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Naïve noncommutative blowups at zero-dimensional schemes ☆

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Abstract

In an earlier paper [D.S. Keeler, D. Rogalski, J.T. Stafford, Naïve noncommutative blowing up, Duke Math. J. 126 (2005) 491–546, MR 2120116], we defined and investigated the properties of the naïve blowup of an integral projective scheme X at a single closed point. In this paper we extend those results to the case when one naïvely blows up X at any suitably generic zero-dimensional subscheme Z. The resulting algebra A has a number of curious properties; for example it is noetherian but never strongly noetherian and the point modules are never parametrized by a projective scheme. This is despite the fact that the category of torsion modules in qgr-A is equivalent to the category of torsion coherent sheaves over X. These results are used in the companion paper [D. Rogalski, J.T. Stafford, A class of noncommutative projective surfaces, in press] to prove that a large class of noncommutative surfaces can be written as naïve blowups.

Keywords: Noncommutative projective geometry; Noncommutative surfaces; Noetherian graded rings; Naïve blowing up

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1. Introduction

The concept of a naïve blowup of a scheme X was introduced in [KRS], where it was also shown that these objects had properties quite unlike their commutative counterparts. Naïve blowups are also used in the companion paper to this one, [RS1], in order to classify a large class of noncommutative algebras. Unfortunately, the algebras considered in [KRS] were obtained by naïvely blowing up a single closed point whereas the applications in [RS1] require one to naïvely blow up any suitably general zero-dimensional subscheme. The aim of this paper is therefore to study this more general case. Before describing the results we need some definitions.

Throughout, k will be an algebraically closed base field. A k-algebra A is called *connected graded* (*cg*) if $A = \bigoplus_{n\geqslant 0} A_n$, where $A_0 = k$ and $\dim_k A_n < \infty$ for each n. Given a cg k-algebra A, the category of noetherian graded A-modules modulo those of finite length is written qgr-A (see p. 799 for more details). A *point module* is a cyclic graded A-module $M = \bigoplus_{n\geqslant 0} M_n$ such that $\dim_k M_n = 1$ for all $n \geqslant 0$. A *point module in* qgr-A is defined to be the image in qgr-A of a cyclic graded A-module $M = \bigoplus_{n\geqslant 0} M_n$, generated in degree zero, such that $\dim_k M_n = 1$ for all $n \gg 0$.

The underlying data for a naïve blowup is as follows. Fix an integral projective scheme X, with automorphism σ and σ-ample sheaf $\mathcal{L}$, as defined in (2.7). Let $Z = Z_{\mathcal{I}} \subset X$ be a zero-dimensional subscheme, with defining ideal $\mathcal{I} \subseteq \mathcal{O}_X$. In a manner reminiscent of the Rees ring construction of the blowup of a commutative scheme, we form the *bimodule algebra* $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma) = \mathcal{O}_X \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots$, where $\mathcal{R}_n = \mathcal{L}_n \otimes_{\mathcal{O}_X} \mathcal{I}_n$, for $\mathcal{L}_n = \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L}$ and $\mathcal{I}_n = \mathcal{I} \cdot \sigma^*\mathcal{I} \cdots (\sigma^{n-1})^*\mathcal{I}$. This bimodule algebra has a natural multiplication and the *naïve blowup algebra of X at Z* is then the algebra of sections

$$R = R(X, Z, \mathcal{L}, \sigma) = \mathrm{H}^0(X, \mathcal{R}) = k \oplus \mathrm{H}^0(X, \mathcal{R}_1) \oplus \mathrm{H}^0(X, \mathcal{R}_2) \oplus \cdots.$$

One can also form $R = R(X, Z, \mathcal{L}, \sigma)$ when Z is the empty set, in which case R is simply *the twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \sigma)$ from [AV] that is so important in noncommutative projective geometry (see [SV], for example). If $B = B(X, \mathcal{L}, \sigma)$ for a σ-ample invertible sheaf $\mathcal{L}$, then B has extremely pleasant properties, among which we mention:

(a) (See [ASZ, Proposition 4.13].) B is *strongly noetherian*; that is, for all commutative noetherian k-algebras C, the ring $B \otimes_k C$ is noetherian.
(b) (See [AV, Theorem 1.3].) qgr-$B \simeq \operatorname{coh} X$, the category of coherent sheaves on X.

(c) At least when B is generated in degree one, the set of point modules for B, both in gr-B and in qgr-B, is parametrized by the scheme X (use [RZ, Theorems 1.1 and 1.2] and [KRS, Proposition 10.2]).
(d) (See [Ye, Theorem 7.3].) B has a balanced dualizing complex in the sense of that paper.

In contrast, the naïve blowup algebra $R(X, Z, \mathcal{L}, \sigma)$ for $Z \neq \emptyset$ has properties quite unlike those just mentioned. Some of these properties are given by the next theorem and will be discussed in more detail later in this introduction.

In order to state the theorem, we need one more definition. A set of closed points $\mathcal{C} \subset X$ is *critically dense* if $\mathcal{C}$ is infinite and every infinite subset of $\mathcal{C}$ has Zariski closure equal to X. A zero-dimensional subscheme $Z \subset X$ is *right saturating*, respectively *saturating* if, for each point $c \in S = \operatorname{Supp} Z$, the set $\{c_i = \sigma^{-i}(c)\colon i \geqslant 0\}$, respectively $\{c_i = \sigma^{-i}(c)\colon i \in \mathbb{Z}\}$ is critically dense. By convention, the empty subscheme is (right) saturating.

Theorem 1.1. *Let X be an integral projective scheme with* $\dim X \geqslant 2$ *and* $\sigma \in \operatorname{Aut}(X)$. *Assume that $\mathcal{L}$ is a σ-ample invertible sheaf on X and that Z is a* (*nonempty*) *zero-dimensional, saturating subscheme of X. If $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma)$ with global sections $R = R(X, Z, \mathcal{L}, \sigma)$, then*:

(1) qgr-$R \simeq$ qgr-$\mathcal{R}$ *and* qgr-$\mathcal{R}$ *is independent of the choice of $\mathcal{L}$.*
(2) *R is always a noetherian domain.*
(3) *R is never strongly noetherian.*
(4) *The category of* (*Goldie*) *torsion objects in* qgr-R, *as defined in Section* 4, *is equivalent to the category of torsion coherent $\mathcal{O}_X$-modules.*
(5) *In particular, the isomorphism classes of simple objects in* qgr-R, *which are also the point modules in* qgr-R, *are in* 1–1 *correspondence with the closed points of X.*
(6) *However, the point modules in* qgr-R *are* not *parametrized by any scheme of locally finite type. When $R_1 \neq 0$, the point modules in* gr-R *are not parametrized by any scheme of locally finite type.*
(7) qgr-R *has finite cohomological dimension. If X is smooth,* qgr-R *has finite homological dimension.*
(8) *If* $\mathrm{H}^1(R) = \operatorname{Ext}^1_{\text{qgr-}R}(R, R)$, *then* $\dim_k \mathrm{H}^1(R) = \infty$. *Consequently, R does not have a balanced dualizing complex.*
(9) *If U is* any *open affine subset of X, then generic flatness, as defined in Section* 7, *fails for the finitely generated $R \otimes \mathcal{O}_X(U)$-module $\mathcal{R}(U) = \bigoplus \mathcal{R}_n(U)$.*

Remark 1.2. All the properties described by Theorem 1.1 pass easily to subrings of finite index; that is, the theorem still holds if one replaces R by any cg k-algebra $R' \subset R$ such that $\dim_k R/R' < \infty$. See Remark 7.7 for the details.

Theorem 1.1 summarizes many of the results of this paper and so its proof takes up much of the paper. Specifically, parts (1) and (2) of the theorem are proved by combining Proposition 2.10 and Theorem 3.1 and their proof takes up most of Sections 2–3. The rest of the paper is then concerned with applying this theorem to get a deeper understanding of the properties of $\mathcal{R}$ and R. In particular, parts (3) and (9) of Theorem 1.1 are proved in Theorem 7.2; part (4) in Theorem 4.10; part (5) by combining Corollary 4.11 and Proposition 7.3; part (6) in Theorem 7.5; part (7) in Theorem 6.9; and part (8) in Theorem 6.2 and Remark 6.11.

The proofs of many of the results intermediary to Theorem 1.1 are very similar to those of the analogous results from [KRS], but there are some significant differences which we now discuss. Part (1) of the theorem is the fundamental tool for studying the naïve blowup algebra $R = R(X, Z, \mathcal{L}, \sigma)$ since it provides the avenue for introducing geometry into that study. In turn, the key step in its proof is the following result.

Proposition 1.3 *(Theorem 3.1). Keep the hypotheses of Theorem* 1.1. *Then the sequence* $\{\mathcal{R}_n = \mathcal{L}_n \otimes \mathcal{I}_n\}$ *is ample in the sense that, for every* $\mathcal{F} \in \operatorname{coh} X$ *and all* $n \gg 0$*, the sheaf* $\mathcal{F} \otimes \mathcal{R}_n$ *is generated by its global sections and satisfies* $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{R}_n) = 0$ *for all* $i > 0$.

The proof of Proposition 1.3 is considerably more subtle than that of its counterpart [KRS, Proposition 4.6]. Even leaving aside the issue of the number of points at which one is naïvely blowing up, the proof in [KRS] only works when $\mathcal{L}$ is both σ-ample and very ample. In contrast, in the application in Theorem 1.4, below, one is only allowed to assume that $\mathcal{L}$ is σ-ample. The proof of Proposition 1.3 in this more general case requires a delicate analysis of the number of points separated by $\mathcal{L}_n$.

As well as its applications in this paper, Proposition 1.3 is also the starting point for the classification in [RS1] of a large class of noncommutative algebras:

Theorem 1.4. *(See [RS1, Theorem 1.1].) Let* A *be a cg noetherian domain that is generated in degree one and assume that the graded quotient ring* $Q(A)$ *has the form* $Q(A) \cong k(Y)[t, t^{-1}; \sigma]$*, where* σ *is induced from an automorphism of the integral projective surface* Y*. Then, up to a finite-dimensional vector space,* A *is isomorphic to either* $B(X, \mathcal{L}, \sigma)$ *or to* $R(X, Z, \mathcal{L}, \sigma)$*, for some projective surface* X *birational to* Y*, with an induced action of* σ*, a* σ*-ample invertible sheaf* $\mathcal{L}$ *and a zero-dimensional saturating subscheme* Z*.*

One striking consequence of this theorem is the fact that the properties described by Theorem 1.1 are not exceptional: as soon as Y has at least one critically dense σ-orbit then Theorem 1.4 and Remark 1.2 imply that, generically, each noetherian cg subalgebra of $k(Y)[t, t^{-1}; \sigma]$ that is generated in degree one has these properties.

A surprising feature of naïve blowups at multiple points concerns torsion extensions. Given a naïve blowup algebra $R = R(X, Z, \mathcal{L}, \sigma)$ satisfying the hypotheses of Theorem 1.1, the *maximal right torsion extension* of R is defined to be the ring $T = T(R) = \{x \in Q(R)\colon xR_{\geqslant n} \subseteq R$ for some $n \geqslant 0\}$. If R is the naïve blowup algebra at a single point (or a twisted homogeneous coordinate ring) then $T(R)/R$ is always finite-dimensional [KRS, Theorem 1.1(8)]. In contrast, if one blows up at multiple points then $T(R)/R$ can be infinite-dimensional (see Example 5.1) and, indeed, the ring R from that example is even an *idealizer subring* $R = \{\theta \in T\colon M\theta \subseteq M\}$ for some left ideal M of T (see the discussion after Lemma 6.7). This implies that naïve blowup algebras can have very nonsymmetric properties. It also implies that the χ_1 condition, as defined in Section 6, will fail for such a ring R. In contrast, [KRS, Theorem 1.1(8)] shows that the χ_1 condition always holds when one naïvely blows up a single point. The details behind these assertions are given in Sections 5 and 6.

In order to describe the maximal right torsion extension of a naïve blowup algebra, and even to prove parts of Theorem 1.1, one needs to pass to a slightly larger class of algebras, called generalized naïve blowup algebras. These algebras are discussed in detail in Section 5 and, because it takes little extra work, much of Theorem 1.1 is actually proved at their level of generality.

We end the introduction by describing the significance of some of the other parts of Theorem 1.1 and we keep the notation and hypotheses from that result. Although part (1) justifies the idea that qgr-$R \simeq$ qgr-$\mathcal{R}$ is a kind of a noncommutative blowup of X, the category qgr-R is in fact much closer to $\operatorname{coh} X$ than it is to $\operatorname{coh} \widetilde{X}$ for the (classical) blowup $\widetilde{X}$ of X at Z. For example, suppose that $k(x)$ is the skyscraper sheaf corresponding to a closed point $x \in \operatorname{Supp} Z$. If one tensors $k(x)$ with the sheaf of Rees rings $\mathcal{R}(X, Z, \mathcal{L}, \mathrm{Id})$ corresponding to $\widetilde{X}$ then, of course, one obtains the $\mathcal{O}_{\widetilde{X}}$-module corresponding to an exceptional divisor on $\widetilde{X}$. In contrast, $k(x) \otimes_{\mathcal{O}_X} \mathcal{R}$ is a finite direct sum of simple objects from qgr-$\mathcal{R}$ (use the computation from [KRS, Proposition 5.3]). Although we take a different approach, this argument can be used to prove much of part (5) of the theorem. This result, together with its generalization in part (4), shows that the differences between the categories qgr-R and $\operatorname{coh} X$ are really quite subtle.

The idea of considering the strongly noetherian condition arose in the work of Artin, Small and Zhang [ASZ,AZ2], who showed that many algebras have this property and that this has a number of important consequences for the algebras in question. Notably, a strongly noetherian graded k-algebra A will always satisfy generic flatness [ASZ, Theorem 0.1]. If A is also generated in degree one then its point modules, both in gr-A and in qgr-A, will be parametrized by a projective scheme (see [AZ2, Corollary E4.5], respectively [KRS, Proposition 10.2]). Thus part (9) and both assertions from part (6) of Theorem 1.1 all imply that R is not strongly noetherian. Finally, parts (7) and (8) of the theorem show contrasting homological properties of qgr-R; in particular, part (8) means that the homological machinery developed by Yekutieli and Zhang in their papers on dualizing complexes cannot easily be applied to the study of R. For an illustration of the complications this causes and the ways in which one can circumvent some of these problems, see Remark 7.4.

Finally, we mention that several peripheral results that are stated but not proved in this paper are proved in full generality in an appendix [RS2] that will be available on the web but not published.

2. Definitions and background material

In this section we set up the appropriate notation relating to the bimodule algebras $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma)$ and their section rings $R = \mathrm{H}^0(X, \mathcal{R})$ and determine, among other things, when $\mathcal{R}$ is noetherian (see Proposition 2.12). With the exception of the proof of that proposition, most of this section is similar to the material in [KRS, Sections 2 and 3], to which the reader is referred for further details.

Fix throughout an integral projective scheme X over an algebraically closed field k. The category of quasi-coherent, respectively coherent, sheaves on X will be written $\mathcal{O}_X$-Mod, respectively $\mathcal{O}_X$-mod. We use the following notation for pullbacks: if $\sigma \in \operatorname{Aut}(X)$ is a k-automorphism of X, and $\mathcal{F} \in \mathcal{O}_X$-mod, then $\mathcal{F}^\sigma = \sigma^*(\mathcal{F})$. We adopt the usual convention that σ acts on functions by $f^\sigma(x) = f(\sigma(x))$, for $x \in X$.

Definition 2.1. A *coherent $\mathcal{O}_X$-bimodule* is a coherent sheaf $\mathcal{F}$ on $X \times X$ such that $Z = \operatorname{Supp} \mathcal{F}$ has the property that both projections $\rho_1, \rho_2 : Z \to X$ are finite morphisms. An *$\mathcal{O}_X$-bimodule* is a quasi-coherent sheaf $\mathcal{F}$ on $X \times X$ such that every coherent $X \times X$-subsheaf is a coherent $\mathcal{O}_X$-bimodule. The left and right $\mathcal{O}_X$-module structures associated to $\mathcal{F}$ are defined to be ${}_{\mathcal{O}_X}\mathcal{F} = (\rho_1)_*\mathcal{F}$ and $\mathcal{F}_{\mathcal{O}_X} = (\rho_2)_*\mathcal{F}$ respectively.

Given $\mathcal{F} \in \mathcal{O}_X$-mod and $\tau, \sigma \in \mathrm{Aut}(X)$, define an $\mathcal{O}_X$-bimodule ${}_\tau\mathcal{F}_\sigma$ by $(\tau, \sigma)_*\mathcal{F}$ where $(\tau, \sigma): X \to X \times X$. We usually write $\mathcal{F}_\sigma$ for ${}_1\mathcal{F}_\sigma$, where 1 is the identity automorphism. The reader may check that ${}_1\mathcal{F}_\sigma$ has left $\mathcal{O}_X$-module structure $\mathcal{F}$ but right $\mathcal{O}_X$-module structure $\mathcal{F}^{\sigma^{-1}}$.

When no other bimodule structure is given, a sheaf $\mathcal{F} \in \mathcal{O}_X$-mod will be assumed to have the bimodule structure ${}_1\mathcal{F}_1$. Thus all sheaves become bimodules, and all tensor products can be thought of as tensor products of bimodules. Unless otherwise stated, when thinking of a bimodule $\mathcal{G}$ as a sheaf, we mean the left $\mathcal{O}_X$-module structure of $\mathcal{G}$. Thus, when we write $\mathrm{H}^i(X, \mathcal{G})$ or say that $\mathcal{G}$ is generated by its global sections we are referring to the left structure of $\mathcal{G}$. Working on the left will have notational advantages, but, as in [KRS], it is otherwise not significant.

The following special case of Van den Bergh's bimodule algebras will form the main objects of interest in this paper.

Definition 2.2. Let $\sigma \in \mathrm{Aut}(X)$. A *graded* $(\mathcal{O}_X, \sigma)$*-bimodule algebra* is an $\mathcal{O}_X$-bimodule $\mathcal{B} = \bigoplus_{n \geqslant 0} \mathcal{B}_n$ with a unit map $1: \mathcal{O}_X \to \mathcal{B}$ and a product map $\mu: \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ satisfying the usual axioms as well as:

(1) For each n, $\mathcal{B}_n \cong {}_1(\mathcal{E}_n)_{\sigma^n}$, for some $\mathcal{E}_n \in \mathcal{O}_X$-mod with $\mathcal{B}_0 = {}_1(\mathcal{O}_X)_1$.
(2) The multiplication map satisfies $\mu(\mathcal{B}_m \otimes \mathcal{B}_n) \subseteq \mathcal{B}_{m+n}$ for all m, n and $1(\mathcal{O}_X) \subseteq \mathcal{B}_0$. Equivalently μ is defined by $\mathcal{O}_X$-module maps $\mathcal{E}_n \otimes \mathcal{E}_m^{\sigma^n} \to \mathcal{E}_{m+n}$ satisfying the appropriate associativity conditions.

We will write $\mathcal{B} = \bigoplus {}_1(\mathcal{E}_n)_{\sigma^n}$ throughout the section.

Definition 2.3. Let $\mathcal{B}$ be a graded $(\mathcal{O}_X, \sigma)$-algebra. A *graded right* $\mathcal{B}$*-module* is a quasi-coherent right $\mathcal{O}_X$-module $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n$ together with a right $\mathcal{O}_X$-module map $\mu: \mathcal{M} \otimes \mathcal{B} \to \mathcal{M}$ satisfying the usual axioms. The *shift of* $\mathcal{M}$ is defined by $\mathcal{M}[n] = \bigoplus \mathcal{M}[n]_i$ with $\mathcal{M}[n]_i = \mathcal{M}_{i+n}$.

The $\mathcal{B}$-module $\mathcal{M}$ is *coherent* (as a $\mathcal{B}$-module) if there is a coherent $\mathcal{O}_X$-module $\mathcal{M}_0$ and a surjective map $\mathcal{M}_0 \otimes \mathcal{B} \to \mathcal{M}$ of ungraded $\mathcal{B}$-modules. Left $\mathcal{B}$-modules are defined similarly and the bimodule algebra $\mathcal{B}$ is *right* (*left*) *noetherian* if every right (left) ideal of $\mathcal{B}$ is coherent. For the algebras that interest us, a more natural definition of coherence will be given in Lemma 2.11.

One can give various different bimodule structures to a graded right $\mathcal{B}$-module $\mathcal{M} = \bigoplus \mathcal{M}_i$ and it will cause no loss of generality to assume that all right $\mathcal{B}$-modules have the form

$$\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} {}_1(\mathcal{G}_n)_{\sigma^n} \quad \text{for some (left) sheaves } \mathcal{G}_n \in \mathcal{O}_X\text{-Mod}. \tag{2.4}$$

The advantage of this choice is that the $\mathcal{B}$-module structure on $\mathcal{M}$ is given by a family of $\mathcal{O}_X$-module maps $\mathcal{G}_n \otimes \mathcal{E}_m^{\sigma^n} \to \mathcal{G}_{n+m}$, again satisfying the appropriate associativity conditions.

Graded right $\mathcal{B}$-modules form an abelian category Gr-$\mathcal{B}$, with homomorphisms graded of degree zero. Its subcategory of coherent modules is denoted gr-$\mathcal{B}$. A graded $\mathcal{B}$-module $\mathcal{M} = \bigoplus \mathcal{M}_i$ is *right bounded* if $\mathcal{M}_i = 0$ for all $i \gg 0$ and *bounded* if $\mathcal{M}_i = 0$ for all but finitely many i. A module $\mathcal{M} \in$ Gr-$\mathcal{B}$ is called *torsion* if every coherent submodule of $\mathcal{M}$ is bounded. Let Tors-$\mathcal{B}$ denote the full subcategory of Gr-$\mathcal{B}$ consisting of torsion modules, and write Qgr-$\mathcal{B}$ for the quotient category Gr-$\mathcal{B}$/Tors-$\mathcal{B}$. The analogous quotient category of gr-$\mathcal{B}$ will be denoted qgr-$\mathcal{B}$. The corresponding categories of left modules will be denoted by $\mathcal{B}$-Gr, etc. Similar definitions and notation apply to modules over a graded ring $A = \bigoplus_{i\geqslant 0} A_i$ and we denote the natural quotient maps by $\pi_{\mathcal{B}}: \text{Gr-}\mathcal{B} \to \text{Qgr-}\mathcal{B}$ and $\pi_A: \text{Gr-}A \to \text{Qgr-}A$. Both maps are written as π if no confusion is possible.

A basic technique for us will be to pass between a bimodule algebra and its ring of sections, and the next theorem, due to Van den Bergh, gives one situation in which this is possible.

Definition 2.5. Suppose that $\{\mathcal{J}_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{O}_X$-bimodules. Then the sequence is *ample* (or, more formally, *right ample*) if the following conditions hold for any $\mathcal{M} \in \mathcal{O}_X$-mod:

(1) $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{J}_n$ is generated by global sections for $n \gg 0$.
(2) $\mathrm{H}^i(X, \mathcal{M} \otimes \mathcal{J}_n) = 0$ for all $i > 0$ and $n \gg 0$.

Theorem 2.6. *Let $\mathcal{B} = \bigoplus \mathcal{B}_i$ be a right noetherian graded $(\mathcal{O}_X, \sigma)$-algebra. Assume that $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ is an ample sequence of $\mathcal{O}_X$-bimodules such that each $\mathcal{B}_n$ is contained in a locally free left $\mathcal{O}_X$-module. Then*:

(1) *The section algebra $B = \mathrm{H}^0(X, \mathcal{B})$ is right noetherian, and there is an equivalence of categories ξ: $\text{qgr-}\mathcal{B} \simeq \text{qgr-}B$ via the inverse equivalences $\mathrm{H}^0(X, -)$ and $- \otimes_B \mathcal{B}$.*
(2) *If $\mathcal{M} \in \text{gr-}\mathcal{B}$ then $\mathrm{H}^0(X, \mathcal{M})$ is a noetherian B-module.*

Proof. (1) This is essentially [VB1, Theorem 5.2]; see [KRS, Theorem 2.12] for more details.
(2) This is Step 3 in the proof of [VB1, Theorem 5.2]. $\square$

An important special case of Definition 2.5 and Theorem 2.6 occurs when $\mathcal{J}_n = \mathcal{B}_n = ({}_1\mathcal{L}_\sigma)^{\otimes n}$ for an invertible sheaf $\mathcal{L}$ on X. We will usually write $\mathcal{L}_\sigma^{\otimes n}$ for $({}_1\mathcal{L}_\sigma)^{\otimes n}$. It is customary to say that

$$\mathcal{L} \text{ is } \sigma\textit{-ample if } \left\{\mathcal{L}_\sigma^{\otimes n}\right\}_{n\geqslant 0} \text{ is an ample sequence of bimodules.} \tag{2.7}$$

We will always write the corresponding bimodule algebra $\bigoplus \mathcal{L}_\sigma^{\otimes n}$ as $\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$ with section algebra $B = B(X, \mathcal{L}, \sigma) = \bigoplus_{n\geqslant 0} \mathrm{H}^0(X, \mathcal{L}_\sigma^{\otimes n})$. This is an equivalent definition of the *twisted homogeneous coordinate ring* of X from [AV]. For more detailed results about $B(X, \mathcal{L}, \sigma)$, see [AV,Ke1].

