R_{\Theta^{-1}} = e$. So e and f are equivalent whenever $\mathscr{C}_T(E)e$ and $\mathscr{C}_T(F)f$ are isomorphic.

Conversely, assume that there exist $x\in e\mathscr{C}_T(E,F)f$ and $y\in f\mathscr{C}_T(F,E)e$ such that $e = xy$ and $f = yx$. Then

$$\begin{array}{ccc} \mathscr{C}_T(E)e & \longrightarrow & \mathscr{C}_T(F)f \\ \varphi & \longmapsto & \varphi x \end{array}$$

and

$$\begin{array}{ccc} \mathscr{C}_T(F)f & \longrightarrow & \mathscr{C}_T(E)e \\ \varphi & \longmapsto & \varphi y \end{array}$$

are well-defined T-linear maps, inverse to each other. So $\mathscr{C}_T(E)e$ and $\mathscr{C}_T(F)f$ are isomorphic whenever e and f are equivalent. $\square$

For any T-module U, we define a map

$$\begin{array}{rccc} m\colon & U & \longrightarrow & T \\ & u & \longmapsto & \displaystyle\bigwedge_{\substack{t\in T\\ t\cdot u=u}} t. \end{array}$$

Lemma 4.13. *Let U be a T-module.*

1. *For all $t\in T$ and $u\in U$, one has $t\cdot u=u$ if and only if $m(u)\leqslant t$.*
2. *The map m maps the set $\mathrm{Irr}(U)$ of join-irreducible elements of U to B.*
3. *Assume that m is order-preserving. For all $t\in T$ and $u,v\in U$, one has $u\leqslant t\cdot v$ if and only if $u=t\cdot u$ and $u\leqslant v$.*
4. *Assume that $U=\mathscr{C}_T(E)e$. Then for all $\varphi\in U$, one has*

$$m(\varphi)=\bigvee_{\alpha\in E}\varphi(\alpha),$$

and for all $\alpha\in E$ and $b\in B$ with $b\leqslant e(\alpha,\alpha)$, one has $m(\delta_\alpha^b e)=b$. In particular, the map m is order-preserving.

Proof. 1. If $t\cdot u=u$, it follows from the definition of m that $m(u)\leqslant t$. Conversely, if $m(u)\leqslant t$, then $u=m(u)\cdot u\leqslant t\cdot u\leqslant u$ by Lemma 4.2. So $t\cdot u=u$.

2. Let $\omega\in\mathrm{Irr}(U)$. One has

$$\omega=m(\omega)\cdot\omega=\left(\bigvee_{\substack{b\in B\\ b\leqslant m(\omega)}} b\right)\cdot\omega=\bigvee_{\substack{b\in B\\ b\leqslant m(\omega)}} b\cdot\omega.$$

Since ω is join-irreducible, there exists $b\leqslant m(\omega)$ such that $b\cdot\omega=\omega$. Then $m(\omega)\leqslant b$, whence $m(\omega)=b\in B$.

3. Assume that $u\leqslant t\cdot v$. Since $t\cdot(t\cdot v)=t\cdot v$, one has $m(t\cdot v)\leqslant t$. Since m is order-preserving, one has $m(u)\leqslant m(t\cdot v)\leqslant t$, so $t\cdot u=u$. Moreover $u\leqslant v$ by Lemma 4.2. Conversely, if $u=t\cdot u$ and $u\leqslant v$, then $u=t\cdot u\leqslant t\cdot v$.

4. Since $m(\varphi)\cdot\varphi=\varphi$, one has $m(\varphi)\wedge\varphi(\alpha)=\varphi(\alpha)$, so $\varphi(\alpha)\leqslant m(\varphi)$ for all $\alpha\in E$. Hence

$$\bigvee_{\alpha\in E}\varphi(\alpha)\leqslant m(\varphi).$$

Conversely, for all $\beta\in E$, one has

$$\left(\bigvee_{\alpha\in E}\varphi(\alpha)\right)\wedge\varphi(\beta)=\varphi(\beta).$$

This means that

$$\left(\bigvee_{\alpha\in E}\varphi(\alpha)\right)\cdot\varphi=\varphi,$$

so

$$m(\varphi)\leqslant\bigvee_{\alpha\in E}\varphi(\alpha).$$

Now if $\alpha\in E$ and $b\in B$ are such that $b\leqslant e(\alpha,\alpha)$, then

$$m(\delta^b_\alpha e)=\bigvee_{\beta\in E}\delta^b_\alpha e(\beta)=\bigvee_{\beta\in E}b\wedge e(\alpha,\beta),$$

so that one has both $m(\delta^b_\alpha e)\leqslant b$ and $m(\delta^b_\alpha e)\geqslant b\wedge e(\alpha,\alpha)=b$. $\square$

Now let U be a fixed finite T-module. We write $\Omega=\mathrm{Irr}(U)$. For all $u\in U$ and $\omega\in\Omega$, we define

$$d(u,\omega)=\bigvee_{\substack{b\in B\\ b\leqslant m(\omega)\\ b\cdot\omega\leqslant u}}b.$$

Then $d\in\mathscr{C}_T(U,\Omega)$.

Lemma 4.14.

1. *For all $u\in U$ and $\omega\in\Omega$, one has $d(u,\omega)\leqslant m(\omega)$.*
2. *For all $\omega\in\Omega$, one has $d(\omega,\omega)=m(\omega)$.*
3. *For all $\omega,\xi\in\Omega$, one has $\xi\leqslant\omega$ if and only if $d(\omega,\xi)=m(\xi)$.*

Proof. 1. It is an immediate consequence of the definition of d.

2. If $\omega\in\Omega$, then $b\cdot\omega\leqslant\omega$ for all $b\in B$, in view of Lemma 4.2. Then

$$d(\omega,\omega)=\bigvee_{\substack{b\leqslant m(\omega)\\ b\cdot\omega\leqslant\omega}}b=\bigvee_{b\leqslant m(\omega)}b=m(\omega).$$

3. If $\xi\leqslant\omega$, then $b\cdot\xi\leqslant\xi\leqslant\omega$ for all $b\in B$. Then

$$d(\omega,\xi)=\bigvee_{\substack{b\leqslant m(\xi)\\ b\cdot\xi\leqslant\omega}}b=\bigvee_{b\leqslant m(\xi)}b=m(\xi).$$

Conversely, if $d(\omega,\xi)=m(\xi)$, then $d(\omega,\xi)\cdot\xi=\xi$. But

$$d(\omega, \xi) \cdot \xi = \bigvee_{\substack{b \leqslant m(\xi) \\ b \cdot \xi \leqslant \omega}} b \cdot \xi \leqslant \omega,$$

so $\xi \leqslant \omega$. □

Lemma 4.15. *For all $u \in U$, one has*

$$u = \bigvee_{\omega \in \Omega} d(u, \omega) \cdot \omega.$$

Proof. Let $u \in U$. For all $\omega \in \Omega$, one has $d(u, \omega) \cdot \omega \leqslant u$ by definition of d, so

$$\bigvee_{\omega \in \Omega} d(u, \omega) \cdot \omega \leqslant u.$$

Conversely, if $\omega \in \Omega$ is such that $\omega \leqslant u$, then $m(\omega) \in B$, thus

$$m(\omega) \leqslant \bigvee_{\substack{b \leqslant m(\omega) \\ b \cdot \omega \leqslant u}} b = d(u, \omega).$$

So

$$u = \bigvee_{\omega \leqslant u} \omega = \bigvee_{\omega \leqslant u} m(\omega) \cdot \omega \leqslant \bigvee_{\omega \leqslant u} d(u, \omega) \cdot \omega \leqslant \bigvee_{\omega \in \Omega} d(u, \omega) \cdot \omega.$$

These two inequalities prove the lemma. □

We write $f = d|_{\Omega \times \Omega} \in \mathscr{C}_T(\Omega, \Omega)$.

Lemma 4.16. *One has $df = f$. In particular, $f \in \mathscr{C}_T(\Omega, \Omega)$ is idempotent.*

Proof. Let $u \in U$. By Lemma 4.14, one has

$$df(u, \omega) = \bigvee_{\xi \in \Omega} d(u, \xi) \wedge f(\xi, \omega) \leqslant \bigvee_{\xi \in \Omega} d(u, \xi) \wedge m(\omega) \leqslant m(\omega).$$

By Lemma 4.15, one has

$$\begin{aligned} u &= \bigvee_{\xi \in \Omega} d(u, \xi) \cdot \xi = \bigvee_{\xi \in \Omega} d(u, \xi) \cdot \left(\bigvee_{\omega \in \Omega} d(\xi, \omega) \cdot \omega \right) \\ &= \bigvee_{\omega \in \Omega} \left(\bigvee_{\xi \in \Omega} d(u, \xi) \wedge f(\xi, \omega) \right) \cdot \omega = \bigvee_{\omega \in \Omega} df(u, \omega) \cdot \omega. \end{aligned}$$

In particular, $df(u,\omega)\cdot\omega \leqslant u$ for all $\omega \in \Omega$. So

$$df(u,\omega) = \bigvee_{b \leqslant df(u,\omega)} b \leqslant \bigvee_{\substack{b \leqslant m(\omega) \\ b\cdot\omega \leqslant u}} b = d(u,\omega),$$

that is, $df \leqslant d$.

Conversely, define $\delta_m \in \mathscr{C}_T(\Omega,\Omega)$ by $\delta_m(\omega,\xi) = m(\omega) \wedge \Delta_\Omega(\omega,\xi)$ for all $\omega,\xi \in \Omega$. By Lemma 4.14, one has $\delta_m \leqslant f$, so $d\delta_m \leqslant df$. Lemma 4.14 again implies that $d\delta_m = f$, so $f \leqslant df$. □

Proposition 4.17. *The map*

$$\begin{array}{rccc} \Theta\colon & \mathscr{C}_T(\Omega)f & \longrightarrow & U \\ & \varphi & \longmapsto & \bigvee_{\omega\in\Omega} \varphi(\omega)\cdot\omega \end{array}$$

is T-linear and surjective.

Proof. It is straightforward to check that Θ is T-linear. By Lemmas 4.16 and 4.15, one has $d(u,-) \in \mathscr{C}_T(\Omega)f$ and $\Theta(d(u,-)) = u$, so Θ is surjective. □

Definition 4.18. A T-module U is *preregular* if it fulfills the following axioms.

- The map m is order-preserving.
- For all $b \in B$ and $\omega \in \mathrm{Irr}(U)$ such that $b \leqslant m(\omega)$, one has $b\cdot\omega \in \mathrm{Irr}(U)$.
- For all $b \in B$ and $\omega \in \mathrm{Irr}(U)$ such that $b \leqslant m(\omega)$, one has $m(b\cdot\omega) = b$.

Proposition 4.19. *Any regular T-module is both preregular and distributive.*

Proof. The notions of preregularity and distributivity are invariant by isomorphism thanks to Lemma 4.5. Then it is enough to prove the proposition for a module of type $\mathscr{C}_T(E)e$. It is a straightforward computation to check that such modules are distributive. Lemma 4.13 implies that the map m is order-preserving. Moreover, Lemma 4.7 ensures that any join-irreducible element in $\mathscr{C}_T(E)e$ is of type $\delta_\alpha^b e$ for some $\alpha \in E$ and $b \in B$ with $b \leqslant e(\alpha,\alpha)$. Then $m(\delta_\alpha^b e) = b$ thanks to Lemma 4.13. Now for all $c \leqslant b$ and for all $\beta \in E$, one has $(c\cdot\delta_\alpha^b)e(\beta) = c \wedge b \wedge e(\alpha,\beta) = c \wedge e(\alpha,\beta) = \delta_\alpha^c e(\beta)$. So $c\cdot\delta_\alpha^b e = \delta_\alpha^c e$. Since $c \in B$ and $c \leqslant b \leqslant e(\alpha,\alpha)$, and thanks to Lemma 4.7, $\delta_\alpha^c e$ is a join-irreducible element of $\mathscr{C}_T(E)e$. Then $m(\delta_\alpha^c e) = c$ thanks to Lemma 4.13 again. Hence the three axioms defining a preregular module are verified by $\mathscr{C}_T(E)e$. □

We now focus on proving the converse, so we assume that U is finite, preregular and distributive.

We first recall the following well-known property, which holds in any distributive lattice A: if a is a join-irreducible element of A, I a nonempty set and $(a_i)_{i\in I}$ a family of elements of A such that

$$a \leqslant \bigvee_{i\in I} a_i,$$

then there exists $i \in I$ such that $a \leqslant a_i$. In the sequel, we shall use it without reference.

Lemma 4.20. *For all $t \in T$, $u, v \in U$ and $\omega \in \Omega$, the following properties hold.*

1. $d(u, \omega) \leqslant m(u)$.
2. $d(t \cdot u, \omega) = t \wedge d(u, \omega)$.
3. $d(u \vee v, \omega) = d(u, \omega) \vee d(v, \omega)$.

Proof. 1. Let $b \in B$ be such that $b \leqslant m(\omega)$ and $b \cdot \omega \leqslant u$. Preregularity implies that $m(b \cdot \omega) = b$ and $m(b \cdot \omega) \leqslant m(u)$, so $b \leqslant m(u)$. Hence $d(u, \omega)$ is a join of elements smaller than $m(u)$, whence $d(u, \omega) \leqslant m(u)$.

2. Let $b \in B$ be such that $b \leqslant d(t \cdot u, \omega)$. Then $b \leqslant m(\omega)$ and $b \cdot \omega \leqslant t \cdot u \leqslant u$, so $b \leqslant d(u, \omega)$. Moreover, $b \leqslant m(t \cdot u)$ by the previous assertion, and $m(t \cdot u) \leqslant t$ since $t \cdot (t \cdot u) = t \cdot u$. So $b \leqslant t$, and then $b \leqslant t \wedge d(u, \omega)$. Conversely, let $b \in B$ be such that $b \leqslant t \wedge d(u, \omega)$. Then $b \leqslant t$, $b \leqslant m(\omega)$ and $b \cdot \omega \leqslant u$, so $b \cdot \omega = (t \wedge b) \cdot \omega = t \cdot (b \cdot \omega) \leqslant t \cdot u$. Hence $b \leqslant d(t \cdot u, \omega)$. Then $\{b \in B, b \leqslant d(t \cdot u, \omega)\} = \{b \in B, b \leqslant t \wedge d(u, \omega)\}$. In other words, $d(t \cdot u, \omega) = t \wedge d(u, \omega)$.