We now turn to a second special case of bimodule algebras; that of naïve blowups. For this we need the following assumptions, which will also be fixed for the rest of the section.

Assumptions 2.8. Fix an integral projective scheme X. Fix $\sigma \in \mathrm{Aut}(X)$, an invertible sheaf $\mathcal{L}$ on X and let $\mathcal{I} = \mathcal{I}_Z$ denote the sheaf of ideals defining a subscheme Z that is either zero-dimensional or empty. Let $S = \operatorname{Supp} Z$. We always assume that each $p \in S$ has infinite order under σ. Our convention on automorphisms from the beginning of the section means that $\mathcal{I}^{\sigma^i} = \mathcal{I}_{\sigma^{-i}(Z)}$, and so $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}^{\sigma^i} = \sigma^{-i}(S)$.

Mimicking classical blowing up we set

$$\mathcal{I}_n = \mathcal{I}\mathcal{I}^\sigma \cdots \mathcal{I}^{\sigma^{n-1}}, \qquad \mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} \quad \text{and} \quad \mathcal{R}_n = {}_1(\mathcal{I}_n \otimes \mathcal{L}_n)_{\sigma^n},$$

where all tensor products are over $\mathcal{O}_X$. From this data we define a bimodule algebra $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma) = \bigoplus_{n=0}^{\infty} \mathcal{R}_n$ with *naïve blowup algebra* $R = R(X, Z, \mathcal{L}, \sigma) = \mathrm{H}^0(X, \mathcal{R}) = \bigoplus_{n \geqslant 0} \mathrm{H}^0(X, \mathcal{R}_n)$. These algebras $\mathcal{R}$ and R are called *nontrivial* if $Z \neq \emptyset$. By [KRS, Lemma 2.9],

$$\mathcal{R}(X, Z, \mathcal{L}, \sigma)^{\mathrm{op}} \cong \mathcal{R}\big(X, \sigma(Z), \mathcal{L}^{\sigma^{-1}}, \sigma^{-1}\big), \tag{2.9}$$

where $\mathcal{R}^{\mathrm{op}}$ denotes the opposite bimodule algebra in the obvious sense (see [KRS, Definition 2.8] for the formal definition). Thus any result proved on the right can immediately be transferred to the left.

We note that, in distinction to the situation in [KRS] where Z is a single reduced point, $\mathcal{I}_n = \mathcal{I}\mathcal{I}^{\sigma} \cdots \mathcal{I}^{\sigma^{n-1}}$ need not equal $\mathcal{I} \otimes \mathcal{I}^{\sigma} \otimes \cdots \otimes \mathcal{I}^{\sigma^{n-1}}$; indeed, in general the latter sheaf need not even be an ideal sheaf. This also means that the multiplication map $\mathcal{R}_m \otimes \mathcal{R}_n \to \mathcal{R}_{m+n}$ is not an isomorphism of sheaves. In generalizing many of the results in [KRS] this is merely an annoyance, but in Sections 5 and 6 it will make a significant difference to the results themselves.

Proposition 2.10. *Given invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$, then there is an equivalence of categories* $\text{Gr-}\mathcal{R}(X, Z, \mathcal{L}, \sigma) \sim \text{Gr-}\mathcal{R}(X, Z, \mathcal{L}', \sigma)$.

Proof. The proof of [KRS, Proposition 3.5] can be used without change. □

Lemma 2.11. *A module $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n \in \text{Gr-}\mathcal{R}$ is coherent if and only if the following conditions hold*:

(1) *Each $\mathcal{M}_n$ is a coherent $\mathcal{O}_X$-module, with $\mathcal{M}_n = 0$ for $n \ll 0$.*
(2) *The natural map $\mu_n : \mathcal{M}_n \otimes \mathcal{R}_1 \to \mathcal{M}_{n+1}$ is surjective for $n \gg 0$.*

Proof. The proof of [KRS, Lemma 3.9] can be used essentially unchanged. (The only difference is that the morphism ϕ_1 in the commutative diagram on [KRS, p. 504] will now be a surjection rather than an isomorphism, but this does not affect the proof.) □

The next result, which is the main result of this section, determines when the bimodule algebra $\mathcal{R}$ is noetherian.

Proposition 2.12. *Keep the hypotheses of* (2.8). *Then*:

(1) *The bimodule algebra $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma)$ is right noetherian if and only if Z is right saturating and noetherian if and only if Z is saturating.*
(2) *If $\mathcal{R}$ is not right noetherian, there exists an infinite ascending chain of coherent right ideals of $\mathcal{R}$ with nontorsion factors.*

Proof. (1) By Proposition 2.10, the result is independent of the choice of $\mathcal{L}$ and we choose $\mathcal{L} = \mathcal{O}_X$. We start by assuming that Z is right saturating. An arbitrary right ideal $\mathcal{G}$ of $\mathcal{R}$ is given by a sequence of bimodules $\mathcal{G}_i = {}_1(\mathcal{H}_i)_{\sigma^i} \subseteq \mathcal{R}_i$, where $\mathcal{H}_i \subseteq \mathcal{I}_i$ is an ideal sheaf, such that the multiplication maps $\mathcal{R}_i \otimes \mathcal{R}_1 \to \mathcal{R}_{i+1}$ restrict to maps $\mu_i : \mathcal{G}_i \otimes \mathcal{R}_1 \to \mathcal{G}_{i+1}$ for all $i \geqslant 0$. By

Lemma 2.11, $\mathcal{G}$ is coherent if and only if μ_i is surjective for all $i \gg 0$. Note that the image of μ_i is ${}_1(\mathcal{H}_i\mathcal{I}^{\sigma^i})_{\sigma^{i+1}}$. Equivalently, we are given ideal sheaves

$$\mathcal{H}_i\mathcal{I}^{\sigma^i} \subseteq \mathcal{H}_{i+1} \subseteq \mathcal{I}\mathcal{I}^{\sigma}\cdots\mathcal{I}^{\sigma^i} \quad \text{for all } i \geqslant 0, \tag{2.13}$$

and $\mathcal{G}$ is coherent if and only if $\mathcal{H}_i\mathcal{I}^{\sigma^i} = \mathcal{H}_{i+1}$ for $i \gg 0$. To avoid trivialities, we assume that $\mathcal{G} \neq 0$.

Pick r such that $\mathcal{H}_r \neq 0$. Since Z is right saturating, $\operatorname{Supp}\mathcal{O}_X/\mathcal{H}_r$ contains at most finitely many points from $\{s_i = \sigma^{-i}(s) \mid s \in S,\ i \geqslant 0\}$. Let d be the largest integer such that $\sigma^d(s) = t$ for some $s, t \in S$ and pick $m \geqslant r + d$ such that $(\operatorname{Supp}\mathcal{O}_X/\mathcal{H}_r) \bigcap \sigma^{-j}(S) = \emptyset$ for all $j \geqslant m$. Put $T = \bigcup_{i=m-d}^{m-1} \sigma^{-i}(S)$, and let Y be the open subscheme $X \setminus T$ of X.

Set $\mathcal{U} = \mathcal{H}_{m-d}|_Y$, and for $n \geqslant m$ set $\mathcal{W}_n = \mathcal{H}_n|_Y$, as well as $\mathcal{J}_n = \mathcal{I}^{\sigma^n}|_Y$ and $\mathcal{V}_n = \prod_{i=m}^{n-1} \mathcal{I}^{\sigma^i}|_Y$ (with the convention that the product of an empty set of ideal sheaves equals $\mathcal{O}_Y$). By (2.13) and the choice of m, the ideal sheaves $\mathcal{H}_{m-d}$ and $\prod_{i=m}^{n-1}\mathcal{I}^{\sigma^i}$ are comaximal in $\mathcal{O}_X$, so certainly $\mathcal{U}$ and $\mathcal{V}_n$ are comaximal in $\mathcal{O}_Y$. By induction, (2.13) shows that $\mathcal{H}_{m-d}(\prod_{i=m}^{n-1}\mathcal{I}^{\sigma^i})(\prod_{i=m-d}^{m-1}\mathcal{I}^{\sigma^i}) \subseteq \mathcal{H}_n \subseteq \prod_{i=m}^{n-1}\mathcal{I}^{\sigma^i}$. Restricting to Y, we get $\mathcal{U} \cap \mathcal{V}_n = \mathcal{U}\mathcal{V}_n \subseteq \mathcal{W}_n \subseteq \mathcal{V}_n$. Thus [KRS, Lemma 3.8] implies that $\mathcal{Z}_n = \mathcal{U} + \mathcal{W}_n$ is maximal among ideal sheaves $\mathcal{Z}$ satisfying $\mathcal{Z}\mathcal{V}_n \subseteq \mathcal{W}_n$. Since $\mathcal{Z}_n\mathcal{V}_{n+1} = \mathcal{Z}_n\mathcal{V}_n\mathcal{J}_n \subseteq \mathcal{W}_n\mathcal{J}_n \subseteq \mathcal{W}_{n+1}$, this implies that $\mathcal{Z}_n \subseteq \mathcal{Z}_{n+1}$ for all $n \geqslant m$. Thus we may pick $n_0 \geqslant m$ such that $\mathcal{Z}_n = \mathcal{Z}_{n+1}$ for all $n \geqslant n_0$. For all such n, [KRS, Lemma 3.8] implies that $\mathcal{W}_{n+1} = \mathcal{Z}_n\mathcal{V}_{n+1} = \mathcal{Z}_n\mathcal{V}_n\mathcal{J}_n = \mathcal{W}_n\mathcal{J}_n$. In other words, $\mathcal{H}_{n+1}|_Y = \mathcal{H}_n\mathcal{I}^{\sigma^n}|_Y$ for all $n \geqslant n_0$.

We need to extend this last equation to all of X. If $t \in T$, then $t \notin \sigma^{-n}(S)$ for $n \gg 0$. Looking locally at t this means that $(\mathcal{H}_n)_t = (\mathcal{H}_n\mathcal{I}^{\sigma^n})_t \subseteq (\mathcal{H}_{n+1})_t$ and hence that $(\mathcal{H}_n)_t = (\mathcal{H}_{n+1})_t$ for $n \gg 0$. Since T is finite, there exists a single integer $n_1 \geqslant n_0$ such that for $n \geqslant n_1$ we have $(\mathcal{H}_n\mathcal{I}^{\sigma^n})_x = (\mathcal{H}_{n+1})_x$ locally at every $x \in T$. By the last paragraph $\mathcal{H}_n\mathcal{I}^{\sigma^n} = \mathcal{H}_{n+1}$ for such n and so $\mathcal{G}$ is coherent and $\mathcal{R}$ is right noetherian.

Conversely, suppose that $\mathcal{I}$ is not right saturating. Write $\mathcal{I} = \mathcal{J}^{(1)}\mathcal{J}^{(2)}\cdots\mathcal{J}^{(r)}$ with $S_i = \operatorname{Supp}\mathcal{O}_X/\mathcal{J}^{(i)}$ so that $S = \bigcup S_i$ partitions S into elements of distinct σ-orbits. Pick $s \in S$ such that the set $\{s_i\}_{i\geqslant 0}$ is not critically dense. By renumbering we may assume that $s \in S_1$ and we may further assume that $S_1 \subset \{\sigma^d(s),\ d \geqslant 0\}$. Now choose an infinite set A of nonnegative integers such that the Zariski closure of $\{s_i\}_{i\in A}$ is a proper closed subset W of X.

Set $\mathcal{J} = \mathcal{J}^{(1)}$, and put $\mathcal{J}_n = \mathcal{J}\mathcal{J}^{\sigma}\cdots\mathcal{J}^{\sigma^{n-1}}$ as usual. We claim that $\mathcal{K} = \bigcap_{i\in A}\mathcal{J}^{\sigma^i}$ is nonzero. To see this, let $\mathcal{M}_t$ be the ideal sheaf defining a closed point $t \in S_1$; thus $\prod_{t\in S_1}(\mathcal{M}_t)^e \subseteq \mathcal{J}$ for some $e \geqslant 1$. The closure of $\{\sigma^{-i}(S_1) \mid i \in A\}$ equals $W' = \bigcup \sigma^d(W)$, where the union is over the finite set $\{d \in \mathbb{Z} \mid \sigma^d(s) \in S_1\}$. Hence $W' \neq X$ and the ideal sheaf $\mathcal{M}_{W'}$ defining W' satisfies $(\mathcal{M}_{W'})^e \subseteq \mathcal{K}$. Thus $\mathcal{K} \neq 0$.

Set $\mathcal{H}_n = \mathcal{K} \cap \mathcal{I}_n$ for $n \geqslant 0$, and observe that $\mathcal{G} = \bigoplus \mathcal{G}_n = \bigoplus {}_1(\mathcal{H}_n)_{\sigma^n}$ is a right ideal of $\mathcal{R}$. Write $\mathfrak{m}_{s_n}$ for the maximal ideal in the local ring $\mathcal{O}_{X,s_n}$. Pick $n \in A$ and note that $s_n \notin \operatorname{Supp}\mathcal{O}_X/\mathcal{J}_n = \bigcup_{i=0}^{n-1}\sigma^{-i}(S_1)$ but $s_n \in \operatorname{Supp}\mathcal{O}_X/\mathcal{J}^{\sigma^n}$ by the choice of s. Equivalently, $(\mathcal{J}_n)_{s_n} = (\mathcal{O}_X)_{s_n}$ but $(\mathcal{J}_{n+1})_{s_n} = (\mathcal{J}^{\sigma^n})_{s_n} \subseteq \mathfrak{m}_{s_n}$. Thus $\mathcal{K}_{s_n} \cap (\mathcal{J}_n)_{s_n} = \mathcal{K}_{s_n}$ and $\mathcal{K}_{s_n} \cap (\mathcal{J}_{n+1})_{s_n} = \mathcal{K}_{s_n} \cap (\mathcal{J}^{\sigma^n})_{s_n} = \mathcal{K}_{s_n}$ as $n \in A$. By Nakayama's lemma

$$\left(\mathcal{H}_n\mathcal{I}^{\sigma^n}\right)_{s_n} = \left(\mathcal{H}_n\mathcal{J}^{\sigma^n}\right)_{s_n} \subseteq \mathcal{K}_{s_n}\mathfrak{m}_{s_n} \subsetneq \mathcal{K}_{s_n} = \mathcal{K}_{s_n} \cap (\mathcal{J}_{n+1})_{s_n} = \mathcal{K}_{s_n} \cap (\mathcal{I}_{n+1})_{s_n} = (\mathcal{H}_{n+1})_{s_n}.$$

Since this happens for infinitely many n, the right ideal $\mathcal{G}$ is not coherent and $\mathcal{R}$ is not right noetherian.

By (2.9), $\mathcal{R}^{\mathrm{op}} \cong \mathcal{R}(X, \sigma(Z), \mathcal{L}^{\sigma^{-1}}, \sigma^{-1})$ and so the analogous result on the left (that $\mathcal{R}$ is left noetherian if and only if $\{\sigma^i(c)\colon i \geqslant 0\}$ is critically dense for each $c \in \operatorname{Supp} Z$) follows from the one on the right. The result for noetherian algebras is then obvious. This completes the proof of (1).

(2) If $\mathcal{R}$ is not noetherian, let $\mathcal{G} = \bigoplus \mathcal{G}_n$ be the noncoherent right ideal defined above. Set $\mathcal{M}^n = \sum_{0 \leqslant i \leqslant n} \mathcal{G}_i \mathcal{R}$; thus $(\mathcal{M}^n)_j = ((\mathcal{K} \cap \mathcal{I}_n)\mathcal{I}^{\sigma^n} \cdots \mathcal{I}^{\sigma^{j-1}})_{\sigma^j}$ for $j \geqslant n$. This gives a chain of coherent right ideals $\mathcal{M}^0 \subseteq \mathcal{M}^1 \subseteq \cdots \subseteq \mathcal{R}$. As before, looking locally at a point s_n for $n \in A$ shows that $(\mathcal{M}^n)_j \subsetneq (\mathcal{M}^{n+1})_j$ for all $n \in A$ and all $j \geqslant n+1$. Thus the subsequence $\{\mathcal{M}^n\colon n \in A\}$ gives the desired chain of right ideals. $\square$

3. Ampleness

We maintain the hypotheses from (2.8). The main aim of this section (Theorem 3.1) is to prove, in considerable generality, that the sequence of bimodules $\{\mathcal{R}_n = (\mathcal{I}_n \otimes \mathcal{L}_n)_{\sigma^n}\}$ is ample in the sense of Definition 2.5. Combined with the results of Section 2 this will prove parts (1) and (2) of Theorem 1.1. This section differs significantly from the proof of ampleness in [KRS], because we are proving a much stronger result. First, we need much stronger estimates of the number of points in a σ-orbit that can be separated by the sheaf $\mathcal{L}_n$. Secondly, for applications in [RS1] we need to prove the result for a σ-ample sheaf $\mathcal{L}$, whereas [KRS] only proved the result when $\mathcal{L}$ was also very ample.

Here is our goal:

Theorem 3.1. *Keep Assumptions* 2.8, *and assume in addition that* $\dim X \geqslant 2$, *that $\mathcal{L}$ is σ-ample, and that $\mathcal{I}$ defines a subscheme Z of X such that each point of $S = \operatorname{Supp} Z$ lies on a dense σ-orbit. Then*:

(1) *$\{\mathcal{R}_n = (\mathcal{I}_n \otimes \mathcal{L}_n)_{\sigma^n}\}$ is an ample sequence.*
(2) *Assume in addition that Z is saturating, and set $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma)$. Then the naïve blowup algebra $R = \mathrm{H}^0(X, \mathcal{R})$ is noetherian and there is an equivalence of categories ξ*: qgr-$\mathcal{R} \simeq$ qgr-R *via the inverse equivalences* $\mathrm{H}^0(X, -)$ *and* $- \otimes_R \mathcal{R}$. *Similarly,* $\mathcal{R}$-qgr $\simeq R$-qgr.

We begin with a useful combinatorial notion and some preliminary results related to it.

Definition 3.2. For us, the natural numbers $\mathbb{N}$ contain 0 and we write $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$. A subset $S \subset \mathbb{N}$ is called *sparse* if for every $m \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ there exists $N(m) \in \mathbb{N}_+$ such that $|S \cap \mathbb{N}_n| \leqslant n/m$ for all $n \geqslant N(m)$. If S is a sparse set, then any monotonically increasing function $N: \mathbb{N}_+ \to \mathbb{N}_+$ satisfying this condition is called a *bounding function* for S.

Lemma 3.3. *Let $T_1, T_2, \ldots, T_d \subset \mathbb{N}$ be sparse sets with respective bounding functions $N_1, N_2, \ldots, N_d$. Then $T = \bigcup_{i=1}^d T_i$ is sparse, with one bounding function N given by* $N(m) = \max_{1 \leqslant i \leqslant d} N_i(md)$.

Proof. If $n \geqslant \max_{1 \leqslant i \leqslant d} N_i(md)$, then $|T \cap \mathbb{N}_n| \leqslant \sum_{i=1}^d |T_i \cap \mathbb{N}_n| \leqslant \sum_{i=1}^d n/(md) = n/m$. $\square$

For any $S \subset \mathbb{N}$ and $i \geqslant 1$, define $S + i = \{s + i \mid s \in S\} \subset \mathbb{N}$ and set $S_d = \bigcup_{i=1}^d ((S+i) \cap S)$ for $d \geqslant 1$. In other words, S_d consists of those numbers $s \in S$ such that some integer $s' \in [s-d, s-1]$ also lies in S.

Lemma 3.4. *Let $S \subset \mathbb{N}$ and suppose that, for all $d \geqslant 1$, the sets $S_d \subset \mathbb{N}$ are sparse, with respective bounding functions N_d. Then S is also sparse, with one bounding function being $N(m) = \max(3m, N_{3m}(3m))$.*

Proof. Write $T^c = \mathbb{N} \setminus T$ for the complement of a set $T \subseteq \mathbb{N}$. Fix $d \geqslant 1$ and suppose that $s \in S \setminus S_d$. Then as long as $s \geqslant d$, we know that $\{s-1, s-2, \ldots, s-d\} \subset S^c$. If $t \neq s \in S \setminus S_d$ is another natural number with $t \geqslant d$, then $|s-t| > d$ and so $\{t-1, t-2, \ldots, t-d\} \subset S^c$, with

$$\{s-1, s-2, \ldots, s-d\} \cap \{t-1, t-2, \ldots, t-d\} = \emptyset.$$

Also, there is at most one $u \in S \setminus S_d$ with $0 \leqslant u < d$. Combining these observations, we conclude that

$$\left|S^c \cap \mathbb{N}_n\right| \geqslant d\big(\left|(S \setminus S_d) \cap \mathbb{N}_n\right| - 1\big) \quad \text{for each } n \geqslant 1. \tag{3.5}$$

Now use the formulae$|S \cap \mathbb{N}_n| + |S^c \cap \mathbb{N}_n| = n$ and $|S \cap \mathbb{N}_n| = |(S \setminus S_d) \cap \mathbb{N}_n| + |S_d \cap \mathbb{N}_n|$ to transform (3.5) into

$$|S \cap \mathbb{N}_n| \leqslant \frac{n + d + d|S_d \cap \mathbb{N}_n|}{d+1} \quad \text{for each } n \geqslant 1. \tag{3.6}$$

For a given $m > 0$, take $d = 3m$ in this calculation. If $n \geqslant \max(3m, N_{3m}(3m))$, then

$$|S \cap \mathbb{N}_n| \leqslant \frac{n + 3m + 3m\,|S_{3m} \cap \mathbb{N}_n|}{3m+1} \leqslant \frac{n + 3m + 3m(n/3m)}{3m+1} \leqslant \frac{3n}{3m+1} \leqslant \frac{n}{m}.$$

So $N(m) = \max(3m, N_{3m}(3m))$ defines a bounding function for S and hence S is sparse. □

Define a *reduced d-cycle W on X* to be a formal sum $W = \sum W_i$, where the W_i are distinct integral closed subschemes of X of dimension d. Equivalently, we may identify W with the reduced and equidimensional closed subscheme $\bigcup W_i$. In the proof of the next proposition we will want to induct on the degrees of cycles on X. To do this, fix some closed immersion $\iota : X \hookrightarrow \mathbb{P}^s$ and define the degree of a reduced d-cycle W in X to be the intersection number

$$i\big(H^{\cdot d} \cdot W; \mathbb{P}^s\big) = i\big(\overbrace{H \cdot H \cdots H}^{d} \cdot W; \mathbb{P}^s\big),$$

where H is a hyperplane of $\mathbb{P}^s$. This is, of course, dependent on the given embedding.

We would like to thank Mark Gross for his helpful suggestions concerning the next result.

Lemma 3.7. *Let X be an integral projective scheme of dimension $\geqslant 2$, with $\sigma \in \operatorname{Aut}(X)$ and a fixed projective embedding $\iota : X \to \mathbb{P}^s$ which will be used to measure degrees. Then there exists $M \geqslant 2$ such that $\deg \sigma(Z) \leqslant M \deg Z$ for all integral closed subschemes $Z \subseteq X$.*

Proof. Let $\dim Z = d$. It will be convenient to interpret degrees in terms of the intersection theory on X, as developed in [Kl]. Thus, write $\mathcal{N} = \iota^* \mathcal{O}_{\mathbb{P}^s}(1)$ and let $(\mathcal{N}^{\cdot d} \cdot \sigma(Z))_X$ denote the intersection number from [Kl]; by [Kl, Proposition 5, p. 298 and Corollary 3, p. 301] this equals $i(H^{\cdot d} \cdot \sigma(Z); \mathbb{P}^s)$. Next, by Bertini's theorem [Ju, Théorème I.6.10(2,3)], we may choose generic hyperplanes H_i on $\mathbb{P}^s$, so that $\sigma(Z) \cap H_1 \cap \cdots \cap H_d$ is a reduced set of points and each $E_i = H_i \cap X$ is a reduced and irreducible subscheme of X.

By [Kl, Remark 1, p. 301] applied to $\mathbb{P}^s$, $\deg \sigma(Z) = \#(\sigma(Z) \cap H_1 \cap \cdots \cap H_d)$, the number of points in this intersection, and hence $\deg \sigma(Z) = \#(\sigma(Z) \cap E_1 \cap \cdots \cap E_d)$. This is also the number of points in $(Z \cap \sigma^{-1}(E_1) \cap \cdots \cap \sigma^{-1}(E_d))$ and, of course, each $\sigma^{-1}(E_i)$ is a reduced and irreducible subscheme of X and hence of $\mathbb{P}^s$. Thus, by Bezout's theorem [Fu, Example 8.4.6], $\deg \sigma(Z) \leqslant \deg Z(\prod_{i=1}^{d} \deg \sigma^{-1}(E_i))$. Finally, by [Kl, Remark 2, p. 301], the degree of the divisor $\sigma^{-1}(E_i)$ regarded as a subscheme of X equals its degree regarded as an element of the complete linear system $|\mathcal{O}_X(\sigma^{-1}(E_i))| = |\sigma^*(\mathcal{N})|$. In other words, $\deg \sigma(Z) \leqslant M \deg Z$, where $M = (\deg \sigma^*(\mathcal{N}))^d$. $\square$

Let $p_1, \ldots, p_n$ be closed points on a projective scheme X. Then an invertible sheaf $\mathcal{L}$ *separates* $\{p_1, \ldots, p_n\}$ if, for each $1 \leqslant i \leqslant n$, there is a section $s_i \in \mathrm{H}^0(X, \mathcal{L})$ such that $s_i(p_i) \neq 0$ but $s_i(p_j) = 0$ for $j \neq i$.