3. Let $b \in B$ be such that $b \leqslant m(\omega)$ and $b \cdot \omega \leqslant u \vee v$. Preregularity implies that $b \cdot \omega \in \mathrm{Irr}(U)$, so $b \cdot \omega \leqslant u$ or $b \cdot \omega \leqslant v$ since U is distributive. Then $\{b \in B, b \leqslant m(\omega), b \cdot \omega \leqslant u \vee v\} \subseteq \{b \in B, b \leqslant m(\omega), b \cdot \omega \leqslant u\} \cup \{b \in B, b \leqslant m(\omega), b \cdot \omega \leqslant v\}$. The converse inclusion being obvious, one has

$$\bigvee_{\substack{b\in B\\ b\leqslant m(\omega)\\ b\cdot\omega\leqslant u\vee v}} b = \bigvee_{\substack{b\in B\\ b\leqslant m(\omega)\\ b\cdot\omega\leqslant u}} b \vee \bigvee_{\substack{b\in B\\ b\leqslant m(\omega)\\ b\cdot\omega\leqslant v}} b,$$

that is, $d(u \vee v, \omega) = d(u, \omega) \vee d(v, \omega)$. $\square$

Proposition 4.21. *If U is finite, preregular and distributive, then Θ is bijective and its inverse is*

$$\begin{array}{rccc} H\colon & U & \longrightarrow & \mathscr{C}_T(\Omega)f \\ & u & \longmapsto & d(u, -). \end{array}$$

Proof. We already saw in the proof of Proposition 4.17 that $\Theta \circ H = \mathrm{Id}_U$, without any assumption about U.

Now if U is finite, preregular and distributive, then Lemma 4.20 ensures that H is T-linear. Then for all $\varphi \in \mathscr{C}_T(\Omega)f$, one has

$$H \circ \Theta(\varphi) = H\left(\bigvee_{\omega\in\Omega} \varphi(\omega)\cdot\omega\right) = \bigvee_{\omega\in\Omega} \varphi(\omega)\cdot H(\omega) = \bigvee_{\omega\in\Omega} \varphi(\omega) \wedge d(\omega,-)$$
$$= \bigvee_{\omega\in\Omega} \varphi(\omega)\wedge f(\omega,-) = \varphi f = \varphi,$$

since $\varphi \in \mathscr{C}_T(\Omega)f$.

Hence $H \circ \Theta = \mathrm{Id}_{\mathscr{C}_T(\Omega)f}$, and H is indeed the inverse of Θ. □

Corollary 4.22. *A finite T-module is regular if and only if it is preregular and distributive.*

Remark 4.23. When $T = \{0,1\}$ is the two-elements lattice, finite $\{0,1\}$-modules are just finite lattices. Propositions 4.17 and 4.21 thus generalize the classical theory of lattices. Indeed, any $\{0,1\}$-module is preregular. So our results ensure that a finite lattice, that is, a $\{0,1\}$-module, is regular if and only if it is distributive. This generalizes the well-known characterization of finite distributive lattices as those that are fully determined by their join-irreducible elements, which is stated for example in [19, Theorem 3.4.1].

Remark 4.24. Let E be a finite set and $e \in \mathscr{C}_T(E,E)$ be an idempotent. The set of join-irreducible elements of the T-module $\mathscr{C}_T(E)e$ has no reason to be in bijection with E. However, Proposition 4.21 ensures that $\mathrm{Irr}(\mathscr{C}_T(\Omega)f)$ and Ω are isomorphic as posets. More precisely, the join-irreducible elements of $\mathscr{C}_T(\Omega)f$ are exactly the elements of type

$$d(\omega,-) = f(\omega,-) = \delta_\omega f$$

with $\omega \in \Omega$. According to Lemma 4.7, elements of type $\delta_\alpha^b f$ with $\alpha \in \Omega$, $b \in B$ and $b \leqslant f(\alpha,\alpha)$ are join-irreducible. Then, such elements must be of type δ_ω for $\omega \in \Omega$. Indeed one has

$$\delta_\alpha^b f = b \wedge f(\alpha,-) = f(b\cdot\alpha,-) = \delta_{b\cdot\alpha} f,$$

and $b \cdot \alpha \in \Omega$ thanks to preregularity.

5. Presheaves

We consider *presheaves* of finite posets on B, that is, functors $B^{\mathrm{op}} \to \mathbf{ord}$, where B^{op} is the opposite of the category associated to the poset B and $\mathbf{ord}$ is the category of finite posets and order-preserving maps.

In this section, we build a correspondence between regular T-modules and presheaves. It will lead to a correspondence between idempotents and presheaves.

Let E be a finite set, let $e \in \mathscr{C}_T(E,E)$ be an idempotent and let $U = \mathscr{C}_T(E)e$. Write $\Omega = \mathrm{Irr}(U)$ for the set of join-irreducible elements of U. For all $b \in B$, write ${}^\sharp\Omega(b) = \Omega \cap m^{-1}(b)$. For all $b, c \in B$ such that $c \leqslant b$, Proposition 4.19 implies that the map

$$\begin{array}{rccc} {}^\sharp\Omega^b_c : & {}^\sharp\Omega(b) & \longrightarrow & {}^\sharp\Omega(c) \\ & \omega & \longmapsto & c \cdot \omega \end{array}$$

is well-defined. Then one can check that ${}^\sharp\Omega \colon B^{\mathrm{op}} \to \mathbf{ord}$ is a presheaf.

Conversely, let $\mathscr{O}$ be a presheaf of finite posets on B and let

$${}^\flat\mathscr{O} = \bigsqcup_{b \in B} \mathscr{O}(b).$$

For all $\omega \in {}^\flat\mathscr{O}$, we write $n(\omega)$ for the unique element $b \in B$ such that $\omega \in \mathscr{O}(b)$. One can check that ${}^\flat\mathscr{O}$ is a poset ordered by

$$\xi \leqslant \omega \text{ if } \begin{cases} n(\xi) \leqslant n(\omega) \\ \xi \leqslant \mathscr{O}^{n(\omega)}_{n(\xi)}(\omega). \end{cases}$$

Let $U = I_\downarrow({}^\flat\mathscr{O})$ be the lattice of lower subsets of ${}^\flat\mathscr{O}$. By classical lattice theory, one has $\mathrm{Irr}(U) \cong {}^\flat\mathscr{O}$ as posets. More precisely, the element $\omega \in {}^\flat\mathscr{O}$ corresponds to the interval $]-,\omega]_{{}^\flat\mathscr{O}}$ in U. We may identify these elements when convenient, and we will write $]-,\omega]$ instead of $]-,\omega]_{{}^\flat\mathscr{O}}$.

For all $t \in T$ and $u \in U$, we write

$$t \cdot u = \bigcup_{\substack{b \in B \\ b \leqslant t}} u \cap \mathscr{O}(b).$$

One can then check that U is a T-module for this action of T.

Lemma 5.1. *Let $b \in B$ and let $\omega \in {}^\flat\mathscr{O}$. If $b \leqslant n(\omega)$, then $b \cdot \omega = \mathscr{O}^{n(\omega)}_b(\omega)$.*

Proof. One has

$$\begin{aligned} b \cdot \omega = b \cdot]-,\omega] &= \bigcup_{c \leqslant b}]-,\omega] \cap \mathscr{O}(c) = \bigcup_{c \leqslant b} \Big]-, \mathscr{O}^{n(\omega)}_c(\omega)\Big] \cap \mathscr{O}(c) \\ &= \bigcup_{c \leqslant b} \Big]-, \mathscr{O}^b_c\left(\mathscr{O}^{n(\omega)}_b(\omega)\right)\Big] \cap \mathscr{O}(c) = \bigcup_{c \leqslant b} \Big]-, \mathscr{O}^{n(\omega)}_b(\omega)\Big] \cap \mathscr{O}(c). \end{aligned}$$

Now if $\Big]-, \mathscr{O}^{n(\omega)}_b(\omega)\Big] \cap \mathscr{O}(c) \neq \varnothing$, then $c \leqslant b$. For this reason

$$\bigcup_{c \leqslant b} \Big]-, \mathscr{O}^{n(\omega)}_b(\omega)\Big] \cap \mathscr{O}(c) = \bigcup_{c \in B} \Big]-, \mathscr{O}^{n(\omega)}_b(\omega)\Big] \cap \mathscr{O}(c) = \Big]-, \mathscr{O}^{n(\omega)}_b(\omega)\Big] = \mathscr{O}^{n(\omega)}_b(\omega). \quad \square$$

Corollary 5.2. *Let $b \in B$ and let $\omega \in {}^{\flat}\mathscr{O}$. If $b \leqslant n(\omega)$, then $b \cdot \omega \in {}^{\flat}\mathscr{O}$ and $n(b \cdot \omega) = b$.*

Lemma 5.3. *For all $t \in T$ and $u \in U$, one has $t \cdot u = u$ if and only if*

$$u \subseteq \bigcup_{\substack{b \in B \\ b \leqslant t}} \mathscr{O}(b).$$

Proof. This is straightforward. $\square$

Corollary 5.4. *For all $\omega \in {}^{\flat}\mathscr{O}$, one has $n(\omega) = m(\omega)$, that is, $m|_{{}^{\flat}\mathscr{O}} = n$.*

Proof. If $t \cdot \omega = \omega$, then there exists $b \leqslant t$ such that $\omega \in \mathscr{O}(b)$ by Lemma 5.3, so $n(\omega) \leqslant t$. Conversely, if $n(\omega) \leqslant t$, then

$$\omega =]-, \omega] \subseteq \bigcup_{b \leqslant n(\omega)} \mathscr{O}(b) \subseteq \bigcup_{b \leqslant t} \mathscr{O}(b),$$

hence $t \cdot \omega = \omega$.

So

$$m(\omega) = \bigwedge_{t \cdot \omega = \omega} t = \bigwedge_{t \geqslant n(\omega)} t = n(\omega). \quad \square$$

Corollary 5.5. *The map m is order-preserving.*

Proof. Let $u, v \in U$ be such that $u \subseteq v$. Since $m(v) \cdot v = v$, Lemma 5.3 implies that

$$u \subseteq v \subseteq \bigcup_{b \leqslant m(v)} \mathscr{O}(b).$$

The same lemma then implies that $m(v) \cdot u = u$. So $m(u) \leqslant m(v)$ by Lemma 4.13. $\square$

Proposition 5.6. *The T-module $U = I_{\downarrow}({}^{\flat}\mathscr{O})$ is regular.*

Proof. By definition, U is a distributive lattice. Corollaries 5.2, 5.4 and 5.5 imply that it is preregular, so it is regular by Proposition 4.21. $\square$

Lemma 5.7. *Let U be a regular T-module and let $\Omega = \mathrm{Irr}(U)$. One has ${}^{\flat\sharp}\Omega = \Omega$ as posets.*

Proof. It is obvious that ${}^{\flat\sharp}\Omega = \Omega$ as sets, and it is straightforward to check, using the regularity of U, that the order relations are the same. $\square$

Lemma 5.8. *Let $\mathscr{O}$ be a presheaf a finite posets on B. One has ${}^{\sharp\flat}\mathscr{O} = \mathscr{O}$ as presheaves.*

Proof. It is straightforward to check that for all $b \in B$, one has ${}^{\sharp\flat}\mathscr{O}(b) = \mathscr{O}(b)$ as posets. Lemma 5.1 then implies that for all $b, c \in B$ such that $c \leqslant b$, one has ${}^{\sharp\flat}\mathscr{O}^b_c = \mathscr{O}^b_c$. So ${}^{\sharp\flat}\mathscr{O} = \mathscr{O}$ as presheaves. □

Lemma 5.9. *Let U and V be two regular T-modules. If U and V are isomorphic, then the presheaves ${}^{\sharp}\operatorname{Irr}(U)$ and ${}^{\sharp}\operatorname{Irr}(V)$ are isomorphic.*

Proof. Let $\Theta\colon U \to V$ be a T-linear isomorphism. Lemma 4.5 implies that for all $b \in B$, the map

$$\begin{array}{rccc} \theta_b\colon & {}^{\sharp}\operatorname{Irr}(U)(b) & \longrightarrow & {}^{\sharp}\operatorname{Irr}(V)(b) \\ & \omega & \longmapsto & \Theta(\omega) \end{array}$$

is well-defined and is an isomorphism of posets. The fact that Θ is T-linear implies that $\theta\colon {}^{\sharp}\operatorname{Irr}(U) \to {}^{\sharp}\operatorname{Irr}(V)$ is a natural transformation. Hence the presheaves ${}^{\sharp}\operatorname{Irr}(U)$ and ${}^{\sharp}\operatorname{Irr}(V)$ are indeed isomorphic. □

Lemma 5.10. *Let $\mathscr{O}$ and $\mathscr{P}$ be two presheaves of finite posets on B. If $\mathscr{O}$ and $\mathscr{P}$ are isomorphic, then the T-modules $I_{\downarrow}({}^{\flat}\mathscr{O})$ and $I_{\downarrow}({}^{\flat}\mathscr{P})$ are isomorphic.*

Proof. Let $\theta\colon \mathscr{O} \to \mathscr{P}$ be an isomorphism of presheaves. We also denote by θ, with a slight abuse, the induced map ${}^{\flat}\mathscr{O} \to {}^{\flat}\mathscr{P}$. It is clearly a bijection. It is straightforward to check that θ maps lower subsets of ${}^{\flat}\mathscr{O}$ to lower subsets of ${}^{\flat}\mathscr{P}$, and similarly that θ^{-1} maps lower subsets of ${}^{\flat}\mathscr{P}$ to lower subsets of ${}^{\flat}\mathscr{O}$. Hence we have a bijective map

$$\begin{array}{rccc} \Theta\colon & I_{\downarrow}({}^{\flat}\mathscr{O}) & \longrightarrow & I_{\downarrow}({}^{\flat}\mathscr{P}) \\ & u & \longmapsto & \theta(u). \end{array}$$

It is then straightforward to check that Θ is a T-linear map, whence a T-linear isomorphism by Lemma 4.5. Hence the T-modules $I_{\downarrow}({}^{\flat}\mathscr{O})$ and $I_{\downarrow}({}^{\flat}\mathscr{P})$ are isomorphic. □

Lemma 5.11. *Any regular T-module U is isomorphic to $I_{\downarrow}(\operatorname{Irr}(U))$.*

Proof. By classical lattice theory and because U is distributive,

$$\begin{array}{ccc} I_{\downarrow}(\operatorname{Irr}(U)) & \longrightarrow & U \\ u & \longmapsto & \displaystyle\bigvee_{\omega \in u} \omega \end{array}$$

is a lattice isomorphism. It remains to prove that it is a T-linear map: let $t \in T$ and $a \in I_{\downarrow}(\operatorname{Irr}(U))$. For all $\xi \in t \cdot a$, there exists $b \in B$ with $b \leqslant t$ such that $\xi \in a \cap {}^{\sharp}\operatorname{Irr}(U)(b)$. In particular, $m(\xi) = b$, so

$$\xi = b \cdot \xi \leqslant t \cdot \xi \leqslant t \cdot \bigvee_{\omega \in a} \omega.$$

Hence

$$\bigvee_{\omega \in t\cdot a} \omega \leqslant t \cdot \bigvee_{\omega \in a} \omega.$$

Conversely, let $\xi \in \Omega$ be such that

$$\xi \leqslant t \cdot \bigvee_{\omega \in a} \omega = \bigvee_{\omega \in a} t \cdot \omega.$$

Since U is distributive, there exists $\omega \in a$ such that $\xi \leqslant t \cdot \omega$. So $\xi \leqslant \omega$, hence $\xi \in a$ because a is a lower subset of $\mathrm{Irr}(U)$ and $\omega \in a$. Moreover $m(\xi) \leqslant m(t \cdot \omega) \leqslant t$. Hence $\xi \in t \cdot a$, so

$$\xi \leqslant \bigvee_{\omega \in t\cdot a} \omega.$$

Hence

$$t \cdot \bigvee_{\omega \in a} \omega \leqslant \bigvee_{\omega \in t\cdot a} \omega. \quad \square$$

Lemma 5.12. *Any presheaf of finite posets $\mathscr{O}$ on B is isomorphic to ${}^{\sharp}\mathrm{Irr}(I_{\downarrow}({}^{\flat}\mathscr{O}))$.*

Proof. By classical lattice theory, the map

$$\begin{array}{rccc} \theta\colon & {}^{\flat}\mathscr{O} & \longrightarrow & \mathrm{Irr}(I_{\downarrow}({}^{\flat}\mathscr{O})) \\ & \omega & \longmapsto &]-,\omega] \end{array}$$

is an isomorphism of posets. For all $b \in B$, this isomorphism maps $\mathscr{O}(b)$ to ${}^{\sharp}\mathrm{Irr}(I_{\downarrow}({}^{\flat}\mathscr{O}))(b)$. Write $\theta_b = \theta|_{\mathscr{O}(b)}$. Lemma 5.1 implies that $\theta_c \mathscr{O}_c^b = {}^{\sharp}\mathrm{Irr}(I_{\downarrow}({}^{\flat}\mathscr{O}))_c^b \theta_c$. In other words, $\theta\colon \mathscr{O} \to I_{\downarrow}({}^{\flat}\mathscr{O})$ is a morphism. Each θ_b is obviously bijective, hence $\mathscr{O}$ is isomorphic to ${}^{\sharp}\mathrm{Irr}(I_{\downarrow}({}^{\flat}\mathscr{O}))$. $\square$

Let E be a finite set and let $e \in \mathscr{C}_T(E,E)$ be an idempotent. We write $\mathscr{O}(e) = {}^{\sharp}\mathrm{Irr}(\mathscr{C}_T(E)e)$. Conversely, let $\mathscr{O}$ be a presheaf of finite posets on B and let $U = I_{\downarrow}({}^{\flat}\mathscr{O})$, so that $\mathrm{Irr}(U) \cong {}^{\flat}\mathscr{O}$. We write $e(\mathscr{O}) \in \mathscr{C}_T({}^{\flat}\mathscr{O}, {}^{\flat}\mathscr{O})$ for the idempotent obtained in Lemma 4.16 for the T-module U.

Proposition 5.13. *The set of equivalence classes of idempotent T-generalized correspondences is in bijection with the set of isomorphism classes of presheaves of finite posets on B. More precisely, if we denote both equivalence and isomorphism classes by square brackets, then the maps $[e] \mapsto [\mathscr{O}(e)]$ and $[\mathscr{O}] \mapsto [e(\mathscr{O})]$ are well-defined and inverse to each other.*

Proof. By Proposition 4.12, Lemma 5.9 and Lemma 5.10, the maps $[e] \mapsto [\mathscr{O}(e)]$ and $[\mathscr{O}] \mapsto [e(\mathscr{O})]$ are well-defined.

Let E be a finite set and let $e \in \mathscr{C}_T(E,E)$ an idempotent. Since $\mathscr{O}(e) = {}^\sharp \operatorname{Irr}(\mathscr{C}_T(E)e)$ by definition, one has

$$\mathscr{C}_T(E)e \cong I_\downarrow(\operatorname{Irr}(\mathscr{C}_T(E)e)) = I_\downarrow({}^\flat\mathscr{O}(e)) \cong \mathscr{C}_T(\operatorname{Irr}(I_\downarrow({}^\flat\mathscr{O}(e))))e(\mathscr{O}(e)),$$

using successively Lemma 5.11, Lemma 5.7 and Proposition 4.21. Then Proposition 4.12 ensures that $[e(\mathscr{O}(e))] = [e]$.

Conversely, let $\mathscr{O}$ be a presheaf of finite posets on B. One has

$$\mathscr{O} \cong {}^\sharp \operatorname{Irr}(I_\downarrow({}^\flat\mathscr{O})) \cong {}^\sharp \operatorname{Irr}(\mathscr{C}_T({}^\flat\mathscr{O})e(\mathscr{O})) \cong \mathscr{O}(e(\mathscr{O})),$$

using successively Lemma 5.12, Proposition 4.21 and Lemma 5.9. Hence $[\mathscr{O}(e(\mathscr{O}))] = [\mathscr{O}]$. $\square$

6. Automorphism groups

Let E be a finite set and let $e \in \mathscr{C}_T(E,E)$ be an idempotent. We write $G_e = (e\mathscr{C}_T(E,E)e)^\times$ for the group of invertible elements of the monoid $e\mathscr{C}_T(E,E)e$.

Let U be a T-module. We write $G_U = \mathbf{Aut}_T(U)$ for the group of T-linear automorphisms of U.

Let $\mathscr{O}$ be a presheaf of finite posets on B. We write $G_\mathscr{O} = \mathbf{Aut}(\mathscr{O})$ for the group of natural automorphisms of the functor $\mathscr{O}$.

Proposition 6.1. *Let E be a finite set and let $e \in \mathscr{C}_T(E,E)$ be an idempotent. We write $U = \mathscr{C}_T(E)e$ and $\mathscr{O} = {}^\sharp \operatorname{Irr}(U)$. Then $G_e \cong G_U \cong G_\mathscr{O}$.*

Proof. We first prove that $G_e \cong G_U$. Consider the maps

$$\begin{array}{rccc} \mathfrak{g}\colon & G_e & \longrightarrow & G_U \\ & g & \longmapsto & \left(\begin{array}{ccc} U & \to & U \\ \varphi & \mapsto & \varphi g^{-1} \end{array}\right) \end{array} \tag{6.a}$$

and

$$\mathfrak{h}\colon\ G_U \longrightarrow G_e$$

defined by

$$\mathfrak{h}(\Theta)(\alpha,\beta) = \Theta^{-1}(\delta_\alpha e)(\beta) \tag{6.b}$$

For all $g \in G_e$ and $\varphi \in U$, one has $\varphi g^{-1} \in U$ since $g^{-1}e = g^{-1}$. For all $\varphi, \psi \in U$, $t \in T$ and $\alpha \in E$, one has

$$\begin{aligned}\mathfrak{g}(g)(\varphi\vee\psi)(\alpha) &= (\varphi\vee\psi)g^{-1}(\alpha) = \bigvee_{\beta\in E}(\varphi\vee\psi)(\beta)\wedge g^{-1}(\beta,\alpha)\\ &= \left(\bigvee_{\beta\in E}\varphi(\beta)\wedge g^{-1}(\beta,\alpha)\right)\vee\left(\bigvee_{\beta\in E}\psi(\beta)\wedge g^{-1}(\beta,\alpha)\right)\\ &= \varphi g^{-1}(\alpha)\vee\psi g^{-1}(\alpha) = (\varphi g^{-1}\vee\psi g^{-1})(\alpha) = (\mathfrak{g}(g)(\varphi)\vee\mathfrak{g}(g)(\psi))(\alpha)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{g}(g)(t\cdot\varphi)(\alpha) &= (t\cdot\varphi)g^{-1}(\alpha) = \bigvee_{\beta\in E}(t\cdot\varphi)(\beta)\wedge g^{-1}(\beta,\alpha) = \bigvee_{\beta\in E} t\wedge\varphi(\beta)\wedge g^{-1}(\beta,\alpha)\\ &= t\wedge\bigvee_{\beta\in E}\varphi(\beta)\wedge g^{-1}(\beta,\alpha) = t\wedge(\varphi g^{-1})(\alpha) = t\wedge\mathfrak{g}(g)(\varphi)(\alpha)\\ &= (t\cdot\mathfrak{g}(g)(\varphi))(\alpha).\end{aligned}$$

Hence $\mathfrak{g}(g)\in\mathbf{End}_T(U)$. Clearly $\mathfrak{g}(e)=\mathrm{Id}_U$ and $\mathfrak{g}(gh)=\mathfrak{g}(g)\circ\mathfrak{g}(h)$ for all $g,h\in G_e$. So $\mathfrak{g}(g)$ is invertible and $\mathfrak{g}(g)^{-1}=\mathfrak{g}(g^{-1})\in\mathbf{End}_T(U)$. In particular, $\mathfrak{g}(g)\in G_U$.

Now for all $\Theta\in G_U$ and $\alpha,\beta\in E$, one has

$$\begin{aligned}e\mathfrak{h}(\Theta)e(\alpha,\beta) &= \bigvee_{\gamma,\varepsilon\in E} e(\alpha,\gamma)\wedge\mathfrak{h}(\Theta)(\gamma,\varepsilon)\wedge e(\varepsilon,\beta) = \bigvee_{\gamma,\varepsilon\in E} e(\alpha,\gamma)\wedge\Theta^{-1}(\delta_\gamma e)(\varepsilon)\wedge e(\varepsilon,\beta)\\ &= \bigvee_{\gamma\in E} e(\alpha,\gamma)\wedge\left(\bigvee_{\varepsilon\in E}\Theta^{-1}(\delta_\gamma e)(\varepsilon)\wedge e(\varepsilon,\beta)\right) = \bigvee_{\gamma\in E} e(\alpha,\gamma)\wedge\Theta^{-1}(\delta_\gamma e)(\beta)\\ &= \Theta^{-1}\left(\bigvee_{\gamma\in E} e(\alpha,\gamma)\cdot(\delta_\gamma e)\right)(\beta).\end{aligned}$$

For all $\zeta\in E$, one has

$$\begin{aligned}\left(\bigvee_{\gamma\in E} e(\alpha,\gamma)\cdot(\delta_\gamma e)\right)(\zeta) &= \bigvee_{\gamma,\eta\in E} e(\alpha,\gamma)\wedge\delta_\gamma(\eta)\wedge e(\eta,\zeta)\\ &= \bigvee_{\gamma\in E} e(\alpha,\gamma)\wedge e(\gamma,\zeta)\\ &= e^2(\alpha,\zeta) = e(\alpha,\zeta) = \delta_\alpha e(\zeta).\end{aligned}$$

So

$$e\mathfrak{h}(\Theta)e(\alpha,\beta) = \Theta^{-1}(\delta_\alpha e)(\beta) = \mathfrak{h}(\Theta)(\alpha,\beta).$$

Hence $\mathfrak{h}(\Theta)\in e\mathscr{C}_T(E,E)e$. Moreover, for all $\Theta,H\in G_U$ and $\alpha,\beta\in E$, one has

$$\mathfrak{h}(\Theta)\mathfrak{h}(H)(\alpha,\beta) = \bigvee_{\gamma\in E} \mathfrak{h}(\Theta)(\alpha,\gamma) \wedge \mathfrak{h}(H)(\gamma,\beta) = \bigvee_{\gamma\in E} \Theta^{-1}(\delta_\alpha e)(\gamma) \wedge H^{-1}(\delta_\gamma e)(\beta)$$
$$= H^{-1}\left(\bigvee_{\gamma\in E} \Theta^{-1}(\delta_\alpha e)(\gamma)\cdot \delta_\gamma e\right)(\beta).$$

For all $\zeta \in E$, one has

$$\left(\bigvee_{\gamma\in E} \Theta^{-1}(\delta_\alpha e)(\gamma)\cdot \delta_\gamma e\right)(\zeta) = \bigvee_{\gamma\in E} \Theta^{-1}(\delta_\alpha e)(\gamma) \wedge e(\gamma,\zeta) = \Theta^{-1}(\delta_\alpha e)e(\zeta) = \Theta^{-1}(\delta_\alpha e)(\zeta).$$

So

$$\mathfrak{h}(\Theta)\mathfrak{h}(H)(\alpha,\beta) = H^{-1}(\Theta^{-1}(\delta_\alpha e))(\beta) = \mathfrak{h}(\Theta\circ H)(\alpha,\beta).$$

Hence $\mathfrak{h}(\Theta\circ H) = \mathfrak{h}(\Theta)\mathfrak{h}(H)$. Clearly $\mathfrak{h}(\mathrm{Id}_U) = e$, so $\mathfrak{h}(\Theta)$ is invertible and $\mathfrak{h}(\Theta)^{-1} = \mathfrak{h}(\Theta^{-1}) \in e\mathscr{C}_T(E,E)e$. In particular, $\mathfrak{h}(\Theta) \in G_e$.

We proved that $\mathfrak{g}$ and $\mathfrak{h}$ are well-defined. It remains to prove that they are inverse to each other. For all $g \in G_e$ and $\alpha,\beta \in E$, one has

$$\mathfrak{h}(\mathfrak{g}(g))(\alpha,\beta) = \mathfrak{g}(g)^{-1}(\delta_\alpha e)(\beta) = \mathfrak{g}(g^{-1})(\delta_\alpha e)(\beta) = \delta_\alpha e g(\beta) = \delta_\alpha g(\beta) = g(\alpha,\beta).$$

Hence $\mathfrak{h}\circ\mathfrak{g} = \mathrm{Id}_{G_e}$.

Conversely, for all $\Theta \in G_U$, $\varphi \in U$ and $\alpha \in E$, one has

$$\mathfrak{g}(\mathfrak{h}(\Theta))(\varphi)(\alpha) = \varphi\mathfrak{h}(\Theta)^{-1}(\alpha) = \bigvee_{\beta\in E} \varphi(\beta) \wedge \mathfrak{h}(\Theta^{-1})(\beta,\alpha) = \bigvee_{\beta\in E} \varphi(\beta) \wedge \Theta(\delta_\beta e)(\alpha)$$
$$= \Theta\left(\bigvee_{\beta\in E} \varphi(\beta)\cdot\delta_\beta e\right)(\alpha) = \Theta(\varphi e)(\alpha) = \Theta(\varphi)(\alpha).$$

Hence $\mathfrak{g}\circ\mathfrak{h} = \mathrm{Id}_{G_U}$, and finally $G_e \cong G_U$.