In Proposition 3.11 we will use intersection theory to estimate the number of points in a σ-orbit on X that can be separated by $\mathcal{L}_n$, in the notation of Assumptions 2.8. The main idea, which is provided by the next proposition, is to study hyperplane sections $W = H \cap X$ of a closed immersion $\iota : X \hookrightarrow \mathbb{P}^s$, and study the size of the sets $W \cap \{\sigma^i(x) \mid i \geqslant 0\}$ for $x \in X$ lying on a dense σ-orbit. These sets need not be finite, so we actually show that $\{i \geqslant 0 \mid \sigma^i(x) \in W\}$ is sparse, and estimate its bounding function.

Proposition 3.8. *Let X be an integral projective scheme of dimension $\geqslant 2$ with automorphism α and an immersion $\iota : X \to \mathbb{P}^s$ that will be used to measure degrees. Then, for any $e \geqslant 1$ and $0 \leqslant d < \dim X$, there exists a bounding function $N_{d,e}(m)$, depending on (X, α, ι), with the following property*:

> *For any $x \in X$ for which $P = \{\alpha^i(x) \mid i \in \mathbb{Z}\}$ is dense in X, and any reduced d-cycle Z on X with* $\deg Z \leqslant e$, *the set $S = \{i \geqslant 0 \mid \alpha^i(x) \in Z\}$ is sparse with bounding function $N_{d,e}(m)$.*

Proof. If Z is a reduced 0-cycle with $\deg Z \leqslant e$, then $\deg Z$ is just the number of points in Z and so $\#S \leqslant e$ for any $x \in X$. It therefore suffices to take $N_{0,e} = e$. Now, for some $0 < d < \dim X$, suppose by induction that $N_{c,f}$ has been constructed for every $0 \leqslant c < d$ and $f \geqslant 1$. By taking supremums, we can assume that $N_{c,f} = N_{b,f}$ for all $0 \leqslant b \leqslant c < d$ and that $N_{c,f} \leqslant N_{c,f+1}$ for all f.

Fix $x \in X$ such that $P = \{\alpha^i(x) \mid i \in \mathbb{Z}\}$ is dense, let Y be an irreducible d-dimensional subvariety of X with $\deg Y \leqslant e$, and define $S = S^Y = \{i \geqslant 0 \mid \alpha^i(x) \in Y\}$. Fix some $a \geqslant 1$ and define the set S_a as before Lemma 3.4. Since $S + j = \{\ell \geqslant j \colon \alpha^\ell(x) \in \alpha^j(Y)\}$, clearly $S_a \subseteq \{i \geqslant 0 \mid \alpha^i(x) \in \bigcup_{k=1}^{a}(Y \cap \alpha^k(Y))\}$. For each k, write $(Y \cap \alpha^k(Y)) = C_{k1} \cup C_{k2} \cup \cdots \cup C_{k,r_k}$ where the C_{kj} are distinct irreducible components.

Suppose first that $Y = \alpha^k(Y)$ for some $k \geqslant 1$. In this case, if $S \neq \emptyset$, then certainly $Y \cap P \neq \emptyset$ and so P is entirely contained in $\bigcup_{i=1}^{k} \alpha^i(Y)$. This contradicts the density of P. Therefore, $S = \emptyset$, which is trivially sparse and any bounding function whatsoever will do.

Thus we may assume that $Y \neq \alpha^k(Y)$ for $k \geqslant 1$. In particular, since Y is irreducible, $\dim C_{ij} < \dim Y$ for all i, j. Let $M \geqslant 2$ denote the constant from Lemma 3.7. Then Bezout's Theorem [Fu, Example 8.4.6] implies that, for any $i \leqslant r_k$,

$$\deg C_{ki} \leqslant \sum_{j=1}^{r_k} \deg C_{kj} \leqslant (\deg Y)\big(\deg \alpha^k(Y)\big) \leqslant (\deg Y)\big(M^k \deg Y\big) \leqslant M^a e^2. \tag{3.9}$$

For each k, j set $T_{kj} = \{i \geqslant 0 \mid \alpha^i(x) \in C_{kj}\}$. Now apply induction, recalling our assumptions on the $N_{c,f}$ for $c < d$. This implies that each T_{kj} is sparse and that we can use $N_{c,b(a)}(m)$, where $c = d-1$ and $b(a) = M^a e^2$, as its bounding function. By definition, $\deg C_{kj} \geqslant 1$ for each k, j and so (3.9) also implies that $r_k \leqslant b(a)$. Therefore, as $S_a = \bigcup T_{kj}$, Lemma 3.3 implies that S_a is a sparse set with bounding function $\widetilde{N}_a(m) = N_{c,b(a)}(ab(a)m)$. By Lemma 3.4, $S = S^Y$ is a sparse set with bounding function $\widetilde{N}(m) = \max\{3m, \widetilde{N}_{3m}(3m)\} = \max\{3m, N_{c,b(3m)}(9m^2 b(3m))\}$. Notice that this bounding function is independent both of x and of Y (and also works for an irreducible subscheme Y satisfying $Y = \alpha^k(Y)$).

Finally, let Z be an arbitrary reduced d-cycle with $\deg Z \leqslant e$ and set $S' = \{i \geqslant 0 \mid \alpha^i(x) \in Z\}$. Write $Z = Y_1 \cup \cdots \cup Y_r$, where the Y_j are distinct irreducible components of dimension d. By the multilinearity of the intersection form, $\deg Z = \sum_{i=1}^r \deg Y_i$ and so $1 \leqslant \deg Y_j \leqslant e$ for each j and hence $r \leqslant e$. For each j, set $U_j = S^{Y_j} = \{i \geqslant 0 \mid \alpha^i(x) \in Y_j\}$. By the conclusion of the last paragraph, each U_j is sparse with bounding function $\widetilde{N}(m)$. Since $S' = \bigcup_{j=1}^r U_j$ with $r \leqslant e$, Lemma 3.3 implies that S' is sparse with bounding function $N(m) = \widetilde{N}(em)$. This depends only on e and the previously constructed bounding functions, and all constructions are independent of the choice of x. Thus we may take $N_{d,e} = N$, completing the induction. $\square$

The following easy fact was observed in the proof of [KRS, Lemma 4.4].

Lemma 3.10. *Suppose that an invertible sheaf $\mathcal{L}$ separates the closed points $\{p_1, \ldots, p_n\}$ on a projective scheme X. If $\mathcal{N}$ is a very ample invertible sheaf then $\mathcal{N} \otimes \mathcal{L}$ also separates $\{p_1, \ldots, p_n\}$.*

We next use the estimates from Proposition 3.8 to get good bounds on the separation of points. The idea behind the proof is similar to that of [KRS, Proposition 4.6], although for the applications in this paper we need much more efficient bounds than those given in [KRS].

Proposition 3.11. *Let X be an integral projective scheme with $\dim X \geqslant 2$, $\sigma \in \operatorname{Aut}(X)$, and a σ-ample invertible sheaf $\mathcal{L}$. Then, for any real number $\delta > 0$, there exists $M = M(\delta) \geqslant 0$ with the following property*:

> *Let $x \in X$ be a closed point, such that $\{\sigma^i(x)\colon i \in \mathbb{Z}\}$ is dense and write $P_n = \{\sigma^{-i}(x)\colon 0 \leqslant i \leqslant n-1\}$ for each $n \geqslant 1$. Then, for all $n \geqslant M$ and all $p \in \mathbb{Z}$, the sheaf $\mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^p}$ separates P_n.*

Proof. Fix $\delta > 0$. As observed at the beginning of the proof of [KRS, Lemma 4.4], $\mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^p}$ separates P_n if and only if $\mathcal{L}_{\lfloor \delta n \rfloor}$ separates $\sigma^p(P_n) = \{\sigma^{-i}(y)\colon 0 \leqslant i \leqslant n-1\}$, where $y = \sigma^p(x)$. Since M will be chosen independently of the point x, it therefore suffices to prove the result when $p = 0$. By [AV, Proposition 3.2], $\mathcal{L}_r$ is very ample for all $r \gg 0$. Fix some such $\mathcal{L}_r$, set $d + 1 = \dim_k \mathrm{H}^0(X, \mathcal{L}_r)$ and write $\tau : X \hookrightarrow \mathbb{P}^d$ for the closed immersion associated to a fixed basis of $\mathrm{H}^0(X, \mathcal{L}_r)$. More generally, for any $j \in \mathbb{Z}$ write $\tau_j = \tau\sigma^j : X \hookrightarrow \mathbb{P}^d$, which is the closed immersion associated to an appropriate basis of $\mathrm{H}^0(X, \mathcal{L}_r^{\sigma^j})$.

Given a closed point $y \in X$ such that $\{\sigma^i(y)\colon i \in \mathbb{Z}\}$ is dense, a hyperplane H of $\mathbb{P}^d$ and $j \in \mathbb{Z}$, set $S(y, H, j) = \{i \geqslant 0\colon \sigma^{-i}(y) \in \tau_j^{-1}(H)\}$. For such a hyperplane H it follows, for example from Bezout's Theorem that $\tau^{-1}(H) = H \cap X$, thought of as a reduced $(\dim X - 1)$-cycle, satisfies $\deg \tau^{-1}(H) \leqslant \deg X$. Thus, by Proposition 3.8 (applied with $\alpha = \sigma^{-1}$), there is a fixed bounding function $N(m)$ such that, independently of H and y, the set $S(y, H, 0)$ is sparse with bounding function N. Notice that $\sigma^{j-i}(y) \in \tau^{-1}(H)$ if and only if $\sigma^{-i}(y) \in \sigma^{-j}(\tau^{-1}(H)) = \tau_j^{-1}(H)$ and so $S(y,H,j) = S(\sigma^j(y),H,0)$. Since $N(m)$ was chosen independently of y, we conclude that each $S(y,H,j)$ is sparse with bounding function $N(m)$. To rephrase, fix $m>0$ and set $M_1 = N(m)$. Then for all $n \geqslant M_1$ and $j \in \mathbb{Z}$, at most n/m of the points in the set P_n lie on a single hyperplane section $\tau_j^{-1}(H)$. We emphasize that the choice of M_1 is independent of y.

A set of closed points $\{y_1,\dots,y_t\} \subset \mathbb{P}^d$ is called *linearly general* if the smallest linear subspace containing the points has dimension $t-1$. Suppose P is a set of closed points in $\mathbb{P}^d$ such that $|P| \geqslant d+1$, and such that P is not entirely contained in any hyperplane of $\mathbb{P}^d$. Then given $y \in P$, an easy inductive argument shows that we can find d other points $y_1,\dots,y_d \in P$ such that $\{y,y_1,\dots,y_d\}$ is linearly general in $\mathbb{P}^d$. Now choose $n \geqslant M_1$ and pick any $z \in P_n$. Suppose first that $n > n/m$. By the previous paragraph we can pick a set of points $T_0 \subseteq P_n \setminus \{z\}$ with $|T_0| = d$ such that $\tau(T_0 \cup \{z\})$ is linearly general in $\mathbb{P}^d$. If $n-d > n/m$, repeat the process to find $T_1 \subseteq P_n \setminus (\{z\} \cup T_0)$ with $|T_1| = d$ such that $\tau_r(\{z\} \cup T_1)$ is linearly general in $\mathbb{P}^d$. Continue this process inductively as long as possible. This partitions P_n into disjoint subsets

$$P_n = \{z\} \cup T_0 \cup \cdots \cup T_{k-1} \cup V$$

where V contains $q \leqslant n/m$ elements and, for each i, $|T_i| = d$ and $\tau_{ir}(\{z\} \cup T_i)$ is linearly general in $\mathbb{P}^d$.

Fix $0 \leqslant j \leqslant k-1$. Since $\tau_{jr}(\{z\} \cup T_j)$ is linearly general in $\mathbb{P}^d$, we can find a hyperplane H in $\mathbb{P}^d$ such that $\tau_{jr}(T_j) \subseteq H$ but $\tau_{jr}(z) \notin H$. Since the morphism $\tau_{jr}: X \hookrightarrow \mathbb{P}^d$ is defined via a basis of $\mathrm{H}^0(X, \mathcal{L}_r^{\sigma^{jr}})$ this is equivalent to the existence of a section $s_j \in \mathrm{H}^0(X, \mathcal{L}_r^{\sigma^{jr}})$ with $s_j(z) \neq 0$ but $s_j(y) = 0$ for all $y \in T_j$. Now $\mathcal{L}_r^{\sigma^{(k+i)r}}$ is very ample for all $i \geqslant 0$ and so it separates any pair of points. Thus, if $V = \{v_0,\dots,v_{q-1}\}$ then, for each $i \leqslant q-1$, we can also find $t_i \in \mathrm{H}^0(X, \mathcal{L}_r^{\sigma^{(k+i)r}})$ such that $t_i(z) \neq 0$ but $t_i(v_i) = 0$. Consequently,

$$\mathcal{L}_{(k+q)r} = \mathcal{L}_r \otimes \mathcal{L}_r^{\sigma^r} \otimes \cdots \otimes \mathcal{L}_r^{\sigma^{(k+q-1)r}}$$

has a section $s = s_0 \otimes \cdots \otimes s_{k-1} \otimes t_0 \otimes \cdots \otimes t_{q-1}$ with $s(z) \neq 0$ but $s(y) = 0$ for all $y \in P_n \setminus \{z\}$. Since $z \in P_n$ was arbitrary, we conclude that $\mathcal{L}_{(k+q)r}$ separates P_n for any $n \geqslant M_1$.

It remains to convert this into the assertion that $\mathcal{L}_{\lfloor \delta n \rfloor}$ separates P_n for n large. By [Ke1, Theorem 6.1], $d = d(r) = \dim_k \mathrm{H}^0(X, \mathcal{L}_r) - 1$ grows at least quadratically in r. So we may choose r large enough so that $r/d < \delta/3$ and then take $m \geqslant 3r/\delta$ in the argument above. Since $n = kd + 1 + q$, this ensures that $(k+q)r < nr/d + qr \leqslant \frac{2}{3}\delta n$. Set $f(n) = \lfloor \delta n \rfloor - (k+q)r$. Since $\lim_{n\to\infty} f(n) = \infty$ and $\mathcal{L}$ is σ-ample, $\mathcal{L}_{f(n)}$ is very ample for $n \gg 0$, say for $n \geqslant M_2$. Since $\mathcal{L}_{(k+q)r}$ separates P_n for all $n \geqslant M_1$, it follows from Lemma 3.10 that the sheaf $\mathcal{L}_{\lfloor \delta n \rfloor} = \mathcal{L}_{(k+q)r} \otimes \mathcal{L}_{f(n)}$ also separates P_n for all $n \geqslant M = M_1 + M_2$. $\square$

For the proof of Theorem 3.1 we will need the following concept of Castelnuovo–Mumford regularity. Let X be a projective scheme with a very ample invertible sheaf $\mathcal{N}$. Then a coherent sheaf $\mathcal{F}$ is called *m-regular with respect to $\mathcal{N}$* if $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{N}^{\otimes m-i}) = 0$ for all $1 \leqslant i \leqslant \dim X$. If $\mathcal{F}$ is m-regular with respect to $\mathcal{N}$ then it is also $m+1$-regular with respect to $\mathcal{N}$ (see [La, Theorem 1.8.5 and Remark 1.8.14]). The *regularity* $\operatorname{reg}_{\mathcal{N}} \mathcal{F}$ *of $\mathcal{F}$ with respect to $\mathcal{N}$* is then defined to be the minimum m for which $\mathcal{F}$ is m-regular with respect to $\mathcal{N}$. We will delete reference to the sheaf $\mathcal{N}$ if it is understood.

We need the following three results, the first of which provides a useful sheaf for measuring regularity.

Lemma 3.12. *(See [Fj, Theorem 1, p. 520].) Let X be a projective scheme. Then there exists a very ample sheaf $\mathcal{N}$ on X such that $\mathrm{H}^i(X, \mathcal{L} \otimes \mathcal{N}) = 0$ for every very ample invertible sheaf $\mathcal{L}$ and integer $i \geqslant 1$.*

Lemma 3.13. *(See [Ke2, Proposition 2.7].) Let X be a projective scheme with very ample invertible sheaf $\mathcal{N}$. Then there is a constant C (depending on X and $\mathcal{N}$) with the following property:*

> *If $\mathcal{F}, \mathcal{G}$ are coherent sheaves on X such that the closed set where both $\mathcal{F}$ and $\mathcal{G}$ fail to be locally free has dimension $\leqslant 2$, then $\operatorname{reg}_{\mathcal{N}}(\mathcal{F} \otimes \mathcal{G}) \leqslant \operatorname{reg}_{\mathcal{N}} \mathcal{F} + \operatorname{reg}_{\mathcal{N}} \mathcal{G} + C$.*

Corollary 3.14. *Let X be a projective scheme with very ample invertible sheaf $\mathcal{N}$. Let $\{\mathcal{F}_n\}$ be a sequence of coherent sheaves on X, such that, for each n, the closed set where $\mathcal{F}_n$ is not locally free has dimension at most 2. Then $\{\mathcal{F}_n\}$ is an ample sequence if and only if $\lim_{n\to\infty} \operatorname{reg}_{\mathcal{N}} \mathcal{F}_n = -\infty$.*

Proof. Suppose that $\lim_{n\to\infty} \operatorname{reg} \mathcal{F}_n = -\infty$ and let $\mathcal{G}$ be a coherent sheaf. Then Lemma 3.13 implies that $\operatorname{reg}(\mathcal{G} \otimes \mathcal{F}_n) \leqslant \operatorname{reg} \mathcal{F}_n + (\operatorname{reg} \mathcal{G} + C)$ for some constant C. Thus $\lim_{n\to\infty} \operatorname{reg}(\mathcal{G} \otimes \mathcal{F}_n) = -\infty$ and so $\mathrm{H}^i(X, \mathcal{G} \otimes \mathcal{F}_n) = 0$ for $n \gg 0$ and all $i > 0$. By Mumford's theorem [La, Theorem 1.8.5 and Remark 1.8.14] $\mathcal{G} \otimes \mathcal{F}_n$ is also generated by its sections for $n \gg 0$. Hence $\{\mathcal{F}_n\}$ is ample.

Conversely, suppose that $\{\mathcal{F}_n\}$ is an ample sequence and pick $m \in \mathbb{Z}$. Then it follows that $\mathrm{H}^i(X, \mathcal{N}^{\otimes m} \otimes \mathcal{F}_n) = 0$ for all $i \geqslant 1$ and $n \gg 0$. Therefore, $\operatorname{reg} \mathcal{F}_n \leqslant m + \dim X$ for all $n \gg 0$ which, since m is arbitrary, implies that $\lim_{n\to\infty} \operatorname{reg} \mathcal{F}_n = -\infty$. $\square$

One advantage of regularity is that it gives a convenient way to rephrase Proposition 3.11.

Corollary 3.15. *Assume the hypotheses and notation from Proposition 3.11, let the closed point $x \in X$ have ideal sheaf $\mathcal{J}$ and pick a very ample invertible sheaf $\mathcal{N}$ by Lemma 3.12. Then, for any real number $\delta > 0$ and function $f : \mathbb{N} \to \mathbb{Z}$, we have $\operatorname{reg}_{\mathcal{N}}(\mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}} \otimes \mathcal{J}_n) \leqslant \dim X + 1$ for all $n \gg 0$.*

Proof. Note that $P_n = \{\sigma^{-i}(x)\colon 0 \leqslant i \leqslant n-1\}$ has ideal sheaf $\mathcal{J}_n = \mathcal{J}\mathcal{J}^{\sigma} \cdots \mathcal{J}^{\sigma^{n-1}}$. By [AV, Proposition 3.2], the sheaf $\mathcal{L}_{\lfloor \delta n \rfloor}$, and hence the sheaf $\mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}}$, is very ample for $n \gg 0$. Thus Lemma 3.12 implies that $\mathrm{H}^1(X, \mathcal{N}^{\otimes m} \otimes \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}}) = 0$ for all $n \gg 0$ and $m \geqslant 1$. Pick M by Proposition 3.11; thus, for all $p \in \mathbb{Z}$ and all $n \geqslant M$, $\mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^p}$ separates P_n. This holds, in particular, for $p = f(n)$ and so $\mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}}$ separates P_n for $n \geqslant M$. As $\mathcal{N}$ is very ample, Lemma 3.10 implies that $\mathcal{N}^{\otimes m} \otimes \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}}$ also separates P_n for all $n \gg 0$. This in turn implies that the canonical map

$$\mathrm{H}^0\big(X, \mathcal{N}^{\otimes m} \otimes \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}}\big) \to \mathrm{H}^0\big(X, \mathcal{N}^{\otimes m} \otimes \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}} \otimes \mathcal{O}_X/\mathcal{J}_n\big)$$

is surjective for $n \gg 0$ (see the last paragraph of the proof of [KRS, Lemma 4.4]). From the long exact sequence in cohomology, we conclude that $\mathrm{H}^1(X, \mathcal{N}^{\otimes m} \otimes \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}} \otimes \mathcal{J}_n) = 0$ for $n \gg 0$ and $m \geqslant 1$.

The higher cohomology groups are much easier to deal with. Indeed, fix $r \geqslant 2$ and $m \geqslant 1$ and write $\mathcal{F} = \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}} \otimes \mathcal{N}^{\otimes m}$ for some $n \gg 0$. Recall that $\mathcal{O}_X/\mathcal{J}_n$ is supported at a set of dimension 0 and that $\mathcal{L}$ is σ-ample. Thus from the exact sequence

$$\mathrm{H}^{r-1}(X, \mathcal{F} \otimes \mathcal{O}_X/\mathcal{J}_n) \to \mathrm{H}^r(X, \mathcal{F} \otimes \mathcal{J}_n) \to \mathrm{H}^r(X, \mathcal{F}) \to \mathrm{H}^r(X, \mathcal{F} \otimes \mathcal{O}_X/\mathcal{J}_n),$$

one obtains $\mathrm{H}^r(X, \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}} \otimes \mathcal{N}^{\otimes m} \otimes \mathcal{J}_n) \cong \mathrm{H}^r(X, \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}} \otimes \mathcal{N}^{\otimes m})$. But, for $n \gg 0$, the sheaf $\mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}}$ is very ample and so $\mathrm{H}^r(X, \mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}} \otimes \mathcal{N}^{\otimes m}) = 0$ by the choice of $\mathcal{N}$. Altogether, this ensures that $\operatorname{reg}(\mathcal{L}_{\lfloor \delta n \rfloor}^{\sigma^{f(n)}} \otimes \mathcal{J}_n) \leqslant \dim X + 1$ for $n \gg 0$, as claimed. $\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. (1) We need to prove that the sequence $\{\mathcal{R}_n = \mathcal{L}_n \otimes \mathcal{I}_n\}$ is ample. All regularities in the proof will be taken with a respect to a very ample sheaf $\mathcal{N}$ satisfying the conclusion of Lemma 3.12. Pick a sequence of closed points $s_1, s_2, \dots, s_d$ in S, in general with repeats, such that if $\mathcal{J}^{(i)}$ is the ideal sheaf of the reduced point s_i, then $\mathcal{J} = \mathcal{J}^{(1)}\mathcal{J}^{(2)} \cdots \mathcal{J}^{(d)} \subseteq \mathcal{I}$. Write

$$\mathcal{G}_n = \mathcal{L}_n \otimes \mathcal{J}_n^{(1)}\mathcal{J}_n^{(2)} \cdots \mathcal{J}_n^{(d)} \quad \text{and} \quad \mathcal{H}_n = \mathcal{L}_n \otimes \mathcal{J}_n^{(1)} \otimes \mathcal{J}_n^{(2)} \otimes \cdots \otimes \mathcal{J}_n^{(d)}.$$

Now let $r = \lfloor n/2d \rfloor$ and $s = n - dr$ and decompose $\mathcal{H}_n$ as

$$\mathcal{H}_n = \mathcal{L}_s \otimes \left(\mathcal{L}_r^{\sigma^s} \otimes \mathcal{J}_n^{(1)}\right) \otimes \left(\mathcal{L}_r^{\sigma^{r+s}} \otimes \mathcal{J}_n^{(2)}\right) \otimes \cdots \otimes \left(\mathcal{L}_r^{\sigma^{(d-1)r+s}} \otimes \mathcal{J}_n^{(d)}\right).$$

By Corollary 3.15, $\operatorname{reg}(\mathcal{L}_r^{\sigma^{(ir+s)}} \otimes \mathcal{J}_n^{(i+1)}) \leqslant \dim X + 1$ for each $i \geqslant 0$ and all $n \gg 0$. By Lemma 3.13 there therefore exists a constant C depending only on X and $\mathcal{N}$ such that $\operatorname{reg} \mathcal{H}_n \leqslant \operatorname{reg} \mathcal{L}_s + (\dim X + 1 + C)d$. As $\{\mathcal{L}_n\}$ is an ample sequence, Corollary 3.14 implies that $\lim_{n\to\infty} \operatorname{reg} \mathcal{L}_n = -\infty$. Since $s \to \infty$ as $n \to \infty$ this implies that $\lim_{n\to\infty} \operatorname{reg} \mathcal{H}_n = -\infty$.