We now prove that $G_U \cong G_{\mathscr{O}}$. Write $\Omega = \mathrm{Irr}(U) = {}^\flat\mathscr{O}$. Consider the maps

$$\begin{array}{rccc} \mathfrak{p}\colon & G_U & \longrightarrow & G_{\mathscr{O}} \\ & \Theta & \longmapsto & (\Theta|_{\mathscr{O}(b)})_{b\in B} \end{array} \tag{6.c}$$

and

$$\mathfrak{q}\colon\ G_{\mathscr{O}} \longrightarrow G_U$$

defined by

$$\mathfrak{q}(\theta)(\varphi) = \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} \theta(\omega) \tag{6.d}$$

For all $\Theta \in G_U$ and $\omega \in \Omega$, one has $\Theta(\omega) \in \Omega$ by Lemma 4.5. Then, by Lemma 4.7, there exist $\alpha \in E$ and $b \in B$ such that $b \leqslant e(\alpha,\alpha)$ and $\omega = \delta_\alpha^b e$. Write $g = \mathfrak{h}(\Theta) = \mathfrak{g}^{-1}(\Theta)$, so that $\Theta(\varphi) = \varphi g^{-1}$ for all $\varphi \in U$. By Lemma 4.13, one has $m(\omega) = b$. The same lemma ensures that

$$\begin{aligned} m(\Theta(\omega)) = m(\omega g^{-1}) &= \bigvee_{\beta\in E} \omega g^{-1}(\beta) = \bigvee_{\beta,\gamma\in E} \omega(\gamma) \wedge g^{-1}(\gamma,\beta) \\ &= \bigvee_{\beta,\gamma\in E} b \wedge e(\alpha,\gamma) \wedge g^{-1}(\gamma,\beta) = b \wedge \bigvee_{\beta\in E} eg^{-1}(\alpha,\beta) = b \wedge \bigvee_{\beta\in E} g^{-1}(\alpha,\beta). \end{aligned}$$

Now

$$b \leqslant e(\alpha,\alpha) = \bigvee_{\beta\in E} g^{-1}(\alpha,\beta) \wedge g(\beta,\alpha),$$

so since $b \in B$ and T is distributive, there exists $\zeta \in E$ such that

$$b \leqslant g^{-1}(\alpha,\zeta) \wedge g(\zeta,\alpha) \leqslant g^{-1}(\alpha,\zeta) \leqslant \bigvee_{\beta\in E} g^{-1}(\alpha,\beta).$$

So $m(\Theta(\omega)) = b = m(\omega)$, hence $\Theta(\mathscr{O}(b)) \subseteq \mathscr{O}(b)$. Since Θ is a T-linear automorphism, one has $\mathscr{O}_c^b \circ \Theta|_{\mathscr{O}(b)} = \Theta|_{\mathscr{O}(c)} \circ \mathscr{O}_c^b$ for all $b, c \in B$ such that $c \leqslant b$. Hence $\mathfrak{p}(\Theta) \in \mathbf{End}(\mathscr{O})$. Clearly $\mathfrak{p}(\mathrm{Id}_U) = \mathrm{Id}_{\mathscr{O}}$ and $\mathfrak{p}(\Theta \circ H) = \mathfrak{p}(\Theta) \circ \mathfrak{p}(H)$ for all $\Theta, H \in G_U$. So $\mathfrak{p}(\Theta)$ is invertible and $\mathfrak{p}(\Theta)^{-1} = \mathfrak{p}(\Theta^{-1}) \in \mathbf{End}(\mathscr{O})$. In particular, $\mathfrak{p}(\Theta) \in G_{\mathscr{O}}$.

Now for all $\theta \in G_{\mathscr{O}}$, $\varphi, \psi \in U$ and $t \in T$, one has

$$\mathfrak{q}(\theta)(\varphi \vee \psi) = \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi\vee\psi}} \theta(\omega) = \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} \theta(\omega) \vee \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\psi}} \theta(\omega) = \mathfrak{q}(\theta)(\varphi) \vee \mathfrak{q}(\theta)(\psi).$$

Moreover, one has

$$\begin{aligned} t \cdot \mathfrak{q}(\theta)(\varphi) &= \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} t \cdot \theta(\omega) = \bigvee_{\substack{\omega,\xi\in\Omega \\ \omega\leqslant\varphi \\ \xi\leqslant t\cdot\theta(\omega)}} \xi = \bigvee_{\substack{\omega,\xi\in\Omega \\ \omega\leqslant\varphi \\ \xi=t\cdot\xi \\ \xi\leqslant\theta(\omega)}} \xi = \bigvee_{\substack{\omega,\chi\in\Omega \\ \omega\leqslant\varphi \\ \theta(\chi)=t\cdot\theta(\chi) \\ \theta(\chi)\leqslant\theta(\omega)}} \theta(\chi) \\ &= \bigvee_{\substack{\omega,\chi\in\Omega \\ \omega\leqslant\varphi \\ \chi=t\cdot\chi \\ \chi\leqslant\omega}} \theta(\chi) = \bigvee_{\substack{\chi\in\Omega \\ \chi\leqslant\varphi \\ \chi=t\cdot\chi}} \theta(\chi) = \bigvee_{\substack{\chi\in\Omega \\ \chi\leqslant t\cdot\varphi}} \theta(\chi) = \mathfrak{q}(\theta)(t \cdot \varphi). \end{aligned}$$

The third and seventh equalities are consequences of Lemma 4.13. Moreover, the fourth equality holds because θ is bijective and the fifth one because θ and θ^{-1} are natural. Hence $\mathfrak{q}(\theta) \in \mathbf{End}_T(U)$. For all $\theta, \eta \in G_{\mathscr{O}}$ and $\varphi \in U$, one has similarly

$$\mathfrak{q}(\theta) \circ \mathfrak{q}(\eta)(\varphi) = \mathfrak{q}(\theta) \left(\bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} \eta(\omega) \right) = \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} \mathfrak{q}(\theta)(\eta(\omega)) = \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} \bigvee_{\substack{\xi\in\Omega \\ \xi\leqslant\eta(\omega)}} \theta(\xi)$$
$$= \bigvee_{\substack{\omega,\chi\in\Omega \\ \omega\leqslant\varphi \\ \eta(\chi)\leqslant\eta(\omega)}} \theta(\eta(\chi)) = \bigvee_{\substack{\chi\in\Omega \\ \chi\leqslant\varphi}} \theta \circ \eta(\chi) = \mathfrak{q}(\theta \circ \eta)(\varphi).$$

Hence, $\mathfrak{q}(\theta \circ \eta) = \mathfrak{q}(\theta) \circ \mathfrak{q}(\eta)$. Clearly $\mathfrak{q}(\mathrm{Id}_{\mathscr{O}}) = \mathrm{Id}_U$, so $\mathfrak{q}(\theta)$ is invertible and $\mathfrak{q}(\theta)^{-1} = \mathfrak{q}(\theta^{-1}) \in \mathbf{End}_T(U)$. In particular $\mathfrak{q}(\theta) \in G_U$.

We proved that $\mathfrak{p}$ and $\mathfrak{q}$ are well-defined. It remains to prove that they are inverse to each other. For all $\Theta \in G_U$ and $\omega \in \Omega$, one has $\mathfrak{p}(\Theta)(\omega) = \Theta(\omega)$. So for all $\varphi \in U$, one has

$$\mathfrak{q}(\mathfrak{p}(\Theta))(\varphi) = \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} \mathfrak{p}(\Theta)(\omega) = \bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} \Theta(\omega) = \Theta \left(\bigvee_{\substack{\omega\in\Omega \\ \omega\leqslant\varphi}} \omega \right) = \Theta(\varphi).$$

Hence $\mathfrak{q} \circ \mathfrak{p} = \mathrm{Id}_{G_U}$.

Conversely, for all $\theta \in G_{\mathscr{O}}$ and $b \in B$, one has $\mathfrak{q}(\theta)|_{\mathscr{O}(b)} = \theta|_{\mathscr{O}(b)}$. So $\mathfrak{p}(\mathfrak{q}(\theta)) = (\mathfrak{q}(\theta)|_{\mathscr{O}(b)})_{b\in B} = (\theta|_{\mathscr{O}(b)})_{b\in B} = \theta$. Hence $\mathfrak{p} \circ \mathfrak{q} = \mathrm{Id}_{G_{\mathscr{O}}}$, and finally $G_U \cong G_{\mathscr{O}}$. □

Let E be a finite set and $\mathfrak{S}(E)$ be the group of permutations of E. For all $\sigma \in \mathfrak{S}(E)$, we consider the T-generalized correspondence $\Delta_\sigma = \Delta_E \circ (\mathrm{Id}_E \times \sigma) \in \mathscr{C}_T(E, E)$. In other words,

$$\Delta_\sigma(\alpha, \beta) = \begin{cases} 1_T & \text{if } \alpha = \sigma(\beta) \\ 0_T & \text{otherwise.} \end{cases}$$

Remark 6.2. One can easily check that

$$\begin{array}{ccc} \mathfrak{S}(E) & \longrightarrow & \mathscr{C}_T(E, E) \\ \sigma & \longmapsto & \Delta_\sigma \end{array}$$

is a morphism of monoids, so that Δ_σ is invertible and $\Delta_\sigma^{-1} = \Delta_{\sigma^{-1}}$ for all $\sigma \in \mathfrak{S}(E)$.

If moreover $e \in \mathscr{C}_T(E, E)$ is idempotent, we write $\mathbf{Aut}(E, e) = \{\sigma \in \mathfrak{S}(E), e\Delta_\sigma = \Delta_\sigma e\}$. In other words, $\sigma \in \mathbf{Aut}(E, e)$ if and only if $e(\sigma(\alpha), \sigma(\beta)) = e(\alpha, \beta)$ for all $\alpha, \beta \in E$.

Lemma 6.3. *Let U be a regular T-module. We write $\Omega = \mathrm{Irr}(U)$ and $f = e(^\sharp\Omega) \in \mathscr{C}_T(\Omega,\Omega)$. Then $G_f = \{f\Delta_\sigma, \sigma \in \mathbf{Aut}(\Omega, f)\}$.*

Proof. Let $\sigma \in \mathbf{Aut}(\Omega, f)$. One has $f(f\Delta_\sigma)f = f^3\Delta_\sigma = f\Delta_\sigma$, since Δ_σ commutes with f and f is idempotent. So $f\Delta_\sigma \in f\mathscr{C}_T(\Omega,\Omega)f$. Clearly $\sigma^{-1} \in \mathbf{Aut}(\Omega, f)$, and similarly $f\Delta_{\sigma^{-1}} \in f\mathscr{C}_T(\Omega,\Omega)f$. Moreover, $(f\Delta_\sigma)(f\Delta_{\sigma^{-1}}) = f^2\Delta_\sigma\Delta_{\sigma^{-1}} = f$. Hence $f\Delta_\sigma \in G_f$ for all $\sigma \in \mathbf{Aut}(\Omega, f)$.

Conversely, let $g \in G_f$. Let $\Theta\colon \mathscr{C}_T(\Omega)f \to U$ be the isomorphism of Propositions 4.17 and 4.21. Then the conjugation Conj_Θ is an isomorphism between $G_{\mathscr{C}_T(\Omega)f}$ and G_U. Write $\Sigma = \mathrm{Conj}_\Theta \circ \mathfrak{g}(g)$, where $\mathfrak{g}\colon G_f \to G_{\mathscr{C}_T(\Omega)f}$ is the isomorphism defined by (6.a) in the proof of Proposition 6.1. So $\Sigma \in G_U$. By Lemma 4.5, $\Sigma(\Omega) \subseteq \Omega$, and in fact $\Sigma(\Omega) = \Omega$ because $|\Sigma(\Omega)| = |\Omega|$ since Σ is injective. So $\sigma = \Sigma|_\Omega^\Omega$ belongs to $\mathfrak{S}(\Omega)$.

Moreover, the fact that Σ is T-linear implies that for all $\omega, \xi \in \Omega$, one has

$$f(\sigma(\omega),\sigma(\xi)) = \bigvee_{\substack{b\leqslant m(\sigma(\xi))\\ b\cdot\sigma(\xi)\leqslant\sigma(\omega)}} b = \bigvee_{\substack{b\leqslant m(\xi)\\ b\cdot\xi\leqslant\omega}} b = f(\omega,\xi).$$

Hence $\sigma \in \mathbf{Aut}(\Omega, f)$.

Then $g = \mathfrak{g}^{-1} \circ \mathrm{Conj}_{\Theta^{-1}}(\Sigma) = \mathfrak{h} \circ \mathrm{Conj}_{\Theta^{-1}}(\Sigma)$, where $\mathfrak{h}$ is defined by (6.b) in the proof of Proposition 6.1. So for all $\omega, \xi \in \Omega$, one has $g(\omega,\xi) = (\Theta^{-1}\Sigma\Theta)^{-1}(\delta_\omega f)(\xi) = (\Theta^{-1}\Sigma^{-1}\Theta)(\delta_\omega f)(\xi)$. Now

$$\Theta(\delta_\omega f) = \bigvee_{\chi\in\Omega} \delta_\omega f(\chi)\cdot\chi = \bigvee_{\chi\in\Omega} f(\omega,\chi)\cdot\chi = \omega$$

thanks to Lemma 4.15. So $g(\omega,\xi) = \Theta^{-1}(\Sigma^{-1}(\omega))(\xi) = f(\sigma^{-1}(\omega),\xi)$. Hence $g = f\Delta_\sigma$. □

7. Parametrization of simple functors

Here we complete the study of Section 2, from which we know that any simple generalized correspondence functor is of the form $S_{E,V}$ where E is a finite set and V is a simple $k\mathscr{C}_T(E,E)$-module.

We need a more precise description of simple $k\mathscr{C}_T(E,E)$-modules. Let E be a finite set, let $e \in \mathscr{C}_T(E,E)$ be an idempotent and V be a simple kG_e-module. We write $M_e = e\mathscr{C}_T(E,E)e$ and we see V as a simple kM_e-module on which $M_e \smallsetminus G_e$ acts by zero. We then consider the $k\mathscr{C}_T(E,E)$-module

$$V_e^\dagger = (k\mathscr{C}_T(E,E)e \otimes_{kM_e} V)/W,$$

where

$$W = \left\{\sum_{i=1}^n \alpha_i \otimes v_i, \forall\beta \in e\mathscr{C}_T(E,E), \sum_{i=1}^n (\beta\alpha_i)\cdot v_i = 0\right\}.$$

We also write $\mathscr{S}_E$ for the class of pairs of type (e, V), where $e \in \mathscr{C}_T(E, E)$ is idempotent and V is a simple kG_e-module.

Then a description of simple $k\mathscr{C}_T(E, E)$-modules is given by the following theorem, which summarizes [10], where it is proved independently of the context of representations of categories. However, it can also be proved using the results of Section 2, as it is done in [14].

Theorem 7.1. *There exists an equivalence relation $\sim$ on $\mathscr{S}_E$ fulfilling the following properties.*

- *If $(e, V) \sim (f, W)$, then e and f are equivalent is the sense of Definition 4.10. Moreover, for all $x \in e\mathscr{C}_T(E, E)f$ and $y \in f\mathscr{C}_T(E, E)e$ such that $e = xy$ and $f = yx$, there exists a k-linear isomorphism $\Phi\colon V \to W$ such that $\Phi(g \cdot v) = (ygx) \cdot \Phi(v)$ for all $g \in G_e$ and $v \in V$.*
- *The map $[(e, V)] \mapsto [V_e^\dagger]$ is a bijection between the quotient set $\mathscr{S}_E/\!\sim$ and the set of isomorphism classes of simple $k\mathscr{C}_T(E, E)$-modules.*

Now let S be a simple generalized correspondence functor and let E be a finite set such that $S(E) \neq 0$. Proposition 2.5 ensures that $S(E)$ is a simple $k\mathscr{C}_T(E, E)$-module. Then Theorem 7.1 ensures that there exist an idempotent $e \in \mathscr{C}_T(E, E)$ and a simple kG_e-module V such that $S(E) \cong V_e^\dagger$. We write $\Omega = \mathrm{Irr}(\mathscr{C}_T(E)e)$ and $f = e(^\sharp\Omega)$. By Proposition 4.21, one has $\mathscr{C}_T(E)e \cong \mathscr{C}_T(\Omega)f$. Then Proposition 4.12 ensures that e and f are equivalent. Hence there exist $x \in e\mathscr{C}_T(E, \Omega)f$ and $y \in f\mathscr{C}_T(\Omega, E)e$ such that $e = xy$ and $f = yx$. Then

$$\begin{array}{ccc} G_f & \longrightarrow & G_e \\ g & \longmapsto & xgy \end{array}$$

is an isomorphism. In particular, this isomorphism makes G_f act on V, which thus can be seen as a simple kG_f-module. Then we have a simple $k\mathscr{C}_T(\Omega, \Omega)$-module $V_f^\dagger$.

Lemma 7.2. *One has $S \cong S_{\Omega, V_f^\dagger}$.*

Proof. By Proposition 2.5, it is enough to prove that $S(\Omega) \cong V_f^\dagger$. But we already know that $S \cong S_{E, V_e^\dagger}$, so that $S(\Omega) \cong S_{E, V_e^\dagger}(\Omega)$. We are going to prove that $S_{E, V_e^\dagger}(\Omega) \cong V_f^\dagger$. Square brackets denote equivalence classes modulo the relevant submodule.

Fix $\alpha \in k\mathscr{C}_T(\Omega, E)$. Consider

$$\begin{array}{cccc} \Phi_\alpha\colon & V_e^\dagger & \longrightarrow & V_f^\dagger \\ & [\gamma \otimes v] & \longmapsto & [\alpha\gamma x \otimes v]\,, \end{array}$$

with $\gamma \in k\mathscr{C}_T(E, E)e$ and $v \in V$. One can check that it is a well-defined k-linear map. Similarly,

$$\begin{array}{rccc} \Phi\colon & S_{E,V_e^\dagger}(\Omega) & \longrightarrow & V_f^\dagger \\ & [\alpha \otimes z] & \longmapsto & \Phi_\alpha(z), \end{array}$$

with $\alpha \in k\mathscr{C}_T(\Omega, E)$ and $z \in V_e^\dagger$, is a well-defined $k\mathscr{C}_T(\Omega,\Omega)$-linear map.

Consider

$$\begin{array}{rccc} \Psi\colon & V_f^\dagger & \longrightarrow & S_{E,V_e^\dagger}(\Omega) \\ & [\beta \otimes v] & \longmapsto & [\beta y \otimes [e \otimes v]] \,, \end{array}$$

with $\beta \in k\mathscr{C}_T(\Omega,\Omega)f$ and $v \in V$. It is also well-defined and $k\mathscr{C}_T(\Omega,\Omega)$-linear.

Then a straightforward computation ensures that both $\Phi\circ\Psi$ and $\Psi\circ\Phi$ are the identity. This proves the lemma. $\square$

Let $\mathscr{O}$ be a presheaf of finite posets on B and let V be a simple $kG_{\mathscr{O}}$-module. We write $\Omega = {}^\flat\mathscr{O}$ and $f = e(\mathscr{O})$. Proposition 6.1 ensures that $G_{\mathscr{O}} \cong G_f$, so that V can be seen as a simple kG_f-module. We then write $S_{\mathscr{O},V} = S_{\Omega,V_f^\dagger}$.

Corollary 7.3. *Any simple functor is of type $S_{\mathscr{O},V}$, where $\mathscr{O}$ is a presheaf of finite posets on B and V a simple $kG_{\mathscr{O}}$-module.*

Let $\mathscr{O}$ and $\mathscr{P}$ be two presheaves of finite posets on B, let V be a simple $kG_{\mathscr{O}}$-module and let W be a simple $kG_{\mathscr{P}}$-module. The pairs $(\mathscr{O},V)$ and $(\mathscr{P},W)$ are called *equivalent*, which we write $(\mathscr{O},V) \simeq (\mathscr{P},W)$, if there exist an isomorphism $\theta\colon \mathscr{O} \to \mathscr{P}$ and a k-linear isomorphism $\Phi\colon V \to W$ such that $\Phi(g\cdot v) = (\theta g \theta^{-1})\cdot\Phi(v)$ for all $g \in G_{\mathscr{O}}$ and $v \in V$.

Proposition 7.4. *Let $\mathscr{O}$ and $\mathscr{P}$ be two presheaves of finite posets on B, let V be a simple $kG_{\mathscr{O}}$-module and let W be a simple $kG_{\mathscr{P}}$-module. If $S_{\mathscr{O},V} \cong S_{\mathscr{P},W}$, then $(\mathscr{O},V) \simeq (\mathscr{P},W)$.*

Proof. Write $\Omega = {}^\flat\mathscr{O}$ and $f = e(\mathscr{O})$ as well as $\Xi = {}^\flat\mathscr{P}$ and $e = e(\mathscr{P})$. Then, one has $S_{\mathscr{O},V} = S_{\Omega,V_f^\dagger}$ and $S_{\mathscr{P},W} = S_{\Xi,W_e^\dagger}$. Let $\zeta\colon S_{\Omega,V_f^\dagger} \to S_{\Xi,W_e^\dagger}$ be an isomorphism.

We first prove that the idempotents e and f are equivalent.

Fix $v \in V \smallsetminus \{0\}$. Since V is a simple kG_f-module, v generates V both as a kG_f-module and as a $kf\mathscr{C}_T(\Omega,\Omega)f$-module. Then $v^\dagger = [f \otimes v]$ generates $V_f^\dagger$ as a $k\mathscr{C}_T(\Omega,\Omega)$-module. Since moreover $f\cdot v^\dagger = v^\dagger$, the map

$$\begin{array}{rccc} \varphi_X\colon & k\mathscr{C}_T(X,\Omega)f & \longrightarrow & S_{\Omega,V_f^\dagger}(X) \\ & \gamma & \longmapsto & \left[\gamma \otimes v^\dagger\right] \end{array}$$

is a surjection for any finite set X. The definition of $S_{\Omega,V_f^\dagger}$ immediately implies that $\varphi\colon k\mathscr{C}_T(-,\Omega) \to S_{\Omega,V_f^\dagger}$ is a natural transformation. Similarly, fixing $w \in W \smallsetminus \{0\}$ gives rise to a surjective morphism $\psi\colon k\mathscr{C}_T(-,\Xi)e \to S_{\Xi,W_e^\dagger}$.

The functor $k\mathscr{C}_T(-,\Omega)f$ is projective as a direct summand of a representable functor. Hence there exists a morphism $\alpha\colon k\mathscr{C}_T(-,\Omega)f \to k\mathscr{C}_T(-,\Xi)e$ such that $\psi\alpha = \zeta\varphi$. Similarly, there exists a morphism $\beta\colon k\mathscr{C}_T(-,\Xi)e \to k\mathscr{C}_T(-,\Omega)f$ such that $\varphi\beta = \zeta^{-1}\psi$.

Write $a = \alpha_\Omega(f) \in k\mathscr{C}_T(\Omega,\Xi)e$ and $b = \beta_\Xi(e) \in k\mathscr{C}_T(\Xi,\Omega)f$. Then

$$\begin{aligned}\left[f\otimes v^\dagger\right] &= \varphi_\Omega(f) = \zeta_\Omega^{-1}\zeta_\Omega\varphi_\Omega(f) = \zeta_\Omega^{-1}\psi_\Omega\alpha_\Omega(f) = \zeta_\Omega^{-1}\psi_\Omega(a) = \zeta_\Omega^{-1}\left(\left[a\otimes w^\dagger\right]\right)\\ &= \zeta_\Omega^{-1}S_{\Xi,W_e^\dagger}(a)\left(\left[e\otimes w^\dagger\right]\right) = S_{\Omega,V_f^\dagger}(a)\zeta_\Xi^{-1}\left(\left[e\otimes w^\dagger\right]\right) = S_{\Omega,V_f^\dagger}(a)\zeta_\Xi^{-1}\psi_\Xi(e)\\ &= S_{\Omega,V_f^\dagger}(a)\varphi_\Xi\beta_\Xi(e) = S_{\Omega,V_f^\dagger}(a)\varphi_\Xi(b) = S_{\Omega,V_f^\dagger}(a)\left(\left[b\otimes v^\dagger\right]\right) = \left[ab\otimes v^\dagger\right].\end{aligned}$$

This equality occurs in $S_{\Omega,V_f^\dagger}(\Omega)$, which is isomorphic to $V_f^\dagger$. Transported in $V_f^\dagger$, this equality becomes $v^\dagger = f\cdot v^\dagger = (ab)\cdot v^\dagger$. This implies that $v = (fab)\cdot v$. Since $M_f \smallsetminus G_f$ acts by 0 on V, one has $fab \notin k(M_f \smallsetminus G_f)$. Decomposing a and b as k-linear combinations of T-generalized correspondences leads to the existence of $c \in \mathscr{C}_T(\Omega,\Xi)e$ and $d \in \mathscr{C}_T(\Xi,\Omega)f$ such that $fcd \in G_f$. Let m be the inverse of fcd in G_f, so that $fcdm = f$. Then one has $f = fceedm$ because $c \in \mathscr{C}_T(\Omega,\Xi)e$. Obviously $fce \in f\mathscr{C}_T(\Omega,\Xi)e$ and $ecm \in e\mathscr{C}_T(\Xi,\Omega)f$. We can decompose e in the same way by a similar argument. Hence Lemma 4.11 implies that f and e are equivalent.

So there exist $x \in f\mathscr{C}_T(\Omega,\Xi)e$ and $y \in e\mathscr{C}_T(\Xi,\Omega)f$ such that $f = xy$ and $e = yx$. We are now going to build an isomorphism $\theta\colon \mathscr{O} \to \mathscr{P}$ thanks to x and y. Proposition 4.12 ensures that $\mathscr{C}_T(\Omega)f \cong \mathscr{C}_T(\Xi)e$. Its proof gives explicit inverse T-linear isomorphisms

$$\begin{array}{ccc}\mathscr{C}_T(\Omega)f & \longleftrightarrow & \mathscr{C}_T(\Xi)e\\ \sigma & \longmapsto & \sigma x\\ \tau y & \longleftarrow\!\shortmid & \tau.\end{array}$$

Remark 4.24 gives inverse isomorphisms of posets

$$\begin{array}{ccc}\mathrm{Irr}(\mathscr{C}_T(\Omega)f) & \longleftrightarrow & \Omega\\ \delta_\omega f & \longleftrightarrow & \omega.\end{array}$$

There exist similar inverse isomorphisms of posets between $\mathrm{Irr}(\mathscr{C}_T(\Xi)e)$ and Ξ.

They give rise to isomorphisms of presheaves $o\colon {}^\sharp\,\mathrm{Irr}(\mathscr{C}_T(\Omega)f) \to {}^\sharp\Omega = \mathscr{O}$ and $\pi\colon {}^\sharp\,\mathrm{Irr}(\mathscr{C}_T(\Xi)e) \to {}^\sharp\Xi = \mathscr{P}$. Now, the T-linear isomorphism $\mathscr{C}_T(\Omega)f \cong \mathscr{C}_T(\Xi)e$ also gives rise to an isomorphism of presheaves $\rho\colon {}^\sharp\,\mathrm{Irr}(\mathscr{C}_T(\Omega)f) \to {}^\sharp\,\mathrm{Irr}(\mathscr{C}_T(\Xi)e)$. We then write $\theta = \pi\rho o^{-1}\colon \mathscr{O} \to \mathscr{P}$.

As we did in the proof of Lemma 5.10, we write again $\theta\colon \Omega \to \Xi$ for the induced isomorphism of posets. In other words, one has $\theta(\omega) = \pi(o^{-1}(\omega)x) = \pi(\delta_\omega x)$ for all $\omega \in \Omega$. Now fix $\omega \in \Omega$. We know that $\delta_\omega x = \rho o^{-1}(\omega) \in \mathrm{Irr}(\mathscr{C}_T(\Xi)e)$. Hence, there exists $\xi \in \Xi$ such that $\delta_\omega x = \delta_\xi e$. Then $\theta(\omega) = \pi(\delta_\omega x) = \pi(\delta_\xi e) = \xi$, that is, $\delta_\omega x = \delta_\xi e = \delta_{\theta(\omega)}e$. Similarly, for all $\xi \in \Xi$, one has $\delta_\xi y = \delta_{\theta^{-1}(\xi)}f$.

We finally prove that there exist an isomorphism $V \to W$ compatible with the actions of $G_{\mathscr{O}}$ and $G_{\mathscr{P}}$.

Write $e^\diamond = e\circ(\theta\times\theta) \in \mathscr{C}_T(\Omega,\Omega)$, $x^\diamond = x\circ(\mathrm{Id}_\Omega\times\theta) \in \mathscr{C}_T(\Omega,\Omega)$ and $y^\diamond = y\circ(\theta\times\mathrm{Id}_\Omega) \in \mathscr{C}_T(\Omega,\Omega)$. It is easy to check that $(e^\diamond)^2 = e^\diamond$, $x^\diamond = fx^\diamond e^\diamond$, $y^\diamond = e^\diamond y^\diamond f$, $x^\diamond y^\diamond = f$ and $y^\diamond x^\diamond = e$. Then

$$\begin{array}{ccc} G_{e^\diamond} & \longrightarrow & G_e \\ g & \longmapsto & g\circ(\theta^{-1}\times\theta^{-1}) \end{array}$$

is a group isomorphism. In particular, W can be seen as a $kG_{e^\diamond}$-module.

Moreover, the functor $S_{\Xi,W_e^\dagger}$ is isomorphic to $S_{\Omega,W_{e^\diamond}^\dagger}$, *via*

$$\begin{array}{rccc} \varepsilon_X\colon & S_{\Xi,W_e^\dagger}(X) & \longrightarrow & S_{\Omega,W_{e^\diamond}^\dagger}(X) \\ & [\alpha\otimes[\beta\otimes w]] & \longmapsto & [\alpha^\diamond\otimes[\beta^\diamond\otimes w]], \end{array}$$

where $\alpha \in k\mathscr{C}_T(X,\Xi)$ and $\beta \in k\mathscr{C}_T(\Xi,\Xi)e$. Here $\alpha^\diamond$ and $\beta^\diamond$ denote the images of α and β by the k-linearizations of the maps

$$\begin{array}{ccc} \mathscr{C}_T(X,\Xi) & \longrightarrow & k\mathscr{C}_T(X,\Omega) \\ \gamma & \longmapsto & \gamma\circ(\mathrm{Id}_X\times\theta) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathscr{C}_T(\Xi,\Xi)e & \longrightarrow & k\mathscr{C}_T(\Omega,\Omega)e^\diamond \\ \gamma & \longmapsto & \gamma\circ(\theta\times\theta) \end{array}$$

respectively. Then

$$\varepsilon_\Omega\zeta_\Omega\colon S_{\Omega,V_f^\dagger}(\Omega)\to S_{\Omega,W_{e^\diamond}^\dagger}(\Omega)$$

is an isomorphism. So $V_f^\dagger \cong W_{e^\diamond}^\dagger$. Theorem 7.1 then ensures that $(f,V)\sim(e^\diamond,W)$. It also ensures that there exists a k-linear isomorphism $\Phi\colon V\to W$ such that $\Phi(g\cdot v) = (y^\diamond g x^\diamond)\cdot\Phi(v)$ for all $g\in G_f$ and $v\in V$.