The multiplication map $\mu : \mathcal{H}_n \to \mathcal{G}_n$ yields a short exact sequence $0 \to \mathcal{K}_n \to \mathcal{H}_n \to \mathcal{G}_n \to 0$ such that each sheaf $\mathcal{K}_n$ is supported on a finite set. For $i > 0$ and $m \in \mathbb{Z}$, the long exact sequence in cohomology therefore shows that $\mathrm{H}^i(X, \mathcal{H}_n \otimes \mathcal{N}^{\otimes m}) = 0 \Leftrightarrow \mathrm{H}^i(X, \mathcal{G}_n \otimes \mathcal{N}^{\otimes m}) = 0$. Hence $\lim_{n\to\infty} \operatorname{reg} \mathcal{G}_n = -\infty$. Finally, the inclusion $\mathcal{J} \subseteq \mathcal{I}$ induces an exact sequence $0 \to \mathcal{G}_n \to \mathcal{R}_n \to \mathcal{C}_n \to 0$ where $\mathcal{C}_n$ is again supported on a finite set. A similar long exact sequence argument implies that $\lim_{n\to\infty} \operatorname{reg} \mathcal{R}_n = -\infty$ and so, by Corollary 3.14, $\{\mathcal{R}_n\}$ is an ample sequence.

(2) By Proposition 2.12(1), $\mathcal{R}$ is right noetherian and by part (1), the sequence $\{\mathcal{R}_n\}$ is ample. Thus all of the hypotheses of Theorem 2.6 are satisfied and so the naïve blowup algebra $R = \mathrm{H}^0(X, \mathcal{R})$ is right noetherian with qgr-$\mathcal{R} \simeq$ qgr-R.

As was noted in (2.9), $\mathcal{R}^{\mathrm{op}} \cong \mathcal{R}(X, \sigma(c), \mathcal{L}^{\sigma^{-1}}, \sigma^{-1})$, by [Ke1, Corollary 5.1], $\mathcal{L}^{\sigma^{-1}}$ is σ^{-1}-ample, and obviously Z is saturating with respect to σ^{-1} as well. Thus the claims on the left follow from those on the right. This completes the proof. $\square$

One can modify Theorem 3.1 so that it also works for curves, but this case is rather uninteresting (see the discussion in [KRS, pp. 511–512]), so it will be ignored.

The results we have proved thus far also give a partial converse to Theorem 3.1.

Proposition 3.16. *Keep the assumptions of* (2.8). *Suppose that* $\dim X \geqslant 2$, *$\mathcal{L}$ is σ-ample, and every point $s \in S = \operatorname{Supp} Z$ lies on a dense σ-orbit, but Z is not right saturating. Then neither $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma)$ nor $R = \mathrm{H}^0(X, \mathcal{R})$ is right noetherian.*

Proof. The proof of [KRS, Proposition 4.8] works without change, except that references to [KRS, Theorem 3.10 and Proposition 4.6] should be replaced by references to Proposition 2.12, respectively Theorem 3.1(1). □

The aim of the paper is to obtain a deeper understanding of the algebras $\mathcal{R}(X, Z, \mathcal{L}, \sigma)$ and $R = \mathrm{H}^0(X, \mathcal{R})$ under the assumptions of Theorem 3.1 and so we make the following assumptions from now on:

Assumptions 3.17. Let X be an integral projective scheme of dimension $d \geqslant 2$. Fix $\sigma \in \operatorname{Aut}(X)$ and a σ-ample invertible sheaf $\mathcal{L}$. Finally assume that $Z \subseteq X$ is a saturating zero-dimensional subscheme of X, or $Z = \emptyset$, with ideal sheaf $\mathcal{I} = \mathcal{I}_Z$. We will always write $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma)$ and $R = \mathrm{H}^0(X, \mathcal{R}) = R(X, Z, \mathcal{L}, \sigma)$. By Theorem 3.1, R is noetherian with qgr-$R \simeq$ qgr-$\mathcal{R}$.

It is often useful to work with connected graded rings that are generated in degree one and we give two ways in which this may be achieved; either by replacing R by a large Veronese ring or by assuming that the invertible sheaf $\mathcal{L}$ is "sufficiently ample."

Proposition 3.18. *Keep the hypotheses of* (3.17). *Then the Veronese ring $R^{(q)} = \bigoplus_{n \geqslant 0} R_{qn}$ is generated in degree 1 for all $q \gg 0$.*

Proof. The argument is quite similar to that of [KRS, Proposition 4.10], but some technical adjustments are needed and so we will give a full proof.

Note that the Veronese ring $R^{(p)}$ is itself a naïve blowup algebra, namely $R^{(p)} \cong R(X, Z_p, \mathcal{L}_p, \sigma^p)$, where Z_p is the 0-dimensional subscheme defined by $\mathcal{I}_p$. Set $\mathcal{J}_n = \mathcal{I}_n \otimes \mathcal{L}_n$ for $n \geqslant 1$. By Theorem 3.1 we may chose $r \geqslant 1$ such that $\mathcal{J}_r$ is generated by its global sections. Thus, there exists a short exact sequence

$$0 \to \mathcal{V} \to \mathrm{H}^0(X, \mathcal{J}_r) \otimes \mathcal{O}_X \to \mathcal{J}_r \to 0,$$

of $\mathcal{O}_X$-modules, for some sheaf $\mathcal{V}$. Since $\{\mathcal{J}_n\}$ and hence $\{\mathcal{J}_n^{\sigma^r}\}$ are ample sequences, there exists n_0 such that $\mathrm{H}^1(X, \mathcal{V} \otimes \mathcal{J}_{nr}^{\sigma^r}) = 0$ for $n \geqslant n_0$. Tensoring the displayed exact sequence on the right with $\mathcal{J}_{nr}^{\sigma^r}$ for $nr \geqslant n_0$ gives an exact sequence

$$0 \to \operatorname{Tor}_1\big(\mathcal{J}_r, \mathcal{J}_{nr}^{\sigma^r}\big) \to \mathcal{V} \otimes \mathcal{J}_{nr}^{\sigma^r} \xrightarrow{\theta} \mathrm{H}^0(X, \mathcal{J}_r) \otimes \mathcal{J}_{nr}^{\sigma^r} \to \mathcal{J}_r \otimes \mathcal{J}_{nr}^{\sigma^r} \to 0.$$

By [KRS, Lemma 3.3], the sheaf $\operatorname{Tor}_1(\mathcal{J}_r, \mathcal{J}_{nr}^{\sigma^r})$ is supported on a finite set of points. Therefore, if $\mathcal{K} = \operatorname{Ker}(\theta)$, then $\mathrm{H}^1(X, \mathcal{K}) = \mathrm{H}^1(X, \mathcal{V} \otimes \mathcal{J}_{nr}^{\sigma^r}) = 0$ for $nr \geqslant n_0$. Taking global sections of the exact sequence

$$0 \to \mathcal{K} \to \mathrm{H}^0(X, \mathcal{J}_r) \otimes \mathcal{J}_{nr}^{\sigma^r} \to \mathcal{J}_r \otimes \mathcal{J}_{nr}^{\sigma^r} \to 0,$$

therefore gives the exact sequence

$$\mathrm{H}^0(X, \mathcal{J}_r) \otimes \mathrm{H}^0\big(X, \mathcal{J}_{nr}^{\sigma^r}\big) \to \mathrm{H}^0\big(X, \mathcal{J}_r \otimes \mathcal{J}_{nr}^{\sigma^r}\big) \to 0. \tag{3.19}$$

Consider the sequence $0 \to \mathrm{Tor}_1(\mathcal{L}_r/\mathcal{J}_r, \mathcal{J}_{nr}^{\sigma^r}) \to \mathcal{J}_r \otimes \mathcal{J}_{nr}^{\sigma^r} \to \mathcal{J}_{(n+1)r} \to 0$. Since $\mathrm{Tor}_1(\mathcal{L}_r/\mathcal{J}_r, \mathcal{J}_{nr}^{\sigma^r})$ is supported on a finite set, $\mathrm{H}^1(X, \mathrm{Tor}_1(\mathcal{L}_r/\mathcal{J}_r, \mathcal{J}_{nr}^{\sigma^r})) = 0$ and so the map $\mathrm{H}^0(X, \mathcal{J}_r \otimes \mathcal{J}_{nr}^{\sigma^r}) \to \mathrm{H}^0(X, \mathcal{J}_{(n+1)r})$ is a surjection. Together with (3.19), this shows that the multiplication map $R_r \otimes R_{nr} \to R_{(n+1)r}$ is surjective. By induction, we obtain $R_{jr}R_{nr} = R_{(n+j)r}$ for all $j \geqslant 1$ and $nr \geqslant n_0$. This implies that for such nr the ring $R^{(nr)}$ is generated in degree one; that is, by R_{nr}. □

Proposition 3.20. *Keep the hypotheses of* (3.17)*, and assume that $\mathcal{L}$ is also ample and generated by its global sections. Then there exists $M \in \mathbb{N}$ such that, for $m \geqslant M$:*

(1) *$\mathcal{I}_n \otimes \mathcal{L}_n^{\otimes m}$ is generated by its global sections for all $n \geqslant 1$.*
(2) *$R(X, Z, \mathcal{L}^{\otimes m}, \sigma)$ is generated in degree* 1.

Proof. This is similar to the proof of [KRS, Proposition 4.12] although, as happened in the proof of Proposition 3.18, one has to contend with some nonzero Tor groups. The details are left to the reader and a full proof can be found in [RS2]. □

The following technical variant of Theorem 3.1(1) is needed in [RS1].

Corollary 3.21. *Keep the hypotheses of* (3.17) *and assume that $\mathcal{L}$ is also very ample. Let $\mathcal{J}$ be an ideal sheaf such that $\mathcal{O}_X/\mathcal{J}$ has finite support. Then there exists a positive integer $M \in \mathbb{N}$ such that $\mathrm{H}^j(X, \mathcal{J}^{\sigma^s} \otimes \mathcal{I}_n \otimes \mathcal{L}_n^{\otimes m}) = 0$ for all $m \geqslant M$, $n \geqslant 1$, $j \geqslant 1$ and $s \in \mathbb{Z}$.*

Proof. By Theorem 3.1(1), the corollary is true for any fixed value of s and m, so the point is to obtain a uniform bound M, which we do by appealing to Castelnuovo–Mumford regularity.

It is easy to see that, for $r \geqslant 0$, there is a constant $E(r)$ with the following property: If $\mathcal{M}$ is a sheaf of ideals such that $\mathcal{O}_X/\mathcal{M}$ has length r then $\mathrm{reg}_{\mathcal{L}}\,\mathcal{M} \leqslant E(r)$ (see [La, Examples 1.8.29 and 1.8.30] for the case of $X = \mathbb{P}^n$). In particular, $\mathrm{reg}_{\mathcal{L}}\,\mathcal{J}^{\sigma^s} \leqslant E$, where E is independent of s.

Pick any constant C and follow the argument used to prove the equation at the top of [KRS, p. 514] for $\mathcal{K}_m = \mathcal{O}_X$. This shows that there exists a constant $M \geqslant 1$ such that $\mathrm{reg}_{\mathcal{L}}\,\mathcal{I}_n \otimes \mathcal{L}_n^{\otimes m} \leqslant -E - C + 1$ for $n \geqslant 1$ and $m \geqslant M$. Together with an application of Lemma 3.13, this implies that $\mathrm{reg}_{\mathcal{L}}\,\mathcal{J}^{\sigma^s} \otimes \mathcal{I}_n \otimes \mathcal{L}_n^{\otimes m} \leqslant 1$ for all $n \geqslant 1$, $m \geqslant M$ and $s \in \mathbb{Z}$. The result follows. □

4. $\mathcal{R}$-modules and equivalences of categories

The hypotheses from Assumptions 3.17 will remain in force throughout this section. One nice consequence of critical density is that it forces modules over $\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma)$ and $R = \mathrm{H}^0(X, \mathcal{R})$ to have a very pleasant structure; indeed in many cases they are just induced from $\mathcal{O}_X$-modules. This will be used in this section to give various equivalences of categories, notably that the category of coherent torsion $\mathcal{O}_X$-modules is equivalent to the subcategory of Goldie torsion modules in qgr-$\mathcal{R}$, as defined below. We also give a natural analogue of the standard fact that, for a blowup $\rho : \widetilde{X} \to X$ at a smooth point x, the schemes $X \setminus \{x\}$ and $\widetilde{X} \setminus \rho^{-1}(x)$ are isomorphic. These results are largely the same as those in [KRS], but some of the proofs need more care since we do not have the identity $\mathcal{I} \otimes \mathcal{I}^{\sigma} = \mathcal{I}\mathcal{I}^{\sigma}$ that was so useful in [KRS].

If A is a noetherian graded domain, a graded A-module M is called *Goldie torsion* (to distinguish this from the notion of torsion already defined) if every homogeneous element of M is killed by some nonzero homogeneous element of A. Equivalently, M is a sum of modules of the form $(A/I)[n]$, for nonzero graded right ideals I. The latter notion passes to all the categories Q we consider; for example a right $\mathcal{R}$-module $\mathcal{M}$ is Goldie torsion if it is a sum of submodules of the form $(\mathcal{R}/\mathcal{K})[n]$ for nonzero right ideals $\mathcal{K}$ of $\mathcal{R}$. Of course, Goldie torsion $\mathcal{O}_X$-modules are just the torsion $\mathcal{O}_X$-modules, as in [Ha, Exercise II.6.12]. We write GT Q for the full subcategory of Goldie torsion modules in Q.

We start by giving some technical results on the structure of Goldie torsion modules. If $\mathcal{N} \in \text{Gr-}\mathcal{R}$, recall from (2.4) that we may write $\mathcal{N} = \bigoplus {}_1(\mathcal{G}_n)_{\sigma^n}$ for some sheaves $\mathcal{G}_n$. It is often convenient to write ${}_1(\mathcal{G}_n)_{\sigma^n} = \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$, where $\mathcal{F}_n = {}_1(\mathcal{F}_n)_1$ has trivial bimodule structure and $\mathcal{L}_\sigma^{\otimes n} = ({}_1\mathcal{L}_\sigma)^{\otimes n}$.

Lemma 4.1.

(1) *Let $\mathcal{N} \in \text{GT}\,\text{gr-}\mathcal{R}$ and write $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{GT}\,\text{gr-}\mathcal{R}$ for some quasi-coherent sheaves $\mathcal{F}_n$. Then there exists a single module $\mathcal{F} \in \text{GT}\,\mathcal{O}_X\text{-mod}$ such that $\mathcal{F}_n = \mathcal{F}$ for all $n \gg 0$.*
(2) *Conversely, if $\mathcal{F} \in \text{GT}\,\mathcal{O}_X\text{-mod}$, then $\bigoplus_{n=0}^{\infty} \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{GT}\,\text{gr-}\mathcal{R}$.*

Proof. The proof of [KRS, Lemma 6.1] goes through with only minor changes (replace [KRS, Lemma 3.9] by Lemma 2.11). □

Lemma 4.2. *If $\mathcal{N} \in \text{gr-}\mathcal{R}$, then there is an exact sequence $0 \to \mathcal{K} \to \mathcal{N} \to \mathcal{T} \to 0$, where $\mathcal{T} \in \text{GT}\,\text{gr-}\mathcal{R}$ is Goldie torsion, and $\mathcal{K} \in \text{gr-}\mathcal{R}$ is a direct sum of shifts of $\mathcal{R}$. In fact we can find a $\mathcal{K}$ such that $\mathcal{K} \cong \bigoplus_{i=0}^{d} \mathcal{R}[-n]$ for some $n \geqslant 0$.*

Proof. This is similar to the proof of the analogous result for finitely generated modules over domains, and so the proof is left to the reader. □

Lemma 4.3. *Suppose that $\mathcal{N} \in \text{Gr-}\mathcal{R}$ and write $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ for some $\mathcal{O}_X$-modules $\mathcal{F}_n$. If $m > 0$, then $\mathcal{N}[m] \cong \bigoplus \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ where $\mathcal{G}_n = (\mathcal{F}_{n+m} \otimes \mathcal{L}_m)^{\sigma^{-m}}$. If $m < 0$ then $\mathcal{N}[m] \cong \bigoplus \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ where $\mathcal{G}_n = (\mathcal{F}_{n+m})^{\sigma^{-m}} \otimes \mathcal{L}_{-m}^{-1}$.*

Proof. If $m > 0$ the result for the bimodule algebra $\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$ follows from [SV, (3.1)] and the same argument works for $\mathcal{R}$. A similar computation then gives the required formula for $m \leqslant 0$. □

In nice cases, the structure maps of an $\mathcal{R}$-module are determined by products instead of tensor products and in order to focus in on this property we make the following definition. Let $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{Gr-}\mathcal{R}$; thus the module structure of $\mathcal{N}$ is determined by the structure morphisms $\theta_n : \mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{F}_{n+1}$ for $n \in \mathbb{Z}$. Then $\mathcal{N}$ is *definable by products in degrees* $\geqslant n_0$ if for all $n \geqslant n_0$, the maps θ_n factor through the multiplication map $\mu_n : \mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \twoheadrightarrow \mathcal{F}_n\mathcal{I}^{\sigma^n}$ in the sense that there exist maps $\rho_n : \mathcal{F}_n\mathcal{I}^{\sigma^n} \to \mathcal{F}_{n+1}$ of $\mathcal{O}_X$-modules such that $\theta_n = \rho_n\mu_n$. We say $\mathcal{N}$ is *definable by products* if some such n_0 exists.

As the next result shows, any coherent module $\mathcal{N}$ is definable by products. For the naïve blowup at a single point, one has the stronger statement that if $\mathcal{N}$ is coherent then μ_n is itself an isomorphism for $n \gg 0$ [KRS, Lemma 6.3]. This stronger result will not always be true in our setting; for instance it fails for $\mathcal{N} = \mathcal{R}$ in the case where Supp Z contains two distinct points from the same σ-orbit.

Lemma 4.4. *Let $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{gr-}\mathcal{R}$. Keep the notation from the discussion above. Then*

(1) *There exists $n_0 \geqslant 0$ such that $\mathcal{N}$ is definable by products in degrees $\geqslant n_0$.*
(2) *There exists $n_1 \geqslant n_0$ such that ρ_n is an isomorphism for $n \geqslant n_1$.*

Proof. To prove that θ_n factors through the map μ_n is equivalent to proving that $\ker \mu_n \subseteq \ker \theta_n$. Tensoring the sequence $0 \to \mathcal{I}^{\sigma^n} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}^{\sigma^n} \to 0$ with $\mathcal{F}_n$ gives the exact sequence

$$0 \to \mathcal{T}or_1^{\mathcal{O}_X}\big(\mathcal{F}_n, \mathcal{I}^{\sigma^n}\big) \xrightarrow{\alpha_n} \mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{F}_n\mathcal{I}^{\sigma^n} \to 0,$$

so we can and will identify $\ker \mu_n$ with $\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{I}^{\sigma^n})$.

Suppose first that $\mathcal{N} = \mathcal{R}[m]$ is a shift of $\mathcal{R}$. Lemma 4.3 implies that $\mathcal{F}_n = \mathcal{I}^{\sigma^{-m}} \cdots \mathcal{I}^{\sigma^{n-1}} \otimes \mathcal{L}_{|m|}^{\alpha}$ for $n > |m|$ and the appropriate α. Then for large n, the map $\theta_n : \mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{F}_{n+1}$ is then none other than the natural map from the tensor product $\mathcal{I}^{\sigma^{-m}} \cdots \mathcal{I}^{\sigma^{n-1}} \otimes \mathcal{I}^{\sigma^n}$ to the product $\mathcal{I}^{\sigma^{-m}} \cdots \mathcal{I}^{\sigma^{n-1}}\mathcal{I}^{\sigma^n}$, tensored by $\mathcal{L}_{|m|}^{\alpha}$. So (1) holds when $\mathcal{N} = \mathcal{R}[m]$.

Now let $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ be an exact sequence in gr-$\mathcal{R}$, where $\mathcal{N}' = \bigoplus \mathcal{F}_n' \otimes \mathcal{L}_\sigma^{\otimes n}$ and $\mathcal{N}'' = \bigoplus \mathcal{F}_n'' \otimes \mathcal{L}_\sigma^{\otimes n}$, with structure maps θ_n' and θ_n'', respectively. Consider the commutative diagram:

$$\begin{array}{ccccc}
\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}_n', \mathcal{I}^{\sigma^n}) & \longrightarrow & \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{I}^{\sigma^n}) & \longrightarrow & \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}_n'', \mathcal{I}^{\sigma^n}) \\
\downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} \\
\mathcal{F}_n' \otimes \mathcal{I}^{\sigma^n} & \longrightarrow & \mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} & \longrightarrow & \mathcal{F}_n'' \otimes \mathcal{I}^{\sigma^n} \\
\downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta''} \\
\mathcal{F}_{n+1}' & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_{n+1}''.
\end{array} \tag{4.5}$$

We first suppose that $\mathcal{N} = \mathcal{N}' \oplus \mathcal{N}''$ and that (1) holds for $\mathcal{N}'$ and $\mathcal{N}''$. In this case, the outside columns of (4.5) are complexes and the rows split. Thus the middle column is also a complex and so (1) holds for $\mathcal{N}$. By induction, we have therefore proved that (1) holds when $\mathcal{N}$ is a direct sum of shifts of $\mathcal{R}$.

By Lemma 4.2, a general coherent module $\mathcal{N}$ fits into an exact sequence $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ where $\mathcal{N}'$ is a sum of shifts of $\mathcal{R}$ and $\mathcal{N}''$ is Goldie torsion. By the last paragraph, the first column of (4.5) is now a complex. By Lemma 4.1 there exists a coherent torsion sheaf $\mathcal{F}''$ such that $\mathcal{F}_n'' = \mathcal{F}''$, say for all $n \gg n_0$. Since Z is saturating, $\operatorname{Supp} \mathcal{F}'' \cap \operatorname{Supp} \mathcal{O}_X/\mathcal{I}^{\sigma^n} = \emptyset$ for $n \gg n_0$ and thus $\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}_n'', \mathcal{I}^{\sigma^n}) = 0$, for such n. A simple diagram chase then shows that the middle column of (4.5) is a complex, proving part (1).

(2) Lemma 2.11 implies that θ_n is surjective for $n \gg 0$, while $\rho_n : \mathcal{F}_n\mathcal{I}^{\sigma^n} \to \mathcal{F}_{n+1}$ is defined for all $n \gg 0$ by part (1). Thus there exists n_0, such that ρ_n is defined and surjective for all $n \geqslant n_0$ and it remains to prove that ρ_n is an isomorphism for $n \gg n_0$.

Pulling back to $\mathcal{F}_{n_0}$, for $n \geqslant n_0$ we may write $\mathcal{F}_n = \mathcal{A}_n/\mathcal{B}_n$, for subsheaves $\mathcal{B}_n \subseteq \mathcal{A}_n \subseteq \mathcal{F}_{n_0}$. Since $\mathcal{F}_{n+1}$ is a homomorphic image of $\mathcal{F}_n\mathcal{I}^{\sigma^n} = (\mathcal{A}_n\mathcal{I}^{\sigma^n} + \mathcal{B}_n)/\mathcal{B}_n$, we find that $\mathcal{B}_{n+1} \supseteq \mathcal{B}_n$ for each $n \geqslant n_0$. Since $\mathcal{F}_{n_0}$ is noetherian, $\mathcal{B}_n = \mathcal{B}_{n+1}$ for all $n \gg n_0$, and hence $\mathcal{F}_n\mathcal{I}^{\sigma^n} \cong \mathcal{F}_{n+1}$ for such n. $\square$

Examples 4.6. (1) Let $\mathcal{F} \in \mathcal{O}_X$-Mod be any quasi-coherent sheaf on X and take $\mathcal{M} = \bigoplus_{n \geqslant 0} \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n}$ in Gr-$\mathcal{R}$. Then $\mathcal{M}$ is easily seen to be definable by products in degrees $\geqslant 0$, even though $\mathcal{M}$ need not be coherent as an $\mathcal{R}$-module.