Now the structure of kG_f-module of V is induced by the inverse of the isomorphism of groups

$$\begin{array}{rccc} \mathfrak{i}\colon & G_{\mathscr{O}} & \longrightarrow & G_f \\ & g & \longmapsto & \mathfrak{h}\mathfrak{q}\,\mathrm{Conj}_{o^{-1}}(g), \end{array}$$

where $\mathfrak{h}\colon G_{\mathscr{C}_T(\Omega)f}\to G_f$ and $\mathfrak{q}\colon G_{\sharp\mathrm{Irr}(\mathscr{C}_T(\Omega)f)}\to G_{\mathscr{C}_T(\Omega)f}$ are the isomorphisms defined respectively by (6.b) and (6.d) in the proof of Proposition 6.1. Similarly, the structure of $kG_{e^\diamond}$-module of W is induced by the isomorphism of groups

$$\begin{array}{rccc} \mathfrak{j}\colon & G_{e^\diamond} & \longrightarrow & G_{\mathscr{P}} \\ & g & \longmapsto & \mathrm{Conj}_\pi\,\mathfrak{p}\mathfrak{g}(g\circ(\theta^{-1}\times\theta^{-1})), \end{array}$$

where $\mathfrak{p}\colon G_{\mathscr{C}_T(\Xi)e}\to G_{\sharp\mathrm{Irr}(\mathscr{C}_T(\Xi)e)}$ and $\mathfrak{g}\colon G_e\to G_{\mathscr{C}_T(\Xi)e}$ are the isomorphisms defined respectively by (6.c) and (6.a) in the proof of Proposition 6.1.

So for all $g\in G_{\mathscr{O}}$ and $v\in V$, one has $\Phi(g\cdot v) = \Phi(\mathfrak{i}(g)\cdot v) = (y^\diamond\mathfrak{i}(g)x^\diamond)\cdot\Phi(v) = \mathfrak{j}(y^\diamond\mathfrak{i}(g)x^\diamond)\cdot\Phi(v)$.

Fix $g \in G_{\mathscr{O}}$, and let us compute $\mathfrak{j}(y^\diamond \mathfrak{i}(g) x^\diamond)$. Write $h = y^\diamond \mathfrak{i}(g) x^\diamond$, so that $\mathfrak{j}(y^\diamond \mathfrak{i}(g) x^\diamond) = \mathfrak{j}(h)$. The latter is an element of $G_{\mathscr{P}}$. For all $\xi \in \Xi = {}^\flat\mathscr{P}$, one has

$$\begin{aligned}
\mathfrak{j}(h)(\xi) &= \pi\left(\mathfrak{pg}(h \circ (\theta^{-1} \times \theta^{-1}))\left(\pi^{-1}(\xi)\right)\right) \\
&= \pi\left(\mathfrak{pg}(h \circ (\theta^{-1} \times \theta^{-1}))\left(\delta_\xi e\right)\right) \\
&= \pi\left(\delta_\xi e\left(h \circ (\theta^{-1} \times \theta^{-1})\right)^{-1}\right),
\end{aligned}$$

using (6.c) and (6.a).

Now $(h \circ (\theta^{-1} \times \theta^{-1}))^{-1} = h^{-1} \circ (\theta^{-1} \times \theta^{-1}) = (y^\diamond \mathfrak{i}(g^{-1}) x^\diamond) \circ (\theta^{-1} \times \theta^{-1})$. This is an element of $\mathscr{C}_T(\Xi, \Xi)$, mapping the element $(\beta, \chi) \in \Xi \times \Xi$ to

$$\begin{aligned}
&\bigvee_{\gamma,\lambda \in \Omega} y^\diamond(\theta^{-1}(\beta), \gamma) \wedge \mathfrak{i}(g^{-1})(\gamma, \lambda) \wedge x^\diamond(\lambda, \theta^{-1}(\chi)) \\
&\quad = \bigvee_{\gamma,\lambda \in \Omega} y(\beta, \gamma) \wedge \mathfrak{q}\operatorname{Conj}_{o^{-1}}(g)(\delta_\gamma f)(\lambda) \wedge x(\lambda, \chi) \\
&\quad = \bigvee_{\lambda \in \Omega} \mathfrak{q}\operatorname{Conj}_{o^{-1}}(g)\left(\bigvee_{\gamma \in \Omega} y(\beta, \gamma) \cdot \delta_\gamma f\right)(\lambda) \wedge x(\lambda, \chi) \\
&\quad = \bigvee_{\lambda \in \Omega} \mathfrak{q}\operatorname{Conj}_{o^{-1}}(g)(\delta_\beta y)(\lambda) \wedge x(\lambda, \chi).
\end{aligned}$$

The first equality above uses the definitions of $y^\diamond$ and $x^\diamond$, and (6.a). The second equality uses the T-linearity of $\mathfrak{q}\operatorname{Conj}_{o^{-1}}(g) \in G_{\mathscr{C}_T(\Omega)f}$, and the third equality uses Lemma 4.8.

Consequently, $\delta_\xi e\left(h \circ (\theta^{-1} \times \theta^{-1})\right)^{-1}$ is the element of $\mathscr{C}_T(\Xi)$ mapping $\chi \in \Xi$ to

$$\begin{aligned}
&\bigvee_{\alpha,\beta \in \Xi} \delta_\xi(\alpha) \wedge e(\alpha, \beta) \wedge (h \circ (\theta^{-1} \times \theta^{-1}))^{-1}(\beta, \chi) \\
&\quad = \bigvee_{\substack{\beta \in \Xi \\ \lambda \in \Omega}} e(\xi, \beta) \wedge \mathfrak{q}\operatorname{Conj}_{o^{-1}}(g)(\delta_\beta y)(\lambda) \wedge x(\lambda, \chi) \\
&\quad = \bigvee_{\lambda \in \Omega} \mathfrak{q}\operatorname{Conj}_{o^{-1}}(g)\left(\bigvee_{\beta \in \Xi} e(\xi, \beta) \cdot \delta_\beta y\right)(\lambda) \wedge x(\lambda, \chi) \\
&\quad = \bigvee_{\lambda \in \Omega} \mathfrak{q}\operatorname{Conj}_{o^{-1}}(g)\,(\delta_\xi y)\,(\lambda) \wedge x(\lambda, \chi) \\
&\quad = \mathfrak{q}\operatorname{Conj}_{o^{-1}}(g)\,(\delta_\xi y)\, x.
\end{aligned}$$

The third equality above uses an argument similar, but not strictly identical, to the one of Lemma 4.8.

Now

$$\begin{aligned}\mathfrak{q}\,\mathrm{Conj}_{o^{-1}}(g)\,(\delta_\xi y) &= \mathfrak{q}(o^{-1}go)\,(\delta_\xi y) = \mathfrak{q}(o^{-1}go)\left(\delta_{\theta^{-1}(\xi)} f\right)\\ &= o^{-1}go\left(\delta_{\theta^{-1}(\xi)} f\right) = o^{-1}g\left(\theta^{-1}(\xi)\right) = o^{-1}g\theta^{-1}(\xi).\end{aligned}$$

Here the third equality uses (6.d).

Finally, we are left with $\mathfrak{j}(y^\diamond \mathfrak{i}(g)x^\diamond)(\xi) = \pi\left(o^{-1}g\theta^{-1}(\xi)x\right) = \pi\rho o^{-1}g\theta^{-1}(\xi) = \theta g\theta^{-1}(\xi)$. In other words, $\mathfrak{j}(y^\diamond \mathfrak{i}(g)x^\diamond) = \theta g\theta^{-1}$. Hence $\Phi(g\cdot v) = (\theta g\theta^{-1})\cdot\Phi(v)$ for all $g \in G_{\mathscr{O}}$ and $v \in V$. Finally, $(\mathscr{O}, V) \simeq (\mathscr{P}, W)$, as was to be proved. $\square$

Corollary 7.3 and Proposition 7.4 imply the following theorem.

Theorem 7.5. *The set of equivalence classes of pairs $(\mathscr{O}, V)$, consisting of a presheaf $\mathscr{O}$ of finite posets on B and a simple module V for its group of automorphisms, is in bijection with the set of isomorphism classes of simple generalized correspondence functors, via the map $[(\mathscr{O}, V)] \mapsto [S_{\mathscr{O},V}]$.*

Remark 7.6. In [6, Theorem 4.7], the simple correspondence functors are parametrized by triples (E, R, V), where E is a finite set, R is an order on E and V a simple $k\,\mathbf{Aut}(E, R)$-module. The finite poset (E, R) can be viewed as a presheaf of finite posets over $\{1\} = \mathrm{Irr}(\{0, 1\})$. Hence, our Theorem 7.5 generalizes this result.

However, an additional property is proved in [6]: in the triple (E, R, V) which corresponds to a simple correspondence functor S, the set E is minimal among sets fulfilling $S(E) \neq 0$.

This property does not necessarily hold here. We always have $S_{\mathscr{O},V}({}^\flat\mathscr{O}) \neq 0$, but for any subset E of ${}^\flat\mathscr{O}$ generating the T-module $I_\downarrow({}^\flat\mathscr{O})$, one also has $S_{\mathscr{O},V}(E) \neq 0$. In general, there exist several proper subsets of ${}^\flat\mathscr{O}$ generating the T-module $I_\downarrow({}^\flat\mathscr{O})$. See [14] for details.

8. Finiteness conditions: the case of a field

In all this section, we assume that k is a field. The first part of this section is devoted to give upper and lower bounds for the dimension of the evaluations of simple functors. The results of this section generalize those of [6, Sections 8 and 9]. Several proofs are straightforward extensions of those of [6].

Let $\mathscr{O}$ be a presheaf of finite posets on B. We write $\Omega = {}^\flat\mathscr{O}$ and $f = e(\mathscr{O})$. For any finite set X and for any map $\varphi\colon X \to \Omega$, we define two T-generalized correspondences Λ_φ and Γ_φ in the following way. For all $\omega \in \Omega$ and $x \in X$, we set $\Lambda_\varphi(\omega, x) = f(\omega, \varphi(x))$ and $\Gamma_\varphi(x, \omega) = f(\varphi(x), \omega)$, so that $\Lambda_\varphi \in \mathscr{C}_T(\Omega, X)$ and $\Gamma_\varphi \in \mathscr{C}_T(X, \Omega)$.

For any finite set X, we fix a map $\theta_X : X \to B$ such that for all $b \in B$, one has

$$|\theta_X^{-1}(b)| \geqslant \left\lfloor \frac{|X|}{|B|} \right\rfloor,$$

where the corner brackets denote the floor function. In particular,

$$\lim_{|X|\to+\infty} |\theta_X^{-1}(b)| = +\infty$$

for all $b \in B$. We write Φ_X for the set of surjective maps $\varphi\colon X \to \Omega$ such that $m\circ\varphi = \theta_X$.

We recall from Section 6 that $\mathbf{Aut}(\Omega, f)$ denotes the group of permutations $\sigma \in \mathfrak{S}(\Omega)$ such that $f\Delta_\sigma = \Delta_\sigma f$, that is, $f(\sigma(\omega), \sigma(\xi)) = f(\omega, \xi)$ for all $\omega, \xi \in \Omega$. This group acts on Φ_X by composition on the left. Indeed, let $\sigma \in \mathbf{Aut}(\Omega, f)$. For any $\omega \in \Omega$, one has $m(\sigma(\omega)) = f(\sigma(\omega), \sigma(\omega)) = f(\omega, \omega) = m(\omega)$, the first and third equalities being a consequence of Lemma 4.14 and the second one coming from the fact that $\sigma \in \mathbf{Aut}(\Omega, f)$. Hence, for all $\varphi \in \Phi_X$, one has $m \circ \sigma \circ \varphi = m \circ \varphi = \theta_X$, so that $\sigma \circ \varphi \in \Phi_X$.

We fix a set $A_X \subseteq \Phi_X$ of representatives of the set of left orbits $\mathbf{Aut}(\Omega, f)\backslash\Phi_X$.

Lemma 8.1. *Let $\varphi, \psi \in A_X$. If $\Lambda_\psi\Gamma_\varphi = f$, then $\varphi \leqslant \psi$, that is, $\varphi(x) \leqslant \psi(x)$ for all $x \in X$.*

Proof. Let $x \in X$ and $\omega \in \Omega$. Lemmas 4.14 and 4.20 ensure that $f(\xi, \omega) \leqslant m(\xi)$ and $f(\omega, \omega) = m(\omega)$ for all $\omega, \xi \in \Omega$. Then

$$\begin{aligned} f(\varphi(x), \omega) &= m(\varphi(x)) \wedge f(\varphi(x), \omega) = \theta_X(x) \wedge f(\varphi(x), \omega) = m(\psi(x)) \wedge f(\varphi(x), \omega) \\ &= f(\psi(x), \psi(x)) \wedge f(\varphi(x), \omega) \leqslant \bigvee_{y\in X} f(\psi(x), \psi(y)) \wedge f(\varphi(y), \omega) \\ &= \Lambda_\psi\Gamma_\varphi(\psi(x), \omega) = f(\psi(x), \omega). \end{aligned}$$

So Lemma 4.15 implies that

$$\varphi(x) = \bigvee_{\omega\in\Omega} f(\varphi(x), \omega) \cdot \omega \leqslant \bigvee_{\omega\in\Omega} f(\psi(x), \omega) \cdot \omega = \psi(x).$$

Hence $\varphi \leqslant \psi$. $\square$

For all $\varphi, \psi \in A_X$, we write $\varphi \preccurlyeq \psi$ if there exists $\sigma \in \mathbf{Aut}(\Omega, f)$ such that $\sigma \circ \varphi \leqslant \psi$.

Lemma 8.2. *The relation $\preccurlyeq$ is a partial order on A_X.*

Proof. It is obvious that $\preccurlyeq$ is reflexive, since $\mathrm{Id}_\Omega \in \mathbf{Aut}(\Omega, f)$, so we prove that it is transitive and antisymmetric.

Let $\varphi, \psi, \chi \in A_X$ be such that $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \chi$. There exist $\sigma, \tau \in \mathbf{Aut}(\Omega, f)$ such that $\sigma \circ \varphi \leqslant \psi$ and $\tau \circ \psi \leqslant \chi$. Now τ is order preserving. Indeed, for any $\omega, \xi \in \Omega$, the following equivalences hold:

$$\xi \leqslant \omega \iff f(\omega, \xi) = m(\xi) \iff f(\tau(\omega), \tau(\xi)) = m(\tau(\xi)) \iff \tau(\xi) \leqslant \tau(\omega).$$

The first and third equivalences come from Lemma 4.14 while the second one is a consequence of the fact that $\tau \in \mathbf{Aut}(\Omega, f)$, the equality $m \circ \tau = m$ having been proved just before Lemma 8.1. So $\tau \circ \sigma \circ \varphi \leqslant \chi$, whence $\varphi \preccurlyeq \chi$.

Finally, let $\varphi, \psi \in A_X$ be such that $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \varphi$. There exist $\sigma, \tau \in \mathbf{Aut}(\Omega, f)$ such that $\sigma \circ \varphi \leqslant \psi$ and $\tau \circ \psi \leqslant \varphi$. The elements of $\mathbf{Aut}(\Omega, f)$ being order-preserving maps, one has $\tau\sigma \circ \varphi \leqslant \varphi$. Let n be the order of $\tau\sigma$ in the group $\mathbf{Aut}(\Omega, f)$. Then $\varphi = (\tau\sigma)^n \circ \varphi \leqslant \tau\sigma \circ \varphi \leqslant \varphi$. So $\varphi = \tau\sigma \circ \varphi \leqslant \tau \circ \psi \leqslant \varphi$, whence $\varphi = \tau \circ \psi$. This implies that $\varphi = \psi$ since A_X is a set of representatives of $\mathbf{Aut}(\Omega, f)\backslash\Phi_X$. $\square$

Let V be a simple $kG_{\mathscr{O}}$-module. We are interested in the dimensions of the evaluations of the simple functor $S_{\mathscr{O},V}$.

Theorem 8.3. *There exist an integer $r \geqslant 0$ and a constant $c > 0$ such that for any finite set X with $|X| \geqslant r$, one has*

$$c\left(\left(\frac{|\Omega|}{|B|}\right)^{1/|B|}\right)^{|X|} \leqslant \dim_k S_{\mathscr{O},V}(X) \leqslant (T^{|\Omega|})^{|X|}.$$

Proof. The functor $S_{\mathscr{O},V} = S_{\Omega,V_f^\dagger}$ is a quotient of $L_{\Omega,V_f^\dagger}$, which is itself a quotient of $k\mathscr{C}_T(-,\Omega)$, since $V_f^\dagger$ is a simple $k\mathscr{C}_T(\Omega,\Omega)$-module. Then for any finite set X, one has $\dim_k S_{\mathscr{O},V}(X) \leqslant \dim_k k\mathscr{C}_T(X,\Omega) = |T|^{|X||\Omega|} = (T^{|\Omega|})^{|X|}$.

Now fix $v \in V \smallsetminus \{0\}$. Then v generates the simple kG_f-module V, and $v^\dagger = [f \otimes v] \in V_f^\dagger$ generates $V_f^\dagger$.

Let X be a finite set. We are going to prove that the image of the family $(\Gamma_\varphi \otimes v^\dagger)_{\varphi \in A_X}$ in $S_{\mathscr{O},V}(X)$ is linearly independent. Assume that the family of scalars $(\lambda_\varphi)_{\varphi \in A_X} \in k^{A_X}$ is such that

$$\sum_{\varphi \in A_X} \lambda_\varphi \left[\Gamma_\varphi \otimes v^\dagger\right] = 0$$

in $S_{\mathscr{O},V}(X)$. It means that

$$\sum_{\varphi \in A_X} \lambda_\varphi \Gamma_\varphi \otimes v^\dagger \in J_{\Omega,V_f^\dagger}(X).$$

So for all $\beta \in \mathscr{C}_T(\Omega, X)$, one has

$$\sum_{\varphi \in A_X} \lambda_\varphi (\beta\Gamma_\varphi) \cdot v^\dagger = 0.$$

Now the action of an element $\alpha \in \mathscr{C}_T(\Omega,\Omega)$, on $v^\dagger$ is given by $\alpha \cdot v^\dagger = [\alpha f \otimes v]$. So the previous condition implies that for all $\beta \in \mathscr{C}_T(\Omega, X)$, one has

$$\left[\sum_{\varphi \in A_X} \lambda_\varphi(\beta\Gamma_\varphi f) \otimes v\right] = 0$$

in $V_f^\dagger$. One has $\Gamma_\varphi f = \Gamma_\varphi$ for all $\varphi \in A_X$: indeed, for all $x \in X$ and $\omega \in \Omega$, one has

$$\Gamma_\varphi f(x,\omega) = \bigvee_{\xi \in \Omega} f(\varphi(x),\xi) \wedge f(\xi,\omega) = f^2(\varphi(x),\omega) = f(\varphi(x),\omega) = \Gamma_\varphi(x,\omega).$$

Then, by definition of $V_f^\dagger$ one has, for all $\beta \in f\mathscr{C}_T(\Omega, X)$,

$$\sum_{\varphi \in A_X} \lambda_\varphi(\beta\Gamma_\varphi) \cdot v = 0.$$

Non-invertible elements of $f\mathscr{C}_T(\Omega,\Omega)f$ act by zero on V, so for all $\beta \in f\mathscr{C}_T(\Omega, X)$, one has

$$\sum_{\substack{\varphi \in A_X \\ \beta\Gamma_\varphi \in G_f}} \lambda_\varphi(\beta\Gamma_\varphi) \cdot v = 0.$$

Now choose $\beta = \Lambda_\psi$ with $\psi \in A_X$. Then for all $\psi \in A_X$, one has

$$\sum_{\substack{\varphi \in A_X \\ \Lambda_\psi\Gamma_\varphi \in G_f}} \lambda_\varphi(\Lambda_\psi\Gamma_\varphi) \cdot v = 0.$$

Lemma 6.3 implies that we can rewrite this equation as

$$\sum_{\substack{\varphi \in A_X \\ \sigma \in \mathbf{Aut}(\Omega,f) \\ \Lambda_\psi\Gamma_\varphi = f\Delta_\sigma}} \lambda_\varphi(\Lambda_\psi\Gamma_\varphi) \cdot v = 0.$$

One has $\Gamma_\varphi\Delta_{\sigma^{-1}} = \Gamma_{\sigma\circ\varphi}$ for all $\varphi \in A_X$ and $\sigma \in \mathbf{Aut}(\Omega, f)$: indeed, for all $x \in X$ and $\omega \in \Omega$, one has

$$\begin{aligned}\Gamma_\varphi\Delta_{\sigma^{-1}}(x,\omega) &= \bigvee_{\xi \in \Omega} f(\varphi(x),\xi) \wedge \Delta_{\sigma^{-1}}(\xi,\omega) = f(\varphi(x),\sigma^{-1}(\omega)) \\ &= f(\sigma\circ\varphi(x),\omega) = \Gamma_{\sigma\circ\varphi}(x,\omega)\end{aligned}$$

since $\sigma \in \mathbf{Aut}(\Omega, f)$. Then, thanks to Remark 6.2, the fact that $\Lambda_\psi\Gamma_\varphi = f\Delta_\sigma$ is equivalent to $\Lambda_\psi\Gamma_{\sigma\circ\varphi} = f$, which in turn implies that $\varphi \preccurlyeq \psi$, thanks to Lemma 8.1. Let $\overline{\preccurlyeq}$ be a linear extension of $\preccurlyeq$, that is, a total order compatible with $\preccurlyeq$, and let ψ be the smallest element of A_X for $\overline{\preccurlyeq}$. The only $\varphi \in A_X$ appearing in the sum

$$\sum_{\substack{\varphi\in A_X\\ \sigma\in\mathbf{Aut}(\Omega,f)\\ \Lambda_\psi\Gamma_{\sigma\circ\varphi}=f}} \lambda_\varphi(\Lambda_\psi\Gamma_\varphi)\cdot v$$

is ψ itself. So our equation reduces to

$$\sum_{\substack{\sigma\in\mathbf{Aut}(\Omega,f)\\ \Lambda_\psi\Gamma_\psi=f\Delta_\sigma}} \lambda_\psi(\Lambda_\psi\Gamma_\psi)\cdot v=0.$$

But for all $\omega,\xi\in\Omega$, one has

$$\Lambda_\psi\Gamma_\psi(\omega,\xi)=\bigvee_{x\in X} f(\omega,\psi(x))\wedge f(\psi(x),\xi)=\bigvee_{\chi\in\Omega} f(\omega,\chi)\wedge f(\chi,\xi)=f^2(\omega,\xi)=f(\omega,\xi),$$

the second equality being a consequence of the surjectivity of ψ. Then the only $\sigma\in\mathbf{Aut}(\Omega,f)$ appearing in this sum is $\sigma=\mathrm{Id}_\Omega$. Indeed, the fact that $f=f\Delta_\sigma$ means that $f(\omega,\xi)=f(\sigma^{-1}(\omega),\xi)$ for all $\omega,\xi\in\Omega$. This in turn implies that $\sigma^{-1}(\omega)=\omega$ thanks to Lemma 4.15. So our equation again reduces to $\lambda_\psi f\cdot v=0$. Now $f\cdot v\neq 0$ since f is the identity element of G_f and $v\neq 0$. So $\lambda_\psi=0$, and by induction on the total order $\overline{\preccurlyeq}$, we get that $\lambda_\psi=0$ for all $\psi\in A_X$. Consequently, the image in $S_{\mathscr{O},V}(X)$ of the family $(\Gamma_\varphi\otimes v^\dagger)_{\varphi\in A_X}$ is linearly independent. In particular, $\dim_k S_{\mathscr{O},V}(X)\geqslant |A_X|$.

For any sets E and F, write $\mathrm{Surj}(E,F)$ for the set of surjective functions from E to F. Then Φ_X is in bijection with the set

$$\prod_{b\in\mathrm{Im}\, m}\mathrm{Surj}(\theta_X^{-1}(b),m^{-1}(b)),$$

via the maps

$$\begin{array}{ccc}
\Phi_X & \longleftrightarrow & \prod_{b\in\mathrm{Im}\, m}\mathrm{Surj}(\theta_X^{-1}(b),m^{-1}(b))\\
\varphi & \longmapsto & \left(\varphi|_{\theta_X^{-1}(b)}\right)_{b\in\mathrm{Im}\, m}\\
\begin{pmatrix} X & \to & \Omega\\ x & \mapsto & \varphi_{\theta_X(x)}(x)\end{pmatrix} & \longleftarrow & (\varphi_b)_{b\in\mathrm{Im}\, m},
\end{array}$$

which are easily checked to be well-defined inverse bijections.

For all $b\in B$, write $\theta_{X,b}=|\theta_X^{-1}(b)|$ and $m_b=|m^{-1}(b)|$. Then one has

$$|\Phi_X|=\prod_{b\in\mathrm{Im}\, m}|\mathrm{Surj}(\theta_X^{-1}(b),m^{-1}(b))|=\prod_{b\in\mathrm{Im}\, m}\sum_{i=0}^{m_b}(-1)^{m_b-i}\binom{m_b}{i}i^{\theta_{X,b}},$$

the cardinality of the sets of type $\mathrm{Surj}(\theta_X^{-1}(b),m^{-1}(b))$ being computed in [19, Section 1.9]. A precise formula for this cardinality is given by [19, Equation 1.94a].

The action of $\mathbf{Aut}(\Omega, f)$ on Φ_X is free, so

$$\begin{aligned}|A_X| &= \frac{1}{|\mathbf{Aut}(\Omega,f)|}|\Phi_X| = \frac{1}{|\mathbf{Aut}(\Omega,f)|}\prod_{b\in\operatorname{Im} m}\sum_{i=0}^{m_b}(-1)^{m_b-i}\binom{m_b}{i} i^{\theta_{X,b}} \\ &= \frac{1}{|\mathbf{Aut}(\Omega,f)|}\prod_{b\in\operatorname{Im} m} {m_b}^{\theta_{X,b}}\left(1+\sum_{i=0}^{m_b-1}(-1)^{m_b-i}\binom{m_b}{i}\left(\frac{i}{m_b}\right)^{\theta_{X,b}}\right).\end{aligned}$$

For all $b \in \operatorname{Im} m$ and $i \leqslant m_b - 1$, one has $\frac{i}{m_b} < 1$. The way we chose θ_X also implies that

$$\lim_{|X|\to+\infty} \theta_{X,b} = +\infty$$

for all $b \in B$. So each sum appearing in the previous expression approaches 0 when $|X|$ approaches $+\infty$. In particular, there exists an integer $r \geqslant 0$ such that for any finite set X with $|X| \geqslant r$ and for all $b \in \operatorname{Im} m$, one has

$$1+\sum_{i=0}^{m_b-1}(-1)^{m_b-i}\binom{m_b}{i}\left(\frac{i}{m_b}\right)^{\theta_{X,b}} \geqslant \frac{1}{2}.$$

So

$$|A_X| \geqslant \frac{1}{|\mathbf{Aut}(\Omega,f)|}\prod_{b\in\operatorname{Im} m}\frac{1}{2}{m_b}^{\theta_{X,b}} = \frac{1}{2^{|\operatorname{Im} m|}|\mathbf{Aut}(\Omega,f)|}\prod_{b\in\operatorname{Im} m}{m_b}^{\theta_{X,b}}.$$

Then for all $b \in \operatorname{Im} m$, one has

$$\begin{aligned}|A_X| &\geqslant \frac{1}{2^{|\operatorname{Im} m|}|\mathbf{Aut}(\Omega,f)|}{m_b}^{\theta_{X,b}} \geqslant \frac{1}{2^{|\operatorname{Im} m|}|\mathbf{Aut}(\Omega,f)|}{m_b}^{\lfloor |X|/|B|\rfloor} \\ &\geqslant \frac{1}{2^{|\operatorname{Im} m|}|\mathbf{Aut}(\Omega,f)|}{m_b}^{|X|/|B|-1}.\end{aligned}$$

In other words,

$$\left(2^{|\operatorname{Im} m|}|\mathbf{Aut}(\Omega,f)||A_X|\right)^{|B|/(|X|-|B|)} \geqslant m_b.$$

This inequality remains true for $b \notin \operatorname{Im}(m)$. So we can sum these inequalities for $b \in B$ to get

$$|B|\left(2^{|\operatorname{Im} m|}|\mathbf{Aut}(\Omega,f)||A_X|\right)^{|B|/(|X|-|B|)} \geqslant \sum_{b\in B} m_b = |\Omega|.$$

It implies that

$$|A_X| \geqslant \frac{1}{2^{|\mathrm{Im}\, m|}|\mathbf{Aut}(\Omega, f)|}\left(\frac{|\Omega|}{|B|}\right)^{|X|/|B|-1} = \frac{|B|}{2^{|\mathrm{Im}\, m|}|\mathbf{Aut}(\Omega, f)||\Omega|}\left(\left(\frac{|\Omega|}{|B|}\right)^{1/|B|}\right)^{|X|}.$$

We get the claimed lower bound by writing

$$c = \frac{|B|}{2^{|\mathrm{Im}\, m|}|\mathbf{Aut}(\Omega, f)||\Omega|}. \quad \square$$

The previous theorem has several interesting consequences in the study of finitely generated functors.

Theorem 8.4. *A generalized correspondence functor F is finitely generated if and only if there exist an integer $r \geqslant 0$ and constants $a, p > 0$ such that $\dim_k F(X) \leqslant ap^{|X|}$ for any finite set X with $|X| \geqslant r$.*

Proof. It is essentially the same as the proof of [6, Theorem 8.4].

Assume that F is finitely generated. Proposition 3.9 ensures that there exist finite sets E and I such that F is a quotient of $k\mathscr{C}_T(-, E)^{\oplus I}$. So for any finite set X, one has $\dim_k F(X) \leqslant |I|(|T|^{|E|})^{|X|}$.