(2) In general, however, a noncoherent $\mathcal{R}$-module $\mathcal{M} \in$ Gr-$\mathcal{R}$ need not be definable by products. For example, take $\mathcal{R} = \mathcal{R}(\mathbb{P}^2, Z, \mathcal{L}, \sigma)$ where $Z = p$ is a single reduced saturating point with ideal sheaf $\mathcal{I}$ and set $\mathcal{N} = \mathcal{O}_X/\mathcal{I} \oplus (\mathcal{I}/\mathcal{I}^2) \otimes \mathcal{L}_\sigma \oplus 0 \oplus 0 \cdots$, where the structure map $\theta_0 : (\mathcal{O}_X/\mathcal{I}) \otimes \mathcal{I} \to \mathcal{I}/\mathcal{I}^2$ is the natural isomorphism. Then $\mathcal{N}$ is clearly not definable by products in degrees $\geqslant 0$ and so $\mathcal{N}[-n]$ is not definable by products in degrees $\geqslant n$. Thus $\bigoplus_{n \geqslant 0} \mathcal{N}[-n]$ is not definable by products in any degree.

For $\mathcal{R}$-modules that are definable by products, the homomorphism groups also have a nice form and the next lemma collects the relevant facts. Recall from Section 2 that the map from Gr-$\mathcal{R}$ to Qgr-$\mathcal{R}$ is denoted π.

Lemma 4.7. *Suppose that $\mathcal{M} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in$ gr-$\mathcal{R}$ is coherent, and that $\mathcal{N} = \bigoplus \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in$ Gr-$\mathcal{R}$ is definable by products.*

(1) *For some n_1, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Qgr}\text{-}\mathcal{R}}\big(\pi(\mathcal{M}), \pi(\mathcal{N})\big) \cong \lim_{n \geqslant n_1} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n).$$

(2) *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are coherent Goldie torsion modules and, by Lemma* 4.1, *write $\mathcal{F}_n = \mathcal{F}$ and $\mathcal{G}_n = \mathcal{G}$ for $n \gg 0$. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Qgr}\text{-}\mathcal{R}}\big(\pi(\mathcal{M}), \pi(\mathcal{N})\big) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Before beginning the proof we need to explain the direct limit appearing in part (1). By Lemma 4.4, pick n_1 so that $\rho_n : \mathcal{F}_n \mathcal{I}^{\sigma^n} \to \mathcal{F}_{n+1}$ is an isomorphism for all $n \geqslant n_1$. We can also assume that $\mathcal{N}$ is definable by products in degrees $\geqslant n_1$, so the structure maps $\theta'_n : \mathcal{G}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{G}_{n+1}$ factor through maps $\rho'_n : \mathcal{G}_n \mathcal{I}^{\sigma^n} \to \mathcal{G}_{n+1}$ for $n \geqslant n_1$. Given $\alpha \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n)$ where $n \geqslant n_1$, then α restricts to a morphism $\alpha' \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n \mathcal{I}^{\sigma^n}, \mathcal{G}_n \mathcal{I}^{\sigma^n})$, and so α' induces a map $\alpha'' = \rho'_n \circ \alpha' \circ \rho_n^{-1} \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_{n+1}, \mathcal{G}_{n+1})$.

Proof of Lemma 4.7. (1) The definition of homomorphisms in quotient categories implies that

$$\mathrm{Hom}_{\mathrm{Qgr}\text{-}\mathcal{R}}\big(\pi(\mathcal{M}), \pi(\mathcal{N})\big) = \lim_{n \to \infty} \mathrm{Hom}_{\mathrm{Gr}\text{-}\mathcal{R}}(\mathcal{M}_{\geqslant n}, \mathcal{N}), \tag{4.8}$$

whenever $\mathcal{M} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ is coherent (see, for example, [VB2, p. 31]). On the other hand, we claim that there are natural vector space maps

$$\mathrm{Hom}_{\mathrm{Gr}\text{-}\mathcal{R}}(\mathcal{M}_{\geqslant n}, \mathcal{N}) \xrightarrow{\phi_n} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}_n, \mathcal{N}_n) \xrightarrow{\psi_n} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n). \tag{4.9}$$

Indeed, if $f \in \mathrm{Hom}_{\mathrm{Gr}\text{-}\mathcal{R}}(\mathcal{M}_{\geqslant n}, \mathcal{N})$, then f is a morphism of right $\mathcal{O}_X$-modules, so define $\phi_n(f)$ to be the restriction of f to $\mathcal{M}_n$. The map ψ_n is the natural isomorphism obtained by tensoring with $(\mathcal{L}_\sigma^{\otimes n})^{-1}$.

We will prove that $\psi_n \circ \phi_n$ is an isomorphism for $n \geqslant n_1$. Given $g \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n)$ for $n \geqslant n_1$, then g induces a unique map

$$\mathcal{M}_{n+r} \cong \mathcal{F}_n \mathcal{I}_r^{\sigma^n} \otimes \mathcal{L}_\sigma^{\otimes(n+r)} \to \mathcal{G}_n \mathcal{I}_r^{\sigma^n} \otimes \mathcal{L}_\sigma^{\otimes(n+r)} \to \mathcal{G}_{n+r} \otimes \mathcal{L}_\sigma^{\otimes(n+r)} = \mathcal{N}_{n+r},$$

for any $r \geqslant 0$. These piece together to give an $\mathcal{R}$-module map $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}_{\geqslant n}, \mathcal{N})$ and in this way we define a morphism $\tau_n : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}_{\geqslant n}, \mathcal{N})$ by setting $\tau_n(g) = f$. Obviously $\psi_n \phi_n \tau_n(g) = g$ and so $\psi_n \phi_n$ is surjective. Moreover, for $n \geqslant n_1$ any element in $\mathrm{Hom}_{\mathrm{Gr}\text{-}\mathcal{R}}(\mathcal{M}_{\geqslant n}, \mathcal{N})$ is easily seen to be determined by its restriction to degree n, because $\mathcal{M}_{n+r} = \mathcal{M}_n \mathcal{R}_r$ for all $r \geqslant 0$. It follows that $\psi_n \phi_n$ is injective and so $\psi_n \phi_n$ is an isomorphism for $n \geqslant n_1$ as claimed.

The isomorphisms $\psi_n \phi_n$ are compatible with the maps in the direct limits, and so they induce an isomorphism $\lim_{n\to\infty} \mathrm{Hom}_{\mathrm{Gr}\text{-}\mathcal{R}}(\mathcal{M}_{\geqslant n}, \mathcal{N}) \to \lim_{n \geqslant n_1} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n)$. So we are done by (4.8).

(2) The nth map in the direct limit $\lim_{n \geqslant n_1} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n)$ is an isomorphism for all large n and so the direct limit stabilizes at $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Thus part (2) is a special case of part (1). □

One of the most important cases of Lemma 4.7(1) occurs when $\mathcal{M} = \mathcal{R}$. In this case, [KRS, Lemma 6.4] shows that part (1) holds for all modules $\mathcal{N}$ provided one naïvely blows up a single point. However, when one blows up more than one point at once, then Lemma 4.7(1) can fail for a general module $\mathcal{N}$. Since the example is a little technical we will omit it, although it can be found in [RS2].

It is now easy to define an equivalence of categories between $\mathrm{GT}\,\mathrm{qgr}\text{-}R$ and $\mathrm{GT}\,\mathcal{O}_X$-mod, thereby proving Theorem 1.1(4).

Theorem 4.10. *Keep the hypotheses from* (3.17). *Then there are equivalences of categories*

$$\mathrm{GT}\,\mathrm{Qgr}\text{-}R \simeq \mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R} \simeq \mathrm{GT}\,\mathcal{O}_X\text{-Mod},$$

which restrict to equivalences $\mathrm{GT}\,\mathrm{qgr}\text{-}R \simeq \mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R} \simeq \mathrm{GT}\,\mathcal{O}_X$-mod. *This equivalence is given by mapping* $\mathcal{F} \in \mathrm{GT}\,\mathcal{O}_X$-Mod *to* $\pi(\bigoplus \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n}) \in \mathrm{Qgr}\text{-}\mathcal{R}$.

Proof. The proof of [KRS, Theorem 6.7] goes through with the following minor changes. Specifically, the references to Theorem 4.1, Lemma 6.1 and Lemma 6.4 of [KRS] given in that proof should be replaced by references to Theorem 3.1(2), Lemma 4.1 and Lemma 4.7, respectively. (In the proof, Lemma 4.7 is only applied to coherent $\mathcal{R}$-modules and so, by Lemma 4.4, the hypotheses of Lemma 4.7 are satisfied.) □

If $k(x)$ is the skyscraper sheaf at a closed point $x \in X$, set $\bar{x} = \bigoplus_{n \geqslant 0}(k(x) \otimes \mathcal{L}_\sigma^{\otimes n}) \in \mathrm{Gr}\text{-}\mathcal{R}$ and write $\widetilde{x} = \pi(\bar{x}) \in \mathrm{Qgr}\text{-}\mathcal{R}$. By Lemma 4.1(2) $\bar{x} \in \mathrm{GT}\,\mathrm{gr}\text{-}\mathcal{R}$ and so $\widetilde{x} \in \mathrm{qgr}\text{-}\mathcal{R}$. Combined with Proposition 7.3, the next result proves Theorem 1.1(5).

Corollary 4.11. *Keep the hypotheses from* (3.17). *Then*:

(1) *There is a* (1–1) *correspondence between the closed points* $x \in X$ *and isomorphism classes of simple objects in* $\mathrm{qgr}\text{-}\mathcal{R}$ *given by* $x \mapsto \widetilde{x}$.

(2) *The simple objects in* qgr-R *are the images of finitely generated* R*-modules* $M \in$ gr-R *with Hilbert series* $(1-t)^{-1}$.
(3) *If* R *is generated in degree one then the simple objects in* qgr-R *are the images of shifts of point modules.*

Proof. (1) Clearly $\widetilde{x}$ is the image of $k(x)$ under the equivalence from Theorem 4.10 and so the simple objects in qgr-$\mathcal{R}$ are exactly these $\widetilde{x}$.

(2) Given $\widetilde{x}$ for a closed point $x \in X$, Theorem 2.6(2) shows that $M(x) = \mathrm{H}^0(X, \widetilde{x}) \in$ gr-R and so $M(x)_n = \mathrm{H}^0(X, k(x) \otimes \mathcal{L}_\sigma^{\otimes n}) \cong \mathrm{H}^0(X, k(x))$ is one-dimensional for all $n \geqslant 0$. By Theorem 2.6(2), again, the image of $M(x)$ in qgr-R corresponds to $\widetilde{x}$ under the equivalence qgr-$R \simeq$ qgr-$\mathcal{R}$.

(3) Take $M(x)$ as in part (2) and suppose that it is generated in degrees $\leqslant r_0$. Since R is generated in degree one it follows that $M(x)_n = M(x)_{r_0} R_{n-r_0}$, for all $n \geqslant r_0$. Thus $M(x)_{\geqslant r_0}$ is a shifted point module. □

Unfortunately the module $M(x)$ constructed in the proof of Corollary 4.11(2) will not be cyclic when $x \in \bigcup_{n \geqslant 0} \operatorname{Supp} \sigma^{-n}(Z)$ and so one needs a more subtle argument to define the simple objects in qgr-R in terms of modules generated in degree zero. The details are given in Section 7.

If $\rho : \widetilde{X} \to X$ is the (classical) blowup of X at a smooth point x, then it is standard that $X \setminus \{x\} \cong \widetilde{X} \setminus \rho^{-1}(x)$. The final result of this section gives the analogous result for qgr-$\mathcal{R}$, although we have to remove whole σ-orbits rather than just isolated points. Thus, define C_X to be the smallest localizing subcategory of $\mathcal{O}_X$-Mod containing all the modules $\{k(c) \mid c \in \bigcup_{i \in \mathbb{Z}} \sigma^i(S)\}$, where $S = \operatorname{Supp} \mathcal{O}_X / \mathcal{I}$. Similarly, write $C_{\mathcal{R}}$ for the localizing subcategory of Qgr-$\mathcal{R}$ generated by the modules $\widetilde{c}$ for $c \in \bigcup_{i \in \mathbb{Z}} \sigma^i(S)$.

Proposition 4.12. *Assuming* (3.17), *there is an equivalence of categories* $\mathcal{O}_X$-Mod$/C_X \simeq$ Qgr-$\mathcal{R}/C_{\mathcal{R}}$.

Proof. The proof of this result is similar to that of [KRS, Preposition 6.9], but since the result is peripheral, we leave the details to the interested reader. A full proof can be found in [RS2]. □

5. Generalized naïve blowups and torsion extensions

Throughout the section, the hypotheses from Assumptions 3.17 will be maintained. If one naïvely blows up a single point then the corresponding naïve blowup algebra R automatically satisfies χ_1 (see [KRS, Theorem 1.1(8)]). The χ conditions are defined in Section 6, but in this section we will just be interested in the following weaker version: A cg Goldie domain R satisfies *the weak (right)* χ_1*-condition* if, given any cg algebra $R \subseteq S \subseteq Q(R)$ such that S/R is (right) torsion then S/R is finite-dimensional. Remarkably, this can fail when one blows up at more than one point. In order to analyse this situation we need to understand the maximal torsion extensions of a naïve blowup algebra and this leads to a variant of naïve blowups, called generalized naïve blowups. These will be studied in this section and applied to the study of the chi conditions in Section 6.

Here is a simple example of this phenomenon. More examples will appear at the end of the section.

Example 5.1. Fix $X = \mathbb{P}^2$ and $\sigma \in \operatorname{Aut}(X)$ for which there exists a closed point $c = c_0 \in X$ with a critically dense σ-orbit. Write $\mathfrak{m}_{[0]}$ for the sheaf of maximal ideals corresponding to c_0. If $\mathfrak{m}_{[0]} = (x, y)$ locally at c_0, let $\mathcal{M}_{[0]}$ be the sheaf of ideals such that $\mathcal{O}_X/\mathcal{M}_{[0]}$ is supported at c_0 but such that $\mathcal{M}_{[0]} = (x^2, y^2)$ locally at c_0. For $i \in \mathbb{Z}$ set $c_i = \sigma^{-i}(c_0)$ and write $\mathfrak{m}_{[i]} = \mathfrak{m}_{[0]}^{\sigma^i}$ and $\mathcal{M}_{[i]} = \mathcal{M}_{[0]}^{\sigma^i}$. The key property of the $\mathcal{M}_{[i]}$ is that $\mathcal{M}_{[i]} \subsetneq \mathfrak{m}_{[i]}^2$ but $\mathcal{M}_{[i]}\mathfrak{m}_{[i]} = \mathfrak{m}_{[i]}^3$.

Let $\mathcal{I} = \mathfrak{m}_{[0]}\mathcal{M}_{[1]}$ and $\mathcal{H} = \mathfrak{m}_{[0]}\mathfrak{m}_{[1]}^2$. A routine computation shows that

$$\mathcal{I}_n = \mathcal{I}\mathcal{I}^\sigma \cdots \mathcal{I}^{\sigma^{n-1}} = \mathfrak{m}_{[0]}\mathfrak{m}_{[1]}^3 \cdots \mathfrak{m}_{[n-1]}^3\mathcal{M}_{[n]} \quad \text{but}$$
$$\mathcal{H}_n = \mathcal{H}\mathcal{H}^\sigma \cdots \mathcal{H}^{\sigma^{n-1}} = \mathfrak{m}_{[0]}\mathfrak{m}_{[1]}^3 \cdots \mathfrak{m}_{[n-1]}^3\mathfrak{m}_{[n]}^2,$$

and so $\mathcal{H}_n\mathcal{I}_r^{\sigma^n} = \mathcal{I}_{n+r}$ for all $n \geqslant 0$ and $r \geqslant 1$. Thus

$$\mathcal{R} = \mathcal{R}(\mathbb{P}^2, Z_{\mathcal{I}}, \mathcal{L}, \sigma) \subset \mathcal{T} = \mathcal{R}(\mathbb{P}^2, Z_{\mathcal{H}}, \mathcal{L}, \sigma)$$

satisfy $\mathcal{T}\mathcal{R}_{\geqslant 1} \subseteq \mathcal{R}$, despite the fact that $\mathcal{R}_n \neq \mathcal{T}_n$ for all $n \geqslant 1$.

Now take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(m)$ where m is large enough so that $R = R(\mathbb{P}^2, Z_{\mathcal{I}}, \mathcal{L}, \sigma)$ and $T = R(\mathbb{P}^2, Z_{\mathcal{H}}, \mathcal{L}, \sigma)$ are both generated in degree one and each $\mathcal{I}_n \otimes \mathcal{L}_n$ and $\mathcal{H}_n \otimes \mathcal{L}_n$ is generated by its sections (this is possible by Proposition 3.20). Then the conclusion of the last paragraph translates into the statement that $R \subset T$ with $TR_{\geqslant 1} \subset R$ despite the fact that $T_n \neq R_n$ for each $n \geqslant 1$. Clearly, this shows that R does not satisfy the weak χ_1 condition on the right.

The rings $R \subset T$ from Example 5.1 have a number of other interesting properties that will become more evident as we develop the appropriate theory. For example, R is quite asymmetric and does satisfy weak χ_1 on the left (see the discussion immediately before Example 5.16). Examples like this are intimately connected to the theory of idealizer rings; in fact, the ring R is the idealizer in T of the left ideal $TR_{\geqslant 1}$ (see Lemma 6.7 and the discussion thereafter). Although in this example T is itself a naïve blowup algebra, this does not always happen (see Example 5.18), and to cater for examples like that we will need to work with the following more general objects.

Definition 5.2. Keep the hypotheses from Assumptions 3.17. A *generalized naïve sequence* is a sequence $\{\mathbb{I}_n\}_{n\geqslant 0}$ of ideal sheaves on X satisfying the following properties:

(1) $\mathbb{I}_0 = \mathcal{O}_X$ and $\mathbb{I}_m\mathbb{I}_n^{\sigma^m} \subseteq \mathbb{I}_{n+m}$ for all $m, n \geqslant 0$.
(2) There exists a constant $t \geqslant 1$ such that $\mathbb{I}_m\mathbb{I}_n^{\sigma^m} = \mathbb{I}_{n+m}$ for all $m, n \geqslant t$.
(3) For $n \geqslant 0$, the subscheme $\operatorname{Supp}\mathcal{O}_X/\mathbb{I}_n$ is either zero-dimensional and saturating, or empty.

If (2) holds with $t = 1$, then $\{\mathbb{I}_n\}$ is called a *naïve sequence*.

Given this data, we write $\mathcal{S} = \mathcal{S}(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma) = \bigoplus_{n\geqslant 0}(\mathbb{I}_n \otimes \mathcal{L}_n)_{\sigma^n}$. This is easily seen to be a bimodule algebra, which we call a *generalized naïve blowup bimodule algebra*. This notation is justified since, if $\{\mathbb{I}_n\}$ is a naïve sequence then $\mathbb{I}_n = \mathbb{I}_1 \cdots \mathbb{I}_1^{\sigma^{n-1}}$ for all $n \geqslant 0$ and so $\mathcal{S}$ is just the bimodule algebra $\mathcal{R} = \mathcal{R}(X, Z_{\mathbb{I}_1}, \mathcal{L}, \sigma)$. The algebra of sections $\mathrm{H}^0(X, \mathcal{S}) = S(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$ will be called *a generalized naïve blowup algebra*. We call $\mathcal{S}$ or S *nontrivial* if $\mathbb{I}_n \neq \mathcal{O}_X$ for some (and hence all) $n \gg 0$.

As we next show, many of the basic properties of $\mathcal{R}$ generalize easily to $\mathcal{S}$.

Lemma 5.3. *Let $\{\mathcal{I}_n\}_{n\geqslant 0}$ be a generalized naïve sequence and set $\mathcal{S} = \mathcal{S}(X, \{\mathcal{I}_n\}, \mathcal{L}, \sigma)$. Take t as in Definition 5.2(2) and pick any $p \geqslant t$. Then the Veronese bimodule algebra $\mathcal{S}^{(p)} = \bigoplus_{n\geqslant 0}(\mathcal{I}_{np} \otimes \mathcal{L}_{np})_{\sigma^{np}}$ equals $\mathcal{R}(X, Z_{\mathcal{I}_p}, \mathcal{L}_p, \sigma^p)$. Moreover, the sequence $\{(\mathcal{I}_{np} \otimes \mathcal{L}_{np})_{\sigma^{np}}\}$ is ample and both $\mathcal{S}^{(p)}$ and its section ring $S^{(p)} = \bigoplus_{n\geqslant 0} \mathrm{H}^0(X, \mathcal{S}_{np})$ are noetherian.*

Proof. Note that $\mathcal{I}_{np} = \mathcal{I}_p \mathcal{I}_p^{\sigma^p} \cdots \mathcal{I}_p^{\sigma^{np-p}}$ for all $n \geqslant 1$ and so $\mathcal{S}^{(p)} = \mathcal{R}(X, Z_{\mathcal{I}_p}, \mathcal{L}_p, \sigma^p)$ holds by definition. The hypotheses from Assumptions 3.17 (and hence those from 2.8) pass to Veronese subsequences and so the result follows from Theorem 3.1. □

Corollary 5.4. *Let $\{\mathcal{I}_n\}_{n\geqslant 0}$ be a generalized naïve sequence and pick p as in Lemma 5.3. Then:*

(1) *$\{\mathcal{I}_n \otimes \mathcal{L}_n\}_{n\geqslant 0}$ is an ample sequence.*
(2) *$\mathcal{S} = \mathcal{S}(X, \{\mathcal{I}_n\}, \mathcal{L}, \sigma)$ is coherent as a left and right $\mathcal{S}^{(p)}$ module and is a noetherian bimodule algebra.*
(3) *$S = \mathrm{H}^0(X, \mathcal{S})$ is a noetherian ring that is finitely generated as a right or left module over $S^{(p)}$.*
(4) *There exists a constant t' such that $S_m S_n = S_{m+n}$ for all $m, n \geqslant t'$.*

Proof. (1) By Definition 5.2(2), we can choose m_0 such that $\mathcal{I}_m \mathcal{I}_{np}^{\sigma^m} = \mathcal{I}_{m+np}$ for all $m_0 \leqslant m \leqslant m_0 + p$ and $n \geqslant 0$. By Lemma 5.3, the sequence $\{\mathcal{I}_{np} \otimes \mathcal{L}_{np}\}_{n\geqslant 0}$ is ample and hence, for each $m_0 \leqslant m \leqslant m_0 + p$, so is the sequence $\{(\mathcal{I}_{np} \otimes \mathcal{L}_{np})^{\sigma^m}\}_{n\geqslant 0}$. For such m and coherent sheaf $\mathcal{F}$, the natural surjection

$$\mathcal{F} \otimes (\mathcal{I}_m \otimes \mathcal{L}_m) \otimes (\mathcal{I}_{np} \otimes \mathcal{L}_{np})^{\sigma^m} \twoheadrightarrow \mathcal{F} \otimes (\mathcal{I}_{n+mp} \otimes \mathcal{L}_{n+mp})$$

has a finitely supported kernel, from which it follows that $\{\mathcal{I}_{np+m} \otimes \mathcal{L}_{np+m}\}_{n\geqslant 0}$ is also ample. Therefore $\{\mathcal{I}_n \otimes \mathcal{L}_n\}_{n\geqslant 0}$ is ample.

(2) We consider $\mathcal{S}$ as a right $\mathcal{S}^{(p)}$-module via the ungraded inclusion of bimodule algebras $\mathcal{S}^{(p)} \subseteq \mathcal{S}$. By Lemma 2.11, in order to show that $\mathcal{S}$ is a coherent right $\mathcal{S}^{(p)}$-module, it suffices to show that $\mathcal{I}_m \mathcal{I}_{np}^{\sigma^m} = \mathcal{I}_{m+np}$ for all $m \gg 0$ and $n \geqslant 1$. This holds by Definition 5.2(2). It then follows from Lemma 5.3 and [KRS, Proposition 2.10] that $\mathcal{S}$ is right noetherian. The same argument works on the left.

(3) By Lemma 5.3, the hypotheses of Theorem 2.6 are satisfied by $\mathcal{S}^{(p)}$. Thus, by part (2) and Theorem 2.6(2), $S = \mathrm{H}^0(X, \mathcal{S})$ is noetherian as both a right and a left $S^{(p)}$-module.

(4) By Lemma 5.3 $S^{(p)}$ is a naïve blowup algebra for all $p \geqslant t$ and so, by Proposition 3.18, the Veronese ring $S^{(q)}$ is generated in degree one for all large multiples q of p. As S is a finitely generated right $S^{(q)}$-module, this implies that, for some m_0, $S_m S_{qr} = S_{m+qr}$ for all $r \geqslant 0$ and $m \geqslant m_0$. Varying p, we can find two relatively prime integers q_1, q_2 and a single m_0 such that the conclusion of the previous sentence holds for both $q = q_1$ and $q = q_2$. Since any $n \gg 0$ can be written as $n = aq_1 + bq_2$ for some $a, b \geqslant 0$, it follows that $S_m S_n = S_{m+n}$ for $m \geqslant m_0$ and all $n \gg 0$. □

We want to understand the asymptotic behavior of a generalized naïve sequence $\{\mathcal{I}_n\}$, for which we need the following notation.