Conversely, assume that there exist an integer $r \geqslant 0$ and constants $a, p > 0$ such that $\dim_k F(X) \leqslant ap^{|X|}$ for any finite set X with $|X| \geqslant r$. Let P and Q be subfunctors of F such that $Q \subseteq P$ and the quotient P/Q is simple, say $P/Q \cong S_{\mathscr{O},V}$. Write $\Omega = {}^\flat\mathscr{O}$. Theorem 8.3 ensures that there exists a constant $c > 0$ such that

$$c\left(\left(\frac{|\Omega|}{|B|}\right)^{1/|B|}\right)^{|X|} \leqslant \dim_k S_{\mathscr{O},V}(X)$$

whenever $|X|$ is large enough. Thanks to our hypothesis about F, one has

$$c\left(\left(\frac{|\Omega|}{|B|}\right)^{1/|B|}\right)^{|X|} \leqslant ap^{|X|}$$

whenever $|X|$ is large enough. This means that

$$c \leqslant a\left(p\left(\frac{|B|}{|\Omega|}\right)^{1/|B|}\right)^{|X|}.$$

Since $c > 0$, one must have

$$p\left(\frac{|B|}{|\Omega|}\right)^{1/|B|} \geqslant 1,$$

otherwise the right-hand side of the second-to-last inequality would approach zero when $|X|$ approaches $+\infty$. Hence $|\Omega| \leqslant |B|p^{|B|}$.

For any finite set Y such that $|Y| \leqslant |B|p^{|B|}$, choose a basis $(v_i)_{1\leqslant i\leqslant d_Y}$ of $F(Y)$. By Yoneda's lemma, there exist morphisms $\psi_i^Y : k\mathscr{C}_T(-,Y) \to F$ such that ψ_i^Y maps Δ_Y to v_i. Then

$$\psi^Y = \sum_{i=1}^{d_Y} \psi_i^Y : k\mathscr{C}_T(-,Y)^{\oplus d_Y} \longrightarrow F$$

is a surjective morphism. We are going to prove that the sum ψ of the ψ^Y's is surjective too.

Write $G = \mathrm{Im}(\psi)$. Assume that $G \neq F$ and fix a finite set X of minimal cardinality such that $F(X)/G(X) \neq 0$. Let W be a simple $k\mathscr{C}_T(X,X)$-submodule of $F(X)/G(X)$. By Lemma 2.6, $S_{X,W}$ is isomorphic to a subquotient of F/G. By Corollary 7.3, $S_{X,W}$ is of type $S_{\mathscr{O},V}$, so $|{}^\flat\mathscr{O}| \leqslant |B|p^{|B|}$ by the first part of the proof.

Write $f = e(\mathscr{O})$. Then $V_f^\dagger \cong S_{\mathscr{O},V}({}^\flat\mathscr{O}) \cong S_{X,W}({}^\flat\mathscr{O})$ is isomorphic to a subquotient of $F({}^\flat\mathscr{O})/G({}^\flat\mathscr{O})$. So $|X| \leqslant |{}^\flat\mathscr{O}|$ by minimality of $|X|$. Hence $|X| \leqslant |B|p^{|B|}$. But this implies that ψ^X is surjective, and $G(X) = F(X)$: this is a contradiction. Hence ψ is surjective.

So F is isomorphic to a quotient of

$$\bigoplus_{|\Omega|\leqslant|B|p^{|B|}} k\mathscr{C}_T(-,\Omega)^{d_\Omega}.$$

Then by Proposition 3.9, F is finitely generated. □

Theorem 8.5.

1. *Any nonzero finitely generated generalized correspondence functor has a maximal subfunctor.*
2. *Any subfunctor of a finitely generated generalized correspondence functor is also finitely generated.*

Proof. It is essentially the same as the proof of [6, Lemma 9.1].

Let F be a finitely generated generalized correspondence functor.

1. Proposition 3.9 implies that there exists a finite set X such that F is generated by $F(X)$. Let M be a maximal $k\mathscr{C}_T(X,X)$-submodule of $F(X)$. Then $F(X)/M$ is a simple $k\mathscr{C}_T(X,X)$-module. So Proposition 2.6 implies that there exist subfunctors $H \subseteq G \subseteq F$ such that G/H is simple and $G(X) = F(X)$. The last equality implies that $G = F$ since F is generated by $F(X)$. Hence H is a maximal subfunctor of F.
2. Let G be a subfunctor of F. Theorem 8.4 ensures that there exist constants $a, p > 0$ such that $\dim_k F(X) \leqslant ap^{|X|}$ whenever $|X|$ is large enough. The same inequality holds for F, so the same theorem ensures that G is also finitely generated. □

The following theorem is a generalization of both [6, Theorem 9.2], which treats the case $T = \{0, 1\}$, and [11, Theorem 3.2], which treats, with different methods, the case of an infinite field. It states that the category $\mathscr{C}_T$ is of *dimension zero*, according to the terminology of [23].

Theorem 8.6. *Any finitely generated generalized correspondence functor has finite length.*

Proof. It is essentially the same as the proof of [6, Theorem 9.2].

Let F be a finitely generated generalized correspondence functor. By Theorem 8.5, it has a maximal subfunctor F_1, which is also finitely generated. Iterating this construction, we get a sequence $F = F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$ such that for all $i \in \mathbb{N}$, either F_i/F_{i+1} is simple or $F_i = 0$. Fix $i \in \mathbb{N}$ such that F_i/F_{i+1} is simple, say $F_i/F_{i+1} \cong S_{\mathscr{O},V}$. By Theorem 8.4, there exist constants $a, p > 0$ such that $\dim_k F(X) \leqslant ap^{|X|}$, whence $\dim_k S_{\mathscr{O},V}(X) \leqslant ap^{|X|}$, whenever $|X|$ is large enough. By Theorem 8.3, there exists a constant $c > 0$ such that

$$c\left(\left(\frac{|^\flat\mathscr{O}|}{|B|}\right)^{1/|B|}\right)^{|X|} \leqslant \dim_k S_{\mathscr{O},V}(X)$$

whenever $|X|$ is large enough. It implies, just like in the proof of Theorem 8.4, that $|^\flat\mathscr{O}| \leqslant |B|p^{|B|}$.

For all $n \leqslant |B|p^{|B|}$, there is only a finite number of presheaves $\mathscr{O}$ such that $|^\flat\mathscr{O}| = n$ up to isomorphism. For each such presheaf, there is only a finite number of simple $kG_{\mathscr{O}}$-modules up to isomorphism. Hence, there is only a finite number of isomorphism classes of simple functors isomorphic to a subquotient of F. If all the quotients F_i/F_{i+1} were simple, one of them would appear infinitely many times. Then for some presheaf $\mathscr{O}$, the module $V^\dagger_{e(\mathscr{O})}$ would appear infinitely many times as a composition factor of $F(^\flat\mathscr{O})$. This is impossible because evaluations of F are finitely generated by Lemma 3.8. So there exists $i \in \mathbb{N}$ such that $F_i = 0$, hence F has finite length. □

9. The case of a noetherian ring

In this section, we study finiteness conditions and stability properties when k is a noetherian ring. All the results of this section are direct generalizations of those of [6, Sections 11 and 12]. For this reason, we do not give full proofs but we only sketch them and highlight differences with [6]. However, full proofs for the generalized case can be found in [14].

For any generalized correspondence functor F, we write

$$\overline{F}(X) = F(X) \Big/ \sum_{Y \subset X} k\mathscr{C}_T(X, Y)F(Y),$$

the sum being running over proper subsets of X.

Let F be a generalized correspondence functor and let $\mathfrak{p}$ be a prime ideal of k. The *localization* $F_\mathfrak{p}$ of F at $\mathfrak{p}$ is defined by $F_\mathfrak{p}(X) = F(X)_\mathfrak{p}$ for any finite set X.

Lemma 9.1.

1. *$F_\mathfrak{p}$ is a generalized correspondence functor over $k_\mathfrak{p}$.*
2. *If F is finitely generated over k, then $F_\mathfrak{p}$ is finitely generated over $k_\mathfrak{p}$.*
3. *For any finite set X, the $k_\mathfrak{p}\mathscr{C}_T(X,X)$-modules $\overline{F}(X)_\mathfrak{p}$ and $\overline{F}_\mathfrak{p}(X)$ are isomorphic.*

Proof. See [6, Lemma 11.2]. □

Proposition 9.2. *Let F be a generalized correspondence functor and let X be a finite set such that $\overline{F}(X) \neq 0$.*

1. *There exists a prime ideal $\mathfrak{p}$ of k such that $\overline{F_\mathfrak{p}}(X) \neq 0$.*

We fix such a prime ideal $\mathfrak{p}$ and we write $k(\mathfrak{p}) = k_\mathfrak{p}/\mathfrak{p}k_\mathfrak{p}$.

2. *If $F(X)$ is a finitely generated k-module, then there exist subfunctors M and N of $F_\mathfrak{p}$ and a simple $k(\mathfrak{p})\mathscr{C}_T(X,X)$-module W with the following properties.*
   - *$\mathfrak{p}F_\mathfrak{p} \subseteq N \subset M$.*
   - *$M/N \cong S_{X,W}$.*
   - *X has minimal cardinality among sets at which $S_{X,W}$ does not vanish.*
3. *There exist an integer $r \geqslant 1$ and a constant $c > 0$ such that for any finite set Y with $|Y| \geqslant r$, one has*

$$c\left(\left(\frac{|X|}{|B|}\right)^{1/|B|}\right)^{|Y|} \leqslant \dim_{k(\mathfrak{p})} S_{X,W}(Y).$$

Proof. 1. This is a consequence of the injectivity of the map

$$\overline{F}(X) \longrightarrow \prod_{\mathfrak{p}\in\mathrm{Spec}(k)} \overline{F}(X)_\mathfrak{p}$$

and of the isomorphism $\overline{F}(X)_\mathfrak{p} \cong \overline{F_\mathfrak{p}}(X)$ of Lemma 9.1.

2. Write $G = F_\mathfrak{p}/\mathfrak{p}F_\mathfrak{p}$, so that G is a generalized correspondence functor over $k(\mathfrak{p})$. If $\overline{G}(X) = 0$, then Nakayama's lemma implies that

$$F_\mathfrak{p}(X) = \sum_{Y\subset X} k_\mathfrak{p}\mathscr{C}_T(X,Y)F_\mathfrak{p}(Y),$$

that is, $\overline{F_\mathfrak{p}}(X) = 0$. This contradicts our choice of $\mathfrak{p}$, so $\overline{G}(X) \neq 0$. Now $\overline{G}(X)$ is a $k(\mathfrak{p})\mathscr{C}_T(X,X)$-module, and it is finitely generated as $k(\mathfrak{p})$-vector space. Hence $\overline{G}(X)$ has a quotient W which is simple as $k(\mathfrak{p})\mathscr{C}_T(X,X)$-module. Moreover, $k(\mathfrak{p})\mathscr{C}_T(X,Y)\mathscr{C}_T(Y,X)$ acts by 0 on $\overline{G}(X)$ whenever $|Y| < |X|$ so $k(\mathfrak{p})\mathscr{C}_T(X,Y)\mathscr{C}_T(Y,X)$ also acts by 0 on W whenever $|Y| < |X|$. Then W is a quotient of $G(X)$, so Proposition 2.6 implies that there exist subfunctors M and N of $F_\mathfrak{p}$ such that $\mathfrak{p}F_\mathfrak{p} \subseteq N \subset M$ and

$$M/N \cong (M/\mathfrak{p}F_\mathfrak{p})/(N/\mathfrak{p}F_\mathfrak{p}) \cong S_{X,W}.$$

By definition of $J_{X,W}$, since $k(\mathfrak{p})\mathscr{C}_T(X,Y)\mathscr{C}_T(Y,X)$ acts by 0 on W whenever $|Y| < |X|$ and since $S_{X,W}(X) \neq 0$, we deduce that X has minimal cardinality among sets at which $S_{X,W}$ does not vanish.

3. By Corollary 7.3, $S_{X,W}$ is of type $S_{\mathscr{O},V}$. One has $V^\dagger_{e(\mathscr{O})} \cong S_{\mathscr{O},V}({}^\flat\mathscr{O}) \cong S_{X,W}({}^\flat\mathscr{O})$, so $|X| \leqslant |{}^\flat\mathscr{O}|$ by minimality of $|X|$. Theorem 8.3 ensures that there exist an integer $r \geqslant 0$ and a constant $c > 0$ such that

$$c\left(\left(\frac{|{}^\flat\mathscr{O}|}{|B|}\right)^{1/|B|}\right)^{|Y|} \leqslant \dim_{k(\mathfrak{p})} S_{\mathscr{O},V}(Y)$$

whenever $|Y| \geqslant r$. Then

$$c\left(\left(\frac{|X|}{|B|}\right)^{1/|B|}\right)^{|Y|} \leqslant \dim_{k(\mathfrak{p})} S_{X,W}(Y)$$

whenever $|Y| \geqslant r$. □

From now on, we assume that k is noetherian.

Proposition 9.3. *Let F be a generalized correspondence functor and let G be a subfunctor of F. If X and Z are finite sets such that F is generated by $F(X)$ and $\overline{G}(Z) \neq 0$, then $|Z| \leqslant |B||T|^{|B||X|}$.*

Proof. By an argument of finite generation and localization, and using Proposition 9.2, we can assume that k is local, that $F = k\mathscr{C}_T(-,X)^{\oplus t}$ for some integer $t \geqslant 1$ and that G has a subfunctor H such that G/H is simple and Z has minimal cardinality among sets at which G/H does not vanish. We then conclude the proof just like [6, Theorem 11.4]. □

Theorem 9.4. *Let F be a generalized correspondence functor and let G be a subfunctor of F.*

1. *If X and Y are finite sets such that F is generated by $F(X)$ and $|Y| \geqslant |B||T|^{|B||X|}$, then G is generated by $G(Y)$.*
2. *If F is finitely generated, then G is finitely generated.*

Proof. See [6, Corollary 11.5]. □

In the same fashion, we can prove the following stabilization results.

Theorem 9.5. *Let F be a generalized correspondence functor. Assume that there exists a finite set X such that F is generated by $F(X)$.*

1. *For any finite set Y with $|Y| \geqslant |B||T|^{|B||X|}$, one has $F \cong L_{Y,F(Y)}$. Moreover, this isomorphism is given by the counit $L_{Y,F(Y)} \to F$ of the adjunction of Proposition 2.3.*
2. *For any finite set Y with $|Y| \geqslant |B||T|^{|B|^2|T|^{|B||X|}}$, one has $F \cong S_{Y,F(Y)}$.*
3. *For any generalized correspondence functor G, for any finite set Y and for any integer $i \in \mathbb{N}$, there exists an integer $n_i \in \mathbb{N}$ such that the evaluation at Y*

$$\mathbf{Ext}^i_{\mathscr{F}_k(\mathscr{C}_T)}(F,G) \longrightarrow \mathbf{Ext}^i_{k\mathscr{C}_T(Y,Y)}(F(Y),G(Y))$$

   is an isomorphism whenever $|Y| \geqslant n_i$.
4. *For any finite set Y with $|Y| \geqslant |B||T|^{|B|^2|T|^{|B||X|}}$, for any finite set Z and for any $k\mathscr{C}_T(X,X)$-module V, one has*

$$\mathbf{Tor}_1^{k\mathscr{C}_T(Y,Y)}(k\mathscr{C}_T(Z,Y), L_{X,V}(Y)) = 0.$$

Proof. See [6, Section 12]. □

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