Notation 5.5. If $\mathcal{J} \subseteq \mathcal{O}_X$ is a sheaf of ideals, we define the *cosupport* of $\mathcal{J}$ to be $\operatorname{coSupp} \mathcal{J} = \operatorname{Supp} \mathcal{O}_X/\mathcal{J}$. Now consider $\mathcal{J} = \mathcal{I}_1$ with cosupport $W = W_1$. Subdivide $W = \bigcup_{a=1}^{d} W(a)$ so that each $W(a)$ consists of the points in W contained in a single σ-orbit and write $\mathcal{J} = \prod_a ({}_a\mathcal{J})$ for the corresponding decomposition of $\mathcal{J}$. Let $c = c_0(a) \in W(a)$ be the unique element for which $W(a) = \{c_j = \sigma^{-j}(c)\}$, for some *positive* set of integers j. We can then (uniquely) write ${}_a\mathcal{J} = \prod {}_a^\ell\mathcal{J}$, where ${}_a^\ell\mathcal{J}$ is supported at c_ℓ, and ${}_a^\ell\mathcal{J} = \mathcal{O}_X$ if $c_\ell \notin \operatorname{coSupp}\mathcal{J}$. The *width* of ${}_a\mathcal{J}$ is defined to be the maximal j such that $c_j(a)$ appears in $\operatorname{coSupp}{}_a\mathcal{J}$, and the *width* of $\mathcal{J}$ is defined to be $\max\{\operatorname{width}{}_a\mathcal{J} \mid 1 \leqslant a \leqslant d\}$.

Now take $n \geqslant 1$ and $r \geqslant 0$. We now repeat the process of the last paragraph for $\mathcal{I}_n^{\sigma^r}$ and $W_{n,r} = \operatorname{coSupp}\mathcal{I}_n^{\sigma^r}$, except that we use the elements $c_0(a)$ defined for $\mathcal{I}_1$. By induction and the equation $\mathcal{I}_1\mathcal{I}_{n-1}^\sigma \subseteq \mathcal{I}_n$ from Definition 5.2, the σ-orbits defined by $W_{n,r}$ are contained in those coming from W_1. Hence each $W_{n,r} = \bigcup_{a=1}^d W_{n,r}(a)$ and, again, $W_{n,r}(a) = \{c_j = \sigma^{-j}(c)\}$, for some positive set of integers j. Of course, it is quite possible that $c_0(a) \notin W_{n,r}(a)$ or even that some $W_{n,r}(a) = \emptyset$; the extreme case occurs when $\mathcal{I}_1 \neq \mathcal{O}_X$ but $\mathcal{I}_n = \mathcal{O}_X$ for all $n > 1$. A useful observation is that

$$ {}_a^j\big(\mathcal{I}_n^{\sigma^{r+1}}\big) = \big({}_a^{j-1}\mathcal{I}_n^{\sigma^r}\big)^\sigma. \tag{5.6}$$

Lemma 5.7. *Let $\{\mathcal{I}_n\}_{n\geqslant 0}$ be a generalized naïve sequence, fix some $1 \leqslant a \leqslant d$ as in Notation* 5.5 *and define t as in Definition* 5.2*. Let $w = \operatorname{width}\mathcal{I}_1$. Then there exist sheaves of ideals $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, independent of n, such that*

$$ {}_a\mathcal{I}_n = \mathcal{A}\mathcal{B}^{\sigma^w}\mathcal{B}^{\sigma^{w+1}}\cdots\mathcal{B}^{\sigma^{n-1}}\mathcal{C}^{\sigma^n} \quad \textit{for all } n \geqslant M = \max\{w, t\}. \tag{5.8}$$

Moreover, $\mathcal{A} = {}^0\mathcal{A}\cdot{}^1\mathcal{A}\cdots{}^{w-1}\mathcal{A}$ and $\mathcal{C} = {}^0\mathcal{C}\cdots{}^{w-1}\mathcal{C}$ but $\mathcal{B} = {}^0\mathcal{B}$. If $w = 0$, the sheaves $\mathcal{A}$ and $\mathcal{C}$ do not appear.

Proof. Clearly we can replace $\{\mathcal{I}_n\}$ by $\{{}_a\mathcal{I}_n\}$ and so we will drop the subscript a and put $c_i = c_i(a)$. Since $\mathcal{I}_n \supseteq \mathcal{I}_1\mathcal{I}_1^\sigma\cdots\mathcal{I}_1^{\sigma^{n-1}}$ it follows that $\operatorname{coSupp}\mathcal{I}_n \subseteq \{c_0, c_1, \ldots, c_{n+w-1}\}$. For $n, r \geqslant t$, Definition 5.2(2) ensures that ${}^j\mathcal{I}_{r+1}{}^j(\mathcal{I}_n^{\sigma^{r+1}}) = {}^j\mathcal{I}_{r+n+1} = {}^j\mathcal{I}_r{}^j(\mathcal{I}_{n+1}^{\sigma^r})$. If $j > r + w$ then ${}^j\mathcal{I}_{r+1} = {}^j\mathcal{I}_r = \mathcal{O}_X$ and so ${}^j(\mathcal{I}_n^{\sigma^{r+1}}) = {}^j(\mathcal{I}_{n+1}^{\sigma^r})$. By (5.6) this is equivalent to $({}^{k-1}\mathcal{I}_n)^\sigma = {}^k\mathcal{I}_{n+1}$ for $k > w$.

Now consider the equation ${}^j\mathcal{I}_{n+1}{}^j(\mathcal{I}_r^{\sigma^{n+1}}) = {}^j\mathcal{I}_{r+n+1} = {}^j\mathcal{I}_n{}^j(\mathcal{I}_{r+1}^{\sigma^n})$, for $n, r \geqslant t$. If $j < n$ then (5.6) implies that ${}^j(\mathcal{I}_{r+1}^{\sigma^n}) = {}^j(\mathcal{I}_r^{\sigma^{n+1}}) = \mathcal{O}_X$ and so ${}^j\mathcal{I}_{n+1} = {}^j\mathcal{I}_n$. Altogether, if $w < j < n$ then $({}^{j-1}\mathcal{I}_n)^\sigma = {}^j\mathcal{I}_{n+1} = {}^j\mathcal{I}_n$.

Finally, take $n \geqslant \max\{w, t\}$ so that (5.8) makes sense. The previous paragraph certainly implies that $\mathcal{I}_n = \mathcal{A}\mathcal{B}^{\sigma^w}\mathcal{B}^{\sigma^{w+2}}\cdots\mathcal{B}^{\sigma^{n-1}}\mathcal{C}^{\sigma^n}$, where $\mathcal{A} = \mathcal{A}(n)$ and $\mathcal{C} = \mathcal{C}(n)$ are supported on $\{c_0, \ldots, c_{w-1}\}$ but $\mathcal{B} = ({}^k\mathcal{I}_n)^{\sigma^{-k}}$ for any $w \leqslant k \leqslant n - 1$. Thus $\mathcal{B}$ is independent of n. We can certainly write $\mathcal{A} = {}^0\mathcal{A}\cdot{}^1\mathcal{A}\cdots{}^{w-1}\mathcal{A}$ and this decomposition is independent of n simply because ${}^j\mathcal{I}_{n+1} = {}^j\mathcal{I}_n$ for $j < n$. Similarly, the fact that $\mathcal{C} = {}^0\mathcal{C}\cdots{}^{w-1}\mathcal{C}$ independently of n follows from the equation $({}^{k-1}\mathcal{I}_n)^\sigma = {}^k\mathcal{I}_{n+1}$ for $k > w$. $\square$

We now turn to torsion extensions of graded algebras and bimodule algebras. Given a cg Ore domain A with homogeneous quotient ring $Q = Q(A)$, the *maximal right torsion extension of A* is the ring

$$ T(A) = \{x \in Q \mid xA_{\geqslant n} \subseteq A \text{ for some } n \geqslant 0\} \cong \lim_{n\to\infty} \operatorname{Hom}_A\big((A_{\geqslant n})_A, A_A\big). \tag{5.9}$$

The ring $T(A)$ is again a $\mathbb{Z}$-graded Ore domain with quotient ring Q. The algebra A is called *right torsion closed* if $T(A) = A$. The *maximal left torsion extension of* A is defined analogously and written $T^{\ell}(A)$.

We also need the analogues of these definitions for bimodule algebras. Let $\mathcal{K}$ be the constant sheaf of rational functions on X, with the induced action of σ, and fix once and for all an injection $\mathcal{L} \hookrightarrow \mathcal{K}$; thereby giving inclusions $\mathcal{L}_n \subseteq \mathcal{K}$ for each n. Given a generalized naïve sequence $\{\mathcal{I}_m\}_{m\geqslant 0}$, and integers $n, m \geqslant 0$, we define $\mathcal{H}_n(m)$ to be the unique largest subsheaf $\mathcal{H} \subset \mathcal{K}$ such that $\mathcal{H}\mathcal{I}_m^{\sigma^n} \subseteq \mathcal{I}_{n+m}$. Given subsheaves $\mathcal{F}, \mathcal{G} \subseteq \mathcal{K}$, we may identify $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ with the unique largest subsheaf of $\mathcal{K}$ such that $\mathcal{H}\mathcal{F} \subseteq \mathcal{G}$; in this way, we have $\mathcal{H}_n(m) = \mathcal{H}om(\mathcal{I}_m^{\sigma^n}, \mathcal{I}_{n+m})$.

Corollary 5.10. *Let $\{\mathcal{I}_n\}_{n\geqslant 0}$ be a generalized naïve sequence and keep the notation from Lemma* 5.7.

(1) *For any $n \geqslant 0$, the $\mathcal{H}_n(m)$ are sheaves of ideals that are equal for $m \geqslant M = \max\{t, w\}$.*
(2) *For $n, m \geqslant M$, one has ${}_a\mathcal{H}_n(m) = \mathcal{A}\mathcal{B}^{\sigma^w}\mathcal{B}^{\sigma^{w+1}} \cdots \mathcal{B}^{\sigma^{n-1}}\mathcal{D}^{\sigma^n}$, where $\mathcal{A}$ and $\mathcal{B}$ are defined by Lemma* 5.7, *while $\mathcal{D}$ is independent of n and m and satisfies $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{O}_X$.*
(3) *$\{\mathcal{H}_n\}_{n\geqslant 0}$ is a generalized naïve sequence.*

Remark 5.11. Using Corollary 5.10(1), we define $\mathcal{H}_n = \mathcal{H}_n(m)$ for any $m \geqslant \max\{t, w\}$.

Proof. As with Lemma 5.7 we can replace $\{\mathcal{I}_n\}$ by $\{{}_a\mathcal{I}_n\}$ and so we can drop the subscript a.

(1) Take $n \geqslant 0$ and $m \geqslant M = \max\{w, t\}$ and identify $\mathcal{H}_n(m) = \mathcal{H}om(\mathcal{I}_m^{\sigma^n}, \mathcal{I}_{n+m})$. By Definition 5.2(3), each c_i is a smooth point on a scheme of dimension $\geqslant 2$, from which it follows that $\mathcal{H}om(\mathcal{I}_m^{\sigma^n}, \mathcal{I}_{n+m}) \subseteq \mathcal{H}om(\mathcal{I}_m^{\sigma^n}, \mathcal{O}_X) = \mathcal{O}_X$ and so $\mathcal{H}_n(m)$ is an ideal sheaf. It is automatic that $\mathcal{I}_n \subseteq \mathcal{H}_n(m)$ and so $\operatorname{coSupp}\mathcal{H}_n(m) \subseteq \operatorname{coSupp}\mathcal{I}_n \subseteq \{c_0, \dots, c_{n+w-1}\}$, whence ${}^j\mathcal{H}_n(m) = \mathcal{O}_X$ unless $0 \leqslant j \leqslant n + w - 1$. For such j (5.8) shows that ${}^j\mathcal{I}_m^{\sigma^n}$ and ${}^j\mathcal{I}_{n+m}$ are independent of the choice of $m \geqslant M$. The result follows.

(2) Throughout the proof we assume that $n, m \geqslant M$ and write $\mathcal{H}_n = \mathcal{H}_n(m)$. Thus $\mathcal{I}_n\mathcal{I}_m^{\sigma^n} = \mathcal{I}_{n+m}$ and so $\mathcal{H}_n\mathcal{I}_m^{\sigma^n} = \mathcal{I}_{n+m}$. If $0 \leqslant j \leqslant n - 1$ then ${}^j(\mathcal{I}_m^{\sigma^n}) = \mathcal{O}_X$ by definition and hence ${}^j\mathcal{H}_n = {}^j\mathcal{I}_{n+m}$. Combined with Lemma 5.7 and the fact that $\mathcal{H}_n$ is an ideal sheaf, this implies that $\mathcal{H}_n = \mathcal{A}\mathcal{B}^{\sigma^w}\mathcal{B}^{\sigma^{w+1}} \cdots \mathcal{B}^{\sigma^{n-1}}\mathcal{D}_n^{\sigma^n}$, where $\mathcal{D}_n$ is an ideal sheaf cosupported on $\{c_0, \dots, c_{w-1}\}$. Since $\mathcal{I}_n \subseteq \mathcal{H}_n$ it is clear that $\mathcal{C} \subseteq \mathcal{D}_n$. On the other hand, for $j > w$, Lemma 5.7 shows that ${}^j(\mathcal{I}_{m+n}^{\sigma}) = {}^j\mathcal{I}_{m+n+1}$ and so

$$ {}^j\big(\mathcal{H}_n^{\sigma}\big) = \mathcal{H}om\big({}^j\big(\mathcal{I}_m^{\sigma^{n+1}}\big), {}^j\big(\mathcal{I}_{n+m}^{\sigma}\big)\big) = \mathcal{H}om\big({}^j\big(\mathcal{I}_m^{\sigma^{n+1}}\big), {}^j\mathcal{I}_{n+m+1}\big) = {}^j\mathcal{H}_{n+1}. $$

Thus D_n is independent of $n \geqslant M$.

(3) It follows from the definition of $\mathcal{H}_n$ that $\mathcal{H}_n\mathcal{H}_r^{\sigma^n} \subseteq \mathcal{H}_{n+r}$ for all $n, r \geqslant 0$. For $n, r \gg 0$, equality follows easily by combining parts (1) and (2). Since $\mathcal{I}_n \subseteq \mathcal{H}_n \subseteq \mathcal{O}_X$, clearly $\operatorname{coSupp}\mathcal{H}_n$ consists of points lying on critically dense σ-orbits and so $\{\mathcal{H}_n\}_{n\geqslant 0}$ is indeed a generalized naïve sequence. □

It follows from Corollary 5.10 that the sequences $\{\mathcal{H}_n\}$ are well-behaved. Indeed we have:

Corollary 5.12. *Let $\{\mathcal{I}_n\}_{n\geqslant 0}$ be a generalized naïve sequence, set $\mathcal{S} = \mathcal{S}(X, \{\mathcal{I}_n\}, \mathcal{L}, \sigma)$ with section algebra $S = \mathrm{H}(X, \mathcal{S})$ and define $\mathcal{H}_n$ by Remark* 5.11. *Then*:

(1) *If* $\mathcal{T} = \bigoplus_{n\geqslant 0}(\mathcal{H}_n \otimes \mathcal{L}_n)_{\sigma^n}$, *then* $T = \mathrm{H}^0(X, \mathcal{T})$ *is the maximal right torsion extension* $T(S)$ *of* S.
(2) $T(S) = S$ *in high degree if and only if* $\mathcal{H}_n = \mathcal{I}_n$ *for* $n \gg 0$.
(3) T *is a finitely generated left* S*-module such that* $TS_{\geqslant n} \subseteq S$ *for some* $n \geqslant 1$.

Proof. (1) By Corollary 5.10(3), $\mathcal{T}$ is a $(\mathcal{O}_X, \sigma)$-bimodule algebra containing $\mathcal{S}$. The given embedding of $\mathcal{L}$ in $\mathcal{K}$ induces an embedding of $\mathcal{T}$ in the $(\mathcal{O}_X, \sigma)$-bimodule algebra $\widetilde{\mathcal{K}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{K}_{\sigma^n}$ and it follows that $Q(S) \subseteq Q(T) \subseteq K[t, t^{-1}; \sigma] \cong \mathrm{H}^0(X, \widetilde{\mathcal{K}})$ (although we will not need it, these inclusions are actually equalities).

Consider the maximal right torsion extension $T' = T(S) \subset Q(S)$ of S. For $n \geqslant 0$, T'_n generates a sheaf of $\mathcal{O}_X$-modules $\mathcal{T}'_n \subset \mathcal{K}$, which we write as $\mathcal{T}'_n = \mathcal{J}_n\mathcal{L}_n$ for some sheaf $\mathcal{J}_n$. The fact that $(T'/S)_S$ is torsion means that, for any given n, one has $T'_n S_m \subset S$ for all $m \gg 0$. By Corollary 5.4(1), $\mathcal{S}_m$ is generated by its global sections for all $m \gg 0$ and $\mathcal{T}'_n$ is generated by its global sections by definition. Consequently, $\mathcal{T}'_n\mathcal{S}_m \subseteq \mathcal{S}$ for $m \gg 0$; equivalently, $\mathcal{J}_n\mathcal{I}_m^{\sigma^n} \subseteq \mathcal{I}_{n+m}$ for all $m \gg 0$. Thus $\mathcal{J}_n \subseteq \mathcal{H}_n$ and $\mathcal{T}' \subseteq \mathcal{T}$, from which it follows that $T' \subseteq \mathrm{H}^0(X, \mathcal{T}') \subseteq \mathrm{H}^0(X, \mathcal{T}) = T$. Conversely, it is easy to see that $(T/S)_S$ is right torsion, and so $T \subseteq T'$.

(2) Suppose that $T_n = S_n$ for all $n \gg 0$. By Corollary 5.10(3) and Corollary 5.4(1), $\mathcal{T}_n = \mathcal{H}_n \otimes \mathcal{L}_n$ and $\mathcal{S}_n = \mathcal{I}_n \otimes \mathcal{L}_n$ are generated by their respective global sections T_n and S_n for all $n \gg 0$. It follows that $\mathcal{H}_n \otimes \mathcal{L}_n = \mathcal{I}_n \otimes \mathcal{L}_n$ and hence $\mathcal{H}_n = \mathcal{I}_n$ for all $n \gg 0$. The converse follows immediately from part (1).

(3) As in the proof of Corollary 5.10(2), for $n, r \geqslant M$ we have $\mathcal{H}_n\mathcal{I}_r^{\sigma^n} = \mathcal{I}_{r+n}$. Thus $\mathcal{T}_n\mathcal{S}_r \subseteq \mathcal{S}$, and hence $T_nS_r \subseteq S$ by taking sections. Since $(T/S)_S$ is torsion, for each $0 < m < M$ we have $T_mS_r \subseteq S$ for all $r \gg 0$. Thus we can pick a single r such that $TS_{\geqslant r} \subseteq S$. □

Next, we study the maximal right torsion extensions of Veronese rings.

Lemma 5.13. *Keep the hypotheses from Corollary* 5.12 *and set* $T = T(S)$. *Then, for* $q \geqslant 1$ *one has*

(1) *If* $T = T(S)$, *then* $T^{(q)} = T(S^{(q)})$.
(2) S *is equal to* $T = T(S)$ *in large degree if and only if* $S^{(q)}$ *is equal to* $T^{(q)}$ *in large degree.*

Proof. (1) Since $S^{(q)}$ is also a generalized naïve blowup algebra, we may apply Corollary 5.12(1) to find its maximal right torsion extension $T(S^{(q)})$. But it is clear from Corollary 5.10(2) that, for any $n \geqslant 0$,

$$\mathcal{H}_{nq} = \mathcal{H}om\big(\mathcal{I}_m^{\sigma^{nq}}, \mathcal{I}_{m+nq}\big) = \mathcal{H}om\big(\mathcal{I}_{mq}^{\sigma^{nq}}, \mathcal{I}_{mq+nq}\big) \quad \text{for all } m \gg 0.$$

Thus $T^{(q)}$ must be the maximal right torsion extension of $S^{(q)}$.

(2) Suppose that $S_r \neq T_r$ for infinitely many r. Then $\mathcal{H}_r \neq \mathcal{I}_r$ for infinitely many r and so Corollary 5.10(2) implies that $\mathcal{C} \neq \mathcal{D}$ and hence $\mathcal{H}_r \neq \mathcal{I}_r$ for all $r \gg 0$. In particular, $\mathcal{H}_{qr} \neq \mathcal{I}_{qr}$ for all $r \gg 0$ and so Corollary 5.12(2) implies that $S_u^{(q)} = S_{qu} \neq T_{qu} = T_u^{(q)}$ for all $u \gg 0$. The other direction is trivial. □

The previous results all have analogs for left torsion extensions. Indeed, let $\{\mathcal{I}_n\}_{n\geqslant 0}$ be a generalized naïve sequence, and set

$$\mathcal{H}_n^\ell = \text{the unique largest subsheaf } \mathcal{H} \text{ of } \mathcal{K} \text{ such that } \mathcal{I}_m \mathcal{H}^{\sigma^m} \subseteq \mathcal{I}_{n+m}, \\ \text{for some } m \gg 0, \tag{5.14}$$

which is again independent of the choice of $m \gg 0$. Now set $\mathcal{S} = \mathcal{S}(X, \{\mathcal{I}_n\}, \mathcal{L}, \sigma)$ with section algebra S. Then *the maximal left torsion extension* $T^\ell = T^\ell(S)$ of S may be calculated as the section algebra of $\mathcal{T}^\ell = \bigoplus_{n\geqslant 0}(\mathcal{H}_n^\ell \otimes \mathcal{L}_n)_{\sigma^n}$. These facts will be used without particular comment in the next few examples and their proof is left to the reader.

We end the section with a few more examples to illustrate what can happen in the passage from a naïve blowup algebra to its maximal right or left torsion extension and we keep the notation developed in Example 5.1. We first note a few more properties of that example:

Example 5.15. This is a continuation of Example 5.1, and we use the notation set up there. Then it is easy to see that $\mathcal{H}_n^\ell = \mathcal{I}_n$ for all $n \geqslant 1$ (use the fact that $\mathfrak{m}_{[0]}$ is a maximal sheaf of ideals of $\mathcal{O}_X$). Hence $T^\ell = R$ by Corollary 5.12. Using Theorem 6.2, below, this also implies that R satisfies χ_1 on the left but not on the right.

It is easy to modify this example so that the χ_1 condition fails on both sides:

Example 5.16. *In the notation from Example* 5.1, *let* $\mathcal{I} = \mathcal{M}_{[0]}\mathfrak{m}_{[1]}\mathcal{M}_{[2]}$. *Then* $R = R(\mathbb{P}^2, Z_{\mathcal{I}}, \mathcal{L}, \sigma)$ *does not satisfy weak* χ_1 *on either side.*

Set $\widehat{\mathcal{H}} = \mathcal{M}_{[0]}\mathfrak{m}_{[1]}\mathfrak{m}_{[2]}^2$. Then a simple computation shows that $\widehat{\mathcal{H}}\mathcal{I}^\sigma = \mathcal{M}_{[0]}\mathfrak{m}_{[1]}^3\mathfrak{m}_{[2]}^3\mathcal{M}_{[3]} = \mathcal{I}\mathcal{I}^\sigma$. Thus $\mathcal{H} \supseteq \widehat{\mathcal{H}}$ (in fact one has $\mathcal{H} = \widehat{\mathcal{H}}$). On the other hand, ${}^{n+1}\widehat{\mathcal{H}}_n = \mathfrak{m}_{[n+1]}^2 \supsetneq \mathcal{M}_{[n+1]} = {}^{n+1}\mathcal{I}_{n+1}$. By Corollary 5.12, this implies that $T(R)_n \neq R_n$ for all $n \gg 0$. By symmetry, $T^\ell(R)_n \neq R_n$ for $n \gg 0$.

In Example 5.16, it happens that passing to the maximal torsion extension on one side and then the other leads to the same ring; indeed $T(T^\ell(R)) = T^\ell(T(R)) = R(\mathbb{P}^2, Z_{\mathcal{J}}, \mathcal{L}, \sigma)$, where $\mathcal{J} = \mathfrak{m}_{[0]}^2\mathfrak{m}_{[1]}\mathfrak{m}_{[2]}^2$. However, as the next example shows, this does not always happen.

Example 5.17. *There exists a naïve blowup algebra* R *such that* $T(R)$ *and* $T^\ell(R)$ *are distinct infinite-dimensional extensions of* R, *yet both* $T(R)$ *and* $T^\ell(R)$ *are left and right torsion closed.*

In the proof we again use the notation from Example 5.1. We first seek ideals primary to (x, y) with the following properties: $P \supsetneq I \supsetneq K \subsetneq J \subsetneq Q$ and $PJ = IJ = IQ = K$, but

$$P = (K : J) = \{r \in k[x, y]: rJ \subseteq K\},$$

$J = (K : P)$, $Q = (K : I)$, and $I = (K : Q)$. An easy calculation shows that the following ideals have the desired properties,

$$J = (x^6, x^5y, xy^5, y^6) + (x, y)^7 \subset Q = J + (x^3y^3),$$
$$I = (x^4y^2, x^2y^4) + (x, y)^7 \subset P = I + (x^3y^3)$$

and

$$K = (x^{10}y^2, x^9y^3, x^8y^4, x^7y^5, x^5y^7, x^4y^8, x^3y^9, x^2y^{10}) + (x, y)^{13}.$$

(A crucial property of K is that $x^6y^6 \notin K$). As before, for $N = P, Q$, etc., write $\mathcal{N}$ for the sheaf of ideals on $\mathbb{P}^2$ that equals N locally at c_0 and is cosupported at c_0, and put $\mathcal{N}_{[i]} = \mathcal{N}^{\sigma^i}$.

Now take $R = R(\mathbb{P}^2, Z_{\mathcal{F}}, \mathcal{L}, \sigma)$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(m)$ for suitably large m and $\mathcal{F} = \mathcal{I}_{[0]}\mathcal{J}_{[1]}$. Either by direct computation or using Corollary 5.10, one shows using $Q = (K : I)$ that $T = R(\mathbb{P}^2, Z_{\mathcal{G}}, \mathcal{L}, \sigma)$ for $\mathcal{G} = \mathcal{I}_{[0]}\mathcal{Q}_{[1]}$, while a similar left-sided computation using $P = (K : J)$ gives $T^\ell = R(\mathbb{P}^2, Z_{\mathcal{H}}, \mathcal{L}, \sigma)$ for $\mathcal{H} = \mathcal{P}_{[0]}\mathcal{J}_{[1]}$. By Corollary 5.10, the fact that $J = (K : P)$ shows that $T(T^\ell(R)) = T^\ell$, while similarly $I = (K : Q)$ implies that $T^\ell(T(R)) = T$. $\square$

To end the section, we give the promised example where the sequence $\{\mathcal{H}_n\}$ arising from a naïve sequence $\{\mathcal{I}_n\}$ is not a naïve sequence. Thus one does need the theory of generalized naïve sequences.

Example 5.18. Keep the notation introduced in Example 5.1. Define the ideals $M = (x^2, y^2)$, $N = (x^3, y^3)$, $F = N + (x, y)^4$, and $G = (x^4, x^3y, xy^3, y^4)$. As usual given $P = M, N$, etc., let $\mathcal{P}$ be the sheaf of ideals with cosupport c_0 that equals P locally at c_0. Now take $\mathcal{I} = \mathcal{M}_{[0]}\mathcal{N}_{[1]}\mathfrak{m}_{[2]}$ and consider $\mathcal{R} = \mathcal{R}(\mathbb{P}^2, Z_{\mathcal{I}}, \mathcal{L}, \sigma)$ with maximal right torsion extension $\mathcal{T} = \bigoplus(\mathcal{H}_n \otimes \mathcal{L}_n)_{\sigma^n}$.

One computes that $\mathcal{M}_{[0]}\mathcal{N}_{[0]}\mathfrak{m}_{[0]} = \mathfrak{m}_{[0]}^5 = \mathcal{M}_{[0]}\mathcal{F}_{[0]}\mathfrak{m}_{[0]}$, and that this sheaf of ideals equals $\mathcal{B}$ in the notation of Lemma 5.7. With the help of Lemma 5.7 and Corollary 5.10, one then calculates that $\mathcal{H}_1 = \mathcal{M}_{[0]}\mathcal{F}_{[1]}\mathfrak{m}_{[2]}$. It follows that ${}^2(\mathcal{H}_1\mathcal{H}_1^\sigma) = \mathcal{G}_{[2]}$, whereas ${}^2\mathcal{H}_2$ is equal to $\mathfrak{m}_{[2]}^4$. Thus $\mathcal{H}_2 \neq \mathcal{H}_1\mathcal{H}_1^\sigma$.

6. The χ conditions and cohomological conditions

The hypotheses from Assumptions 3.17 remain in force in this section and we first define the χ conditions. Let A be a cg k-algebra and identify k with the factor ring $k = A/A_{\geqslant 1}$. For $n \geqslant 1$, we say that A *satisfies* χ_n *on the right* if $\dim \operatorname{Ext}^i_{\text{mod-}A}(k, M) < \infty$ for all finitely generated graded right A-modules M and all $i \leqslant n$. It is immediate that a ring satisfying χ_1 also satisfies weak χ_1.

As was shown in Section 5, the naïve blowup algebra $R = R(X, Z, \mathcal{L}, \sigma)$ needs not satisfy even weak χ_1 when one naïvely blows up more than one point and this is in marked contrast to the case of blowing up a single point, where χ_1 always holds [KRS, Theorem 1.1(8)]. In this section we continue our study of the χ conditions, showing in particular that the maximal right torsion extension of R will satisfy χ_1 on the right. On the other hand, the higher χ conditions behave the same way whether one naïvely blows up one or more than one point—they always fail. We will also want to consider the χ_1 condition at the level of individual modules for which we need another definition. Recall that $\pi : \text{Gr-}R \to \text{Qgr-}R$ is the natural morphism. For $N \in \text{gr-}R$, we say that *the condition* $\chi_1(N)$ *holds* provided that the natural map

$$N \to \lim_{n\to\infty} \operatorname{Hom}_R\big((R_{\geqslant n})_R, N_R\big) = \bigoplus_m \operatorname{Hom}_{\text{Qgr-}R}\big(\pi(R), \pi(N)[m]\big) \tag{6.1}$$

has a right bounded cokernel, as defined on p. 799. By [AZ1, Propositions 3.11(2) and 3.14(2)], R satisfies χ_1 if and only if condition $\chi_1(N)$ holds for all $N \in \text{gr-}R$.

The main result of this section is the following theorem, which also proves part (8) of Theorem 1.1.

Theorem 6.2. *Keep the hypotheses from* (3.17), *let* $R = R(X, Z, \mathcal{L}, \sigma)$ *and write* $T = T(R)$ *for the maximal right torsion extension of* R. *Then*:

(1) *Condition* $\chi_1(N)$ *holds for all Goldie torsion modules* $N \in \text{gr-}R$.
(2) χ_1 *holds for* R *on the right if and only if* $R = T$ *in large degree.*
(3) *If* R *is a nontrivial naïve blowup algebra, then* χ_2 *fails for* R *on the right*; *indeed* $\mathrm{Ext}^2_{\text{Mod-}R}(k, R)$ *is infinite-dimensional.*
(4) *If* R *is nontrivial, then* $\mathrm{H}^1(\pi(R)) = \mathrm{Ext}^1_{\text{Qgr-}R}(\pi(R), \pi(R))$ *is infinite-dimensional.*

Proof. (1) Fix $N \in \mathrm{GT}\,\text{gr-}R$. We convert (6.1) into a statement about the $\mathcal{R}$-module $\mathcal{N} = N \otimes_R \mathcal{R}$. By the equivalence of categories, Theorem 3.1, $\xi^{-1} \circ \pi_R(N) = \pi_{\mathcal{R}}(\mathcal{N})$ and $N = \mathrm{H}^0(X, \mathcal{N})$ in high degree. Thus (6.1) has a right bounded cokernel if and only if the morphism

$$\mathrm{H}^0(X, \mathcal{N}) \to \bigoplus_m \mathrm{Hom}_{\text{Qgr-}\mathcal{R}}\big(\pi(\mathcal{R}), \pi(\mathcal{N})[m]\big) \tag{6.3}$$

does. Since $N \otimes_R \mathcal{R}$ is Goldie torsion, it will suffice to show that (6.3) has right bounded cokernel for an arbitrary Goldie torsion module $\mathcal{N} \in \mathrm{GT}\,\text{gr-}\mathcal{R}$.

Write $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ and $\mathcal{R} = \bigoplus \mathcal{I}_n \otimes \mathcal{L}_\sigma^{\otimes n}$. For $n \geqslant n_0 \gg 0$, Lemma 4.1 implies that $\mathcal{F}_n = \mathcal{F}_{n+1} = \mathcal{F}$, say. Now fix some $m \geqslant n_0$ and write $\mathcal{N}[m] = \bigoplus \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n}$; thus $\mathcal{G}_n = \mathcal{G}_{n+1} = \mathcal{G} = (\mathcal{F} \otimes \mathcal{L}_m)^{\sigma^{-m}}$ for all $n \geqslant 0$, by Lemma 4.3. By Lemma 4.7(1) we have an isomorphism

$$\mathrm{Hom}_{\text{Qgr-}\mathcal{R}}\big(\pi(\mathcal{R}), \pi(\mathcal{N})[m]\big) \cong \lim_{n\to\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G}_n) = \lim_{n\to\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G}).$$

If we can show, for $m \gg 0$, that the maps in the direct limit $\lim_{n\to\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G})$ are isomorphisms for all $n \geqslant 0$, then we are done, since the zeroth term of the limit is nothing more than $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_0, \mathcal{G}_0) = \mathrm{H}^0(X, \mathcal{G}_0) \cong \mathrm{H}^0(X, \mathcal{F}_m \otimes \mathcal{L}_m) = \mathrm{H}^0(X, \mathcal{N})_m$.

Since $\mathcal{F}$ is a Goldie torsion sheaf, its support $\mathrm{Supp}\,\mathcal{F}$ is a proper closed subset of X. As Z is saturating, each point in $S = \mathrm{Supp}\,Z$ lies on a critically dense σ-orbit and so $\sigma^{-m}(S) \cap \mathrm{Supp}\,\mathcal{F} = \emptyset$ for all $m \gg 0$. Since $\mathrm{Supp}\,\mathcal{G} = \sigma^m(\mathrm{Supp}\,\mathcal{F})$, we can therefore choose $m \gg 0$ such that $\mathrm{Supp}\,\mathcal{G} \cap \sigma^{-j}(S) = \emptyset$ for all $j \geqslant 0$. Now since $\mathrm{Supp}\,\mathcal{I}_n/\mathcal{I}_{n+1} \subseteq \mathrm{Supp}\,\mathcal{O}_X/\mathcal{I}^{\sigma^n} \subseteq \sigma^{-n}(S)$, [KRS, Lemma 7.2(1)] implies that $\mathrm{Hom}(\mathcal{I}_n/\mathcal{I}_{n+1}, \mathcal{G}) = 0 = \mathrm{Ext}^1(\mathcal{I}_n/\mathcal{I}_{n+1}, \mathcal{G})$ for all $n \geqslant 0$ for such large m. This implies that the nth map of the direct limit $\lim_{n\to\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G})$ is an isomorphism for all $n \geqslant 0$ as we needed.

(2) By [AZ1, Proposition 3.14] we need to prove that $\chi_1(N)$ holds for all modules $N \in \text{gr-}R$. This condition clearly holds for N if and only if it holds for a shift $N[r]$. Since N has a filtration by shifts of R and Goldie torsion modules, it suffices to prove the condition in those two cases. When N is Goldie torsion, the result is given by part (1), so assume that $N = R$. Then $\lim_{n\to\infty} \mathrm{Hom}_R(R_{\geqslant n}, N)$ is simply the maximal right torsion extension of R, namely T. Thus in this case the condition demanded by (6.1) is precisely that T and R are equal in large degree.

(3) By [AZ1, (†), p. 274], it suffices to prove that $\dim_k \mathrm{Ext}^1_{\text{Qgr-}R}(\pi(R), \pi(R)) = \infty$. Thus (3) follows from (4).

(4) By Theorem 3.1, this is equivalent to showing that $\dim_k \mathrm{Ext}^1_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{R})) = \infty$. Write $\mathcal{B} = \bigoplus_{n\geqslant 0} \mathcal{L}_\sigma^{\otimes n}$, which we think of as a right $\mathcal{R}$-module. The long exact sequence in Hom

induced from the inclusion $\pi(\mathcal{R}) \subset \pi(\mathcal{B})$ provides the following exact sequence:

$$\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}\big(\pi(\mathcal{R}), \pi(\mathcal{B})\big) \xrightarrow{\phi_1} \operatorname{Hom}_{\text{Qgr-}\mathcal{R}}\big(\pi(\mathcal{R}), \pi(\mathcal{B}/\mathcal{R})\big) \xrightarrow{\phi_2} \operatorname{Ext}^1_{\text{Qgr-}\mathcal{R}}\big(\pi(\mathcal{R}), \pi(\mathcal{R})\big). \quad (6.4)$$

We need to understand the first two terms in (6.4). As in Example 4.6(1), we see that $\mathcal{B}$ is definable by products in all degrees $\geqslant 0$, and thus Lemma 4.7(1) implies that $\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B})) \cong \lim_{n\to\infty} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{O}_X)$. By [KRS, Lemma 7.2(2)], $\operatorname{Ext}^1(\mathcal{O}_X/\mathcal{I}_n, \mathcal{O}_X) = 0$ and so the natural map $\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \operatorname{Hom}(\mathcal{I}_n, \mathcal{O}_X)$ is an isomorphism for all $n \geqslant 0$. Thus $\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = k$.

On the other hand, $\mathcal{B}/\mathcal{R}$ will generally not be definable by products and so we cannot use Lemma 4.7(1) to calculate $\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B}/\mathcal{R}))$ directly. Instead we will examine a simpler submodule of $\mathcal{B}/\mathcal{R}$. By assumption, $Z \neq \emptyset$ and so we may choose some $c \in \operatorname{Supp} Z$ and write $c_i = \sigma^{-i}(c)$ for $i \in \mathbb{Z}$. More than one point of $\operatorname{Supp} Z$ might lie on the same σ-orbit as c, but we can choose the point $c \in \operatorname{Supp} Z$ so that $c_i \notin \operatorname{Supp} Z$ for all $i < 0$. Since $c_{n-1} \in \operatorname{Supp} \mathcal{O}_X/\mathcal{I}_n$ for $n \geqslant 1$, there exists an ideal sheaf $\mathcal{J}_n$ such that $\mathcal{J}_n/\mathcal{I}_n \cong k(c_{n-1})$. Set $\mathcal{M}_n = \sum_{1\leqslant i\leqslant n} \mathcal{J}_i \mathcal{I}_{n-i}^{\sigma^i}$, and notice that for each i we have $\mathcal{J}_i \mathcal{I}_{n-i}^{\sigma^i}/\mathcal{I}_n \cong k(c_{i-1})$, simply because $c_{i-1} \notin \operatorname{Supp} \mathcal{O}_X/\mathcal{I}_{n-i}^{\sigma^i}$. Consequently, for each n we have $\mathcal{M}_n/\mathcal{I}_n \cong \bigoplus_{i=1}^n \mathcal{J}_i \mathcal{I}_{n-i}^{\sigma^i}/\mathcal{I}_n \cong \bigoplus_{i=1}^n k(c_{i-1})$. Now for each $i \geqslant 1$ we can define a right $\mathcal{R}$-module $\mathcal{N}^{(i)} = \bigoplus_{n\geqslant i} \mathcal{J}_i \mathcal{I}_{n-i}^{\sigma^i}/\mathcal{I}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ in the obvious way, and it is clear that $\mathcal{N}^{(i)} \cong \bar{c}_{i-1}[-i]$ in the notation from before Corollary 4.11. Moreover, $\mathcal{N}^{(i)}$ is definable by products in degrees $\geqslant i$. Then $\bigoplus_{n\geqslant 1} \mathcal{M}_n/\mathcal{I}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ is a submodule of $\mathcal{B}/\mathcal{R}$ which is isomorphic to $\bigoplus_{i\geqslant 1} \mathcal{N}^{(i)}$.

Applying Lemma 4.7(1) to $\mathcal{N}^{(i)}$ shows that

$$\begin{aligned}\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}\big(\pi(\mathcal{R}), \pi\big(\mathcal{N}^{(i)}\big)\big) &= \lim_{n\to\infty} \operatorname{Hom}_{\mathcal{O}_X}\big(\mathcal{I}_n, k(c_{i-1})\big) \\ &= \lim_{n\to\infty} \operatorname{Hom}_{\mathcal{O}_{X,c_{i-1}}}\big((\mathcal{I}_n)_{c_{i-1}}, k(c_{i-1})\big).\end{aligned}$$

The final direct limit clearly stabilizes for $n \gg 0$ to something nonzero. It then follows that $\operatorname{Hom}_{\text{qgr-}\mathcal{R}}(\pi(\mathcal{R}), \bigoplus_{i\geqslant 1} \pi(\mathcal{N}^{(i)}))$ is infinite-dimensional over k. By left-exactness of Hom, we see that $\dim_k \operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B}/\mathcal{R})) = \infty$ as well. Then the map ϕ_2 in (6.4) has an infinite-dimensional cokernel, and we are done. $\square$

The characterization of χ_1 given in the theorem easily extends to the case of generalized naïve blowups, as the following corollary shows.

Corollary 6.5. *Let $S = S(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$ be a generalized naïve blowup algebra and write $T = T(S)$ for the maximal right torsion extension of S. Then*:

(1) *T is right torsion closed and satisfies right χ_1.*
(2) *S satisfies right χ_1 if and only if S equals T in large degree.*
(3) *If S is left torsion closed, then so is T. If S satisfies left χ_1 then so does T.*
(4) *If S is a nontrivial generalized naïve blowup algebra then χ_2 fails for S.*

Proof. (1) Let $x \in Q(T) = Q(S)$ be a homogeneous element with $xT_{\geqslant n} \subseteq T$ for some n. For any homogeneous element $z \in T_{\geqslant n}$, we have $xzS_{\geqslant m} \subseteq S$ for some m depending on z. By Corollary 5.4, S is a cg noetherian algebra and so it is finitely generated as an algebra, say in degrees $\leqslant d$. Set $V = (\bigoplus_{i=n}^{n+d} S_i)$. Then there exists a single m such that $(xV)S_{\geqslant m} \subseteq S$. Since $VS_{\geqslant m} \supseteq S_{\geqslant (m+n+d)}$, this implies that $xS_{\geqslant (m+n+d)} \subseteq S$. Thus $x \in T$ and T is right torsion closed. Combining Corollary 5.12(1), Corollary 5.10(3) and Lemma 5.3 shows that $T^{(q)}$ is a naïve blowup algebra for $q \gg 0$. By Lemma 5.13, $T^{(q)}$ is also right torsion closed and so Theorem 6.2(2) shows that $T^{(q)}$ satisfies right χ_1. By Corollary 5.4(3) and [AZ1, Theorem 8.3(1)] T satisfies right χ_1.

(2) If S does not equal T in large degree, then T is an infinite-dimensional right torsion extension of S and so S fails right χ_1. If S does equal T in large degree, then S satisfies right χ_1 by part (1) combined with [AZ1, Lemma 8.2(5)].

(3) Suppose that S is left torsion closed and let $x \in Q(S) = Q(T)$ be a homogeneous element such that $T_{\geqslant m}x \subseteq T$ for some $m \geqslant 0$. By Corollary 5.12(3), $T_{\geqslant m}xS_{\geqslant n} \subseteq TS_{\geqslant n} \subseteq S$ for some n, so in particular $S_{\geqslant m}xS_{\geqslant n} \subseteq S$. Since S is left torsion closed, this implies that $xS_{\geqslant n} \subseteq S$ and hence that $x \in T$. So T is also left torsion closed.

Assume that S satisfies left χ_1. Then $\dim_k T^{\ell}(S)/S < \infty$, whence $T^{\ell}(S) \subseteq T$. By [AZ1, Lemma 8.2(5)] $T^{\ell}(S)$ satisfies left χ_1 and so, replacing S by $T^{\ell}(S)$, we can assume that S is left torsion closed. Then T is left torsion closed by the last paragraph and satisfies left χ_1 by the left-hand analogue of part (2).

(4) By Lemma 5.3, some Veronese ring $S^{(p)}$ is a naïve blowup algebra and it is obviously still nontrivial. By the theorem, χ_2 fails for $S^{(p)}$ and so, by the proof of [AZ1, Proposition 8.7], it also fails for S. $\square$

Remark 6.6. The significance of the χ_1 condition is that it allows one to recover the ring R from qgr-R and to apply the results from [AZ1]. For example, suppose that $R = R(X, Z, \mathcal{L}, \sigma)$ satisfies Assumptions 3.17 and that $R = T(R)$ in large degree. Then it follows from Theorem 3.1 and [AZ1, Theorem 4.5(2)] that *R is equal in large degree to* $\bigoplus_{n\geqslant 0} \mathrm{Hom}_{\mathrm{qgr}\text{-}R}(\pi(R), \pi(R)[n])$. See [KRS, pp. 528–529] for a further discussion.

Let $R = R(X, Z, \mathcal{L}, \sigma)$ satisfy Assumptions 3.17. As Examples 5.16 and 5.17 show, R may well fail χ_1 on both sides. However Corollary 6.5 shows that we can repair this failure without greatly changing the properties of R. Specifically, apply Corollary 6.5 to R on the right to give the algebra $T = T(R)$ that satisfies right χ_1. Then apply the left-sided analogue of this construction to T, giving an algebra $U = T^{\ell}(T)$ which, by Corollary 6.5, will satisfy χ_1 on both sides. In terms of noncommutative geometry these operations are fairly innocuous. Indeed, recall that the noncommutative projective scheme proj-S for a cg k-algebra S is defined to be the pair (qgr-$S, \pi(S)$). By [SZ, Proposition 2.7], one has proj-$R \simeq$ proj-T. The same is not quite true in passing from T to U, although by mimicking the proof of [Ro2, Lemma 3.2], one can show that (qgr-$U, \pi(I_T)) \simeq$ proj-T for an appropriate module I; thus the underlying category will be the same, although the distinguished object may change.

Such phenomena occur elsewhere in noncommutative geometry; for example, by [SZ, Lemma 2.2(iii)] they occur for the idealizer ring R from [SZ, Theorem 2.3]. In fact, using the following observation, we can interpret a number of our examples as idealizer rings.

Lemma 6.7. *Suppose that the cg Ore domain S is left torsion closed in $Q(S)$ and that its maximal right torsion extension $T = T(S)$ satisfies $TS_{\geqslant n} \subseteq S$ for some n. Then S is the* idealizer $S = \{\theta \in Q(S)\colon I\theta \subseteq I\}$ *of the left ideal $I = TS_{\geqslant n}$ of T.*

Proof. Obviously $IS \subseteq I$. Conversely, suppose that $x \in Q(S)$ is homogeneous and $Ix \subseteq I$. Then $S_{\geqslant n}x \subseteq TS_{\geqslant n}x \subseteq TS_{\geqslant n}$, and so x is in the maximal left torsion extension of S, namely S. $\square$

Given a generalized naïve blowup algebra S, then $T^{\ell}(S)$ is left torsion closed by the left-hand analogue Corollary 6.5 and so, by Lemma 6.7, $T^{\ell}(S)$ is an idealizer subring of $T(T^{\ell}(S))$. For example, the ring R from Example 5.1 satisfies $T^{\ell}(R) = R$ (see Example 5.15) and so R is an idealizer ring inside $T = T(R)$; indeed, we showed that $\mathcal{T}\mathcal{R}_{\geqslant 1} \subseteq \mathcal{R}$ from which it follows that R is actually the idealizer of $I = TR_{\geqslant 1} = R_{\geqslant 1}$.

There is a curious contrast between these examples and earlier appearances of idealizer domains in noncommutative geometry in [AS,Ro2,SZ]. Those earlier examples all have the property that no Veronese ring is generated in degree 1 (see [AS, Proposition 6.6], [Ro2, Theorem 8.2(6)] and [SZ, Corollary 3.2]). In contrast, by Lemma 5.3 and Proposition 3.18, any idealizer ring S which is also a generalized naïve blowup algebra will always have some Veronese ring $S^{(q)}$ that is generated in degree 1. In particular, by replacing some such idealizer S by $S^{(q)}$ one obtains an example of an idealizer which is a cg domain generated in degree 1.

Let $S = S(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$ be a nontrivial generalized naïve blowup algebra. We end the section by studying the homological and cohomological dimensions of Qgr-$S \simeq$ Qgr-$\mathcal{S}$. Here, the *global dimension* of Qgr-$\mathcal{S}$ (or Qgr-S) is defined to be

$$\operatorname{gld}(\text{Qgr-}\mathcal{S}) = \sup\bigl\{i \bigm| \operatorname{Ext}^i_{\text{Qgr-}\mathcal{S}}(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \text{Qgr-}\mathcal{S}\bigr\}.$$

The *cohomological dimension* of Qgr-$\mathcal{S}$ (and Qgr-S) is $\operatorname{cd}(\text{Qgr-}\mathcal{S}) = \sup\{\operatorname{cd}(\mathcal{N}) \mid \mathcal{N} \in \text{Qgr-}\mathcal{S}\}$, where $\operatorname{cd}(\mathcal{N}) = \sup\{i \mid \operatorname{Ext}^i_{\text{Qgr-}\mathcal{S}}(\mathcal{S}, \mathcal{N}) \neq 0.\}$

Before stating the theorem, we need the following lemma.

Lemma 6.8.

(1) *Let S be a cg noetherian domain such that, for some $t \geqslant 0$, one has $S_nS_m = S_{n+m}$ for all $n, m \geqslant t$. Then the Veronese ring $S^{(t)}$ is noetherian and* qgr-$S \simeq$ qgr-$S^{(t)}$ *via the functor $M \mapsto M^{(t)}$.*

(2) *Let $S = S(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$ be a generalized naïve blowup algebra. Then there exists a naïve blowup algebra R such that $R = T(R)$ is generated in degree one, satisfies χ_1, and has* qgr-$R \simeq$ qgr-S.

Proof. (1) This is similar to the proof of [AS, Proposition 6.1]. The ring $S^{(t)}$ is noetherian by [AZ1, Proposition 5.10]. Thus, given $M \in$ gr-S, then $M^{(t)} \in$ gr-$S^{(t)}$ and if $N \in$ gr-$S^{(t)}$ then $N \otimes_{S^{(t)}} S \in$ gr-S. Clearly $(N \otimes S)^{(t)} = N$, so consider the kernel and cokernel of the natural map from $M^{(t)} \otimes S \to M$. Either module L is a noetherian S-module satisfying $L^{(t)} = 0$. We claim that L is bounded.

If L is not bounded, pick some $a \in L_r$ such that aS is infinite-dimensional. Pick $u \in \mathbb{N}$ with $ut > r$. For all $m \geqslant 0$ we have $aS_{mt+(ut-r)} \in L^{(t)} = 0$ whence $0 = aS_{mt+(ut-r)}S_v = aS_w$ for all $v \geqslant t$ and $w = v + (mt + ut - r)$. But all integers $w \gg 0$ can be so written, implying that $L_w = 0$ for all $w \gg 0$. Thus L is indeed bounded. It follows routinely that the maps $M \mapsto M^{(t)}$ and $N \mapsto N \otimes_{S^{(t)}} S$ define the equivalence between qgr-S and qgr-$S^{(t)}$.

(2) By Lemma 5.12 and Corollary 5.10(3), $T = T(S)$ is a generalized naïve blowup algebra and, as mentioned after Remark 6.6, qgr-$S \simeq$ qgr-T follows from [SZ, Proposition 2.7]. By Lemma 5.3 and Proposition 3.18, for some $q \gg 0$ the ring $T^{(q)}$ is a naïve blowup algebra that is generated in degree one. By part (1) and Lemma 5.4(4), qgr-$S \simeq$ qgr-$T^{(q)}$ for such q. Fi-

nally, by Lemma 5.13, $T^{(q)}$ is right torsion-closed and so Theorem 6.2 implies that $R = T^{(q)}$ satisfies χ_1. $\square$

Theorem 6.9. *Let $S = S(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$ be a nontrivial generalized naïve blowup algebra. Then one has* $\operatorname{cd}(\text{Qgr-}S) \leqslant \dim X$. *If X is smooth, then* $\operatorname{gld}(\text{Qgr-}S) \leqslant 1 + \dim X$.

Remark 6.10. This result proves Theorem 1.1(7) from the introduction.

Proof. By Corollary 5.4(4) and Lemma 6.8 we can replace S by some large Veronese ring $S^{(p)}$ and so, by Lemma 5.3, assume that $S = R(X, Z_{\mathcal{I}}, \mathcal{L}, \sigma)$ is a naïve blowup algebra.

The proof of the corresponding assertions in [KRS, Theorem 8.2 and Corollary 8.3] now go through with the following minor changes. First, the statement and proof of [KRS, Lemma 7.2] go through unchanged using the definition of $\mathcal{I}_n$ from this paper. Then one should replace, in order of their appearance, [KRS, Theorem 4.1] by Theorem 3.1; [KRS, Lemma 6.1] by Lemma 4.1; [KRS, Lemma 6.4] by Lemma 4.7(1) and Example 4.6; [KRS, Theorem 6.7] by Theorem 4.10; finally, [KRS, Lemma 6.2] by Lemma 4.2. $\square$

In fact, one can prove that $\dim X - 1 \leqslant \operatorname{cd}(\text{Qgr-}S) \leqslant \dim X$ and $\dim X \leqslant \operatorname{gld}(\text{Qgr-}S) \leqslant 1 + \dim X$ in Theorem 6.9. A detailed proof of this assertion can be found in [RS2], but we will not give it here, in part because we conjecture that the correct dimension is $\dim X$ in both cases. In the commutative case, and in contrast to Theorem 6.9, one can easily blow up a nonsingular integral scheme at a zero-dimensional subscheme and obtain a scheme that is singular (see, for example, [EH, Section IV.2.3]).

Remark 6.11. As a final application of Theorem 6.2 note that, by [YZ, Theorem 4.2], Theorem 6.2 implies that a nontrivial naïve blowup algebra R does not have a balanced dualizing complex, in the sense of Yekutieli [Ye]. By [Jg] and Theorem 6.9, it does however have a dualizing complex in the weaker sense of [Jg].

7. Generic flatness and parametrization

The hypotheses from Assumptions 3.17 will be assumed throughout this section. In this final section, we give several further results about the structure of nontrivial naïve blowup algebras $R = R(X, Z, \mathcal{L}, \sigma)$, and, more generally, for nontrivial generalized naïve blowup algebras $R = S(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$. We prove in particular that generic flatness always fails for R and that both the point modules in gr-R and their analogues in qgr-R fail to be parametrized by any scheme of locally finite type. This is in marked contrast to Corollary 4.11(1) which shows that the latter are in 1–1 correspondence with the closed points of X.

We first consider generic flatness, which is defined as follows. If M is a module over a commutative domain C, then M is *generically flat* over C if there exists $f \in C \setminus \{0\}$ such that $M[f^{-1}]$ is a flat $C[f^{-1}]$-module. If A is a cg k-algebra and C is a commutative k-algebra, set $A_C = A \otimes_k C$, regarded as a graded C-algebra.

Lemma 7.1. *Let $\mathcal{R} = \mathcal{S}(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$ and $R = S(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$ be as in Definition 5.2 and suppose that $R' \subseteq R$ is a cg subalgebra such that $\dim_k R/R' < \infty$. Then*:

(1) *There exists $n_0 \geqslant 0$ such that, for any open affine subset $U \subset X$, the $R'_{\mathcal{O}_X(U)}$-module $\mathcal{R}(U)$ is equal in degrees $\geqslant n_0$ to the submodule $1 \cdot R'_{\mathcal{O}_X(U)}$ generated by $1 \in \mathcal{R}(U)_0 = \mathcal{O}_X(U)$.*
(2) *If $R'_1 \neq 0$ then $\mathcal{R}(U) = 1 \cdot R'_{\mathcal{O}_X(U)}$ for some open affine set $U \subset X$.*

Proof. (1) As $\{\mathcal{L}_n \otimes \mathcal{I}_n\}$ is ample, there exists $n_0 \geqslant 0$ such that $\mathcal{R}_n = \mathcal{L}_n \otimes \mathcal{I}_n$ is generated by its sections R_n for all $n \geqslant n_0$. By hypothesis, $R_n = R'_n$ for all $n \gg 0$ so, after possibly increasing n_0, we can assume that $R_n = R'_n$ for all $n \geqslant n_0$, as well. Thus, for any open affine set $U \subset X$, the element $1 \in \mathcal{R}(U)_0$ generates $\mathcal{R}_n(U) = R_n \mathcal{O}_X(U) = 1 \cdot (R'_{\mathcal{O}_X(U)})_n$.

(2) In this case, for each $1 \leqslant m < n_0$ we pick $\alpha_m \in R'_m \setminus \{0\}$, and then we can find an open affine subset $U_m \subset X$ such that $\alpha_m \mathcal{O}_X(U_m) = \mathcal{L}_m(U_m) = (\mathcal{L}_m \otimes \mathcal{I}_m)(U_m)$. So, replace U by $U \cap U_1 \cap \cdots \cap U_{n_0-1}$. $\square$

We can now show that generic flatness fails for some very natural R-modules, thereby proving parts (3) and (9) of Theorem 1.1.

Theorem 7.2. *Let $\mathcal{R} = \mathcal{S}(X, \{\mathcal{I}_n\}, \mathcal{L}, \sigma)$ be a nontrivial generalized naïve blowup bimodule algebra with $R = \mathrm{H}^0(X, \mathcal{R})$. Let V be any open affine subset of X and write $C = \mathcal{O}_X(V)$ and $M = \mathcal{R}(V)$. We regard M as a right R_C-module, with R acting from the right and C from the left.*

Then M is a finitely generated right R_C-module which is not generically flat over C. It follows that R is neither strongly right noetherian nor strongly left noetherian.

Proof. The proof is similar to that of [KRS, Theorem 9.2]. By Lemma 7.1(1), M is finitely generated as a right R_C-module. Any localization $C[f^{-1}]$ of C equals $\mathcal{O}_X(U)$ for an open subset $U \subseteq V$, and so we can always replace V by U in the statement of the result. In particular, in order to prove that M is not generically flat over C, it suffices to prove that M_n is not flat for $n \gg 0$.

Consider the short exact sequence

$$0 \to M_n \to \mathcal{L}_n(V) \to (\mathcal{O}_X/\mathcal{I}_n)(V) \to 0.$$

By nontriviality and the saturation property, the final term is nonzero for $n \gg 0$. Thus, as $\mathcal{O}_X/\mathcal{I}_n$ is zero-dimensional and supported at nonsingular points of X, the C-module $(\mathcal{O}_X/\mathcal{I}_n)(V)$ has projective dimension equal to $\dim X$. Thus, for $n \gg 0$, the C-module M_n has projective dimension equal to $\dim X - 1 \geqslant 1$, as required.

The second assertion of the theorem follows from the first combined with [ASZ, Theorem 0.1]. $\square$

We next turn to the representability of functors, for which we need some notation. Let $S = \bigoplus_{n \geqslant 0} S_n$ be a cg k-algebra and write $\mathcal{P}(S, C)$ for the set of isomorphism classes of graded factors V of S_C with the property that each V_n is a flat C-module of constant rank $h(n) = 1$. Note that $\mathcal{P}(S, k)$ denotes the point modules for S, as defined in the introduction. Moreover, $\mathcal{P}(S, -)$ defines a functor from the category of commutative k-algebras to the category of sets. Following [AZ2, Section E5], we also have an analogue of point modules in qgr-S. Specifically, let $\mathcal{P}'(S, C)$ denote the set of isomorphism classes of graded factors V of S_C with the property that, for $n \gg 0$, the C-module V_n is flat of constant rank $h(n) = 1$. Then write $\mathcal{P}_{\mathrm{qgr}}(S, C)$ for the image of $\mathcal{P}'(S, C)$ in qgr-S_C.

When S is a generalized naïve blowup algebra, it is clear from Corollary 5.4(4) that $\mathcal{P}_{\mathrm{qgr}}(S, k)$ consists of simple objects in qgr-S but, as we show next, the converse is also true. Combined with Corollary 4.11, this completes the proof of Theorem 1.1(5).

Proposition 7.3. *Let $R = S(X, \{\mathcal{I}_n\}, \mathcal{L}, \sigma)$ be a generalized naïve blowup algebra. Then $\mathcal{P}_{\mathrm{qgr}}(R, k)$ is the set of isomorphism classes of simple objects in* qgr-R.

Remark 7.4. A couple of comments about the proof are in order. One would like to claim that the module $M(x)$ constructed in the proof of Corollary 4.11 is a point module, as this would essentially prove the proposition. Unfortunately this is not always true. Instead, we will use the more subtle global sections functor from [AZ1]. Unfortunately, again, this does not behave well for rings that do not satisfy χ_1, as may happen for R (see Example 5.1). So we will have to work simultaneously with R and its maximal right torsion extension $T(R)$.

Proof. As suggested in the remark, we first study modules for $T = T(R) = S(X, \{\mathcal{H}_n\}, \mathcal{L}, \sigma)$, as defined in (5.9), and its associated bimodule algebra $\mathcal{T} = \mathcal{S}(X, \{\mathcal{H}_n\}, \mathcal{L}, \sigma)$. Note that, by Corollaries 5.10(3) and 5.12, T is a generalized naïve blowup algebra so the earlier results of the paper are available to us.

We now follow the proof of [KRS, Proposition 10.7]. Explicitly, for a module $\mathcal{N} \in$ qgr-T, write

$$\Gamma_{\mathrm{AZ}}(\mathcal{N}) = \bigoplus_{m \geqslant 0} \mathrm{Hom}_{\mathrm{qgr}\text{-}T}\big(\pi_T(T), \mathcal{N}[m]\big)$$

for the image of $\mathcal{N}$ under the Artin–Zhang global section functor [AZ1]. By Corollary 6.5, T satisfies χ_1 and so [AZ1, Theorem 4.5(2)] implies that $T = \Gamma_{AZ}(\pi_T(T))$. Thus, by [AZ1, S2, p. 252 and S5, p. 253], $\Gamma_{AZ}(\mathcal{N})$ is a finitely generated T-module that is torsion-free in the sense that it has no finite-dimensional submodules.

Fix a closed point $x \in X$, and recall the notation $\bar{x} = \bigoplus_{n \geqslant 0}(k(x) \otimes \mathcal{L}_\sigma^{\otimes n})$ from before Corollary 4.11. The proof of [KRS, Lemma 6.1](2) shows that $\bar{x} \in$ gr-$\mathcal{T}$ and so, by Theorem 2.6, $\mathrm{H}^0(X, \bar{x})$ is finitely generated as an T-module. Let $\mathcal{N} = \pi_T(\mathrm{H}^0(X, \bar{x}))$, considered as an element of qgr-T, and set $N(x) = \Gamma_{\mathrm{AZ}}(\mathcal{N}) \in$ gr-T. As $\mathrm{H}^0(X, \bar{x})$ is noetherian, its maximum torsion submodule must be finite-dimensional. Thus, as T satisfies χ_1, it follows from [AZ1, (3.12.3) and Proposition 3.14] that the natural map $\mathrm{H}^0(X, \bar{x}) \to N(x)$ is an isomorphism in large degree. In particular, $\dim_k N(x)_m = 1$, for $m \gg 0$.

We next show that $N(x)_0 = \mathrm{Hom}_{\mathrm{qgr}\text{-}T}(\pi(T), \mathcal{N})$ is nonzero. By saturation, we can choose $t \geqslant 1$ such that $x \notin \bigcup_{m \geqslant 0} \mathrm{Supp}\, \mathcal{O}_X/\mathcal{H}_m^{\sigma^t}$. We may also assume that $\mathcal{H}_m \mathcal{H}_n^{\sigma^m} = \mathcal{H}_{m+n}$ for all $m, n \geqslant t$. Then any surjection of sheaves $\theta : \mathcal{H}_t \twoheadrightarrow \mathcal{H}_t/\mathcal{M} \cong k(x)$ induces a canonical surjection $\theta_m : \mathcal{H}_{t+m} = \mathcal{H}_t \mathcal{H}_m^{\sigma^t} \twoheadrightarrow \mathcal{H}_{t+m} + \mathcal{M}/\mathcal{M} \cong k(x)$, and hence a surjection of $\mathcal{O}_X$-modules $\theta_m \otimes \mathrm{Id} : \mathcal{T}_{t+m} = \mathcal{H}_{t+m} \otimes \mathcal{L}_\sigma^{t+m} \twoheadrightarrow k(x) \otimes \mathcal{L}_\sigma^{t+m}$ for all $m \geqslant t$. These are the structure maps for a surjective homomorphism $f : \mathcal{T}_{\geqslant 2t} \to \bar{x}_{\geqslant 2t}$ in gr-$\mathcal{T}$. Finally, by taking global sections and passing to qgr-T, the morphism f induces a nonzero element of $N(x)_0$.

Now consider $N(x)$ as an R-module and fix a nonzero element $a \in N(x)_0$. We claim that aR is not torsion. Indeed, otherwise $aR_{\geqslant r} = 0$ for some $r \geqslant 1$. But, Lemma 5.12(3) implies that T is a finitely generated left R-module and so $T/R_{\geqslant r}T$ is finite-dimensional as a left (and therefore right) k-module. Hence $T_{\geqslant s} \subseteq R_{\geqslant r}T$ for some s. Thus $aT_{\geqslant s} = 0$, contradicting the fact that $aT \subseteq N(x)$ is a torsion-free right T-module.

So, aR is not torsion. By Corollary 5.4(4), there exists u such that $R_m R_n = R_{m+n}$ for all $m, n \geqslant u$. It follows that $aR_n \neq 0$ for all $n \geqslant u$. (To see this, note that if $aR_n = 0$ for some $n \geqslant u$, then $0 = aR_n R_m = aR_{n+m}$ for all $m \geqslant u$, which leads to the contradiction $aR_{\geqslant(n+u)} = 0$.) Since $aR_n \subseteq N(x)_n$ and $\dim_k N(x)_n = 1$ for $n \gg 0$, it follows that $aR_n = N(x)_n$ is 1-dimensional for all $n \gg 0$. In particular, $N(x)$ is a finitely generated right R-module. Moreover, if π_R denotes the natural map gr-$R \to$ qgr-R, then $\pi_R(aR) = \pi_R(N(x)) \in \mathcal{P}_{\mathrm{qgr}}(R,k)$.

Finally, since $\mathrm{H}^0(X,\bar{x})$ and $N(x)$ are isomorphic in large degree, $\pi_R(aR)$ is also equal to $\pi_R(\mathrm{H}^0(X,\bar{x}))$. But, if we use the equivalence of categories, Theorem 2.6, to identify qgr-R with qgr-$\mathcal{R}$, then $\pi_R(\mathrm{H}^0(X,\bar{x})) = \pi_{\mathcal{R}}(\bar{x}) = \widetilde{x}$. If R is a naïve blowup algebra then, by Corollary 4.11(1), the $\widetilde{x}$ are also just the simple objects in qgr-R. In other words, the set of isomorphism classes of simple objects in qgr-R is just $\mathcal{P}_{\mathrm{qgr}}(R,k)$ as is required to prove the theorem. If R is not a naïve blowup algebra, apply Lemma 5.3 to pick $t \geqslant 1$ such that $R^{(t)}$ is one. Then Corollary 4.11(1) can be applied to show that the simple objects in qgr-$R^{(t)}$ are just the images of the closed points in X; that is the objects $\pi(\bigoplus_{n\geqslant 0} k(x) \otimes (\mathcal{L}_t)^{\otimes n}_{\sigma^t}) = \widetilde{x}^{(t)}$, in the notation of this proof. However, by Lemma 6.8(1), qgr-$R \simeq$ qgr-$R^{(t)}$ via the functor $M \mapsto M^{(t)}$. Thus, the simple objects in qgr-R are still the $\widetilde{x}$ for $x \in X$, as we needed. $\square$

Theorem 7.5. *Let $R = S(X, \{\mathcal{I}_n\}, \mathcal{L}, \sigma)$ be a nontrivial generalized naïve blowup algebra and suppose that $R' \subseteq R$ is a cg subalgebra such that* $\dim_k R/R' < \infty$.

(1) *If $R'_1 \neq 0$ then $\mathcal{P}(R',-)$ is not represented by any scheme Y of locally finite type.*
(2) *Whether $R'_1 = 0$ or not, $\mathcal{P}_{\mathrm{qgr}}(R',-)$ is not represented by any scheme Y of locally finite type.*

Remarks 7.6. (1) This proves Theorem 1.1(6).

(2) If $R'_1 = 0$, then part (1) of the theorem will fail. Indeed, in this case, given *any* commutative k-algebra C and R'_C-module M generated in degree zero, then $M_1 = 0$. In other words, there are no point modules for R' and $\mathcal{P}(R',-)$ is represented by the empty scheme.

Proof. The proof is similar to that of [KRS, Theorem 10.4 and Corollary 10.5] and, as there, the idea of the proof is that, for any open affine $U \subset X$, the module $\mathcal{R}(U)$ is "trying but failing" to be the module corresponding to the commutative ring $\mathcal{O}_X(U)$. We need to make this assertion formal.

Assume that $\mathcal{P}(-) = \mathcal{P}(R',-)$ is represented by the scheme Y of locally finite type. Pick an open affine set $U \subset X$ by Lemma 7.1(2), fix a closed point $p \in U \setminus \bigcup_{m\in\mathbb{Z}} \operatorname{Supp} \mathcal{O}_X/\mathcal{I}_1^{\sigma^m}$ and set $C = \mathcal{O}_{X,p}$. Then $(\mathcal{L}_n \otimes \mathcal{I}_n)_p \cong C$ for all n and so $\mathcal{R}_p = \mathcal{R}(U) \otimes_{\mathcal{O}_X(U)} C \cong \bigoplus_{n\geqslant 0} C$. By Lemma 7.1(2), $\mathcal{R}_p$ is generated as an R'_C-module by the element 1 in degree zero, so $\mathcal{R}_p \in \mathcal{P}(C)$. Thus there exists $\theta_p \in \mathcal{P}(C) = \mathrm{Morph}(\operatorname{Spec} C, Y)$ corresponding to $\mathcal{R}_p$.

By the definition of locally finite type [Ha, p.84], we may pick an open affine neighborhood V of $\theta_p(p)$ in Y of finite type over k. Then we get a map of algebras $\theta'_p : \mathcal{O}_Y(V) \to \mathcal{O}_{\operatorname{Spec} C}(\theta_p^{-1}(V))$. Since $\theta_p^{-1}(V)$ is an open set containing p, it is necessarily $\operatorname{Spec} C$ and so $\mathrm{Im}(\theta'_p) \subseteq C$. Since $\mathcal{O}_Y(V)$ is a finitely generated k-algebra and $\mathcal{O}_X(U)$ is a domain, $\theta'_p(\mathcal{O}_Y(V)) \subseteq \mathcal{O}_X(U')$, for some open set $U' \subseteq U$. Since it does no harm to replace U by a smaller open set containing p, we may as well assume that $U = U'$. In other words, we have extended θ_p to a map $\widetilde{\theta}_p \in \mathrm{Morph}(U,Y)$ such that $\theta_p = \widetilde{\theta}_p \circ \pi_p$, where $\pi_p : \operatorname{Spec} C \to U$ is the natural morphism.

By construction, $\widetilde{\theta}_p$ corresponds to a module $M_U \in \mathcal{P}(\mathcal{O}(U))$ with the property that $M_U \otimes_{\mathcal{O}(U)} C \cong \mathcal{R}_p$. But $\mathcal{R}(U)$ is a second finitely generated $R'_{\mathcal{O}(U)}$-module that satisfies $\mathcal{R}(U) \otimes_{\mathcal{O}(U)} C \cong \mathcal{R}_p$. This local isomorphism of R'_C-modules lifts to an isomorphism $M_W = M_U \otimes_{\mathcal{O}(U)} \mathcal{O}(W) \cong \mathcal{R}(W)$ of $R'_{\mathcal{O}(W)}$-modules, for some open affine set $W \subseteq U$. By the definition of $\mathcal{P}$, the $\mathcal{O}(W)$-module $(M_W)_n = (M_U)_n \otimes_{\mathcal{O}(U)} \mathcal{O}(W)$ is flat for all n. On the other hand, for $n \gg 0$, the proof of Theorem 7.2 implies that $(M_W)_n \cong \mathcal{R}(W)_n$ is *not* flat over $\mathcal{O}(W)$. This contradiction proves (1).

(2) To begin, assume that $R'_1 \neq 0$ and consider the proof of part (1). In the final paragraph of that proof, $M_W \in \mathcal{P}(\mathcal{O}(W))$ and so $\pi(M_W)$ certainly lies in $\mathcal{P}_{\mathrm{qgr}}(R', \mathcal{O}(W))$. In contrast, as $\mathcal{R}(W)_n$ is not flat as an $\mathcal{O}(W)$-module for any $n \gg 0$, no tail $\mathcal{R}(W)_{\geqslant n}$ of $\mathcal{R}(W)$ is a flat $\mathcal{O}(W)$-module. Hence $\pi(\mathcal{R}(W))$ cannot belong to $\mathcal{P}_{\mathrm{qgr}}(R', \mathcal{O}(W))$. Thus, the proof of part (1) also proves part (2).

If $R'_1 = 0$ then the same proof works, except that one now uses Lemma 7.1(1) in place of Lemma 7.1(2) and, for each module N that appears in the proof, one ignores the terms N_n for $0 < n < n_0$. $\square$

Remark 7.7. To end the paper we justify the comments made in Remark 1.2. So, assume that the hypotheses (and conclusions) of Theorem 1.1 hold for a naïve blowup algebra $R = R(X, Z, \mathcal{L}, \sigma)$ and let $R' \subseteq R$ be a cg subalgebra such that $\dim R/R' < \infty$. We need to prove that the conclusions of that theorem also hold for R'.

First of all, it is routine that R' is noetherian, proving part (2), while part (3) is trivial. By [SZ, Proposition 2.7], qgr-R' = qgr-R and so parts (1), (4), (7), (8) immediately hold for R'. Part (5) and hence the first part of (6) are also easy exercises. Moreover, the module $\mathcal{R}(U)$ is still a finitely generated $R' \otimes_k \mathcal{O}_X(U)$-module, so part (9) also holds for R'. Thus, it only remains to prove that the point modules for R' are not parameterizable, and this was proved directly in Theorem 7.5.

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Further reading

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